# 4. Theory of Linear Programming: First Steps

## 4.1 Equational Form

In the introductory chapter we explained how each linear program can be converted to the form

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b}.$$

But the simplex method requires a different form, which is usually called the *standard form* in the literature. In this book we introduce a less common, but more descriptive term *equational form*. It looks like this:

**Equational form of a linear program:**

$$\begin{array}{ll} \text{Maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0}. \end{array}$$

As usual, $\mathbf{x}$ is the vector of variables, $A$ is a given $m \times n$ matrix, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ are given vectors, and $\mathbf{0}$ is the zero vector, in this case with $n$ components.

The constraints are thus partly equations, and partly inequalities of a very special form $x_j \ge 0$, $j = 1, 2, \ldots, n$, called **nonnegativity constraints**. (Warning: Although we call this form equational, it contains inequalities as well, and these must not be forgotten!)

Let us emphasize that *all* variables in the equational form have to satisfy the nonnegativity constraints.

In problems encountered in practice we often have nonnegativity constraints automatically, since many quantities, such as the amount of consumed cucumber, cannot be negative.

**Transformation of an arbitrary linear program to equational form.** We illustrate such a transformation for the linear program

$$\begin{array}{ll} \text{maximize} & 3x_1 - 2x_2 \\ \text{subject to} & 2x_1 - x_2 \le 4 \\ & x_1 + 3x_2 \ge 5 \\ & x_2 \ge 0. \end{array}$$

We proceed as follows:

1. In order to convert the inequality $2x_1 - x_2 \leq 4$ to an equation, we introduce a new variable $x_3$, together with the nonnegativity constraint $x_3 \geq 0$, and we replace the considered inequality by the equation $2x_1 - x_2 + x_3 = 4$. The auxiliary variable $x_3$, which won't appear anywhere else in the transformed linear program, represents the difference between the right-hand side and the left-hand side of the inequality. Such an auxiliary variable is called a **slack variable**.
2. For the next inequality $x_1 + 3x_2 \geq 5$ we first multiply by $-1$, which reverses the direction of the inequality. Then we introduce another slack variable $x_4$ with the nonnegativity constraint $x_4 \geq 0$, and we replace the inequality by the equation $-x_1 - 3x_2 + x_4 = -5$.
3. We are not finished yet: The variable $x_1$ in the original linear program is allowed to attain both positive and negative values. We introduce two new, nonnegative, variables $y_1$ and $z_1$, $y_1 \geq 0$, $z_1 \geq 0$, and we substitute for $x_1$ the difference $y_1 - z_1$ everywhere. The variable $x_1$ itself disappears.

The resulting equational form of our linear program is

$$\begin{array}{ll}\text{maximize} & 3y_1 - 3z_1 - 2x_2 \\ \text{subject to} & 2y_1 - 2z_1 - x_2 + x_3 = 4 \\ & -y_1 + z_1 - 3x_2 + x_4 = -5 \\ & y_1 \geq 0,\ z_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.\end{array}$$

So as to comply with the conventions of the equational form in full, we should now rename the variables to $x_1, x_2, \ldots, x_5$.

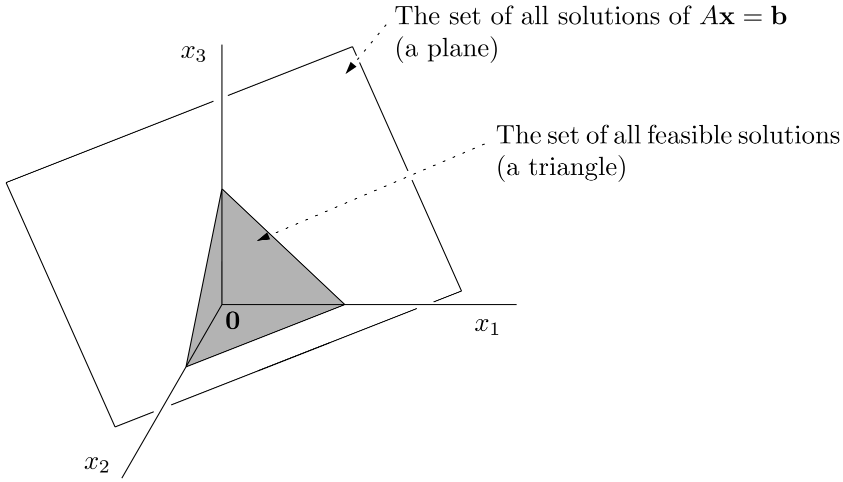
The presented procedure converts an arbitrary linear program with $n$ variables and $m$ constraints into a linear program in equational form with at most $m + 2n$ variables and $m$ equations (and, of course, nonnegativity constraints for all variables).

**Geometry of a linear program in equational form.** Let us consider a linear program in equational form:

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

As is derived in linear algebra, the set of all solutions of the system $A\mathbf{x} = \mathbf{b}$ is an affine subspace $F$ of the space $\mathbb{R}^n$. Hence the set of all feasible solutions of the linear program is the intersection of $F$ with the *nonnegative orthant*, which is the set of all points in $\mathbb{R}^n$ with all coordinates nonnegative.[1] The following picture illustrates the geometry of feasible solutions for a linear program with $n = 3$ variables and $m = 1$ equation, namely, the equation $x_1 + x_2 + x_3 = 1$:

[1] In the plane ($n = 2$) this set is called the nonnegative *quadrant*, in $\mathbb{R}^3$ it is the nonnegative *octant*, and the name *orthant* is used for an arbitrary dimension.




(In interesting cases we usually have more than 3 variables and no picture can be drawn.)

**A preliminary cleanup.** Now we will be talking about solutions of the system $A\mathbf{x} = \mathbf{b}$. By this we mean arbitrary real solutions, whose components may be positive, negative, or zero. So this is not the same as feasible solutions of the considered linear program, since a feasible solution has to satisfy $A\mathbf{x} = \mathbf{b}$ *and* have all components nonnegative.

If we change the system $A\mathbf{x} = \mathbf{b}$ by some transformation that preserves the set of solutions, such as a row operation in Gaussian elimination, it influences neither feasible solutions nor optimal solutions of the linear program. This will be amply used in the simplex method.

**Assumption:** We will consider only linear programs in equational form such that

- the system of equations $A\mathbf{x} = \mathbf{b}$ has at least one solution, and
- the rows of the matrix $A$ are linearly independent.

As an explanation of this assumption we need to recall a few facts from linear algebra. Checking whether the system $A\mathbf{x} = \mathbf{b}$ has a solution is easy by Gaussian elimination, and if there is no solution, the considered linear program has no feasible solution either, and we can thus disregard it.

If the system $A\mathbf{x} = \mathbf{b}$ has a solution and if some row of $A$ is a linear combination of the other rows, then the corresponding equation is redundant and it can be deleted from the system without changing the set of solutions. We may thus assume that the matrix $A$ has $m$ linearly independent rows and (therefore) rank $m$.