

Simple finite element methods in Python

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1 Heat equation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be the computational domain. We suppose to have a disjointed partition of its boundary: $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$. We consider the parabolic equation for the temperature T , heat flux \vec{q} and heat release \dot{q}

$$\left\{ \begin{array}{l} \vec{q} = -k\nabla T \\ \rho C_p \frac{\partial T}{\partial t} + \operatorname{div}(\vec{v}T) + \operatorname{div} \vec{q} = \dot{q} \quad \text{in } \Omega \\ T = T^D \quad \text{in } \Gamma_D \\ k \frac{\partial T}{\partial n} = q^N \quad \text{in } \Gamma_N \\ c_R T + k \frac{\partial T}{\partial n} = q^R \quad \text{in } \Gamma_R \end{array} \right. \quad (1.1)$$

Let $H_f^1 := \left\{ u \in H^1(\Omega) \mid T|_{\Gamma_D} = f \right\}$. The standard weak formulation looks for $T \in H_{T^D}^1$ such that for all $\phi \in H_0^1(\Omega)$

$$\int_{\Omega} \rho C_p \frac{\partial T}{\partial t} \phi - \int_{\Omega} \vec{v}T \cdot \nabla \phi + \int_{\Omega} k \nabla T \cdot \nabla \phi + \int_{\Gamma_R} c_R T \phi + \int_{\Gamma_R \cup \Gamma_N} \vec{v}_n T \phi = \int_{\Omega} \dot{q} \phi + \int_{\Gamma_R} q^R \phi \quad (1.2)$$

We can derive (1.2) from (1.1) by the divergence theorem

$$\int_{\Omega} \operatorname{div} \vec{F} = \int_{\partial\Omega} \vec{F}_n \quad \xLeftrightarrow{F \rightarrow F\phi} \quad \int_{\Omega} (\operatorname{div} \vec{F}) \phi = - \int_{\Omega} \vec{F} \cdot \nabla \phi + \int_{\partial\Omega} \vec{F}_n \phi,$$

which gives with $\vec{F} = \vec{v} + \vec{q}$

$$\int_{\Omega} \operatorname{div}(\vec{v} + \vec{q}) \phi = - \int_{\Omega} \vec{v} \cdot \nabla \phi + \int_{\Omega} k \nabla T \cdot \nabla \phi + \int_{\partial\Omega} \vec{F}_n \phi.$$

Using that ϕ vanishes on Γ_D we have

$$\int_{\partial\Omega} \vec{F}_n \phi = \int_{\Gamma_N \cup \Gamma_R} \vec{F}_n \phi = \int_{\Gamma_N \cup \Gamma_R} \vec{v}_n \phi + \int_{\Gamma_N \cup \Gamma_R} \vec{q}_n \phi,$$

and then with the different boundary conditions, we find

$$\int_{\Gamma_N \cup \Gamma_R} \vec{q}_n \phi = \int_{\Gamma_D} q^N \phi + \int_{\Gamma_R} (\mathbb{R} - c_R T) \phi$$

Appendices

A Finite elements on simplices

A.1 Simplices

We consider an arbitrary non-degenerate simplex $K = (x_0, x_1, \dots, x_d)$. The (signed) volume of K is given by

$$|K| = \frac{1}{d!} \det(x_1 - x_0, \dots, x_d - x_0) = \frac{1}{d!} \det(1, x_0, x_1, \dots, x_d) \quad 1 = (1, \dots, 1)^T. \quad (\text{A.1})$$

The $d+1$ sides S_k (co-dimension one, $d-1$ -simplices or facets) are defined by $S_k = (x_0, \dots, \cancel{x_k}, \dots, x_d)$. The height is $d_k = |P_{S_k} x_k - x_k|$, where P_S is the orthogonal projection on the hyperplane associated to S_k . We have

$$d_k = d \frac{|K|}{|S_k|} \quad (\text{and for } d = 3 \ |S_k| = \frac{1}{2} |u \times v|)$$

A.2 Integration on simplices

Any polynomial in the barycentric coordinates can be integrated exactly.

$$\int_K \prod_{i=1}^{d+1} \lambda_i^{n_i} dv = d! |K| \frac{\prod_{i=1}^{d+1} n_i!}{\left(\sum_{i=1}^{d+1} n_i + d \right)!} \quad (\text{A.2})$$

see [EisenbergMalvern73], [VermolenSegal18].

A.3 Finite elements

The $d + 1$ basis functions of the P^1 (Courant) element are the barycentric coordinates λ_i defined as being affine with respect to the coordinates and $\lambda_i(x_j) = \delta_{ij}$. Their constant gradient is given by

$$\nabla \lambda_i = -\frac{1}{d_i} \vec{n}_i.$$