Compliant plugin

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Abstract

1 Constraint forces

This approach unifies soft and hard constraints. Hard constraints are usually implemented using Lagrange multipliers λ in the following equation:

$$\begin{pmatrix} \mathbf{M} & -\mathbf{J}^T \\ \mathbf{J} & \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{ext} \\ -\phi \end{pmatrix} \tag{1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{J}$ is the Jacobian matrix of the constraint(s), $\mathbf{a}$ is the acceleration, $\mathbf{f}_{ext}$ is the net external force applied to the system, λ is the constraint force and ϕ is the constraint violation. λ and ϕ are vectors with as many entries as scalar constraints. The equation system is typically solved using a Schur complement to compute the constraint forces:

$$\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T\lambda = -\phi - \mathbf{J}\mathbf{M}^{-1}\mathbf{f}_{ext} \tag{2}$$

and then the acceleration is computed as $\mathbf{a} = \mathbf{M}^{-1}(\mathbf{f}_{ext} + \mathbf{J}^T\lambda)$.

In the generalized approach, the constraint forces are considered proportional to the constraint violations:

$$\lambda = \frac{1}{c}\left(\phi + d\dot{\phi}\right) \tag{3}$$

where c is the compliance of the constraints and d its damping ratio. Combined with a time discretization scheme, this leads to an equation system similar to (2) as shown is the next section.

2 Time integration

Our implicit scheme is:

$$\begin{aligned} \mathbf{v}_{n+1} &= \mathbf{v}_n + h\mathbf{M}^{-1}\left(\alpha\mathbf{f}_{n+1} + (1-\alpha)\mathbf{f}_n\right) && (4) \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + h\left(\beta\mathbf{v}_{n+1} + (1-\beta)\mathbf{v}_n\right) && (5) \end{aligned}$$

where index n denotes current values while index $n+1$ denotes next values, α is the implicit velocity factor, and β is the implicit position factor. Let

$$\begin{aligned} \mathbf{\Delta v} = \mathbf{v}_{n+1} - \mathbf{v}_n &= h\mathbf{M}^{-1}\left(\alpha\mathbf{f}_{n+1} + (1-\alpha)\mathbf{f}_n\right) && (6) \\ \mathbf{\Delta x} = \mathbf{x}_{n+1} - \mathbf{x}_n &= h(v + \beta\mathbf{\Delta v}) && (7) \end{aligned}$$

be the velocity and position changes across the time step. The constraint violation ϕ and its Jacobian $\mathbf{J}$ are:

$$\begin{aligned} \mathbf{J} &= \frac{\partial\phi}{\partial\mathbf{x}} && (8) \\ \phi_{n+1} &\simeq \phi_n + \mathbf{J}\mathbf{\Delta x} = \phi_n + h\dot{\phi} + \mathbf{J}h\beta\Delta\mathbf{v} && (9) \\ \dot{\phi}_{n+1} &\simeq \phi_{n+1} + \mathbf{J}\mathbf{\Delta v} && (10) \end{aligned}$$

The corresponding forces are:

$$\begin{aligned} \mathbf{f} &= \mathbf{f}_{ext} + \mathbf{J}^T\lambda && (11) \\ \lambda_i &= -\frac{1}{c_i}(\phi_i + d\dot{\phi}_i) && (12) \end{aligned}$$

where the subscript i denotes a scalar constraint.

The average constraint forces are computed using equations 12, 9 and 10:

$$\begin{aligned}
\bar{\lambda}_i &= \alpha\lambda_{n+1} + (1-\alpha)\lambda_n \\
&= -\frac{1}{c_i}(\alpha\phi + \alpha h\dot{\phi} + \alpha h\beta\mathbf{J}\Delta\mathbf{v} + \alpha d\dot{\phi} + \alpha d\mathbf{J}\boldsymbol{\Delta}\mathbf{v} + (1-\alpha)\phi + (1-\alpha)d\dot{\phi}) \\
&= -\frac{1}{c_i}(\phi + d\dot{\phi} + \alpha h\dot{\phi} + \alpha(h\beta + d)\mathbf{J}\boldsymbol{\Delta}\mathbf{v})
\end{aligned}$$

We can rewrite the previous equation as:

$$\mathbf{J}\boldsymbol{\Delta}\mathbf{v} + \frac{1}{\alpha(h\beta+d)}\mathbf{C}\bar{\lambda} = -\frac{1}{\alpha(h\beta+d)}(\phi + (d+\alpha h)\dot{\phi}) \tag{13}$$

where values without indices denote current values. The complete equation system is:

$$\begin{pmatrix} \frac{1}{h}\mathbf{PM} & -\mathbf{PJ}^T \\ \mathbf{J} & \frac{1}{l}\mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Delta}\mathbf{v} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{Pf}_{ext} \\ -\frac{1}{l}(\phi + (d+\alpha h)\dot{\phi}) \end{pmatrix} \tag{14}$$

where $l = \alpha(h\beta + d)$ The system is singular due to matrix $\mathbf{P}$, however we can use $\mathbf{PM}^{-1}\mathbf{P}$ as inverse mass matrix to compute a Schur complement:

$$\begin{array}{ccc}
\left(h\mathbf{JPM}^{-1}\mathbf{PJ}^T + \frac{1}{l}\mathbf{C}\right)\bar{\lambda} & = & -\frac{1}{l}\left(\phi + (d+h\alpha)\dot{\phi}\right) - h\mathbf{JM}^{-1}\mathbf{f}_{ext} \\
\boldsymbol{\Delta}\mathbf{v} & = & h\mathbf{PM}^{-1}(\mathbf{f}_{ext} + \mathbf{J}^T\bar{\lambda}) \\
\boldsymbol{\Delta}\mathbf{x} & = & h(\mathbf{v} + \beta\boldsymbol{\Delta}\mathbf{v})
\end{array}$$

3 Matrix assembly

The equation system, in its most general form, can be written as:

$$\begin{pmatrix} \mathbf{M} & -\mathbf{J}^T \\ \mathbf{J} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \phi \end{pmatrix} \tag{15}$$

We assemble the 7 terms of equation 15 separately. Figure 3 shows an example of mechanical system and matrix. The independent DOFs are X_a and X_d. State X_b is attached to X_a using a simple mapping, and a mass matrix M_{bb} is defined at this level. State X_c is attached to X_b using a simple mapping, and a compliance matrix $C_{\alpha\alpha}$ (possibly a deformation force) is applied to these DOFs. State X_e is attached to X_a and X_d at the same time, using a MultiMapping. A compliance matrix $C_{\beta\beta}$, possibly an interaction force, is applied to these DOFs, while a mass M_{dd} is applied to X_d. The corresponding equation system has the block structure shown in the right of Figure 3.

The 8 parts of the equation system are shown in Figure 3. The J matrices are the mapping matrices. The bottom row has two mappings, since the state X_e impacted by compliance β depends on two parent states. Offset matrices J_{*0} are shown in the bottom of the figure. They are composed of identity matrices (represented with a diagonal in a block) and null blocks. These matrices are used to shift the indices of local vector and matrices to global numbering.

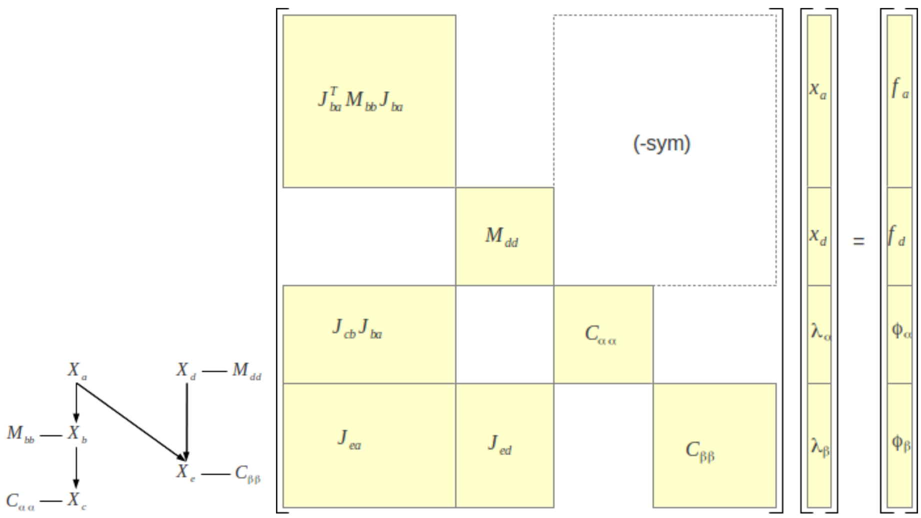

Figure 1: A mechanical system and its matrix equation. Left: scene graph. Right: block view of the corresponding equation system, with non-null blocks highlighted in yellow.

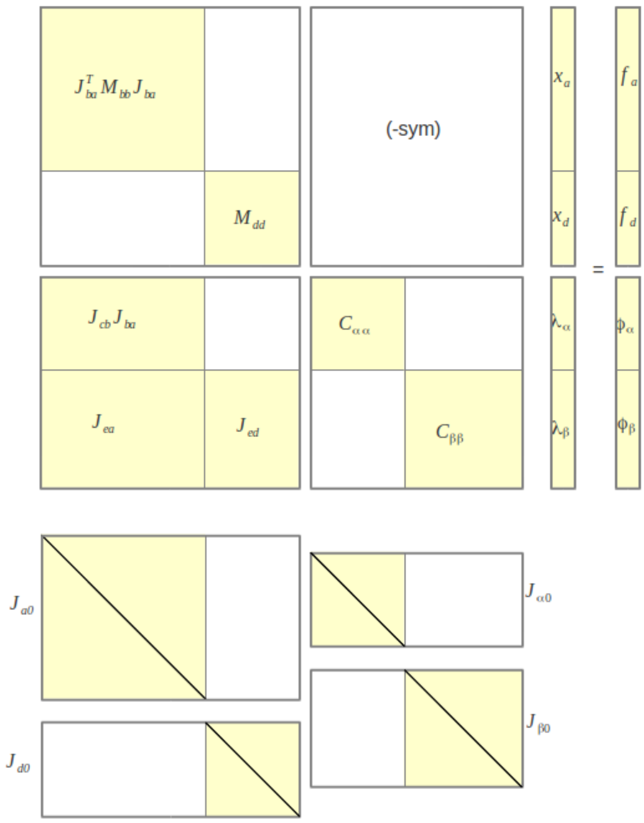

Figure 2: Top: the 8 parts of the equation system. Bottom: the offset matrices used in the assembly.