Simple finite element methods in Python

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A.1 Simplices

We consider an arbitrary non-degenerate simplex $K = (x_0, x_1, \ldots, x_d)$. The (signed) volume of K is given by

$$|K| = \frac{1}{d!}\det(x_1 - x_0, \ldots, x_d - x_0) = \frac{1}{d!}\det(1, x_0, x_1 \ldots, x_d) \quad 1 = (1, \ldots, 1)^T. \tag{A.1}$$

The d+1 sides S_k (co-dimension one, d−1-simplices or facets) are defined by $S_k = (x_0, \ldots, \cancel{x_k}, \ldots, x_d)$. The height is $d_k = |P_{S_k} x_k - x_k|$, where P_S is the orthogonal projection on the hyperplane associated to S_k. We have

$$d_k = d\frac{|K|}{|S_k|} \qquad \text{(and for } d = 3\ |S_k| = \tfrac{1}{2}|u \times v|)$$

A.2 Integration on simplices

Any polynomial in the barycentric coordinates can be integrated exactly.

$$\int_K \prod_{i=1}^{d+1} \lambda_i^{n_i}\, dv = d!|K| \frac{\prod_{i=1}^{d+1} n_i!}{\left(\sum_{i=1}^{d+1} n_i + d\right)!} \tag{A.2}$$

see **[EisenbergMalvern73]**, **[VermolenSegal18]**.

A.3 Finite elements

The $d + 1$ basis functions of the P^1 (Courant) element are the barycentric coordinates λ_i defined as being affine with respect to the coordinates and $\lambda_i(x_j) = \delta_{ij}$. Their constant gradient is given by

$$\nabla \lambda_i = -\frac{1}{d_i}\vec{n_i}.$$