GRAPH THEORY WITH APPLICATIONS

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1 Graphs and Subgraphs

Theorem 1.1

$$\sum_{v \in V} d(v) = 2\varepsilon$$

Corollary 1.1 In any graph, the number of vertices of odd degree is even.

Theorem 1.2 A graph is bipartite if and only if it contains no odd cycle.

2 Trees

Theorem 2.1 In a tree, any two vertices are connected by a unique path.

The converse of this theorem holds for graphs without loops

Theorem 2.2 If G is a tree, then $\varepsilon = \nu - 1$.

Corollary 2.2 Every nontrivial tree has at least two vertices of degree one.

Theorem 2.3 An edge e of G is a cut edge of G if and only if e is contained in no cycle of G.

Theorem 2.4 A connected graph is a tree if and only if every edge is a cut edge.

Corollary 2.4.1 Every connected graph contains a spanning tree.

Corollary 2.4.2 If G is connected, then $\varepsilon \geq \nu - 1$.

Theorem 2.5 Let T be a spanning tree of a connected graph G and let e be an edge of G not in T. Then $T + e$ contains a unique cycle.

Theorem 2.6 Let T be a spanning tree of a connected graph G, and let e be any edge of T. Then

(i) the cotree $\bar{T}$ contains no bond of G;
(ii) $\bar{T} + e$ contains a unique bond of G.

Theorem 2.7 A vertex v of a tree G is a cut vertex of G if and only if $d(v) > 1$.

Corollary 2.7 Every nontrivial loopless connected graph has at least two vertices that are not cut vertices.

Theorem 2.8 If e is a link of G, then $\tau(G) = \tau(G - e) + \tau(G \cdot e)$.

2.2.1. Algorithm. (Prüfer code) Production of $f(T) = (a_1, \ldots, a_{n-2})$.
Input: A tree T with vertex set $S \subseteq \mathbb{N}$.
Iteration: At the ith step, delete the least remaining leaf, and let a_i be the *neighbor* of this leaf. ■

Theorem 2.9 $\tau(K_n) = n^{n-2}$. (Cayley's Formula [1889]).

2.2.4. Corollary. Given positive integers $d_1, \ldots, d_n$ summing to $2n-2$, there are exactly $\frac{(n-2)!}{\prod (d_i - 1)!}$ trees with vertex set $[n]$ such that vertex i has degree d_i, for each i.

Theorem 2.10 Any spanning tree $T^* = G[\{e_1, e_2, \ldots, e_{\nu-1}\}]$ constructed by Kruskal's algorithm is an optimal tree.

3 Connectivity

Theorem 3.1 $\kappa \leq \kappa' \leq \delta$.

Theorem 3.2 A graph G with $\nu \geq 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths.

Corollary 3.2.1 If G is 2-connected, then any two vertices of G lie on a common cycle.

Corollary 3.2.2 If G is a block with $\nu \geq 3$, then any two edges of G lie on a common cycle.

Theorem 3.2 has a generalisation to k-connected graphs, known as *Menger's theorem*: a graph G with $\nu \geq k+1$ is k-connected if and only if any two distinct vertices of G are connected by at least k internally-disjoint paths. There is also an edge analogue of this theorem: a graph G is k-edge-connected if and only if any two distinct vertices of G are connected by at least k edge-disjoint paths.

4 Euler Tours and Hamilton Cycles

Theorem 4.1 A nonempty connected graph is eulerian if and only if it has no vertices of odd degree.

Corollary 4.1 A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

Theorem 4.2 If G is hamiltonian then, for every nonempty proper subset S of V

$$\omega(G-S) \leq |S| \tag{4.1}$$

Theorem 4.3 If G is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, then G is hamiltonian.

Lemma 4.4.1 Let G be a simple graph and let u and v be nonadjacent vertices in G such that

$$d(u) + d(v) \geq \nu \tag{4.5}$$

Then G is hamiltonian if and only if $G + uv$ is hamiltonian.

Lemma 4.4.2 $c(G)$ is well defined.

Theorem 4.4 A simple graph is hamiltonian if and only if its closure is hamiltonian.

Corollary 4.4 Let G be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then G is hamiltonian.

Theorem 4.5 Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of m less than $\nu/2$ for which $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Then G is hamiltonian.

Theorem 4.6 (Chvátal, 1972) If G is a nonhamiltonian simple graph with $\nu \geq 3$, then G is degree-majorised by some $C_{m,\nu}$.

Corollary 4.6 If G is a simple graph with $\nu \geq 3$ and $\varepsilon > \binom{\nu-1}{2} + 1$, then G is hamiltonian. Moreover, the only nonhamiltonian simple graphs with ν vertices and $\binom{\nu-1}{2} + 1$ edges are $C_{1,\nu}$ and, for $\nu = 5$, $C_{2,5}$.

5 Matchings

Theorem 5.1 (Berge, 1957) A matching M in G is a maximum matching if and only if G contains no M-augmenting path.

Theorem 5.2 Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if

$$|N(S)| \geq |S| \quad \text{for all} \quad S \subseteq X \tag{5.2}$$

Corollary 5.2 If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

marriage theorem

Lemma 5.3 Let M be a matching and K be a covering such that $|M|=|K|$. Then M is a maximum matching and K is a minimum covering.

Theorem 5.3 In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Theorem 5.4 G has a perfect matching if and only if

$$o(G-S)\leq|S| \quad \text{for all} \quad S\subset V \tag{5.6}$$

Corollary 5.4 Every 3-regular graph without cut edges has a perfect matching.

6 Edge Colourings

Lemma 6.1.1 Let G be a connected graph that is not an odd cycle. Then

G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two.

Lemma 6.1.2 Let $\mathscr{C}=(E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Theorem 6.1 If G is bipartite, then $\chi'=\Delta$.

Theorem 6.2 If G is simple, then either $\chi'=\Delta$ or $\chi'=\Delta+1$. Vizing (1964)

Lemma 6.3 Let M and N be disjoint matchings of G with $|M|>|N|$. Then there are disjoint matchings M' and N' of G such that $|M'|=|M|-1$, $|N'|=|N|+1$ and $M'\cup N'=M\cup N$.

Theorem 6.3 If G is bipartite, and if $p\geq\Delta$, then there exist p disjoint matchings $M_1, M_2, \ldots, M_p$ of G such that

$$E=M_1\cup M_2\cup\ldots\cup M_p \tag{6.4}$$

and, for $1\leq i\leq p$

$$[\varepsilon/p]\leq|M_i|\leq\{\varepsilon/p\} \tag{6.5}$$

(Note: condition (6.5) says that any two matchings M_i and M_j differ in size by at most one.)

7 Independent Sets and Cliques

Theorem 7.1 A set $S \subseteq V$ is an independent set of G if and only if $V \backslash S$ is a covering of G.

Corollary 7.1 $\alpha + \beta = \nu$.

Theorem 7.2 (Gallai, 1959) If $\delta > 0$, then $\alpha' + \beta' = \nu$.

Theorem 7.3 In a bipartite graph G with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

Theorem 7.4 For any two integers $k \geq 2$ and $l \geq 2$

$$r(k, l) \leq r(k, l-1) + r(k-1, l) \tag{7.7}$$

Theorem 7.5 $r(k, l) \leq \binom{k+l-2}{k-1}$

Theorem 7.6 (Erdös, 1947) $r(k, k) \geq 2^{k/2}$

Corollary 7.6 If $m = \min\{k, l\}$, then $r(k, l) \geq 2^{m/2}$

Theorem 7.7 $r(k_1, k_2, \ldots, k_m) \leq r(k_1 - 1, k_2, \ldots, k_m) + r(k_1, k_2 - 1, \ldots, k_m) + \ldots + r(k_1, k_2, \ldots, k_m - 1) - m + 2$

Corollary 7.7 $r(k_1 + 1, k_2 + 1, \ldots, k_m + 1) \leq \dfrac{(k_1 + k_2 + \ldots + k_m)!}{k_1!\, k_2! \ldots k_m!}$

Theorem 7.8 If a simple graph G contains no K_{m+1}, then G is degree-majorised by some complete m-partite graph H. Moreover, if G has the same degree sequence as H, then $G \cong H$.

Theorem 7.9 If G is simple and contains no K_{m+1}, then $\varepsilon(G) \leq \varepsilon(T_{m,\nu})$. Moreover, $\varepsilon(G) = \varepsilon(T_{m,\nu})$ only if $G \cong T_{m,\nu}$.

Theorem 7.10 Let $(S_1, S_2, \ldots, S_n)$ be any partition of the set of integers $\{1, 2, \ldots, r_n\}$. Then, for some i, S_i contains three integers x, y and z satisfying the equation $x + y = z$.

Theorem 7.11 If $\{x_1, x_2, \ldots, x_n\}$ is a set of diameter 1 in the plane, the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $[n^2/3]$. Moreover, for each n, there is a set $\{x_1, x_2, \ldots, x_n\}$ of diameter 1 with exactly $[n^2/3]$ pairs of points at distance greater than $1/\sqrt{2}$.

8 Vertex Colourings

Theorem 8.1 If G is k-critical, then $\delta \geq k-1$.

Corollary 8.1.1 Every k-chromatic graph has at least k vertices of degree at least $k-1$.

Corollary 8.1.2 For any graph G,

$$\chi \leq \Delta + 1$$

Theorem 8.2 In a critical graph, no vertex cut is a clique.

Corollary 8.2 Every critical graph is a block.

Theorem 8.3 (Dirac, 1953) Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

(i) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of type i $(i = 1, 2)$, and

(ii) both $G_1 + uv$ and $G_2 \cdot uv$ are k-critical (where $G_2 \cdot uv$ denotes the graph obtained from G_2 by identifying u and v).,

Corollary 8.3 Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

$$d(u) + d(v) \geq 3k - 5 \tag{8.1}$$

Theorem 8.4 If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Theorem 8.5 If G is 4-chromatic, then G contains a subdivision of K_4.

Theorem 8.6 If G is simple, then $\pi_k(G) = \pi_k(G-e) - \pi_k(G \cdot e)$ for any edge e of G.

Corollary 8.6 For any graph G, $\pi_k(G)$ is a polynomial in k of degree ν, with integer coefficients, leading term k^ν and constant term zero. Furthermore, the coefficients of $\pi_k(G)$ alternate in sign.

Theorem 8.7 For any positive integer k, there exists a k-chromatic graph containing no triangle.

9 Planar Graphs

Theorem 9.1 K_5 is nonplanar.

Theorem 9.2 A graph G is embeddable in the plane if and only if it is embeddable on the sphere.

Theorem 9.3 Let v be a vertex of a planar graph G. Then G can be embedded in the plane in such a way that v is on the exterior face of the embedding.

Theorem 9.4 If G is a plane graph, then

$$\sum_{f \in F} d(f) = 2\varepsilon$$

Theorem 9.5 If G is a connected plane graph, then

$$\nu - \varepsilon + \phi = 2$$

Corollary 9.5.1 All planar embeddings of a given connected planar graph have the same number of faces.

Corollary 9.5.2 If G is a simple planar graph with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.

Corollary 9.5.3 If G is a simple planar graph, then $\delta \leq 5$.

Corollary 9.5.4 K_5 is nonplanar.

Corollary 9.5.5 $K_{3,3}$ is nonplanar.

Theorem 9.6 If two bridges overlap, then either they are skew or else they are equivalent 3-bridges.

Theorem 9.7 If a bridge B has three vertices of attachment v_1, v_2 and v_3, then there exists a vertex v_0 in $V(B)\backslash V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$, P_i and P_j have only the vertex v_0 in common (see figure 9.10).

Theorem 9.8 Inner (outer) bridges avoid one another.

Theorem 9.9 **An inner bridge that avoids every outer bridge is transferable.**

Lemma 9.10.1 **If G is nonplanar, then every subdivision of G is nonplanar.**

Lemma 9.10.2 **If G is planar, then every subgraph of G is planar.**

Lemma 9.10.3 If G is nonplanar, then at least one of H_1 and H_2 is also nonplanar.

Lemma 9.10.4 Let G be a nonplanar connected graph that contains no subdivision of K_5 or $K_{3,3}$ and has as few edges as possible. Then G is simple and 3-connected.

Theorem 9.10 A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 9.11 **Every planar graph is 5-vertex-colourable.**

Theorem 9.12 The following three statements are equivalent:

(i) every planar graph is 4-vertex-colourable;
(ii) every plane graph is 4-face-colourable;
(iii) every simple 2-edge-connected 3-regular planar graph is 3-edge-colourable.

Theorem 9.13 **Let G be a loopless plane graph with a Hamilton cycle C. Then**

$$\sum_{i=1}^{\nu} (i-2)(\phi_i' - \phi_i'') = 0 \tag{9.3}$$

Theorem 9.14 **If $\tilde{H}$ is G-admissible then, for every bridge B of H, $F(B, \tilde{H}) \neq \emptyset$.**