

Kähler Differentials

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This seminar has as a main purpose to get familiar with the sheaf of relative (Kähler) differentials and the de Rham complex.

1 The module of relative differentials

This section is based on section II.8 in [4] and section 26 in chapter 10 of [6]. Throughout the section all rings will be commutative and have a unit 1_A .

Definition 1.1. Let A be a ring and M an A -module, then a derivation of A into M is an additive map $d : A \rightarrow M$ such that $d(aa') = ada' + a'da$.

If A is a R -algebra for some ring R , then d is called a R -derivation if moreover $d(1_A \cdot R) = 0$.

We denote the group of all such derivations by $\text{Der}(A, M)$ and $\text{Der}_R(A, M)$ respectively.

Remark. A derivation $d : A \rightarrow M$ is not a morphism of A -modules. On the other hand, if d is an R -derivation, then d has a structure of R -module morphism.

Definition 1.2. The *module of relative differential forms* of A over R is an A -module $\Omega_{A/R}$ together with a R -derivation $d : A \rightarrow \Omega_{A/R}$ satisfying the following universal property: for any A -module M and for any R -derivation $d' : A \rightarrow M$, there exists a unique A -module morphism $f : \Omega_{A/R} \rightarrow M$ such that $d' = f \circ d$. I.e. for each A -module M there is an isomorphism:

$$\text{Hom}_A(\Omega_{A/R}, M) \xrightarrow{\cong} \text{Der}_R(A, M) : f \mapsto f \circ d$$

One can construct a module of relative differentials by taking the quotient of the free module generated by the symbols $\{da \mid a \in A\}$ by dividing out the submodule generated by all expressions of the form $d(a + a') - da - da'$, $d(aa') - ada' - a'da$ or $d(1_A \cdot r)$. The derivation is the canonical map sending a to da . The universal property implies that any construction of a module of relative differentials is isomorphic to this one, hence it makes sense to talk about *the* module of relative differentials.

Although the above construction of the module of relative differentials is the

most canonical one for rings, it does not allow the best generalization to schemes. The following result gives an alternative construction which will be the one we generalize to schemes:

Proposition 1.3. *Let A be a R -algebra, let $\Delta : A \otimes_R A \rightarrow A$ be the multiplication morphism $\Delta(a \otimes a') = aa'$ and let $I = \ker(\Delta)$. Consider $A \otimes_R A$ as an A -module via multiplication on the left. Then I/I^2 is the module of relative differentials of A over R where the derivation is given by $d : A \rightarrow I/I^2 : da = 1 \otimes a - a \otimes 1$.*

Proof. This is done in [6, p.182] by showing that the pair $(I/I^2, d)$ satisfies the universal property. We give a sketch of the proof:

1. Let $D : A \rightarrow M$ be an R -derivation. We want to show that there is a unique morphism $f : I/I^2 \rightarrow M$ such that $D = f \circ d$
2. Check that I/I^2 is generated by $\{da \mid a \in A\}$ as an A -module. (Use $x \otimes y = xy \otimes 1 - x(y \otimes 1 - 1 \otimes y)$.) Hence if f exists, it is unique.
3. For each (commutative!) ring S and S -module N , define the *trivial extension of S by N* (notation $S * N$) as the ring $S \oplus N$ where multiplication is given by

$$(s, m)(t, n) = (st, sn + tm)$$

(Note that $N^2 = 0$ if we identify N with $0 \oplus N \subset S * N$.)

4. Denote $B := A \otimes_R A$
5. Define a ring morphism $\phi : B \rightarrow A * M : a \otimes a' \mapsto (aa', aD(a'))$. We have $\phi(I^2) = \phi(I)^2 \subset M^2 = 0$, hence ϕ factors through

$$\bar{\phi} : B/I^2 \rightarrow A * M$$

6. For each element of the form $da \in I/I^2 \subset B/I^2$ we have $\bar{\phi}(da) = (0, D(a))$. Hence the restriction of $\bar{\phi}$ to I/I^2 induces an A -module morphism $f : I/I^2 \rightarrow M$ such that $D = f \circ d$. \square

We now list some results on the module of relative differentials. All proofs can be found in [6, p.182-188]

Proposition 1.4. *Let A and R' be R -algebras and let $A' := A \otimes_R R'$, then $\Omega_{A'/R'} = \Omega_{A/R} \otimes_A A'$*

Theorem 1.5 (First fundamental exact sequence). *Let A be a R -algebra and B be an A -algebra. Then there is a natural exact sequence of B -modules*

$$\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow \Omega_{B/A} \rightarrow 0$$

Proof. Let $\psi : A \rightarrow B$ be the structure morphism, then the B -module morphisms in the above sequence are given by:

$$\begin{aligned} u : \Omega_{B/R} &\rightarrow \Omega_{B/A} : d_{B/R}(b) \mapsto d_{B/A}(b) \\ v : \Omega_{A/R} \otimes_A B &\rightarrow \Omega_{B/R} : d_{A/R}(a) \otimes b \mapsto b \cdot d_{B/R}(\psi(a)) \end{aligned}$$

The only non-trivial thing to check is $\ker(u) = \text{im}(v)$. See [6, Theorem 57] for the details. \square

Theorem 1.6 (Second fundamental exact sequence). *Let A be a R -algebra, let I be an ideal of A and denote $B = A/I$. Then there is a natural exact sequence of B -modules:*

$$I/I^2 \rightarrow \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow 0$$

Proof. The morphism $v : \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}$ is the same as in Theorem 1.5 and it is surjective as $A \rightarrow B$ is surjective. For the first map note that the morphism $\delta_0 : I \rightarrow \Omega_{A/R} \otimes_A B : x \mapsto d_{A/R}(x) \otimes 1$ sends I^2 to 0 and hence factors through a morphism $\delta : I/I^2 \rightarrow \Omega_{A/R} \otimes_A B$. See [6, Theorem 58] for the proof of $\ker(v) = \text{im}(\delta)$. \square

Proposition 1.7. *If A is (a localization of) a finitely generated B -algebra, then $\Omega_{A/B}$ is a finitely generated A -module. In the special case that $A = B[x_1, \dots, x_n]$, $\Omega_{A/B}$ is the free A -module on $\{dx_1, \dots, dx_n\}$.*

Proof. Any set of (algebra-)generators $\{a_i\}_{i \in I}$ for A over B gives rise to a set of (A -module-)generators $\{da_i\}_{i \in I}$ by the fact that $d : A \rightarrow \Omega_{A/B}$ is a derivation. If A is the polynomial algebra $B[x_1, \dots, x_n]$ then we claim that the generators $\{dx_1, \dots, dx_n\}$ for $\Omega_{A/B}$ are linearly independent. For this suppose that $\sum_{i=1}^n P_i \cdot dx_i = 0$ for a collection of polynomials $\{P_1, \dots, P_n\} \subset A$. Now note for each $j \in \{1, \dots, n\}$ that $\frac{\partial}{\partial x_j} : A \rightarrow A$ is a B -derivation and by the universal property gives rise to a morphism $f_j : \Omega_{A/B} \rightarrow A$ such that $\frac{\partial}{\partial x_j} = f_j \circ d_{A/B}$. In particular $f_j(dx_i) = \delta_{ij}$ and we have $0 = f_j(0) = f_j(\sum_{i=1}^n P_i \cdot dx_i) = P_j$. \square

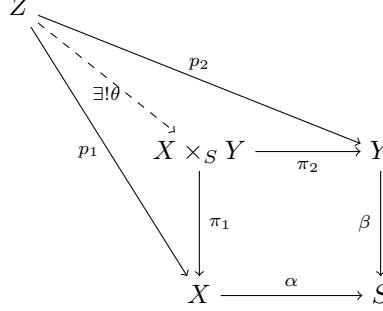
2 The sheaf of relative differentials

We first recall the notion of a fibred product of schemes:

Definition 2.1. Let X, Y, S be schemes and let $\alpha : X \rightarrow S$, $\beta : Y \rightarrow S$ be morphisms. We then define the *fibred product* $X \times_S Y$ of X and Y over S to be the pullback of the diagram

$$\begin{array}{ccc} & Y & \\ & \beta \downarrow & \\ X & \xrightarrow{\alpha} & S \end{array}$$

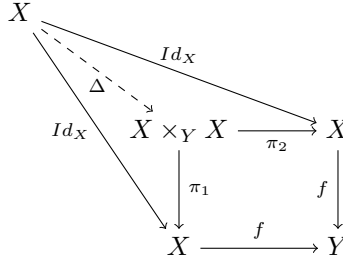
$X \times_S Y$ in the category of all schemes. In particular $X \times_S Y$ is a scheme equipped with *projection* morphisms $\pi_1 : X \times_S Y \rightarrow X$ and $\pi_2 : X \times_S Y \rightarrow Y$ such that $\alpha \circ \pi_1 = \beta \circ \pi_2$ and such that the following universal property holds:



Theorem 2.2. *Let X, Y, S be as above, then $X \times_S Y$ exists and is unique up to unique isomorphism.*

Proof. Uniqueness follows immediately from the universal property. In order to prove the existence we first note that we have existence locally: if $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and $S = \text{Spec}(C)$, then $X \times_S Y$ can be constructed as $\text{Spec}(A \otimes_C B)$. The hard part then lies in checking that this construction can be glued. See [4, Theorem II.3.3] for the details. \square

Now let X, Y be schemes and let $f : X \rightarrow Y$ be a morphism. Then we construct the diagonal morphism $\Delta : X \rightarrow X \times_Y X$ by the universal property of the fibred product:



Δ is an immersion and it is a closed immersion if and only if $f : X \rightarrow Y$ is separated (see [4, §II.4]). As our main application is in the case f is proper (and $Y = \text{Spec}(k)$), we shall always assume the latter. Under these assumptions we can consider the ideal sheaf \mathcal{I} on $X \times_Y X$ defining X , i.e.

$$\mathcal{I} = \ker(\mathcal{O}_{X \times_Y X} \rightarrow \Delta_*(\mathcal{O}_X))$$

(recall: there is a one-one correspondence between sheaves of ideals and closed subschemes [4, Proposition II.5.9])

We then define $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$. This is a quasi-coherent sheaf as \mathcal{I} is quasi-coherent ([4, Proposition II.5.9]).

Remark. As the diagonal in $X \times_Y X$ (i.e. $\text{Supp}(\Delta_* \mathcal{O}_X)$) contains $\text{Supp}(\mathcal{I}/\mathcal{I}^2)$ and $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \text{Id}_X$ one could equally well define $\Omega_{X/Y} = \pi_{1,*}(\mathcal{I}/\mathcal{I}^2)$ or $\Omega_{X/Y} = \pi_{2,*}(\mathcal{I}/\mathcal{I}^2)$.

Remark. $\Omega_{X/Y}$ can be obtained by glueing schemes of the form $\widetilde{\Omega_{A/R}}$. To see this, let $U = \text{Spec}(R)$ be an affine open subset of Y and $V = \text{Spec}(A)$ an affine open subset of X such that $f(V) \subset U$, then $V \times_U V$ is an open subset of $X \times_Y X$ isomorphic to $\text{Spec}(A \otimes_R A)$ and the restriction of $\mathcal{I}/\mathcal{I}^2$ to this affine open subset is given by $\widetilde{I/I^2}$ where I is the kernel of $A \otimes_R A \rightarrow A$. In particular $\Omega_{V/U} = \widetilde{\Omega_{A/R}}$. As $X \times_Y X$ is covered by affine open subsets of the form $V \times_U V$, one can use this data to define $\Omega_{X/Y}$. For more details, see [5, Proposition 6.1.17]. This remark allows us to lift the results on modules of relative differentials to sheaves of relative differentials. Moreover it also provides us with a morphism of $f^{-1}(\mathcal{O}_Y)$ -modules: $d_{X/Y} : \mathcal{O}_X \rightarrow \Omega_{X/Y}$ which is a Y -derivation in the sense that $d(aa') = ada' + a'da$ holds for all local sections.

Proposition 2.3. *The sheaf of relative differentials satisfies the following universal property: for each \mathcal{O}_X -module \mathcal{M} the homomorphism*

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{M}) \rightarrow \text{Der}_Y(\mathcal{O}_X, \mathcal{M}) : g \mapsto g \circ d_{X/Y}$$

is an isomorphism. In particular this universal property defines $\Omega_{X/Y}$ up to unique isomorphism.

Proof. This is a global version of Proposition 1.3 □

Proposition 2.4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{h} & Y \end{array} \quad (1)$$

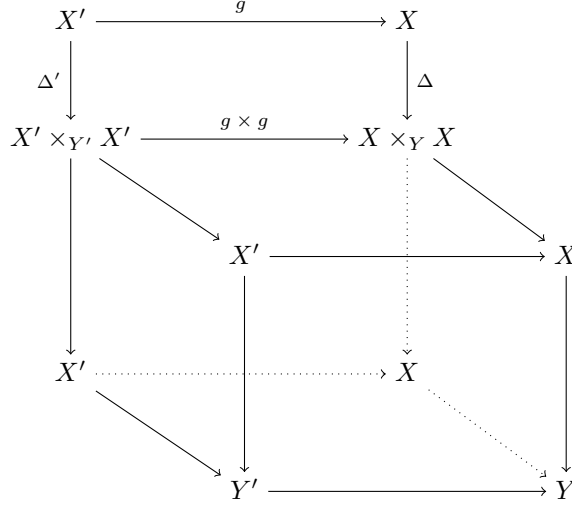
be a commutative square. Then

1. *There is a morphism*

$$g^* \Omega_{X/Y} \rightarrow \Omega_{X'/Y'} \quad (2)$$

2. *if (1) is a pullback diagram (i.e. $X' \cong Y' \times_Y X$) then (2) is an isomorphism.*

Proof. 1. We have the following commutative diagram



In this diagram the morphism $g \times g$ exists by the universal property of the fibred product $X \times_Y X$. This universal property also shows

$$\Delta \circ g = (g \times g) \circ \Delta' \quad (3)$$

All other squares in the diagram commute by construction. We now have a diagram of $\mathcal{O}_{X \times_Y X}$ -modules:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (g \times g)_* \mathcal{I}' & \longrightarrow & (g \times g)_* (\mathcal{O}_{X' \times_{Y'} X'}) & \longrightarrow & (g \times g)_* (\Delta'_* \mathcal{O}_{X'}) = \Delta_* (g_* \mathcal{O}_{X'}) \\
& & & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_{X \times_Y X} & \longrightarrow & \Delta_* \mathcal{O}_X
\end{array}$$

and this diagram commutes by (3), inducing a morphism

$$\mathcal{I} \rightarrow (g \times g)_* \mathcal{I}'$$

which by adjointness gives a morphism

$$(g \times g)^* \mathcal{I} \rightarrow \mathcal{I}'$$

and hence a morphism

$$g^* \Omega_{X/Y} = g^* (\Delta^* (\mathcal{I}/\mathcal{I}^2)) = \Delta'^* ((g \times g)^* (\mathcal{I}/\mathcal{I}^2)) \rightarrow \Delta'^* (\mathcal{I}'/\mathcal{I}'^2) = \Omega_{X'/Y'}$$

2. This is a global version of Proposition 1.4. □

Theorem 2.5 (First fundamental exact sequence). *Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} S$ be morphisms of schemes. Then there is a natural exact sequence of \mathcal{O}_X -modules*

$$f^*\Omega_{Y/S} \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0$$

Proof. The existence of the morphism is given by Proposition 2.4. Exactness of the sequence is a global version of Theorem 1.5. \square

Theorem 2.6 (Second fundamental exact sequence). *Let $X \xrightarrow{f} Y$ be a morphism of schemes and Z a closed subscheme of X with ideal sheaf \mathcal{I} , then there is a natural exact sequence of \mathcal{O}_Z -modules:*

$$\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/Y} \otimes_X \mathcal{O}_Z \rightarrow \Omega_{Z/Y} \rightarrow 0 \quad (4)$$

Proof. This is a global version of Theorem 1.6. \square

Remark. There was a slight abuse of notation in the above theorem: $\mathcal{I}/\mathcal{I}^2$ is actually a sheaf on X , but we consider its restriction to Z . As the support of $\mathcal{I}/\mathcal{I}^2$ is contained inside Z it is customary not to mention this restriction. In a situation like this, one often says that $\mathcal{I}/\mathcal{I}^2$ has a structure of \mathcal{O}_Z -module.

Definition 2.7. If $f : X \rightarrow Y$ is a morphism of schemes, we say f is locally of finite type if Y can be covered by open subsets of the form $U_i = \text{Spec}(R_i)$ such that $f^{-1}(U_i)$ is covered by open subsets $V_{ij} = \text{Spec}(A_{ij})$ where each A_{ij} is a finitely generated R_i -algebra. I.e.

$$A_{ij} = R_i[x_1, \dots, x_{n_{ij}}]/I_{ij}$$

If moreover all A_{ij} can be chosen such that the ideal I_{ij} is finitely generated, we say f is locally of finite presentation.

Remark. In case Y is a locally noetherian scheme, both notions coincide. This happens in our main application where $Y = \text{Spec}(k)$ and X is of finite type over k .

Proposition 2.8. *Let Y be locally Noetherian and $f : X \rightarrow Y$ locally of finite type, then $\Omega_{X/Y}$ is a coherent \mathcal{O}_X -module.*

Proof. As Y is locally Noetherian, so is X . Then $\Omega_{X/Y}$ is quasi-coherent this follows from Proposition 1.7. \square

Theorem 2.9. *Let k be a field and X a nonsingular variety over k of dimension n . Then:*

1. $\Omega_{X/k}$ is a locally free sheaf of rank n
2. For any subvariety $Z \subset X$ defined by a sheaf of ideals \mathcal{I} , the sequence (4) is exact on the left as well:

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/k} \otimes_X \mathcal{O}_Z \rightarrow \Omega_{Z/k} \rightarrow 0$$

Proof. 1. This reduces to a statement of local rings, see [4, Theorem II.8.15].

2. [4, Theorem II.8.17]

\square

3 The de Rham complex

Definition 3.1. We use the short hand notation:

$$\Omega_{X/Y}^i = \begin{cases} \bigwedge^i \Omega_{X/Y} & \text{if } i \geq 1 \\ \mathcal{O}_X & \text{if } i = 0 \\ 0 & \text{if } i < 0 \end{cases}$$

As an immediate corollary of Theorem 2.9 we obtain:

Proposition 3.2. *If X is a nonsingular variety of dimension n , then $\Omega_{X/k}^m = 0$ for all $m > n$.*

Proof. This is a direct application of the ever-reappearing [4, Exercise II.5.16] \square

The de Rham complex is uniquely defined by the following result:

Proposition 3.3. *Let $X \rightarrow Y$ be a morphism of schemes and $\Omega_{X/Y}^i$ be as above. Then there is a unique sequence of $f^{-1}(\mathcal{O}_Y)$ -linear maps $\{d^i : \Omega_{X/Y}^i \rightarrow \Omega_{X/Y}^{i+1}\}$ satisfying the following conditions:*

- $d^0 = d_{X/Y}$ (see Proposition 2.3)
- $d^{i+1} \circ d^i = 0$
- For all $U \subset X$ open, for all $p, q \in \mathbb{N}$ and for all $\omega \in \Gamma(U, \Omega_{X/Y}^p)$, $\omega' \in \Gamma(U, \Omega_{X/Y}^q)$ we have

$$d^{p+q}(\omega \wedge \omega') = d^p(\omega) \wedge \omega' + (-1)^p \omega \wedge d^q(\omega')$$

Proof. One first proves uniqueness: let $U \subset X$ be an affine open subset. Then each element of $\Gamma(U, \Omega_{X/Y}^i)$ is a linear combination of elements of the form $gd^0 f_1 \wedge \dots \wedge d^0 f_i$ where $g, f_1, \dots, f_i \in \Gamma(U, \mathcal{O}_X)$. Using the third condition it follows by induction that

$$d^i(gd^0 f_1 \wedge \dots \wedge d^0 f_i) = d^0 g \wedge d^0 f_1 \wedge \dots \wedge d^0 f_i$$

from which uniqueness follows by the first condition. In order to prove existence we can reduce to the affine setting $X = \text{Spec}(A)$, $Y = \text{Spec}(R)$. By [2, Proposition 14, §III.10.9] and the first condition in the statement, it suffices to construct a map $d^1 : \Omega_{A/R}^1 \rightarrow \Omega_{A/R}^2$ such that $d^1(g \cdot d_{A/R}(f)) = d_{A/R}(g) \wedge d_{A/R}(f)$ holds for all elements $f, g \in A$. This is done in [3, Theorem 16.6.2]. \square

Hence we can write the de Rham complex as the unique complex

$$\Omega_{X/Y}^\bullet := 0 \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega_{X/Y}^1 \xrightarrow{d} \Omega_{X/Y}^2 \xrightarrow{d} \dots$$

Note that if $Y = \text{Spec}(k)$ and X is a nonsingular variety, then the de Rham complex is a finite complex.

Much more information about the de Rham complex can be found in [3, §16].

4 Thickenings

This section is based on §1.1 and §2.3 in [1, Frobenius and Hodge degenerations]

Definition 4.1. We say a morphism of schemes $i : T_0 \rightarrow T$ is a thickening of order n if i is a closed immersion defined by a sheaf of ideals \mathcal{I} , such that $\mathcal{I}^{n+1} = 0$ (and $\mathcal{I}^n \neq 0$?).

Remark. If $i : T_0 \rightarrow T$ is a thickening of order n , then i factors as a sequence of thickenings of order 1:

$$T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_{n-1} \rightarrow T_n$$

where T_m is the closed subscheme of T defined by the sheaf of ideals \mathcal{I}^{m+1} . As a consequence we will only consider thickenings of order 1 in what follows.

Let $j : X \rightarrow Z$ be a closed immersion defined by a sheaf of ideals \mathcal{I} . Then j factors in a unique way as

$$X \xrightarrow{j_1} Z_1 \xrightarrow{h_1} Z$$

where Z_1 is the closed subscheme of Z defined by the sheaf of ideals $\mathcal{I}/\mathcal{I}^2$ and $j_1 : X \rightarrow Z_1$ is a thickening of order 1. X and Z_1 have the same underlying space but the structure sheaf of Z_1 remembers some infinitesimal information around X . One often calls Z_1 the *first infinitesimal neighborhood* of X in Z .

Example 4.2. Let $X = \text{Spec}(k[x])$, $Z = \text{Spec}(k[x, y])$ and $j : X \rightarrow Z$ the closed embedding defined by $k[x, y] \rightarrow k[x, y]/(y) \cong k[x]$. Then Z_1 is given by $\text{Spec}(k[x, y]/(y^2))$. Z_1 and X both have the x -axis as underlying space, but whereas the image of a polynomial $p(x, y)$ under the morphism $\mathcal{O}_Z \rightarrow j_*(\mathcal{O}_X)$ only allows us to recover $p(x, 0)$, the image under $\mathcal{O}_Z \rightarrow (j_1)_*(\mathcal{O}_{Z_1})$ allows us to recover $\frac{\partial p}{\partial y}(x, 0)$ as well.

Example 4.3. Let $f : X \rightarrow Y$ be a (separated) morphism and $\Delta : X \rightarrow X \times_Y X$ the associated closed embedding. Then we can factor Δ as

$$X \xrightarrow{j_1} Z_1 \xrightarrow{h} X \times_Y X$$

and there is an associated short exact sequence of \mathcal{O}_X -modules:

$$0 \rightarrow \Omega_{X/Y} \rightarrow (j_1)^*(\mathcal{O}_{Z_1}) \xrightarrow{h^\#} \mathcal{O}_X \rightarrow 0$$

We have $(\pi_i \circ h) \circ j_1 = \text{Id}_X$ for $i = 1, 2$. Hence the morphisms $\pi_i \circ h : Z_1 \rightarrow X$ provide sections $j_i : \mathcal{O}_X \rightarrow (j_1)^*(\mathcal{O}_{Z_1})$ ($i = 1, 2$) such that $h^\# \circ j_1 = h^\# \circ j_2 = \text{Id}_{\mathcal{O}_X}$. In particular $h^\# \circ (j_2 - j_1) = 0$ and $j_2 - j_1$ defines a morphism $\mathcal{O}_X \rightarrow \Omega_{X/Y}$. This morphism is equal to $d_{X/Y}$ as in Proposition 2.3. (see [1, (1.2.4)]).

5 Smooth, non-ramified and étale morphisms

The following notions shall be used extensively in next week's seminar:

Definition 5.1. Let $f : X \rightarrow Y$ be a morphism of schemes. We say f is *smooth*, respectively *net* = *non-ramified*, respectively *étale* if f is locally of finite type and if for each commutative diagram

$$\begin{array}{ccccc} & & & & X \\ & & & \nearrow^{g_0} & \downarrow f \\ T_0 & \xrightarrow{i} & T & \xrightarrow{\alpha} & Y \end{array}$$

where i is a thickening (of order 1), there exists, locally in the Zariski topology, at least one, respectively at most one, respectively exactly one morphism $g : T \rightarrow X$ such that all triangles commute, i.e. $g \circ i = g_0$ and $f \circ g = \alpha$.

Remark. We can reformulate the above definition using the notion of Y -morphisms and Y -schemes: if X and T are Y -schemes, i.e. there are morphisms $f : X \rightarrow Y$ and $\alpha : T \rightarrow Y$, then $g : T \rightarrow X$ is called a Y -morphism if $f \circ g = \alpha$. We denote the set of Y -morphisms from T to X as $\text{Hom}_Y(T, X)$. We hence say that f is *smooth*, respectively *non-ramified*, respectively *étale* if f is locally of finite presentation and if for all thickenings of Y -schemes $i : T_0 \rightarrow T$ the map

$$\text{Hom}_Y(T, X) \rightarrow \text{Hom}_Y(T_0, X) : g \mapsto g \circ i$$

is surjective, respectively injective, respectively bijective when restricted to Zariski open subsets in a sufficiently fine covering.

Remark. Due to glueing arguments, the definition of an étale (or non-ramified) morphism can be formulated globally: we need not restrict to Zariski open subsets in order to have existence of the unique (respectively at most one) factorization. For smooth morphisms this no longer holds, the existence of a local morphism no longer lifts to global morphisms.

Proposition 5.2. *Composition or base change of smooth, net or étale morphisms is another such morphism.*

Proof. This follows immediately from the definitions. In case of base change: use the universal property of fibred products. \square

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