تمرین: نشان دهید عنصر طول قوس $\ell(z) = \dfrac{c}{\mathrm{Im}(z)}$ تحت

دو تبدیل $-\bar{z},\ -\dfrac{1}{z} \in \mathrm{M\ddot{o}b}(\mathbb{H})$ نیز پایاست.

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① $-\dfrac{1}{z}$:

$$\mathrm{Im}\left(\frac{-1}{f}\right) = \mathrm{Im}\left(\frac{-1}{x+yi}\right) = \mathrm{Im}\left(\frac{-x+yi}{x^2+y^2}\right) = \frac{y}{x^2+y^2} \quad \Leftarrow \quad f(t) = \underbrace{x(t)}_{\mathrm{Re}(f)} + \underbrace{y(t)}_{\mathrm{Im}(f)} i$$

فرض کنید

$$\Rightarrow \quad \mathrm{length}_\rho(f) = \int_a^b \frac{c}{\mathrm{Im}(f)} |f'|\, dt = \int_a^b \frac{c}{y} |f'|\, dt$$

$$\mathrm{length}_\rho\left(\frac{-1}{f}\right) = \int_a^b \frac{c}{\mathrm{Im}\left(-\frac{1}{f}\right)} \left|\left(\frac{-1}{f}\right)'\right| dt = \int_a^b \frac{c\,(x^2+y^2)}{y} \, \frac{|f'|}{x^2+y^2}\, dt$$

$$\left|\frac{1}{f^2}\right| |f'| = \frac{|f'|}{x^2+y^2}$$

$$= \int_a^b \frac{c}{\mathrm{Im}(f)} |f'|\, dt \quad \checkmark$$

② $-\bar{z}$:

$$\mathrm{length}_\rho(-\bar{f}) = \int_a^b \frac{c}{\mathrm{Im}(-\bar{f})} |-\bar{f}'|\, dt$$

$$\mathrm{Im}(-x+yi) = y = \mathrm{Im}(f)$$

$$|-\bar{f}'| = |-(x(t)-y(t)i)'| = |-x'+y'i| = |x'+y'i| = |f'|$$

$$= \int_a^b \frac{c}{\mathrm{Im}(f)} |f'|\, dt \quad \checkmark$$