مثال: صورت مختلط سری فوریه تابع

$$y = e^{x}, \quad -\pi < x < \pi, \quad p = 2\pi$$

را یافته و به کمک آن صورت حقیقی سری فوریه تابع فوق
را بدست آورید.

Complex Fourier:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x} e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-ik)}\,dx$$

$$= \frac{1}{2\pi(1-ik)}\, e^{x(1-ik)}\Big]_{-\pi}^{\pi} = \frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2\pi(1-ik)}$$

$$* \; e^{ik\pi} = \cos(k\pi) + \cancel{i\sin(k\pi)} = (-1)^k$$

$$\Rightarrow \quad c_k = \frac{(-1)^k}{2\pi(1-ik)}\left(e^{\pi} - e^{-\pi}\right) = \frac{(-1)^k\left(e^{\pi} - e^{-\pi}\right)(1+ik)}{2\pi\,(1+k^2)}$$

$$\Rightarrow \quad f(x) = \boxed{\frac{\left(e^{\pi} - e^{-\pi}\right)}{2\pi}\sum_{-\infty}^{\infty}\frac{(-1)^k(1+ik)}{1+k^2}\, e^{ikx}} \qquad -\pi < x < \pi$$

$$(1+ik)e^{ikx} + (1-ik)\,e^{-ikx} =$$

$$\cos kx + \cancel{i\sin kx} + \cancel{ik\cos kx} - k\sin kx + \cos kx - \cancel{i\sin kx} - \cancel{ik\cos kx} - k\sin kx$$

$$= 2\,(\cos kx - k\sin kx)$$

$$* \ \frac{(-1)^k(1+ik)}{1+k^2}\,e^{ikx} + \frac{(-1)^k\,(1-ik)}{1+k^2}\,e^{-ikx} = \frac{2(-1)^k}{1+k^2}\,(\cos kx - k\sin kx)$$

$$\Rightarrow \quad f(x) = \boxed{\frac{e^{\pi}-e^{-\pi}}{2\pi} + \frac{e^{\pi}-e^{-\pi}}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}\,(\cos kx - k\sin kx)}$$

$$-\pi < x < \pi$$

Real Fourier :

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t\cos(kt)\,dt = \frac{1}{\pi}\,\frac{e^t}{1+k^2}(\cos kt + k\sin kt)\Big]_{-\pi}^{\pi}$$

$$\hookrightarrow \int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\cos(bx) + b\sin(bx)) + C$$

$$= \frac{1}{\pi}\,\frac{(e^{\pi}-e^{-\pi})}{1+k^2}\,(-1)^k$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^t\sin(kt)\,dt = \frac{1}{\pi}\,\frac{e^t}{1+k^2}(\sin kt - k\cos kt)\Big]_{-\pi}^{\pi}$$

$$\hookrightarrow \int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\sin(bx) - b\cos(bx)) + C$$

$$= \frac{1}{\pi}\,\frac{(e^{\pi} - e^{-\pi})}{1+k^2}\left(-k(-1)^k\right)$$

$$\Rightarrow \quad f(x) = \boxed{\frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{1}^{\infty} \frac{(-1)^k (e^{\pi} - e^{-\pi})}{\pi(1+k^2)}\left(\cos kx - k \sin kx\right)}$$

$$-\pi < x < \pi$$

که با قبلی یکی است

آناهیتا صدری