Simple finite element methods in Python

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1 Heat equation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be the computational domain. We suppose to have a disjoined partition of its boundary: $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$. We consider the parabolic equation for the temperature T, heat flux $\vec{q}$ and heat release $\dot{q}$

Heat equation (strong formulation)

$$\begin{cases} \vec{q} = -k\nabla T & \\ \rho C_p \dfrac{\partial T}{\partial t} + \operatorname{div}(\vec{v}T) + \operatorname{div}\vec{q} = \dot{q} & \text{in } \Omega \\ T = T^D & \text{in } \Gamma_D \\ k\dfrac{\partial T}{\partial n} = q^N & \text{in } \Gamma_N \\ c_R T + k\dfrac{\partial T}{\partial n} = q^R & \text{in } \Gamma_R \end{cases} \tag{1.1}$$

Heat equation (weak formulation)

Let $H^1_f := \left\{ u \in H^1(\Omega) \,\middle|\, T_{|\Gamma_D} = f \right\}$. The standard weak formulation looks for $T \in H^1_{T^D}$ such that for all $\phi \in H^1_0(\Omega)$

$$\int_\Omega \rho C_p \frac{\partial T}{\partial t}\phi - \int_\Omega \vec{v}T \cdot \nabla\phi + \int_\Omega k\nabla T \cdot \nabla\phi + \int_{\Gamma_R} c_R T\phi + \int_{\Gamma_R \cup \Gamma_N} \vec{v}_n T\phi = \int_\Omega \dot{q}\phi + \int_{\Gamma_R} q^R\phi \tag{1.2}$$

We can derive (1.2) from (1.1) by the divergence theorem

$$\int_\Omega \operatorname{div}\vec{F} = \int_{\partial\Omega} \vec{F}_n \quad \overset{F \to F\phi}{\Longrightarrow} \quad \int_\Omega (\operatorname{div}\vec{F})\phi = -\int_\Omega \vec{F} \cdot \nabla\phi + \int_{\partial\Omega} \vec{F}_n\phi,$$

which gives with $\vec{F} = \vec{v} + \vec{q}$

$$\int_\Omega \operatorname{div}(\vec{v} + \vec{q})\,\phi = -\int_\Omega \vec{v} \cdot \nabla\phi + \int_\Omega k\nabla T \cdot \nabla\phi + \int_{\partial\Omega} \vec{F}_n\phi.$$

Using that ϕ vanishes on Γ_D we have

$$\int_{\partial\Omega} \vec{F}_n\phi = \int_{\Gamma_N \cup \Gamma_R} \vec{F}_n\phi = \int_{\Gamma_N \cup \Gamma_R} \vec{v}_n\phi + \int_{\Gamma_N \cup \Gamma_R} \vec{q}_n\phi,$$

and then with the different boundary conditions, we find

$$\int_{\Gamma_N \cup \Gamma_R} \vec{q}_n\phi = \int_{\Gamma_D} q^N\phi + \int_{\Gamma_R} \left(q^R - c_R T\right)\phi$$

Appendices

A Finite elements on simplices

A.1 Simplices

We consider an arbitrary non-degenerate simplex $K = (x_0, x_1, \dots, x_d)$. The (signed) volume of K is given by

$$|K| = \frac{1}{d!}\det(x_1 - x_0, \dots, x_d - x_0) = \frac{1}{d!}\det(1, x_0, x_1 \dots, x_d) \quad 1 = (1, \dots, 1)^{\mathsf{T}}. \tag{A.1}$$

The d+1 sides S_k (co-dimension one, d−1-simplices or facets) are defined by $S_k = (x_0, \dots, \cancel{x_k}, \dots, x_d)$. The height is $d_k = |P_{S_k} x_k - x_k|$, where P_S is the orthogonal projection on the hyperplane associated to S_k. We have

$$d_k = d\frac{|K|}{|S_k|} \qquad \text{(and for } d = 3\ |S_k| = \tfrac{1}{2}|u \times v|\text{)}$$

A.2 Integration on simplices

Any polynomial in the barycentric coordinates can be integrated exactly.

$$\int_K \prod_{i=1}^{d+1} \lambda_i^{n_i}\, dv = d!|K| \frac{\prod\limits_{i=1}^{d+1} n_i!}{\left(\sum\limits_{i=1}^{d+1} n_i + d\right)!} \tag{A.2}$$

see **[EisenbergMalvern73]**, **[VermolenSegal18]**.

A.3 Finite elements

The $d + 1$ basis functions of the P^1 (Courant) element are the barycentric coordinates λ_i defined as being affine with respect to the coordinates and $\lambda_i(x_j) = \delta_{ij}$. Their constant gradient is given by

$$\nabla\lambda_i = -\frac{1}{d_i}\vec{n_i}.$$