حل تمرین سری اول DS

آناهیتا حیدری

۱ - فرض کنید هزینه زمان اجرای تابع F1 به ازای ورودی i از مرتبه Θ(i^k) است. اثبات کنید هزینه زمان اجرای تابع زیر از مرتبه Θ(n^(k+1)) است. توجه. برای اثبات f∈Θ(g) یک بار اثبات کنید f∈O(g) و یک بار اثبات کنید f∈Ω(g)

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Function F2(n)
  for i = 1 to n
    F1(i)
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$\longrightarrow \text{cost} = c_i\, i^k$

$$\text{cost}[F2(n)] = \sum_{i=1}^{n} c_i\, i^k = f \quad \text{where } c_i \text{ is a constant}$$

(1) $f \in O(n^{k+1})$

اثبات $\rightarrow$ $f = \sum_{i=1}^{n} c_i\, i^k \leqslant \sum_{i=1}^{n} c_i\, n^k = n^k \sum_{i=1}^{n} c_i \leq n^k \sum_{i=1}^{n} \text{Max}\{c_i\}$

$$= n^k \cdot n \,\text{Max}\{c_i\} = n^{k+1} c' \in O(n^{k+1})$$

(2) $f \in \Omega(n^{k+1})$

اثبات $\rightarrow$ $f = \sum_{i=1}^{n} c_i\, i^k \geqslant \sum_{i=1}^{n} \min\{c_i\}\, i^k = \min\{c_i\} \sum_{i=1}^{n} i^k$

لم (نامساوی توان‌ها) به برای $x_i \in \mathbb{R}^+$ و $k \in \mathbb{N}$ داریم:

$$\frac{\sum_{i=1}^{n} x_i^k}{n} \geqslant \left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^k$$

$$\Rightarrow \min\{c_i\} \sum_{i=1}^{n} i^k \geqslant \min\{c_i\} \cdot n \cdot \left[\frac{\sum_{i=1}^{n} i}{n}\right]^k = \min\{c_i\} \cdot n \cdot \left(\frac{n \cdot (n+1)}{2n}\right)^k$$

$$= \min\{c_i\} \cdot n \cdot \left[\frac{n+1}{2}\right]^k \geqslant \frac{\min\{c_i\}}{2^k} \cdot n \cdot n^k = C' n^{k+1}$$

$$\in \Omega(n^{k+1})$$

$(1), (2) \Rightarrow f \in \Theta(n^{k+1})$

۲- مشابه مسئله ۱، اما این بار فرض کنید زمان اجرای تابع F1 به ازای ورودی i از مرتبه $\Theta(\log i)$ است و اثبات کنید زمان اجرای F2 از مرتبه $\Theta(n \log n)$ است.

$$\text{cost}[F2(n)] = \sum_{i=1}^{n} c_i \log i = f$$

(1) $f \in O(n \log n)$

اثبات $f = \sum_{i=1}^{n} c_i \log i \leqslant \sum_{i=1}^{n} \max\{c_i\} \log n = n \cdot \max\{c_i\} \log n$

$$= c' n \log n \in O(n \log n)$$

(2) $f \in \Omega(n \log n)$

اثبات $\Rightarrow$ $f = \sum_{i=1}^{n} c_i \log i \geqslant \sum_{i=1}^{n} \min\{c_i\} \log i = \min\{c_i\} \sum_{i=1}^{n} \log i$

$$\geqslant \min\{c_i\} \sum_{i=\frac{n}{2}}^{n} \log i \geqslant \min\{c_i\} \sum_{i=\frac{n}{2}}^{n} \log \frac{n}{2}$$

$$\geqslant \min\{c_i\} \cdot \frac{n}{2} \cdot \log \frac{n}{2} = c' n \left[\log n - \log 2 \right]$$

$$\in \Omega(n \log n)$$

$$* \left[\sum_{i=1}^{n} \log i = \log(n!) \Longrightarrow \log(n!) \in \Theta(n \log n) \right]$$

$(1), (2) \Rightarrow f \in \Theta(n \log n)$

3-2 *Relative asymptotic growths*

Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	✓	✓	✗	✗	✗
b.	n^k	c^n	✓	✓	✗	✗	✗
c.	$\sqrt{n}$	$n^{\sin n}$	✗	✗	✗	✗	✗
d.	2^n	$2^{n/2}$	✗	✗	✓	✓	✗
e.	$n^{\lg c}$	$c^{\lg n}$	✓	✗	✓	✗	✓
f.	$\lg(n!)$	$\lg(n^n)$	✓	✗	✓	✗	✓

c. $n^{\sin n}$ حالت تناوبی محدود بین n و $\frac{1}{n}$ دارد

d. $\lim_{n\to\infty} \frac{\sqrt{2}^{\,n}}{2^n} = \lim_{n\to\infty} \frac{\sqrt{2}^{\,n} \ln\sqrt{2}}{2^n \ln 2} = \frac{1}{2} \lim_{n\to\infty} \frac{2^{\frac{n}{2}}}{2^n} \Rightarrow = 0$

e. $\lg n^{\lg c} \stackrel{?}{=} \lg c^{\lg n} \iff \lg c \lg n \stackrel{?}{=} \lg n \lg c$ ✓

f. اثبات در قسمت قبل

3-3 Ordering by asymptotic growth rates

a. Rank the following functions by order of growth; that is, find an arrangement $g_1, g_2, \ldots, g_{30}$ of the functions satisfying $g_1 = \Omega(g_2)$, $g_2 = \Omega(g_3)$, ..., $g_{29} = \Omega(g_{30})$. Partition your list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$.

$\Theta(1)$: $n^{1/\lg n}$

$f = n^{1/\lg n} \iff \lg f = \frac{1}{\lg n} \lg n = 1 \iff f = 2$

$\Theta(\lg(\lg^* n))$

$\lim_{n\to\infty} \frac{\lg \lg^* n}{\lg^* n} = \lim_{a\to\infty} \frac{\lg a}{a} = 0$

$\Theta(\lg^* n)$: $\lg^*(\lg n)$

$\lim_{n\to\infty} \frac{\lg^* n}{2^{\lg^* n}} = \lim_{a\to\infty} \frac{a}{2^a} = 0$

$\lg^*(\lg n) = \min\{i \geq 0 : \lg^i \lg n \leq 1\} = \min\{i \geq 0 : \lg^{i+1} n \leq 1\}$

$= \min\{j-1 \geq 0 : \lg^j n \leq 1\} = \lg^* n - 1$

$\Theta(2^{\lg^* n})$

$2^{\lg^* n} \;?\; \lg^{(2)} n \rightarrow \lg^* n \;?\; \lg^{(3)} n \rightarrow \lg^* n < \lg^{(3)} n$

$\Theta(\lg^{(2)} n)$: $\ln \ln n$

$\lim_{n\to\infty} \frac{\lg^{(2)} n}{\sqrt{\lg n}} = \lim_{a\to\infty} \frac{\lg a}{\sqrt{a}} = 0$

$\Theta(\sqrt{\lg n})$ $\qquad$ $\lim\limits_{n\to\infty} \dfrac{\sqrt{\lg n}}{\lg n} = 0$

$\Theta(\lg n)$: $\ln n$ $\qquad$ $\lim\limits_{n\to\infty} \dfrac{\lg n}{\lg^2 n} = \lim\limits_{n\to\infty} \dfrac{1}{\lg n} = 0$

$\Theta(\lg^2 n)$ $\qquad$ $\lim\limits_{n\to\infty} \dfrac{\lg^2 n}{2^{\sqrt{2\lg n}}} = \lim\limits_{a\to\infty} \dfrac{a^2}{2^{\sqrt{2a}}} = 0$

$\Theta(2^{\sqrt{2\lg n}})$ $\qquad$ $2^{\sqrt{2\lg n}} \ ? \ \sqrt{n} \to \sqrt{2\lg n} \ ? \ \frac{1}{۲}\lg n \to \sqrt{۲}\, a < \frac{1}{۲} a^2$

$\Theta(\sqrt{n})$: $\sqrt{2}^{\lg n}$ $\qquad$ $\sqrt{2}^{\lg n} = 2^{\frac{1}{2}\lg n} = 2^{\lg n^{\frac{1}{2}}} = \sqrt{n}$

$\Theta(n)$: $2^{\lg n}$

$\Theta(n\lg n)$: $\lg(n!)$ $\qquad \longrightarrow \qquad$ اثبات در سوال اول

$\Theta(n^2)$: $4^{\lg n}$ $\qquad$ $4^{\lg n} = 2^{2\lg n} = n^2$

$\Theta(n^3)$ $\qquad$ $\lim\limits_{n\to\infty} \dfrac{n^3}{(\lg n)!} = \dfrac{2^{3\lg n}}{2^{\lg((\lg n)!)}} = \dfrac{2^{3\lg n}}{2^{\lg n \lg^{(2)} n}} = 0$

$\Theta((\lg n)!)$ $\qquad$ $\lim\limits_{n\to\infty} \dfrac{(\lg n)!}{\lg n^{\lg n}} \leq \lim\limits_{n\to\infty} \dfrac{\left(\frac{\lg n}{2}\right)^{\frac{\lg n}{2}} \cdot \lg n^{\frac{\lg n}{2}}}{\lg n^{\lg n}} = \dfrac{\left(\frac{\lg n}{2}\right)^{\frac{\lg n}{2}}}{\lg n^{\frac{\lg n}{2}}}$

$= \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^{\frac{\lg n}{2}} = 0$

$\Theta((\lg n)^{\lg n})$: $n^{\lg \lg n}$

$n^{\lg \lg n} \ ? \ \left(\frac{3}{2}\right)^n \rightarrow (\lg \lg n) \cdot \lg n \ ? \ n \lg\left(\frac{3}{2}\right)$

$< \lg n \cdot \lg n \ ? \ n \lg\left(\frac{3}{2}\right)$

$< n^{\varepsilon} \ ? \ n \lg\left(\frac{3}{2}\right) \Rightarrow n^{\lg \lg n} \in o\left(\frac{3}{2}\right)^n$

$\Theta\left(\left(\frac{3}{2}\right)^n\right)$

$\Theta(2^n)$

$\Theta(n \cdot 2^n)$

$$\lim \frac{n \cdot 2^n}{e^n} = \lim \frac{n \cdot 2^n}{\left(\frac{e}{2}\right)^n \cdot 2^n} = \lim \frac{n}{\left(\frac{e}{2}\right)^n} = 0$$

$\Theta(e^n)$

$$\lim \frac{e^n}{e^{\ln n!}} = \lim \frac{e^n}{e^{n \ln n}} = 0$$

$\Theta(n!)$

$\Theta((n+1)!)$

$$\lim \frac{(n+1)!}{2^{2^n}} = \frac{2^{\lg (n+1)!}}{2^{2^n}} = \frac{2^{(n+1)\lg(n+1)}}{2^{2^n}} = 0$$

$\Theta(2^{2^n})$

$$\lim_{n \to \infty} \frac{2^{2^n}}{2^{2^{n+1}}} = \lim_{a \to \infty} \frac{a}{a^2} = 0$$

$\Theta(2^{2^{n+1}})$

b. Give an example of a single nonnegative function $f(n)$ such that for all functions $g_i(n)$ in part (a), $f(n)$ is neither $O(g_i(n))$ nor $\Omega(g_i(n))$.

$$f(n) = |\sin n| \cdot 2^{2^{2^n}}$$