مثال: مسأله SL زیر را در نظر بگیرید

$$y'' + \lambda y = 0, \qquad 0 \le x \le \pi$$

با $\quad y(0)=0, \quad y'(\pi)=0$

اگر $\lambda \le 0$ مسأله SL چه رخ خواهد داد؟

$r^2 + \lambda = 0$

① $\lambda = 0$:

$$r^2 = 0 \Rightarrow r = 0 \Rightarrow y = c_1 + c_2 x$$

$$\left.\begin{aligned} y(0) &= c_1 = 0 \\ y'(\pi) &= c_2 = 0 \end{aligned}\right\} \Rightarrow y = 0$$

جواب غیربدیهی ندارد $\Leftarrow$ $\lambda = 0$ مقدار ویژه نیست

② $\lambda < 0$:

$$r^2 = \underbrace{-\lambda}_{+} \Rightarrow r = \pm\sqrt{-\lambda} \Rightarrow y = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$$

$$y(0) = c_1 + c_2 = 0$$

$$y'(\pi) = c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\,\pi} - c_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\,\pi} = c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\,\pi} + c_1\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\,\pi} = 0$$

اگر $c_1 \neq 0$ $\Rightarrow e^{\sqrt{-\lambda}\,\pi} = e^{-\sqrt{-\lambda}\,\pi} \Rightarrow \lambda = 0$

$\Rightarrow c_1 = 0 \Rightarrow y = 0 \Rightarrow$ λ مقدار ویژه نیست

مثال: معادله زیر را در نظر بگیرید (معادله اویلر)

$$x^2 y'' + x y' + \lambda y = 0 \qquad 1 \leq x \leq e \checkmark$$

$$y(1) = 0, \quad y(e) = 0$$

جواب معادله اویلر ([illegible]) به صورت

$$y(x) = C_1 x^{i\sqrt{\lambda}} + C_2 x^{-i\sqrt{\lambda}}$$

$$r^2 + \lambda = 0$$

① $\lambda = 0$:

$$r = 0 \implies y = c_1 |x|^r + c_2 |x|^r \ln|x| \qquad (x \neq 0)$$

$$= c_1 + c_2 \ln|x|$$

$$\left.\begin{array}{l} y(1) = c_1 = 0 \\ y(e) = c_2 = 0 \end{array}\right\} \implies y = 0 \implies$$

$\lambda = 0$ مقدار ویژه نیست

② $\lambda < 0$:

$$r = \pm\sqrt{-\lambda} \implies y = c_1 |x|^{\sqrt{-\lambda}} + c_2 |x|^{-\sqrt{-\lambda}}$$

$$y(1) = c_1 + c_2 = 0$$

$$y(e) = c_1 e^{\sqrt{-\lambda}} - c_1 e^{-\sqrt{-\lambda}} = 0 \Rightarrow$$ ولی $c_1 \neq 0$ $$e^{\sqrt{-\lambda}} = e^{-\sqrt{-\lambda}} \Rightarrow \lambda = 0 \quad \times$$

$$\Rightarrow c_1 = 0 \Rightarrow y = 0$$

λ مقدار ویژه نیست

③ $\lambda > 0$:

$$r = \pm i\sqrt{\lambda} = \alpha \pm i\beta$$

$$\Rightarrow y = c_1 |x|^{\alpha} \cos(\beta \ln|x|) + c_2 |x|^{\alpha} \sin(\beta \ln|x|)$$

$$= c_1 \cos(\sqrt{\lambda} \ln|x|) + c_2 \sin(\sqrt{\lambda} \ln|x|)$$

$$y(1) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0$$

$$y(e) = c_2 \sin(\sqrt{\lambda}) = 0 \qquad c_2 \neq 0 \Rightarrow \sin(\sqrt{\lambda}) = 0$$

که λ مقدار ویژه شود

$$\Rightarrow \sqrt{\lambda} = k\pi \Rightarrow \lambda = k^2\pi^2 \qquad k \in \mathbb{N}$$

$$C_1 x^{i\sqrt{\lambda}} + C_2 x^{-i\sqrt{\lambda}} = C_1\left(\cos(\sqrt{\lambda}\ln x) + i\sin(\sqrt{\lambda}\ln x)\right)$$

$$+ C_2\left(\cos(-\sqrt{\lambda}\ln x) + i\sin(-\sqrt{\lambda}\ln x)\right)$$

$$= \underbrace{(C_1 + C_2)}_{c_1} \cos(\sqrt{\lambda}\ln x) + \underbrace{(C_1 - C_2)i}_{c_2} \sin(\sqrt{\lambda}\ln x) \checkmark$$

مثال: دستگاه SL زیر را در نظر بگیرید

$$y'' + \lambda y = 0 \qquad 0 \leq x \leq 1$$

$$y(0) = 0$$

$$y(1) + h y'(1) = 0, \quad h > 0$$

مقادیر ویژه و توابع ویژه آن را مشخص کنید.

$$\frac{a_2}{a_1} = 0 \Rightarrow p = e^0 = 1$$

$$\frac{a_3}{a_1} = \lambda \Rightarrow q = \lambda$$

$$s = 1$$

p و s روی $[0,1]$ مثبت هستند $\Leftarrow$ دستگاه RSL است

① $\lambda = 0$: (مثال قبل)

$$y = 0 \quad \begin{cases} y(0) = c_1 = 0 \\ y(1) + h y'(1) = c_2 + h c_2 = 0 \xRightarrow{h>0} c_2 = 0 \end{cases}$$

② $\lambda < 0$:

$$y = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$$

$$y(0) = c_1 + c_2 = 0$$

$$y(1) + h y'(1) =$$

$$: \lambda < 0 \quad ②$$

$$c_1 e^{\sqrt{-\lambda}} - c_1 e^{-\sqrt{-\lambda}} + h c_1 \sqrt{-\lambda}\, e^{\sqrt{-\lambda}} + h c_1 \sqrt{-\lambda}\, e^{-\sqrt{-\lambda}} = 0$$

$$= c_1 \Big(\underbrace{e^{\sqrt{-\lambda}}(1 + h\sqrt{-\lambda}) + e^{-\sqrt{-\lambda}}(-1 + h\sqrt{-\lambda})}_{g}\Big)$$

$$g > e^{\sqrt{-\lambda}}(1+h\sqrt{-\lambda}) - e^{\sqrt{-\lambda}}(1+h\sqrt{-\lambda}) = 0 \implies c_1 = 0 \implies y = 0$$

$$r = \pm i\sqrt{\lambda} \implies y = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x) \quad : \lambda > 0 \quad ③$$

$$y(0) = c_1 = 0$$

$$y(1) + h y'(1) = c_2 \sin(\sqrt{\lambda}) + h\sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}) = 0$$

$$c_2 \neq 0 \implies \boxed{\sin(\sqrt{\lambda}) = -h\sqrt{\lambda}\cos(\sqrt{\lambda})}$$

$$\boxed{y = c_2 \sin(\sqrt{\lambda}\, x)}$$

آناهیتا حیدری