Connections on Principal Fibre Bundles

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Chapter 1

Principal Fibre Bundles

We are going to closely follow the following: [1] and [4]. We will use figures from: [2] and [3]. We also consulted the book: [5].

1.1 Introduction

There is a useful concept in manifolds. Start with two manifolds M_1, M_2 and build from them a new manifold, using the product topology: $M_1 \times M_2$. A fibre bundle is a useful generalization of this concept.

Product Manifolds: A Visual Picture

One way to interpret a product manifold is to place a copy of M_1 at each point of M_2. Alternatively, we could be placing a copy of M_2 at each point of M_1.

Example

Take $M_1 = S^1$, $M_2 = (0, 1)$. The product topology here just gives a piece of a cylinder.

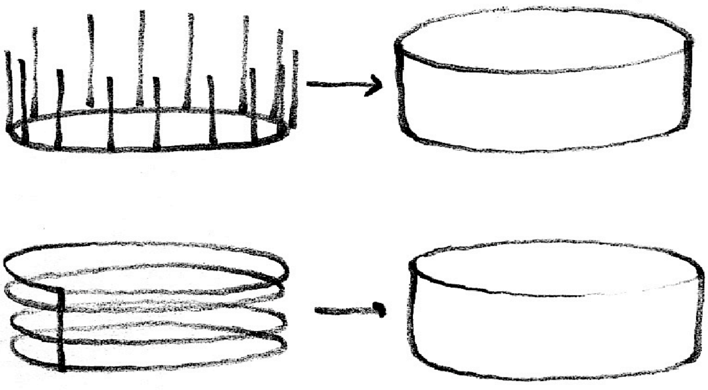

Figure 1.1: Placing a line segment at each point of a circle and placing a circle at each point of a line segment.

A More General Concept

A fibre bundle is an object closely related to this idea. In any local neighbourhood, a fibre bundle looks like $M_1 \times M_2$. Globally, however, a fibre bundle is generally not a product manifold.

Example: Moebius band

We can create the Moebius band by starting with the circle S^1 and at each point on the circle attaching a copy of the open interval $(0,1)$, but in a nontrivial manner. Instead of just attaching a bunch of parallel intervals to the circle, our intervals perform a 180^o twist as we go around.

Another Way to Think About This

Consider a rectangular strip. This can of course be seen as the product of two line segments. If now one wants to join two opposite edges of the strip to turn one of the line segments into a circle, there are two ways to go about this.

The first possibility is to join the two edges in a straightforward way to form a cylinder C. Then $C = S^1 \times (0,1)$. The second way is the more interesting one. Before gluing the edges together, perform a twist on one of them to arrive at a Moebius strip M_0. Locally, along each open subset $U \subset S^1$, the Moebius strip still looks like a product, $M_0 = U \times (0,1)$. Globally, however, there is no unambiguous and continuous way to write a point $m \in M_0$ as a cartesian pair $(s,t) \in S^1 \times (0,1)$.

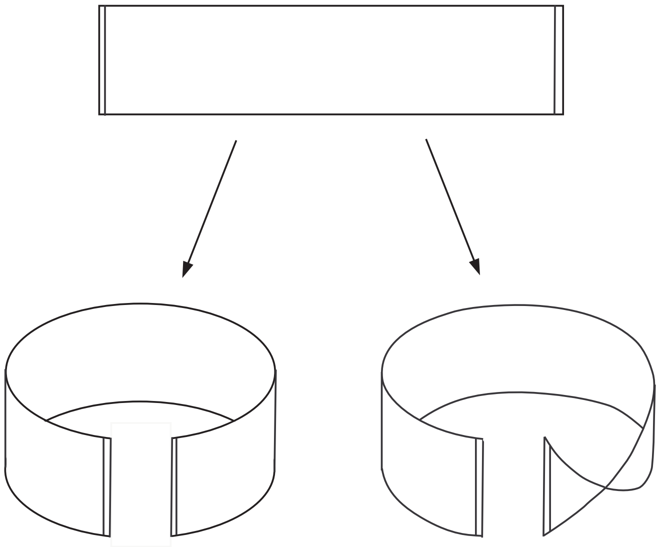

Figure 1.2: Gluing together a rectangular strip to arrive at a Moebius strip or a piece of a cylinder.

1.2 Vector Bundles

Definition 1. *Let M be a smooth manifold. The product manifold $M \times \mathbb{R}^n$ is called an n-dimensional* **product bundle** *over M. The projection $pr_1 : M \times \mathbb{R}^n \to M$ on the first factor is a surjective smooth map called the* **projection** *of the bundle. For a point m in M, the inverse image $pr_1^{-1}(m) = \{m\} \times \mathbb{R}^n$ is called the* **fibre** *over m. It has a natural vector space structure such that the projection $pr_2 : \{m\} \times \mathbb{R}^n \to \mathbb{R}^n$ on the second factor is a linear isomorphism, i.e. with operations $(m, u) + (m, v) = (m, u + v)$ and $k(m, u) = (m, ku)$ for vectors $u, v \in \mathbb{R}^n$ and $k \in \mathbb{R}$.*

The product bundle is an example of a more general bundle called a vector bundle. A vector bundle is a smooth manifold which looks locally like a product bundle.

Definition 2. *Let M be a smooth manifold. A smooth manifold E is called an n-dimensional* **vector bundle** *over M if the following three conditions are satisfied:*
(i) There is a surjective smooth map $\pi : E \to M$ which is called the **projection** *of the* **total space** *E onto the* **base space** *M.*
(ii) For each point $m \in M$, $\pi^{-1}(m)$ is an n-dimensional vector space called the **fibre** *over m.*

(iii) For each point $m \in M$ there is an open neighbourhood U around m and a diffeomorphism $t : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "t"] \arrow[dr, "\pi_U"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

is commutative, and t is a linear isomorphism on each fibre. The pair $(t, \pi^{-1}(U))$ is called a **local trivialization** *around m.*

We often use the projection $\pi : E \to M$ insstead of the total space E to denote the bundle in order to indicate more of the bundle struture in the notation. The fibre $\pi^{-1}(m)$ is also denoted E_m.
A family of local trivializations $\{(t_\alpha, \pi^{-1}(U_\alpha))|\alpha \in A\}$, where $\{U_\alpha|\alpha \in A\}$ is an open cover of M, is called a **trivializing cover** of E.

1.3 Fibre Bundles

Definition 3. *Let M and F be smooth manifolds, and let G be a Lie group which acts on F effectively on the left. A smooth manifold E is called a* **fibre bundlle** *over M with* **fibre** *F and* **structure group** *G if the following three conditions are satisfied:*
(i) There is a surjective smooth map $\pi : E \to M$ which is called the **projection** *of the* **total space** *E onto the* **base space** *M. For each point $m \in M$ the inverse image $\pi^{-1}(m)$ is called the* **fibre over** *m.*
(ii) We have a family of G-related local trivializations $\mathcal{T} = \{(t_\alpha, \pi^{-1}(U_\alpha))|\alpha \in A\}$, in the sense defined below, such that $\{U_\alpha|\alpha \in A\}$is an open cover of M. By a **local trivialization** *we mean a pair $(t, \pi^{-1}(U))$, where U is an open subset of M and $t : \pi^{-1}(U) \to U \times F$ is a diffeomorphism such that the diagram*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "t"] \arrow[dr, "\pi_U"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

*is commutative. Two local trivializations $(t, \pi^{-1}(U))$ and $(s, \pi^{-1}(V))$ are said to be G-***related** *if there is a smooth map $\phi : U \cap V \to G$, called the* **transition map***, so that*

$$s \circ t^{-1}(m, v) = (m, \phi(m)v) \tag{1.1}$$

for every $m \in U \cap V$ and $v \in F$.
(iii) The family $\mathcal{T}$ is maximal in the sense that if $(t, \pi^{-1}(U))$ is a local trivial-

ization which is G-related to every local trivialization in $\mathcal{T}$, then $(t, \pi^{-1}(U)) \in \mathcal{T}$.

1.4 Principal Fibre Bundles

Definition 4. *Let M be a smooth manifold and G be a Lie group. Then a* **principal fibre bundle** *P over M with* **structure group** *G is a fibre bundle $\pi : P \to M$ where the fibre F coincides with the structure group G which is acting on F by left translation. The principal bundle P is also denoted by $P(M, G)$ and is called a* **principal G-bundle** *over M.*

If $\pi : P \to M$ is a principal G-bundle, we may define an operation of G on P on the right in the following way.
Let $u \in P_m$ (i.e., $u \in \pi^{-1}(m) \subset P$) and $g \in G$, and choose a local trivialization $(t, \pi^{-1}(U))$ in P with $m \in U$.
If $t(u) = (m, h)$, we define $ug \in P_m$ so that $t(ug) = (m, hg)$, i.e.,

$$\pi(ug) = \pi(u) \text{ and } t_m(ug) = t_m(u)g \text{ when } m = \pi(u) \in U \tag{1.2}$$

where $t_m : P_m \to G$ is defined by $t_m = pr_2 \circ t|_{P_m}$.
We must show that the operation of G does not depend on the local trivialization $(t, \pi^{-1}(U))$. So let $(s, \pi^{-1}(V))$ be another local trivialization with $m \in V$ which is G-related to $(t, \pi^{-1}(U))$ with transition map $\phi : U \cap V \to G$.
Let us also use the following fact.

Remark 1. *Let $(t, \pi^{-1}(U))$ and $(s, \pi^{-1}(V))$ be two G-related local trivializations in the fibre bundle $\pi : E \to M$ with transition map $\phi : U \cap V \to G$. Then we have that*

$$s_m \circ t_m^{-1}(v) = \phi(m)v \tag{1.3}$$

for every $m \in U \cap V$ and $v \in F$, i.e. the diffeomorphism $s_m \circ t_m^{-1} : F \to F$ coincides with the operation of the group element $\phi(m) \in G$ on the fibre F for every $m \in U \cap V$. Since G acts effectively on F, we know that $\phi(m)$ is uniquely determined by $s_m \circ t_m^{-1}$.
Moreover, we have that

$$s_m(u) = \phi(m)t_m(u) \tag{1.4}$$

when $m \in U \cap V$ and $u \in E_m$.

Then we have that

$$s_m(ug) = \phi(m)t_m(ug) = \phi(m)[t_m(u)g] = [\phi(m)t_m(u)]g = s_m(u)g \tag{1.5}$$

showing that we have a well-defined map $v : P \times G \to P$ with $v(u, g) = ug$ for $u \in P$ and $g \in G$.

Now we have that

$$u(g_1 g_2) = (ug_1)g_2 \text{ and } ue = u \tag{1.6}$$

for every $u \in P$ and $g_1, g_2 \in G$, since

$$t_m(u(g_1g_2)) = t_m(u)(g_1g_2) = (t_m(u)g_1)g_2 = t_m(ug_1)g_2 = t_m((ug_1)g_2)$$
and $t_m(ue) = t_m(u)e = t_m(u)$ when $u \in P_m$

Also, as the map

$$t \circ v \circ (t^{-1} \times \mathrm{id}) : U \times G \times G \to U \times G$$

is given by

$$t \circ v \circ (t^{-1} \times \mathrm{id})(m, h, g) = (m, hg) \tag{1.7}$$

for $m \in U$ and $h, g \in G$, it follows that v is smooth and hence is an operation of G on P on the right.

We see that G acts freely on P, and that the orbits of G are the fibres of P.
Indeed, if $ug = u$ for $u \in P_m$, then $t_m(u)g = t_m(u)$ which implies that $g = e$, therefore G acts freely on P.
Moreover, given $u_1, u_2 \in P_m$ there is a $g \in G$ with $t_m(u_1)g = t_m(u_2)$ so that $u_1 g = u_2$. So the orbits of G are the fibres of P.

Using these results, we are ready to give an alternative definition of a principal fibre bundle.

Definition 5. *A* **principal fibre bundle** *consists of the following data:*
· *a manifold P, called the* **total space**
· *a Lie group G acting freely on P on the right:*

$P \times G \to P$
$(p, g) \mapsto pg$ *(or sometimes $R_g p$)*

where by a free action we mean that the stabilizer of every point is trivial, i.e. that every element of G (except the identity) moves every point in P.
We will also assume that the space of orbits $M = P/G$ is a manifold called the **base** *and the natural map $\pi : P \to M$ taking a point to its orbit is a*

smooth surjection.
For every $m \in M$, the submanifold $\pi^{-1}(m) \subset P$ is called the **fibre over** *m.*
Further, this data will be subject to the condition of **local triviality** *that M admits an open cover $\{U_\alpha\}$ and G-equivariant diffeomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that the following diagram commutes:*

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "pr_1"] \\
& U_\alpha &
\end{tikzcd}$$

i.e., $\psi_\alpha(p) = (\pi(p), g_\alpha(p))$, for some G-equivariant map $g_\alpha : \pi^{-1}(U_\alpha) \to G$ which is a fibrewise diffeomorphism.
Equivariance means that $g_\alpha(pg) = g_\alpha(p)g$.

We say that the bundle is **trivial** if there exists a diffeomorphism $\Psi : P \to M \times G$ such that $\Psi(p) = (\pi(p), \psi(p))$ and such that $\psi(pg) = \psi(p)g$.

A **section** is a smooth map $s : M \to P$ such that $\pi \circ s = \text{id}$. In other words, it is a smooth assignment to each point in the base of a point in the fibre over it.

***Exercise* 1:** Show that a principal fibre bundle admits a section if and only if it is trivial.

Solution. Let (P, π, M, G) be a principal fibre bundle over M and let $s : M \to P$ be a section.
Since the right action is transitive and free, any element $p \in P$ is uniquely written as $s(m)a$ for some $m \in M$ and $a \in G$.
Define a map $\Psi : P \to M \times G$ by

$$\Psi : s(m)a \mapsto (m, a) \tag{1.8}$$

Let us also define $\psi(s(m)a) = a$. Also note that $\pi(s(m)a) = \pi(s(m)) = m$.
So $\Psi(s(m)a) = (\pi(s(m)a), \psi(s(m)a)) = (m, a)$, where $\psi((s(m)a)g) = \psi(s(m)(ag)) = ag = \psi(s(m)a)g$.
So we have found a diffeomorphism $\Psi : P \to M \times G$ such that $\Psi(p) = (\pi(p), \psi(p))$ and such that $\psi(pg) = \psi(p)g$, so the bundle is trivial.

Conversely, suppose the bundle is trivial. Let $\Psi^{-1} : M \times G \to P$ be the inverse of the diffeomorphism Ψ. Take a fixed element $g \in G$. Then $s_g : M \to P$ defined by $s_g(m) = \Psi^{-1}(m, g)$ is a section, since

$$\begin{aligned}
&\text{pr}_1 \circ \Psi(p) = \pi(p) \Leftrightarrow \text{pr}_1 \circ \Psi \circ s_g(m) = \pi \circ s_g(m) \Leftrightarrow \\
&\text{pr}_1 \circ \Psi \circ \Psi^{-1}(m, g) = \pi \circ s_g(m) \Leftrightarrow \text{pr}_1(m, g) = \pi \circ s_g(m) \Leftrightarrow \\
&\pi \circ s_g(m) = m \Leftrightarrow \pi \circ s = \text{id}
\end{aligned} \tag{1.9}$$

$\square$

Nevertheless, since P is locally trivial, local sections do exist.
In fact, there are local sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ canonically associated to the trivialization, defined so that for every $m \in U_\alpha$, $\psi_\alpha(s_\alpha(m)) = (m, e)$, where $e \in G$ is the identity element.
In other words, $g_\alpha \circ s_\alpha : U_\alpha \to G$ is the constant function sending every point to the identity.
Conversely, a local section s_α allows us to identify the fibre over m with G.
Indeed, given any $p \in \pi^{-1}(m)$, there is a unique group element $g_\alpha(p) \in G$ such that $p = s_\alpha(m)g_\alpha(p)$.
On nonempty overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have two ways of trivializing the bundle:
For $m \in U_{\alpha\beta}$ and $p \in \pi^{-1}(m)$, we have $\psi_\alpha(p) = (m, g_\alpha(p))$ and $\psi_\beta(p) = (m, g_\beta(p))$, whence there is $\bar{g}_{\alpha\beta}(p) \in G$ such that $g_\alpha(p) = \bar{g}_{\alpha\beta}(p)g_\beta(p)$.
In other words, $\bar{g}_{\alpha\beta}(p) = g_\alpha(p)g_\beta(p)^{-1}$.
In fact, $\bar{g}_{\alpha\beta}(p)$ is constant on each fibre:

$$\begin{aligned}\bar{g}_{\alpha\beta}(pg) &= g_\alpha(pg)g_\beta(pg)^{-1} \\ &= g_\alpha(p)gg^{-1}g_\beta(p)^{-1} \text{ due to equivariance of } g_\alpha, g_\beta \\ &= g_\alpha(p)g_\beta(p)^{-1} = \bar{g}_{\alpha\beta}(p)\end{aligned} \tag{1.10}$$

In other words, $\bar{g}_{\alpha\beta}(p) = g_{\alpha\beta}(\pi(p))$ for some function $g_{\alpha\beta} : U_{\alpha\beta} \to G$.
These **transition functions** obey the following **cocycle conditions**:

$$g_{\alpha\beta}(m)g_{\beta\alpha}(m) = e, \qquad \forall m \in U_{\alpha\beta} \tag{1.11}$$

Proof. $g_{\alpha\beta}(m)g_{\beta\alpha}(m) = g_\alpha(p)g_\beta(p)^{-1}g_\beta(p)g_\alpha(p)^{-1} = g_\alpha(p)g_\alpha(p)^{-1} = e$ $\square$

$$g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = e \qquad \forall m \in U_{\alpha\beta\gamma} \tag{1.12}$$

Proof.

$$\begin{aligned}&g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = g_\alpha(p)g_\beta(p)^{-1}g_\beta(p)g_\gamma(p)^{-1}g_\gamma(p)g_\alpha(p)^{-1} \\ &= g_\alpha(p)g_\alpha(p)^{-1} = e\end{aligned}$$

$\square$

where $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$.

***Exercise* 2:** Show that on double overlaps, the canonical sections s_α are related by

$$s_\beta(m) = s_\alpha(m)g_{\alpha\beta}(m), \qquad \forall m \in U_{\alpha\beta} \tag{1.13}$$

Solution. Let $m \in U_{\alpha\beta}$. Take $p \in \pi^{-1}(m)$. Then there is a unique group element $g_\alpha(p) \in G$ such that $p = s_\alpha(m)g_\alpha(p)$ and another unique group element $g_\beta(p) \in G$ such that $p = s_\beta(m)g_\beta(p)$ (since $m \in U_\alpha$ and $m \in U_\beta$). Therefore,

$$\begin{aligned} &s_\beta(m)g_\beta(p) = s_\alpha(m)g_\alpha(p) \Leftrightarrow s_\beta(m) = s_\alpha(m)g_\alpha(p)g_\beta(p)^{-1} \\ &= s_\alpha(m)g_{\alpha\beta}(m) \end{aligned} \tag{1.14}$$

Since $m \in U_{\alpha\beta}$ was arbitrary, QED. □

One can reconstruct the bundle from an open cover $\{U_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$ obeying the cocycle conditions as follows:

$$P = \bigsqcup_\alpha (U_\alpha \times G)/\sim \tag{1.15}$$

where $(m, g) \sim (m, g_{\alpha\beta}(m)g)$, $\forall m \in U_{\alpha\beta}$ and $g \in G$.
Notice that π is induced by the projection onto the first factor and the action of G on P is induced by right multiplication on G, both of which are preserved by the equivalence relation, which uses left multiplication by the transition functions. (Associativity of group multiplication guarantees that right and left multiplications commute).

Example: Moebius band
The boundary of the Moebius band is an example of a nontrivial principal $\mathbb{Z}_2$-bundle. $\mathbb{Z}_2$ is the multiplicative group comprising 1 and -1. This can be described as follows. Let $S^1 \subset \mathbb{C}$ denote the complex numbers of unit modulus and let $\pi : S^1 \to S^1$ be the map defined by $z \mapsto z^2$. Then the fibre $\pi^{-1}(z^2) = \{\pm z\}$ consists of two points. A global section would correspond to choosing a square-root function smoothly on the unit circle.

Remark 2. *(Branch points of $z^{1/2}$)*
Let $z = re^{i\theta}$. Then $z^{1/2} = r^{1/2}e^{\frac{i}{2}\theta}$. If we perform a circuit of the origin by setting $z = \varepsilon e^{it}$ and then letting t increase from 0 to 2π, then the value of $z^{1/2}$ does not return to its initial value $\varepsilon^{1/2}$, but switches to $-\varepsilon^{1/2}$. Thus there is a branch point at $z = 0$. There is also a branch point at infinity, since $(\frac{1}{z})^{1/2}$ has a branch point at $z = 0$.

This does not exist, however, since any definition of $z^{1/2}$ always has a branch cut from the origin out to the point at infinity.
Therefore the bundle is not trivial.
In fact, if the bundle were trivial, the total space would be disconnected, being two distinct copies of the circle. However, building a paper model of

the Moebius band, one quickly sees that its boundary is connected.
We can understand this bundle in terms of the local data as follows.
Cover the circle by two overlapping open sets: U_1 and U_2. Their intersection is the disjoint union of two intervals in the circle: $V_1 \sqcup V_2$.
Let $g_i : V_i \to \mathbb{Z}_2$ denote the transition functions which are constant since V_i are connected and $\mathbb{Z}_2$ is discrete, so we can think of $g_i \in \mathbb{Z}_2$. There are no triple overlaps, so the cocycle condition is vacuously satisfied.
It is an easy exercise to check that the resulting bundle is trivial if and only if $g_1 = g_2$.

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