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Kocherlakota Satya Narayana Murthy

PREFACE

The first attempt at bringing together the statistical community deeply involved in the study and applications of discrete distributions was the International Symposium held at Montreal in 1963. Since then, the subject has received much attention on a global scale. The intervening period of nearly three decades has proved to be an explosive era, leading to the extensions of old results. More important it has been an exciting period of discovery of new distributions with applications in a variety of areas. New techniques relating to the estimation of the parameters and testing of hypotheses have been suggested. Characterizing properties of the distributions have also been presented. The computer age has made its imprint by providing the technology needed to generate random observations from a variety of populations, leading to the simulation of diverse types of populations.

In 1986 the present authors made an attempt at bringing together, as was done in 1963, the international community working in the various areas of discrete distributions. This was achieved by putting out a special issue of the *Communications in Statistics, Theory and Methods* with about 450 pages of contributions from around the world. However, we felt that a comprehensive documentation of the vast amount of literature that has accumulated over the past several years in the area of discrete distributions needed to be undertaken. Of particular interest to us was the development of the multidimensional discrete distributions. Unfortunately, we discovered that the general multivariate analytical tools, that have become commonplace in the normal case, are not available for the discrete distributions. What has been examined in this set-up are the structural properties of the multidimensional distributions. Also, in most multidimensional generalizations the distributions are of the "homogeneous" type in the sense of A. W. Kemp. Most of the non-homogeneous type of probability generating functions have to be restricted to the bivariate situations to facilitate mathematical tractability. This, in addition to the literature extant in inference and practical applications, led us to the decision to restrict the coverage in the text to the bivariate discrete distributions.

It should be made amply clear at the outset that this decision to limit the study to the bivariate case does not imply that we ignore or propose the ignoring of the multivariate distributions. To this end, we have widened the bibliography by incorporating relevant results in the multivariate situations. Hopefully, this will make the restricted coverage more palatable to the prospective user.

A general introduction to the structural properties of discrete distributions is presented in Chapter 1. These include the various generating functions and their relationships to the probability generating function; the moment relationships; polynomial representations of the bivariate probability functions; basic ideas of compounding and generalizing; and the computer simulations of random variables.

In Chapter 2 a general discussion of the problems of statistical inference is given with emphasis placed on specific techniques pertinent to the discrete case.

The book is laid out so that various chapters beginning with the third chapter are independent of each other. Although there is a common thread stretching across these topics, their treatment is very much individualized. The arrangement of the material exploits their interrelationships and thereby facilitates an easy understanding of the development of the distributions.

As in the univariate case, the Bernoulli trials form the basic building blocks of the distributions arising in the bivariate case. This is the lead-off topic of the text in Chapter 3, giving a natural access to the bivariate binomial distributions.

In Chapter 4 the bivariate Poisson distribution is introduced and studied both as a limit of the bivariate binomial and in its own right as a stochastic process.

Inverse sampling and waiting times in Bernoulli trials are considered in Chapter 5 under the general heading of the bivariate negative binomial distribution. The genesis of the distribution from the point of view of compounding is also discussed, although the general topic of generalized and compound distributions is deferred to a later chapter.

Sampling from a finite population without replacement is the thrust of Chapter 6 giving rise to the bivariate hypergeometric distributions of different forms. Inverse sampling from such populations is also examined in some detail in this chapter.

The bivariate logarithmic series distribution is derived as the Fisher-limit to the bivariate negative binomial distribution in Chapter 7. Various other models leading to this distribution and its modified version are also presented in this chapter.

The study of distributions when the parameter is subject to variability has been of interest in a variety of situations. Of particular interest for this purpose is the Poisson distribution; one reason for this seems to be that the parameter of the distribution has a natural tendency to be variable over the population. This has led to a rich class of distributions of practical value arising out of such compounding. These problems relating to compounding with the bivariate Poisson distribution are considered in the most general setting in Chapter 8.

The final chapter of the book deals with the distributions of more recent origin developed in connection with accident theory. These are the bivariate Waring and 'short' distributions. In addition, the bivariate generalized power series distribution is also discussed briefly in this chapter.

In most of the cases, the latest techniques for computer simulation of the distributions considered are discussed in some detail. While all attempts were made to include practical data in the relevant sections, in some instances we have had to compensate for its lack by using simulated data.

While we make no apology for the topics discussed in the book, it should be mentioned that we do feel that but for constraints of space we would have included several more contributions. To this extent we regret not having incorporated the material produced by a number of authors known to us personally. To them our sincere regrets. To others not known to us too we extend our personal regrets. The value of the manuscript has no doubt been enhanced by the inclusion of the vast amount of literature in the bibliography. This is made more useful by the provision of the keywords and phrases. A detailed index for the extended bibliography is appended to further enhance its usefulness.

A final word of thanks to all the authors who gave us hours of pleasure derived from going over their publications. The University of Manitoba is to be thanked for providing the necessary amenities as well as the study/research leaves to both of us. Without these, the work would have been much delayed.

Subrahmaniam Kocherlakota
Kathleen Kocherlakota

CONTENTS

NOTATION

$B(\nu, \theta)$ binomial distribution on ν trials with probability of success θ

$b(x; \nu, \theta)$ binomial probability function at x of $B(\nu, \theta)$

BVB Type I Type I bivariate binomial distribution defined in (3.3.2)

BVB Type II Type II bivariate binomial distribution defined in (3.4.2)

$P(\lambda)$ Poisson distribution with parameter λ

$p(x; \lambda)$ probability function at x of $P(\lambda)$

$BVP(\lambda_1, \lambda_2, \lambda_3)$ bivariate Poisson distribution defined in (4.3.4)

$NB(r, \theta)$ negative binomial distribution defined in (5.1.1)

$NTA(\lambda, \theta)$ Neyman Type A distribution obtained by $P(\theta\phi) \underset{\phi}{\wedge} P(\lambda)$ or $P(\lambda) \vee P(\theta)$

$IG(\mu, \lambda)$ univariate inverse Gaussian distribution defined in (8.5.3)

$UVWD(\alpha, \kappa; \beta)$ univariate Waring distribution defined in (9.2.8)

$BVWD(\alpha; \tau, \kappa; \beta)$ bivariate Waring distribution defined in (9.2.7)

$UVSD(\Phi, \lambda, \theta)$ univariate 'short' distribution defined in (9.3.2)

BVSD bivariate 'short' distributions defined in (9.3.6), (9.3.7), (9.3.8), (9.3.9)

GPSD univariate generalized power series distribution defined in (9.4.1)

BVGPSD bivariate generalized power series distribution defined in (9.4.3)

$L(\underline{\theta}; \underline{x})$ likelihood function of $\underline{\theta}$ given the observation $\underline{x}$

log x natural logarithm of x

Γ information matrix: $\left\{- E \left[\frac{\partial^2 \log L}{\partial \theta_i \, \partial \theta_j}\right]\right\}$ or $\left\{E \left[\frac{\partial \log L}{\partial \theta_i} \, \frac{\partial \log L}{\partial \theta_j}\right]\right\}$

Γ^{-1} asymptotic variance matrix of $\hat{\underline{\theta}}$

$\sim$ distributed as

$\approx$ asymptotically distributed as

pf probability function

pdf probability density function

pgf probability generating function

mgf moment generating function

cgf cumulant generating function

1

PRELIMINARIES

In this chapter we shall be considering the basic properties of bivariate discrete distributions. As is well known, central to the study of probability distributions is the generating function. In the case of discrete distributions, the probability generating function plays a vital role as, in most of the situations, it is much simpler to handle than is the moment generating function. However, in addition to the probability generating function we will define a variety of generating functions of which the importance and usefulness will be made clear in the context in which they appear. The relationships between the different types of moments will also be developed. The structure of bivariate discrete distributions has been studied by several authors by a canonical representation. These ideas will be introduced in this chapter. Specifics will be referred to a later section appropriate to the distribution under consideration. Finally, the computer age has been instrumental in making the abstraction of theory a reality. To this end, some general ideas concerning the computer generation of random samples from bivariate discrete distributions will also be presented in this chapter with, once again, the specifics being left to be given in the appropriate sections.

In the following, unless otherwise stated, we will be considering the joint distribution of the random variables X and Y. They will be assumed to have the probability mass function $f(x, y)$ at the point (x, y) with $(x, y) \in T$, a subset of the Cartesian product of the set of nonnegative integers on the real line. In this case the pair (X, Y) will be said to have bivariate discrete distribution over T with the probability function $f(x, y)$.

1.1 Generating functions

In the study of random variables a variety of generating functions have been introduced to facilitate the summarization of the properties in a compact but manageable functional form. There are a number of such properties that are of interest in general. Thus correspondingly, there are a number of generating functions defined, each giving rise to at least one specific property of the random variable. In this section we will define several such functions useful for developing the relationships between the various moment features of the random variables.

Definition 1.1.1 Probability generating function

The probability generating function (pgf) of the pair of random variables (X, Y) with probability function f(x, y) is the $E[t_1^X t_2^Y]$.

We shall denote the pgf by $\Pi(t_1, t_2)$. From the definition it is readily seen that

$$\Pi(t_1, t_2) = \sum_{(x,y)\in T} t_1^x t_2^y f(x, y) . \tag{1.1.1}$$

From equation (1.1.1) it is obvious that the series is absolutely convergent in the unit rectangle $|t_1| \le 1$, $|t_2| \le 1$. As such the pgf can be differentiated with respect to these variables at (0, 0) any number of times. It can also be shown that the pgf is unique in the sense of a one-to-one relationship to the probability function (pf). For determining the pf, given the pgf, we can either

(i) expand $\Pi(t_1, t_2)$ in powers of t_1 and t_2. Then the coefficient of $t_1^x t_2^y$ will be the probability function f(x, y) at (x, y),

or

(ii) use the fact that the pgf can be differentiated any number of times with respect to t_1 and t_2 and evaluated at (0, 0) yielding

$$f(x, y) = \frac{1}{x!}\frac{1}{y!}\frac{\partial^{x+y}}{\partial t_1^x \partial t_2^y}\, \Pi(t_1, t_2)\Big|_{t_1=0,t_2=0} . \tag{1.1.2}$$

As its name implies, the probability generating function gives rise to the probability function of the random variable. However, our interest is often in the joint moments $\mu'_{r,s} = E(X^rY^s)$ of the random variables X and Y. For this purpose we are led to the definition of the moment generating function (mgf). Unfortunately, the existence of the mgf is dependent on the existence of the moments. To ensure that it does exist, we use the restricted definition given by Hogg and Craig (1978, p. 77).

Definition 1.1.2 Moment generating function

The moment generating function (mgf) of the pair of random variables X and Y is the $E[\exp(t_1X + t_2Y)]$ provided the expectation exists for (t_1, t_2) lying in the rectangle $-h_1 < t_1 < h_1$, $-h_2 < t_2 < h_2$ where h_1 and h_2 are positive real numbers.

The moment generating function is represented by $M(t_1, t_2)$. From the definition,

$$M(t_1, t_2) = \sum_{x,y} \exp[t_1x + t_2y]\, f(x, y). \tag{1.1.3}$$

As pointed out, this definition of the mgf assumes that all the moments do exist. If, however, the mgf does not exist, it is necessary to deal with the characteristic function defined over the complex plane. For details on the difficulties in this connection and other related problems we refer to Stuart and Ord (1987).

Expanding the exponential functions in (1.1.3), it is possible to write

$$M(t_1, t_2) = \sum_{r,s} \frac{t_1^r}{r!}\frac{t_2^s}{s!}\mu'_{r,s}\,. \tag{1.1.4}$$

From this expansion and the fact that the order of differentiation and summation can be interchanged, it is possible to establish the following rules for the determination of the moments, given the mgf:

(i) $\mu'_{r,s}$ is the coefficient of $\frac{t_1^r}{r!}\frac{t_2^s}{s!}$ in the expansion of the mgf in powers of t_1 and t_2.

(ii) $\mu'_{r,s}$ is given by the mixed partial derivative

$$\frac{\partial^{r+s}}{\partial t_1^r \partial t_2^s} M(t_1, t_2)\Big|_{t_1=0, t_2=0}.$$

Comparing (1.1.1) and (1.1.3) we have the relationship

$$M(t_1, t_2) = \Pi[\exp t_1, \exp t_2].$$

Cumulants generating function

As in the univariate case the cumulants generating function, and consequently, the cumulants are defined by the relations

$$K(t_1, t_2) = \log M(t_1, t_2) \tag{1.1.5}$$

$$= \sum_{r} \sum_{s} \frac{t_1^r}{r!} \frac{t_2^s}{s!} \kappa_{r,s}. \qquad (r,s)\neq(0,0) \tag{1.1.6}$$

In (1.1.6) the quantity $\kappa_{r,s}$ is called the cumulant of order (r, s).

Factorial moments and cumulants

In discrete distributions the factorial moments and factorial cumulants play an important role in describing the distribution as well as in the procedures for estimation.

The factorial moment generating function of (X, Y) is defined by the equation

$$G(t_1, t_2) = \Pi(t_1+1, t_2+1). \tag{1.1.7}$$

Substituting for the right hand side of (1.1.7), expanding the binomials and summing with respect to the variables x and y, we have

$$G(t_1, t_2) = \sum_{r,s} \frac{t_1^r}{r!} \frac{t_2^s}{s!} \mu_{[r,s]}, \tag{1.1.8}$$

where $\mu_{[r,s]}$ stands for $E[X^{[r]}Y^{[s]}]$ and $x^{[r]} = x(x-1)(x-2) \ldots (x-r+1)$. This expected value is called the factorial moment of order (r, s).

Thus it is readily seen that the factorial moment $\mu_{[r,s]}$ is generated as the coefficient of $\frac{t_1^r}{r!} \frac{t_2^s}{s!}$ in the power series expansion of the function $G(t_1, t_2)$. Another representation of the factorial moment is

$$\mu_{[r,s]} = \frac{\partial^{r+s}}{\partial t_1^r t_2^s} \Pi(t_1+1,\, t_2+1)\Big|_{t_1=0, t_2=0} . \tag{1.1.9}$$

The factorial cumulants generating function is defined by the relationship

$$\begin{aligned} H(t_1, t_2) &= \log G(t_1, t_2) \\ &= \sum_r \sum_s{}_{(r,s)\neq(0,0)} \frac{t_1^r}{r!}\frac{t_2^s}{s!} \kappa_{[r,s]} . \end{aligned} \tag{1.1.10}$$

The quantity $\kappa_{[r,s]}$ is called the factorial cumulant of order (r, s).

Relationships among moments and cumulants

The question that arises is whether and, if so, how these moments and cumulants are interrelated. Before we answer this question, let us introduce the central moment of order r in the univariate case as $E[(X-\mu_x)^r]$, where μ_x is the expected value of X. Similarly the (r, s)th central moment in the bivariate case is $\mu_{r,s} = E[(X-\mu_x)^r(Y-\mu_y)^s]$. The following procedure is suggested by David and Barton (1962) to determine the relationship between the central moments and those defined earlier around zero . Consider the formal representation

$$\mu(r^4) = \mu'(r^4) - 4\,\mu'(r^3)\,\mu'(r) + 6\,\mu'(r^2)\,[\mu'(r)]^2 - 3\,[\mu'(r)]^4.$$

Operate on both sides of this equation by $s\frac{\partial}{\partial r}$ and cancel out the coefficient 4 from both sides of the resulting equation. Then changing the order of differentiation and summation, we have

$$\begin{aligned} \mu_{3,1} &= \mu'_{3,1} - 3\mu'_{2,1}\mu'_{1,0} - \mu'_{3,0}\mu'_{0,1} + 3\mu'_{1,1}\mu'^2_{1,0} \\ &\quad + 3\mu'_{2,0}\mu'_{1,0}\mu'_{0,1} - 3\mu'^3_{1,0}\mu'_{0,1} . \end{aligned} \tag{1.1.11}$$

This procedure can be applied repeatedly to obtain further relationships. Thus operating on $\mu[r^3s]$ by $s\frac{\partial}{\partial r}$ and cancelling out the factor 3, we have

$$\mu_{2,2} = \mu'_{2,2} - 2[\mu'_{2,1}\mu'_{0,1} + \mu'_{1,2}\mu'_{1,0}] + \mu'^2_{1,0}\mu'_{0,2} + \mu'^2_{0,1}\mu'_{2,0}$$

$$+ 4\mu'_{1,1}\mu'_{1,0}\mu'_{0,1} - 3\mu'^2_{1,0}\mu'^2_{0,1}. \qquad (1.1.12)$$

This method can also be applied to set up the relationships between the moments and the cumulants. For example, recalling the expression for the fourth raw moment in terms of the cumulants

$$\mu'_4 = \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4,$$

we can write formally

$$\mu'(r^4) = \kappa(r^4) + 4\kappa(r^3)\kappa(r) + 3[\kappa(r^2)]^2 + 6\kappa(r^2)[\kappa(r)]^2 + [\kappa(r)]^4. \qquad (1.1.13)$$

Treating this as a function of r and operating on it by $s\frac{\partial}{\partial r}$ and then replacing the mixed products by their corresponding moments, we have

$$\mu'_{3,1} = \kappa_{3,1} + 3\kappa_{2,1}\kappa_{1,0} + \kappa_{3,0}\kappa_{0,1} + 3\kappa_{1,1}\kappa_{2,0} + 3\kappa_{1,1}\kappa_{1,0}^2$$
$$+ 3\kappa_{2,0}\kappa_{1,0}\kappa_{0,1} + \kappa_{1,0}^3\kappa_{0,1}.$$

If the means are zero, then this equation simplifies to $\mu_{3,1} = \kappa_{3,1} + 3\kappa_{1,1}\kappa_{2,0}$. Using the same technique and treating the means equal to zero, we have the reverse relation $\kappa_{3,1} = \mu_{3,1} - 3\mu_{1,1}\mu_{2,0}$.

Similarly, the operation with $(s\frac{\partial}{\partial r})^2$ on (1.1.13) gives, upon setting the means equal to zero, $\mu_{2,2} = \kappa_{2,2}+2\kappa_{1,1}^2+\kappa_{2,0}\kappa_{0,2}$. The reverse relation can also be shown to be $\kappa_{2,2} = \mu_{2,2} - 2\mu_{1,1}^2 - \mu_{2,0}\mu_{0,2}$. Reference may be made to Stuart and Ord (1987, p. 104) for further examples in the computation of the relationships of this nature.

Employing expansion techniques, Kocherlakota (1991) has studied the relationship between the factorial cumulants and the ordinary cumulants. It is shown that

$$\kappa_{[1,1]} = \mu_{1,1} = \kappa_{1,1}$$
$$\kappa_{[2,1]} = \mu_{2,1} - \mu_{1,1} = \kappa_{2,1} - \kappa_{1,1}$$

$$\kappa_{[3,1]} = \mu_{3,1} - 3\,\mu_{2,1} + 2\,\mu_{1,1} - 3\,\mu_{2,0}\mu_{1,1}$$
$$= \kappa_{3,1} - 3\,\kappa_{2,1} + 2\,\kappa_{1,1}$$
$$\kappa_{[2,2]} = \mu_{2,2} - \mu_{2,1} - \mu_{1,2} + \mu_{1,1} - 2\,\mu_{1,1}^2 - \mu_{2,0}\,\mu_{0,2}$$
$$= \kappa_{2,2} - \kappa_{2,1} - \kappa_{1,2} + \kappa_{1,1}$$
$$\kappa_{[4,1]} = \mu_{4,1} - 6\,\mu_{3,1} + 11\,\mu_{2,1} - 6\,\mu_{1,1} - 4\,\mu_{1,1}\,\mu_{3,0} + 18\,\mu_{2,0}\,\mu_{1,1} - 6\,\mu_{2,0}\,\mu_{2,1}$$
$$= \kappa_{4,1} - 6\,\kappa_{3,1} + 11\,\kappa_{2,1} - 6\,\kappa_{1,1}$$
$$\kappa_{[3,2]} = \mu_{3,2} - 3\,\mu_{2,2} - \mu_{3,1} + 3\,\mu_{2,1} + 2\,\mu_{1,2} - 2\,\mu_{1,1} - \mu_{0,2}\,(\,\mu_{3,0} - 3\,\mu_{2,0}\,) - 3\,\mu_{2,0}\,(\,\mu_{1,2} - \mu_{1,1}\,) - 6\,\mu_{1,1}\,(\,\mu_{2,1} - \mu_{1,1}\,)$$
$$= \kappa_{3,2} - 3\,\kappa_{2,2} - \kappa_{3,1} + 3\,\kappa_{2,1} + 2\,\kappa_{1,2} - 2\,\kappa_{1,1}$$
$$\kappa_{[5,1]} = \kappa_{5,1} - 10\,\kappa_{4,1} + 35\,\kappa_{3,1} - 50\,\kappa_{2,1} + 24\,\kappa_{1,1}$$
$$\kappa_{[4,2]} = \kappa_{4,2} - 6\,\kappa_{3,2} - \kappa_{4,1} + 6\,\kappa_{3,1} + 11\,\kappa_{2,2} - 6\,\kappa_{1,2} - 11\,\kappa_{2,1} + 6\,\kappa_{1,1}$$
$$\kappa_{[3,3]} = \kappa_{3,3} - 3\,\kappa_{3,2} - 3\,\kappa_{2,3} + 2\,\kappa_{3,1} + 9\,\kappa_{2,2} + 2\,\kappa_{1,3} - 6\,\kappa_{2,1} - 6\,\kappa_{1,2} + 4\,\kappa_{1,1}.$$

The inverse relations are correspondingly given by

$$\kappa_{1,1} = \kappa_{[1,1]}$$
$$\kappa_{2,1} = \kappa_{[2,1]} + \kappa_{[1,1]}$$
$$\kappa_{3,1} = \kappa_{[3,1]} + 3\,\kappa_{[2,1]} + \kappa_{[1,1]}$$
$$\kappa_{2,2} = \kappa_{[2,2]} + \kappa_{[2,1]} + \kappa_{[1,2]} + \kappa_{[1,1]}$$
$$\kappa_{4,1} = \kappa_{[4,1]} + 6\,\kappa_{[3,1]} + 7\,\kappa_{[2,1]} + \kappa_{[1,1]}$$
$$\kappa_{3,2} = \kappa_{[3,2]} + \kappa_{[3,1]} + 3\,\kappa_{[2,2]} + 3\,\kappa_{[2,1]} + \kappa_{[1,2]} + \kappa_{[1,1]}$$
$$\kappa_{5,1} = \kappa_{[5,1]} + 10\,\kappa_{[4,1]} + 25\,\kappa_{[3,1]} + 15\,\kappa_{[2,1]} + \kappa_{[1,1]}$$
$$\kappa_{4,2} = \kappa_{[4,2]} + 6\,\kappa_{[3,2]} + \kappa_{[4,1]} + 6\,\kappa_{[3,1]} + 7\,\kappa_{[2,2]} + 7\,\kappa_{[2,1]} + \kappa_{[1,2]} + \kappa_{[1,1]}$$
$$\kappa_{3,3} = \kappa_{[3,3]} + 3\,(\,\kappa_{[3,2]} + \kappa_{[2,3]}\,) + 9\,\kappa_{[2,2]} + \kappa_{[1,3]} + \kappa_{[3,1]} + 3\,(\,\kappa_{[1,2]} + \kappa_{[2,1]}\,) + \kappa_{[1,1]}.$$

David and Barton (1962, p. 51) show that the factorial and ordinary moments have the relationship

$$\kappa_{[r]} = \sum_{i=0}^{r} s(r, i)\, \kappa_r, \qquad \kappa_r = \sum_{i=0}^{r} S(r, i)\, \kappa_{[r]}, \tag{1.1.14}$$

where s(r, i) and S(r, i) are Stirling numbers of the first and second kind, respectively. [See, for example, Berg (1988), p. 776.] An examination of the above equations shows that the expression for $\kappa_{[r,u]}$ in terms of $\kappa_{t,v}$ for $t \le r$, $v \le u$ has the coefficient s(r, t) s(u, v). On the other hand the expression for $\kappa_{r,u}$ in terms of $\kappa_{[t,v]}$ for $t \le r$, $v \le u$ has the coefficient S(r, t) S(u, v).

1.2 Convolutions

Some times bivariate distributions can be generated by convolutions of random variables. Thus if we consider $X = X_1+X_3$ and $Y = X_2+X_3$ with X_1, X_2 and X_3 being independently distributed, then the random variables X and Y are jointly distributed. The method, termed the trivariate reduction method, is discussed in Mardia (1970).

Let the pgf's of the random variables under consideration be $\Pi_i(t)$, i=1, 2, 3. Then the joint pgf of (X, Y) is

$$\Pi(t_1, t_2) = \Pi_1(t_1)\, \Pi_2(t_2)\, \Pi_3(t_1 t_2). \tag{1.2.1}$$

Similarly, if the mgf of X_i, i = 1, 2, 3 is $M_i(t)$ then the mgf of (X, Y) is seen to be

$$M(t_1, t_2) = M_1(t_1)\, M_2(t_2)\, M_3(t_1+t_2). \tag{1.2.2}$$

Probability function

From the joint pgf (1.2.1), by successive differentiation with respect to the arguments, we have

$$\Pi^{(r,s)}(t_1, t_2) = \sum_{i=0}^{r} \sum_{h=0}^{s} \sum_{k=0}^{\min(r-i,s-h)} \frac{r!s!}{i!h!k!(r-i-k)!(s-h-k)!}\, t_1^{s-h-k} t_2^{r-i-k}.$$
$$\Pi_1^{(i)}(t_1)\, \Pi_2^{(h)}(t_2)\, \Pi_3^{[r+s-i-h-k]}(t_1 t_2). \tag{1.2.3}$$

Dividing (1.2.3) by r! s! and setting $t_1 = t_2 = 0$, we get the joint pf of

(X, Y) as

$$P\{X=r, Y=s\} = \sum_{k=0}^{\min(r,s)} f_1(r-k)f_2(s-k)f_3(k), \tag{1.2.4}$$

where $f_i(w)$ is the pf of the random variable X_i. This equation can also be obtained directly from the definition of the random variables X and Y.

Moments

Setting $t_1 = t_2 = 1$ in (1.2.3) yields

$$\mu_{[r,s]} = \sum_{i=0}^{r} \sum_{h=0}^{s} \sum_{k=0}^{\min(r-i,s-h)} \frac{r!s!}{i!h!k!(r-i-k)!(s-h-k)!} \mu^{(1)}_{[r-i-k]} \mu^{(2)}_{[s-h-k]} \mu^{(3)}_{[i+h+k]}. \tag{1.2.5}$$

The moment generating function (1.2.2) gives the (raw) moments

$$\mu'_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \binom{r}{i}\binom{s}{j} \mu_i^{(1)'} \mu_j^{(2)'} \mu_{r+s-i-j}^{(3)'}. \tag{1.2.6}$$

Since

$$E\left[e^{t_1(X-\mu_X) + t_2(Y-\mu_Y)}\right] = E\left[e^{t_1(X_1-\mu_1)} e^{t_2(X_2-\mu_2)} e^{(t_1+t_2)(X_3-\mu_3)}\right],$$

the equation for the central moments is

$$\mu_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \binom{r}{i}\binom{s}{j} \mu_i^{(1)} \mu_j^{(2)} \mu_{r+s-i-j}^{(3)}. \tag{1.2.7}$$

Factorial Cumulants

The factorial cumulant generating function is in this case

$$H(t_1,t_2) = \log \pi_1(1+t_1) + \log \pi_2(1+t_2) + \log \pi_3\{(1+t_1)(1+t_2)\}. \tag{1.2.8}$$

Now

$$\log \pi_1(1+t_1) = \sum_{r=1}^{\infty} \frac{t_1^r}{r!} \kappa^{(1)}_{[r]}, \qquad \log \pi_2(1+t_2) = \sum_{s=1}^{\infty} \frac{t_2^s}{s!} \kappa^{(2)}_{[s]},$$

and

$$\log \pi_3\{(1+t_1)(1+t_2)\} = \sum_{u=1}^{\infty} \frac{(t_1+t_2+t_1t_2)^u}{u!} \kappa^{(3)}_{[u]}$$

$$= \sum_{u=1}^{\infty} \frac{\kappa^{(3)}_{[u]}}{u!} \sum_{i=0}^{u} \sum_{\substack{j=0 \\ i+j\le u}}^{u} \frac{u!\, t_1^{u-j}\, t_2^{u-i}}{i!j!(u-i-j)!} \, .$$

Hence

$$H(t_1,t_2) = \sum_{r=1}^{\infty} \frac{t_1^r}{r!} \kappa^{(1)}_{[r]} + \sum_{s=1}^{\infty} \frac{t_2^s}{s!} \kappa^{(2)}_{[s]} + \sum_{u=1}^{\infty} \frac{\kappa^{(3)}_{[u]}}{u!} \sum_{i=0}^{u} \sum_{\substack{j=0 \\ i+j\le u}}^{u} \frac{u!\, t_1^{u-j}\, t_2^{u-i}}{i!j!(u-i-j)!} \, . \tag{1.2.9}$$

The marginal factorial cumulants are readily seen to be

$$\kappa_{[r,0]} = \kappa^{(1)}_{[r]} + \kappa^{(3)}_{[r]} \, , \qquad \kappa_{[0,s]} = \kappa^{(1)}_{[s]} + \kappa^{(3)}_{[s]} .$$

Also the last term of (1.2.9) can be written as

$$\sum_{r=1}^{\infty} \frac{t_1^r}{r!} \sum_{s=1}^{\infty} \frac{t_2^s}{s!} \sum_{u=\max(r,s)}^{r+s} \frac{r!s!}{(u-r)!(u-s)!(r+s-u)!} \kappa^{(3)}_{[u]} . \tag{1.2.10}$$

Hence, for $r \neq 0$, $s \neq 0$,

$$\kappa_{[r,s]} = \sum_{u=\max(r,s)}^{r+s} \frac{r!s!}{(u-r)!(u-s)!(r+s-u)!} \kappa^{(3)}_{[u]}$$

$$= \kappa_{[s,r]} \, . \tag{1.2.11}$$

Cumulants

Using (1.2.2), the cumulant generating function is seen to be

$$K(t_1,t_2) = \log M_1(t_1) + \log M_2(t_2) + \log M_3(t_1 + t_2)$$

$$= \sum_{r=1}^{\infty} \frac{t_1^r}{r!} \kappa^{(1)}_{r} + \sum_{s=1}^{\infty} \frac{t_2^s}{s!} \kappa^{(2)}_{s} + \sum_{u=1}^{\infty} \frac{(t_1+t_2)^u}{u!} \kappa^{(3)}_{u} . \tag{1.2.12}$$

Therefore, the cumulants are seen to be

$$\kappa_{r,0} = \kappa_r^{(1)} + \kappa_r^{(3)}, \quad \kappa_{0,s} = \kappa_s^{(2)} + \kappa_s^{(3)}, \quad \kappa_{r,s} = \kappa_{r+s}^{(3)} = \kappa_{s,r}. \qquad (1.2.13)$$

Factorial cumulants in terms of cumulants

Recalling that $\kappa_{[u]} = \sum_{j=0}^{u} s(u, j)\, \kappa_j$ and $\kappa_u = \sum_{j=0}^{u} S(u, j)\, \kappa_{[j]}$, we can also write

$$\left.\begin{aligned} \kappa_{[r,t]} &= \sum_{u=\max(r,t)}^{r+t} \frac{r!t!}{(u-r)!(u-t)!(r+t-u)!} \sum_{j=0}^{u} s(u, j)\kappa_j^{(3)} \\ \kappa_{r,t} &= \sum_{j=0}^{r+t} S(r+t, j)\kappa_{[j]}^{(3)} \end{aligned}\right\}. \qquad (1.2.14)$$

Example 1.2.1: From the relations between ordinary and factorial cumulants, we have

$$\kappa_{[1,1]} = \kappa_2^{(3)} = \kappa_{[2]}^{(3)}$$

$$\kappa_{[2,1]} = \kappa_3^{(3)} - \kappa_2^{(3)} = \kappa_{[3]}^{(3)} + 2\kappa_{[2]}^{(3)}$$

$$\kappa_{[3,1]} = \kappa_4^{(3)} - 3\,\kappa_3^{(3)} + 2\kappa_2^{(3)} = \kappa_{[4]}^{(3)} + 3\kappa_{[3]}^{(3)}$$

$$\kappa_{[2,2]} = \kappa_4^{(3)} - 2\,\kappa_3^{(3)} + \kappa_2^{(3)} = \kappa_{[4]}^{(3)} + 4\,\kappa_{[3]}^{(3)}$$

$$\kappa_{[4,1]} = \kappa_5^{(3)} - 6\,\kappa_4^{(3)} + 11\,\kappa_3^{(3)} - 6\,\kappa_2^{(3)} = \kappa_{[5]}^{(3)} + 4\,\kappa_{[4]}^{(3)}$$

$$\kappa_{[3,2]} = \kappa_5^{(3)} - 4\,\kappa_4^{(3)} + 5\,\kappa_3^{(3)} - 2\,\kappa_2^{(3)} = \kappa_{[5]}^{(3)} + 6\,\kappa_{[4]}^{(3)} + 6\,\kappa_{[3]}^{(3)}$$

$$\kappa_{[5,1]} = \kappa_6^{(3)} - 10\,\kappa_5^{(3)} + 35\,\kappa_4^{(3)} - 50\,\kappa_3^{(3)} + 24\,\kappa_2^{(3)} = \kappa_{[6]}^{(3)} + 5\,\kappa_{[5]}^{(3)}$$

$$\kappa_{[4,2]} = \kappa_6^{(3)} - 7\,\kappa_5^{(3)} + 17\,\kappa_4^{(3)} - 17\,\kappa_3^{(3)} + 6\,\kappa_2^{(3)} = \kappa_{[6]}^{(3)} + 8\,\kappa_{[5]}^{(3)} + 12\,\kappa_{[4]}^{(3)}$$

$$\begin{aligned} \kappa_{[3,3]} &= \kappa_6^{(3)} - 6\,\kappa_5^{(3)} + 13\,\kappa_4^{(3)} - 12\,\kappa_3^{(3)} + 4\,\kappa_2^{(3)} \\ &= \kappa_{[6]}^{(3)} + 9\,\kappa_{[5]}^{(3)} + 18\,\kappa_{[4]}^{(3)} + 6\,\kappa_{[3]}^{(3)} \end{aligned}$$

1.3 Marginal and conditional distributions

It is of interest to consider the marginal and conditional distributions. The former describes the individual behavior of each of the random variables. On the other hand, the interdependence of the random variables is brought out by the conditional distributions. Thus a determination of the conditional distribution in a form that can be given simple interpretations is useful. These will be shown in a later section to be helpful in the computer simulation of the bivariate distribution. One aspect of the interrelationships that is useful in prediction is the regression function.

It will be shown here that it is possible to relate the pgf's of the conditional distributions with the pgf of the joint distribution. Further, It will be seen that in most situations the resulting representation leads to an interesting interpretation of the conditional distribution. Although the usual method of determining the conditional pf directly from the joint pf and the marginal pf's can be used, this will entail the determination of the joint probability function. Unfortunately, in the case of the bivariate discrete distributions this is not always a trivial problem. Also the resulting conditional pf does not lend itself to useful interpretation, being quite intractable in most of the cases that arise in practice. On the other hand, it will be seen that the use of the pgf in determining the conditional distribution without having to find the bivariate pf is much more general. It is also helpful in giving a greater insight into the nature of the conditional distributions. The resulting pgf is seen in most situations to yield general, but simple, techniques for the computer simulation of the bivariate distribution.

Marginal distributions

Let the joint pf of (X, Y) be f(x, y). Then the marginal probability functions are $g(x) = \sum_{y} f(x, y)$ and $h(y) = \sum_{x} f(x, y)$. The generating functions of the marginal distributions can be determined from the joint generating functions. Thus the marginal pgf's are

$$\left.\begin{aligned}\Pi_x(t) &= \sum_x g(x)t^x = \sum_x t^x \sum_y f(x, y) = \sum_x \sum_y f(x, y)t^x \\ &= \Pi(t, 1) \\ \Pi_y(t) &= \sum_y h(y)t^y = \sum_y t^y \sum_x f(x, y) = \sum_x \sum_y f(x, y)t^y \\ &= \Pi(1, t)\end{aligned}\right\}. \tag{1.3.1}$$

Similarly from the definition of mgf it is possible to see that the mgf's of the marginal distributions are $M_x(t) = M(t,0)$ and $M_y(t) = M(0,t)$.

Conditional distributions

By definition, the conditional pf of Y, given X=x, is

$$f(y|x) = \frac{f(x, y)}{g(x)}. \tag{1.3.2}$$

The following result due to Subrahmaniam (1966) gives the pgf of this conditional distribution in terms of the joint pgf of (X, Y).

Theorem 1.3.1

Let $\Pi(t_1, t_2)$ be the joint pgf of (X, Y). Then the pgf of the conditional distribution of Y given X=x is

$$\Pi_y(t|x) = \frac{\Pi^{(x,0)}(0, t)}{\Pi^{(x,0)}(0, 1)}, \tag{1.3.3}$$

where

$$\Pi^{(x,y)}(u, v) = \frac{\partial^{x+y}}{\partial t_1^x \partial t_2^y} \Pi(t_1, t_2)\bigg|_{t_1=u, t_2=v}.$$

Proof:

By definition the conditional pgf is

$$\Pi_y(t|x) = \sum_y f(y|x)t^y = \frac{\sum_y f(x, y)t^y}{\sum_y f(x, y)}. \tag{1.3.4}$$

From the definition of $\Pi(t_1, t_2)$ it can be readily seen that

$$\Pi^{(x,0)}(t_1, t_2) = \sum_{r,s} \frac{r!}{(r-x)!} t_1^{r-x} t_2^s f(r, s);$$

therefore,

$$\Pi^{(x,0)}(0, t) = x! \sum_s f(x, s)\, t^s, \qquad \Pi^{(x,0)}(0, 1) = x! \sum_s f(x, s),$$

which yield the result in (1.3.3). □

The theorem can be used to determine the regression of Y on X as

Corollary 1.3.1

Regression of Y on X is

$$E[Y|X=x] = \frac{\Pi^{(x,1)}(0, 1)}{\Pi^{(x,0)}(0, 1)} . \qquad (1.3.5)$$

Proof.

By definition

$$E[Y|X=x] = \frac{\partial}{\partial t} \Pi(t|x)\Big|_{t=1}$$

$$= \frac{\partial}{\partial t} [\Pi^{(x,0)}(0, t)/\Pi^{(x,0)}(0, 1)]\Big|_{t=1},$$

which is the result in (1.3.5). □

Definition 1.3.1 Independence

Random variables X and Y are said to be independent if and only if $f(x, y) = g(x)h(y)$ for all $(x, y) \in T$.

A property of the pgf that characterizes independence is given by the following theorem.

Theorem 1.3.2

Let the joint pgf of the random variables X and Y be $\Pi(t_1, t_2)$ with marginal pgf's $\Pi_x(t)$ and $\Pi_y(t)$, respectively. Then X and Y are independent if and only if $\Pi(t_1, t_2) = \Pi_x(t_1)\, \Pi_y(t_2)$.

Proof:

If X and Y are independent, then

$$\Pi(t_1, t_2) = \sum_r \sum_s t_1^r t_2^s f(r, s) = \sum_r \sum_s t_1^r t_2^s g(r)\, h(s) = \sum_r t_1^r g(r) \sum_s t_2^s h(s)$$

$$= \Pi_x(t)\, \Pi_y(t). \tag{1.3.6}$$

On the other hand, if $\Pi(t_1, t_2) = \Pi_x(t_1)\, \Pi_y(t_2)$, then substituting for the appropriate expressions in terms of the summations, we have

$$\sum_r \sum_s t_1^r t_2^s f(r, s) = \sum_r t_1^r g(r) \sum_s t_2^s h(s) = \sum_r \sum_s t_1^r t_2^s g(r) h(s). \tag{1.3.7}$$

Equating like powers of t_1 and t_2 on the two sides of the equation (1.3.7), we have $f(r, s) = g(r)h(s)$ for all $(r, s) \in T$. □

1.4 Sum and difference of the random variables

If the joint pgf of (X, Y) is $\Pi(t_1, t_2)$ then the distribution of the sum $Z = X + Y$ has the pgf

$$\Pi_z(t) = E[t^Z] = E[t^{X+Y}] = \Pi(t, t). \tag{1.4.1}$$

Similarly, it is possible to see that the pgf of the difference $W = X-Y$ has the pgf

$$\Pi_w(t) = E[t^W] = E[t^{X-Y}] = \Pi(t, -t). \tag{1.4.2}$$

Kemp (1981) has considered the conditional distribution of the random variable X given the sum and difference. We will examine these problems in the following sections.

1.4.1 Conditional distribution of X given the sum

The joint pgf of X and Z is

$$\Pi_{x,z}(t_1, t_2) = E[t_1^X t_2^Z] = E[t_1^X t_2^{X+Y}] = \Pi(t_1 t_2, t_2). \tag{1.4.3}$$

Similarly, the joint pgf of Y and Z is

$$\Pi_{Y,Z}(t_1, t_2) = E[t_1^Y t_2^Z] = E[t_1^Y t_2^{X+Y}] = \Pi(t_2, t_1t_2). \tag{1.4.4}$$

To find the conditional pgf of X given the sum Z = z there are two possible techniques available:

(i) From Theorem 1.3.1 we know that

$$\Pi_X(t|z) = \frac{\left.\dfrac{\partial^z}{\partial t_2^z}\Pi(t_1t_2, t_2)\right|_{t_1=t,t_2=0}}{\left.\dfrac{\partial^z}{\partial t_2^z}\Pi(t_1t_2, t_2)\right|_{t_1=1,t_2=0}}. \tag{1.4.5}$$

(ii) The following approach is simpler in some instances. Writing f(r, s) for the probability function of (X, Z), we have

$$\Pi_{X,Z}(t_1, t_2) = \sum_{r,s} t_1^r t_2^s f(r, s) = \sum_s t_2^s \sum_r t_1^r f(r, s). \tag{1.4.6}$$

In this equation, the coefficient of t_2^s is $\sum_r t_1^r f(r, s)$. The marginal probability of {Z = z} can be found as the coefficient of t_2^s in $\Pi_{X,Z}(1, t_2)$. Therefore the conditional pgf of X given Z = z is

$$\Pi_X(t|z) = \sum_r \frac{f(r, z)}{f(., z)} t^r = \frac{\sum_r f(r, z)t^r}{f(., z)}.$$

Or

$$\Pi_X(t|z) = \frac{\text{coeff. of } t_2^z \text{ in } \sum_{r,s} t_2^s t^r f(r, s)}{\text{coeff. of } t_2^z \text{ in } \sum_{r,s} t_2^s t^r f(r, s) \text{ at } t=1}, \tag{1.4.7}$$

where f(., s) denotes the marginal pf of Z. Hence, the conditional pgf is the ratio