$$u_{xx} + u_{yy} = x + 2y \qquad 0 < x < \pi \qquad 0 < y < \pi$$

B.C. for y, $0 \leq x \leq \pi$
$$\begin{cases} u(x,0) = x \\ u(x,\pi) = 2 \end{cases}$$

B.C. for x, $0 \leq y \leq \pi$
$$\begin{cases} u(0,y) = y \\ u(\pi,y) = \cos y \end{cases}$$

ابتدا B.C. را برای $x = 0, \pi$ همگن می‌کنیم.

$$u = v + w$$

$$\begin{cases} v(0,y) = 0 \Rightarrow w(0,y) = y \xrightarrow{w = Ax + B} B = y \\ v(\pi,y) = 0 \Rightarrow w(\pi,y) = \cos y \longrightarrow A\pi + y = \cos y \Rightarrow A = \dfrac{\cos y - y}{\pi} \end{cases}$$

$$\Rightarrow \boxed{w(x,y) = \frac{\cos y - y}{\pi}\, x + y}$$

$$v_{xx} + v_{yy} = -w_{xx} - w_{yy} + x + 2y = \frac{x \cos y}{\pi} + x + 2y = F(x,y)$$

$$① \begin{cases} v(0,y) = 0 \\ v(\pi,y) = 0 \end{cases} \qquad ② \begin{cases} v(x,0) = x - w(x,0) = x - \dfrac{x}{\pi} = f(x) \\ v(x,\pi) = 2 - w(x,\pi) = \dfrac{-1-\pi}{\pi}\, x + \pi = g(x) \end{cases}$$

$u = XY$

اگر $F=0 \Rightarrow$: Laplace Equation

$$X''Y + XY'' = 0 \xRightarrow{/XY \neq 0} \frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

$* \quad X'' - \lambda X = 0$

برای $\lambda = 0$ و $\lambda > 0$ جواب غیربدیهی ندارد

$\Rightarrow \lambda < 0:$

$$X = a \cos \sqrt{-\lambda}\, x + b \sin \sqrt{-\lambda}\, x$$

$$X(0) = a = 0$$

$$X(\pi) = b \sin \sqrt{-\lambda}\, \pi = 0 \Rightarrow \sqrt{-\lambda}\, \pi = n\pi \Rightarrow \lambda_n = -n^2 \qquad n = 1, 2, \dots$$

$$\Rightarrow \boxed{X_n = b_n \sin nx} \qquad \Rightarrow u = \sum_{n=1}^{\infty} Y_n (\sin nx)$$

$$u_{xx} + u_{yy} = \sum \left(Y_n'' - n^2 Y_n \right) \sin nx = F(x,y)$$

$$\Rightarrow Y_n'' - n^2 Y_n = \frac{2}{\pi} \int_0^{\pi} F \sin nx \, dx$$

$$\xRightarrow{ODE} Y_n = c_n \cosh ny + d_n \sinh ny + Y_n^*$$

$$u(x,0) = \sum \left(c_n + Y_n^*(0) \right) \sin nx = f(x)$$

$$\Rightarrow c_n + Y_n^*(0) = \frac{2}{\pi} \int_0^{\pi} f \sin nx \, dx \Rightarrow \boxed{c_n = \frac{2}{\pi} \int_0^{\pi} f \sin nx \, dx - Y_n^*(0)}$$

$$v(x,\pi) = \sum \left(c_n \cosh n\pi + d_n \sinh n\pi + \varphi_n^*(\pi) \right) \sin nx = g(x)$$

$$\Rightarrow c_n \cosh n\pi + d_n \sinh n\pi + \varphi_n^*(\pi) = \frac{2}{\pi} \int_0^\pi g \sin nx \, dx$$

$$\Rightarrow \boxed{d_n = \frac{\frac{2}{\pi} \int_0^\pi g \sin nx \, dx - c_n \cosh n\pi - \varphi_n^*(\pi)}{\sinh(n\pi)}}$$

$$u(x,y) = v(x,y) + w(x,y)$$

آناهیتا صیدری