Magnetic Materials

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Stereographic Projection

Let us start from a change of variables. The magnetization has unit length $|\mathbf{m}| = 1$ so this change of variables can be considered as a stereographic projection from the unit sphere to the complex plane. Let us define the stereographic projection as

$$\Omega = \frac{m_1 + im_2}{1 + m_3} \tag{1}$$
$$\overline{\Omega} = \frac{m_1 - im_2}{1 + m_3} \tag{2}$$

We want to express the magnetization $\mathbf{m}$ in terms of the stereographic projection variables Ω, $\overline{\Omega}$.
To do this, we start from

$$\Omega\overline{\Omega} = \frac{m_1 + im_2}{1 + m_3}\frac{m_1 - im_2}{1 + m_3} = \frac{(m_1 + im_2)(m_1 - im_2)}{(1 + m_3)^2} = \frac{m_1^2 + m_2^2}{(1 + m_3)^2}$$

Then,

$$1 + \Omega\overline{\Omega} = \frac{(1 + m_3)^2 + m_1^2 + m_2^2}{(1 + m_3)^2} = \frac{1 + 2m_3 + m_1^2 + m_2^2 + m_3^2}{(1 + m_3)^2} = \frac{2(1 + m_3)}{(1 + m_3)^2} = \frac{2}{1 + m_3}$$

since $|\mathbf{m}|^2 = m_1^2 + m_2^2 + m_3^2 = 1$.
But this implies that

$$1 + m_3 + \Omega\overline{\Omega} + m_3\Omega\overline{\Omega} = 2 \Leftrightarrow m_3(1 + \Omega\overline{\Omega}) = 1 - \Omega\overline{\Omega} \Leftrightarrow m_3 = \frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}}$$

Next, we compute

$$\begin{aligned}
&\Omega + \overline{\Omega} = \frac{m_1 + im_2 + m_1 - im_2}{1 + m_3} = \frac{2m_1}{1 + m_3} \\
&\Rightarrow \Omega + \overline{\Omega} + m_3(\Omega + \overline{\Omega}) = 2m_1 \\
&\Rightarrow \Omega + \overline{\Omega} + \left(\frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}}\right)(\Omega + \overline{\Omega}) = 2m_1 \\
&\Leftrightarrow \Omega + \overline{\Omega} + \left(\frac{\Omega + \overline{\Omega} - \Omega\overline{\Omega}(\Omega + \overline{\Omega})}{1 + \Omega\overline{\Omega}}\right) = 2m_1 \\
&\Leftrightarrow \frac{\Omega + \overline{\Omega} + \Omega\overline{\Omega}(\Omega + \overline{\Omega}) + \Omega + \overline{\Omega} - \Omega\overline{\Omega}(\Omega + \overline{\Omega})}{1 + \Omega\overline{\Omega}} = 2m_1 \\
&\Leftrightarrow m_1 = \frac{\Omega + \overline{\Omega}}{1 + \Omega\overline{\Omega}}
\end{aligned}$$

Finally,

$$
\begin{aligned}
&\Omega = \frac{m_1 + im_2}{1 + m_3} \Leftrightarrow \Omega + \Omega m_3 = m_1 + im_2 \Leftrightarrow \\
&\Omega + \Omega\left(\frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}}\right) = \frac{\Omega + \overline{\Omega}}{1 + \Omega\overline{\Omega}} + im_2 \Leftrightarrow \\
&\frac{\Omega(1 + \Omega\overline{\Omega}) + \Omega - \Omega^2\overline{\Omega} - \Omega - \overline{\Omega}}{1 + \Omega\overline{\Omega}} = im_2 \Leftrightarrow \\
&\frac{\Omega + \Omega^2\overline{\Omega} + \Omega - \Omega^2\overline{\Omega} - \Omega - \overline{\Omega}}{1 + \Omega\overline{\Omega}} = im_2 \Leftrightarrow \\
&m_2 = \frac{1}{i}\frac{\Omega - \overline{\Omega}}{1 + \Omega\overline{\Omega}}
\end{aligned}
$$

Summarizing the results,

$$
m_1 = \frac{\Omega + \overline{\Omega}}{1 + \Omega\overline{\Omega}} \tag{3}
$$

$$
m_2 = \frac{1}{i}\frac{\Omega - \overline{\Omega}}{1 + \Omega\overline{\Omega}} \tag{4}
$$

$$
m_3 = \frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}} \tag{5}
$$

Having expressed the magnetization in terms of the stereographic projection complex variables we want to express the energy functionals in terms of the complex variables as well.
Let us consider only one dimension. The energy functional is comprised of the exchange energy, the anisotropy energy and the Dzyaloshinskii-Moriya energy:

$$
E(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{R}} \partial_x\mathbf{m} \cdot \partial_x\mathbf{m}\,\mathrm{d}x + \frac{\kappa}{2}\int_{\mathbb{R}} (1 - m_3^2)\,\mathrm{d}x + \lambda\int_{\mathbb{R}} \mathbf{m} \cdot (\nabla \times \mathbf{m})\,\mathrm{d}x
$$

We will consider two cases. First, we will consider only the exchange and anisotropy energies and obtain the domain wall solution. Next, we will consider the exchange and Dzyaloshinskii-Moriya energies and obtain the spiral solution. We will also show that the energy only due to the exchange interaction and the anisotropy is non-negative (for any solution), while when we add the Dzyaloshinskii-Moriya interaction, the spiral solution corresponds to negative energy, and thus has a lower energy than the domain wall solution. We want $E(\Omega, \overline{\Omega})$. We will start with the following calculations, using Equations 3, 4, 5:

$$\partial_x m_1 = \frac{(\Omega_x + \overline{\Omega}_x)(1+\Omega\overline{\Omega}) - (\overline{\Omega}_x\Omega + \Omega_x\overline{\Omega})(\Omega+\overline{\Omega})}{(1+\Omega\overline{\Omega})^2}$$
$$\partial_x m_2 = \frac{1}{i}\frac{(\Omega_x - \overline{\Omega}_x)(1+\Omega\overline{\Omega}) - (\overline{\Omega}_x\Omega + \Omega_x\overline{\Omega})(\Omega-\overline{\Omega})}{(1+\Omega\overline{\Omega})^2}$$
$$\partial_x m_3 = \frac{-(\overline{\Omega}_x\Omega + \Omega_x\overline{\Omega})(1+\Omega\overline{\Omega}) - (\overline{\Omega}_x\Omega + \Omega_x\overline{\Omega})(1-\Omega\overline{\Omega})}{(1+\Omega\overline{\Omega})^2}$$
$$= -\frac{(\overline{\Omega}_x\Omega + \Omega_x\overline{\Omega})(1+\Omega\overline{\Omega}+1-\Omega\overline{\Omega})}{(1+\Omega\overline{\Omega})^2}$$
$$= -\frac{2(\overline{\Omega}_x\Omega + \Omega_x\overline{\Omega})}{(1+\Omega\overline{\Omega})^2}$$

These equations simplify to

$$\partial_x m_1 = \frac{\Omega_x(1-\overline{\Omega}^2) + \overline{\Omega}_x(1-\Omega^2)}{1+\Omega\overline{\Omega})^2} \tag{6}$$
$$\partial_x m_2 = \frac{1}{i}\frac{\Omega_x(1+\overline{\Omega}^2) - \overline{\Omega}_x(1+\Omega^2)}{(1+\Omega\overline{\Omega})^2} \tag{7}$$
$$\partial_x m_3 = -2\frac{\Omega_x\overline{\Omega} + \overline{\Omega}_x\Omega}{(1+\Omega\overline{\Omega})^2} \tag{8}$$

We will first treat the exchange energy by considering its argument:

$$\partial_x \mathbf{m}\cdot\partial_x\mathbf{m} = (\partial_x m_1)^2 + (\partial_x m_2)^2 + (\partial_x m_3)^2$$
$$= \frac{\Omega_x^2(1-\overline{\Omega}^2)^2 + 2\Omega_x\overline{\Omega}_x(1-\overline{\Omega}^2)(1-\Omega^2) + \overline{\Omega}_x^2(1-\Omega^2)^2}{(1+\Omega\overline{\Omega})^4}$$
$$-\frac{\Omega_x^2(1+\overline{\Omega}^2)^2 - 2\Omega_x\overline{\Omega}_x(1+\overline{\Omega}^2)(1+\Omega^2) + \overline{\Omega}_x^2(1+\Omega^2)^2}{(1+\Omega\overline{\Omega})^4}$$
$$+4\frac{\Omega_x^2\overline{\Omega}^2 + 2\Omega_x\overline{\Omega}_x\Omega\overline{\Omega} + \overline{\Omega}_x^2\Omega^2}{(1+\Omega\overline{\Omega})^4}$$

Expanding, cancelling and simplifying,

$$\partial_x\mathbf{m}\cdot\partial_x\mathbf{m} = 4\frac{\Omega_x\overline{\Omega}_x}{(1+\Omega\overline{\Omega})^2} \tag{9}$$

Now we will treat the anisotropy term

$$(1-m_3^2) = 1 - \frac{(1-\Omega\overline{\Omega})^2}{(1+\Omega\overline{\Omega})^2} = 4\frac{\Omega\overline{\Omega}}{(1+\Omega\overline{\Omega})^2} \tag{10}$$

This is enough to find the domain wall solution, but let us obtain the Dzyaloshinskii-Moriya energy in terms of Ω, $\overline{\Omega}$ as well in order to use it later to find the spiral solution.
For the Dzyaloshinskii-Moriya energy we will consider the one dimension to be the z-direction. Then,

$$\begin{aligned}
E_{DM} &= \lambda \int_{\mathbb{R}} \mathbf{m}\cdot(\nabla\times\mathbf{m})\,\mathrm{d}z = \lambda\int_{\mathbb{R}}(m_2\partial_z m_1 - m_1\partial_z m_2)\,\mathrm{d}z \\
&= \lambda\int_{\mathbb{R}}\left[\frac{1}{i}\frac{\Omega-\overline{\Omega}}{1+\Omega\overline{\Omega}}\frac{\Omega_z(1-\overline{\Omega}^2)+\overline{\Omega}_z(1-\Omega^2)}{(1+\Omega\overline{\Omega})^2} - \frac{\Omega+\overline{\Omega}}{1+\Omega\overline{\Omega}}\frac{1}{i}\frac{\Omega_z(1+\overline{\Omega}^2)-\overline{\Omega}_z(1+\Omega^2)}{(1+\Omega\overline{\Omega})^2}\right]\mathrm{d}z \\
&= \frac{\lambda}{i}\int_{\mathbb{R}}\frac{\Omega_z[-2\overline{\Omega}-2\Omega\overline{\Omega}^2]+\overline{\Omega}[2\Omega+2\overline{\Omega}\Omega^2]}{(1+\Omega\overline{\Omega})^3}\,\mathrm{d}z \\
&= \frac{2\lambda}{i}\int_{\mathbb{R}}\frac{\overline{\Omega}_z\Omega-\Omega_z\overline{\Omega}}{(1+\Omega\overline{\Omega})^2}\,\mathrm{d}z
\end{aligned}$$

Now we will recover a standard result from the calculus of variations, namely the natural boundary conditions in a simple case when the variation is not necessarily fixed at the boundaries. This is a variable end point problem, a special case of which can be stated as follows. Let us have a functional $E(\overline{\Omega})$, where $\overline{\Omega}=\overline{\Omega}(x)$. Among all curves whose end points lie on two given vertical lines $x=a$ and $x=b$, find the curve for which the functional has an extremum. For us, a and b until now have been $-\infty$ and ∞ but we can consider finite a and b as well. This problem means that $\overline{\Omega}$ can only lie on the verticals at $\pm\infty$. These are the simplest variable end point boundary conditions we can get, that is why we recover this result.
Consider the following functional

$$E(\overline{\Omega}) = \int_{\mathbb{R}} F(\overline{\Omega},\overline{\Omega}_x)\,\mathrm{d}x$$

We will calculate the variation δE of the functional, which is the linear part of ΔE where

$$\Delta E = E(\overline{\Omega}+h) - E(\overline{\Omega})$$

where h is an arbitrary test function. Therefore, and using a Taylor expansion,

$$\begin{aligned}
\Delta E &= \int_{\mathbb{R}}[F(\overline{\Omega}+h,\overline{\Omega}_x+h_x)-F(\overline{\Omega},\overline{\Omega}_x)]\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\left(\frac{\partial F}{\partial\overline{\Omega}}h + \frac{\partial F}{\partial(\overline{\Omega}_x)}h_x\right)\mathrm{d}x + \ldots
\end{aligned}$$

and

$$\delta E = \int_{\mathbb{R}} \left(\frac{\partial F}{\partial \overline{\Omega}} h + \frac{\partial F}{\partial (\overline{\Omega}_x)} h_x \right) \mathrm{d}x$$

Now the test function h does not necessarily vanish at the boundaries, therefore

$$\begin{aligned} \delta E &= \int_{\mathbb{R}} \left(\frac{\partial F}{\partial \overline{\Omega}} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial (\overline{\Omega}_x)} \right) \right) h \, \mathrm{d}x + \left[\frac{\partial F}{\partial (\overline{\Omega}_x)} h \right]_{-\infty}^{+\infty} \\ &= \int_{\mathbb{R}} \left(\frac{\partial F}{\partial \overline{\Omega}} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial (\overline{\Omega}_x)} \right) \right) h \, \mathrm{d}x + \left. \frac{\partial F}{\partial (\overline{\Omega}_x)} \right|_{x=\infty} h(\infty) - \left. \frac{\partial F}{\partial (\overline{\Omega}_x)} \right|_{x=-\infty} h(-\infty) \end{aligned} \tag{11}$$

First we consider functions h such that $h(\infty) = h(-\infty) = 0$. Then

$$\delta E = 0 \Rightarrow \frac{\partial F}{\partial \overline{\Omega}} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial (\overline{\Omega}_x)} \right) = 0$$

Since for a specific class of test functions, for $\overline{\Omega}$ to be a solution to this specific variable end point problem it has to satisfy this, the Euler, equation, for an arbitrary test function, it necessarily has to satisfy the Euler equation again. But if $\overline{\Omega}$ satisfies the Euler equation, in other words if it is an extremal, then the integral in Equation 11 vanishes, therefore,

$$\begin{aligned} &\delta E = 0 \Leftrightarrow \left. \frac{\partial F}{\partial (\overline{\Omega}_x)} \right|_{x=\infty} h(\infty) - \left. \frac{\partial F}{\partial (\overline{\Omega}_x)} \right|_{x=-\infty} h(-\infty) \\ &\Rightarrow \left. \frac{\partial F}{\partial (\overline{\Omega}_x)} \right|_{x=\infty} = 0 \text{ and } \left. \frac{\partial F}{\partial (\overline{\Omega}_x)} \right|_{x=-\infty} = 0 \end{aligned} \tag{12}$$

since h is arbitrary. These conditions we call natural boundary conditions. Let us now get back to finding the domain wall solution. The energy functional comprised of the exchange interaction and the anisotropy is, as we have found,

$$\begin{aligned} E_{ea}(\Omega, \overline{\Omega}) &= \frac{1}{2} \int_{\mathbb{R}} \left[4 \frac{\Omega_x \overline{\Omega}_x}{(1 + \Omega\overline{\Omega})^2} + 4\kappa \frac{\Omega\overline{\Omega}}{(1 + \Omega\overline{\Omega})^2} \right] \mathrm{d}x = \\ &= \int_{\mathbb{R}} \left[\frac{2\Omega_x \overline{\Omega}_x + 2\kappa \Omega\overline{\Omega}}{(1 + \Omega\overline{\Omega}))^2} \right] \mathrm{d}x \end{aligned}$$

Let us name

$$F(\Omega, \overline{\Omega}, \Omega_x, \overline{\Omega}_x) = \frac{2\Omega_x \overline{\Omega}_x + 2\kappa \Omega\overline{\Omega}}{(1 + \Omega\overline{\Omega})^2}$$

Then,

$$\begin{aligned}
&\frac{\delta E_{ea}}{\delta \overline{\Omega}} = 0 \Leftrightarrow \frac{\partial F}{\partial \overline{\Omega}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial (\overline{\Omega}_x)}\right) = 0 \Leftrightarrow \\
&\frac{-2\Omega_{xx}(1+\Omega\overline{\Omega}) + 4\Omega_x^2\overline{\Omega} + 2\kappa\Omega(1-\Omega\overline{\Omega})}{(1+\Omega\overline{\Omega})^3} = 0 \Leftrightarrow \\
&\Omega_{xx}(1+\Omega\overline{\Omega}) - 2\Omega_x^2\overline{\Omega} - \kappa\Omega(1-\Omega\overline{\Omega}) = 0
\end{aligned} \tag{13}$$

since $\Omega\overline{\Omega} \neq -1$ ($\Omega\overline{\Omega} = \frac{m_1^2+m_2^2}{(1+m_3)^2} \geq 0$).
Let us make the ansatz $\Omega = ae^{\lambda x}$, where $a \in \mathbb{C}$, $\lambda \in \mathbb{R}$.
Then, $\overline{\Omega} = \overline{a}e^{\lambda x}$, $\Omega_x = a\lambda e^{\lambda x}$, $\Omega_{xx} = a\lambda^2 e^{\lambda x}$.
Therefore, Equation 13 becomes

$$\begin{aligned}
&a\lambda^2 e^{\lambda x}(1+|a|^2 e^{2\lambda x}) - 2a^2\lambda^2 e^{2\lambda x}\overline{a}e^{\lambda x} - \kappa a e^{\lambda x} + \kappa a e^{\lambda x}|a|^2 e^{2\lambda x} = 0 \Leftrightarrow \\
&e^{3\lambda x}(-\lambda^2 a|a|^2 + \kappa a|a|^2) + e^{\lambda x}(a\lambda^2 - \kappa a) = 0
\end{aligned}$$

Since $e^{3\lambda x} \neq 0$ and $e^{\lambda x} \neq 0$ and $e^{3\lambda x} > 0$ and $e^{\lambda x} > 0$, the following must vanish

$$\begin{aligned}
&-\lambda^2|a|^2 a + \kappa a|a|^2 = 0 \Rightarrow -\lambda^2 + \kappa = 0 \Rightarrow \lambda = \pm\sqrt{\kappa} \\
&a\lambda^2 - \kappa a = 0 \Rightarrow \lambda^2 - \kappa = 0 \Rightarrow \lambda = \pm\sqrt{\kappa}
\end{aligned}$$

So $a \in \mathbb{C}$ can be anything.
Then we have $\Omega = ae^{\pm\sqrt{\kappa}x}$, $\overline{\Omega} = \overline{a}e^{\pm\sqrt{\kappa}x}$ and we can find the magnetization $\mathbf{m}$.

$$m_1 = \frac{\Omega + \overline{\Omega}}{1+\Omega\overline{\Omega}} = \frac{(a+\overline{a})e^{\pm\sqrt{\kappa}x}}{1+|a|^2 e^{\pm 2\sqrt{\kappa}x}}$$

Let us take $a = 1$. Then $m_1 = \frac{2e^{\pm\sqrt{\kappa}x}}{1+e^{\pm 2\sqrt{\kappa}x}}$.
For the negative square root,

$$m_1 = \frac{2e^{-\sqrt{\kappa}x}}{1+e^{-2\sqrt{\kappa}x}} = \mathrm{sech}(\sqrt{\kappa}x) \tag{14}$$

But also for the positive square root, since $\mathrm{sech}(x) = \mathrm{sech}(-x)$, we have the same solution for m_1.
Moreover,

$$m_2 = \frac{1}{i}\frac{\Omega - \overline{\Omega}}{1+\Omega\overline{\Omega}} = \frac{1}{i}\frac{(a-\overline{a})e^{\pm\sqrt{\kappa}x}}{1+|a|^2 e^{\pm 2\sqrt{\kappa}x}}$$

By choosing $a = 1$, $m_2 = 0$.
Then,

$$m_3 = \frac{1 - \Omega\overline{\Omega}}{1 + \Omega\overline{\Omega}} = \frac{1 - |a|^2 e^{\pm 2\sqrt{\kappa}x}}{1 + |a|^2 e^{\pm 2\sqrt{\kappa}x}}$$

By choosing $a = 1$, $m_3 = \frac{1 - e^{\pm 2\sqrt{\kappa}x}}{1 + e^{\pm 2\sqrt{\kappa}x}}$.
For the negative square root,

$$m_3 = \frac{1 - e^{-2\sqrt{\kappa}x}}{1 + e^{-2\sqrt{\kappa}x}} = \tanh\left(\sqrt{\kappa}x\right) \tag{15}$$

For the positive square root, since $\tanh(-x) = -\tanh(x)$, $m_3 = -\tanh\left(\sqrt{\kappa}x\right)$.
Now we move on to the spiral solution.
The energy functional that includes the exchange and the Dzyaloshinskii-Moriya energies is

$$E(\Omega, \overline{\Omega}) = 2\int_{\mathbb{R}} \frac{\Omega_z \overline{\Omega}_z}{(1 + \Omega\overline{\Omega})^2}\,\mathrm{d}z + \frac{2\lambda}{i}\int_{\mathbb{R}} \frac{\overline{\Omega}_z \Omega - \Omega_z \overline{\Omega}}{(1 + \Omega\overline{\Omega})^2}\,\mathrm{d}z$$

The Euler equation is $\frac{\partial F}{\partial \overline{\Omega}} - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial F}{\partial(\overline{\Omega}_z)}\right) = 0$ where F is everything inside the integral as usual.
Evaluating the Euler equation we arrive at

$$\Omega_{zz} - 2\frac{(\Omega_z)^2\overline{\Omega}}{1 + \Omega\overline{\Omega}} + \frac{2\lambda}{i}\frac{\Omega_z(1 - \Omega\overline{\Omega})}{1 + \Omega\overline{\Omega}} = 0 \tag{16}$$

Let us make the ansatz $\Omega = \Omega_0 e^{i\lambda z}$, where $\Omega_0 \in \mathbb{C}$ and λ is the DM constant.
Then $\overline{\Omega} = \overline{\Omega_0} e^{-i\lambda z}$, $\Omega_z = i\lambda\Omega_0 e^{i\lambda z}$, $\overline{\Omega}_z = -i\lambda\overline{\Omega_0} e^{-i\lambda z}$, $\Omega_{zz} = -\lambda^2\Omega_0 e^{i\lambda z}$.
We put these into Equation 16 and after a few calculations we get

$$\Omega_0 e^{i\lambda z}\left(2\lambda^2\left(\frac{1}{1 + |\Omega_0|^2}\right) - \lambda^2\right) = 0$$

Since we have assumed that $\Omega_0 \neq 0$ and that $\lambda \neq 0$, and we know that $e^{i\lambda z} \neq 0$,

$$\frac{1}{1 + |\Omega_0|^2} = \frac{1}{2} \Leftrightarrow |\Omega_0|^2 = 1 \Leftrightarrow |\Omega_0| = 1$$

and the ansatz satisfies the differential equation.
Let us also look at the natural boundary conditions. Now we assume that our space is finite and we integrate over an arbitrary range from $z = a$ to

$z = b$. As will become evident, we only have to look at one natural boundary condition and use that $|\Omega_0| = 1$

$$\begin{aligned}
&\frac{\partial F}{\partial(\overline{\Omega}_z)}\bigg|_{z=b} = \frac{2\Omega_z}{(1+\Omega\overline{\Omega})^2} + \frac{2\lambda}{i}\frac{\Omega}{(1+\Omega\overline{\Omega})^2}\bigg|_{z=b} = \\
&= \frac{2i\lambda\Omega_0 e^{i\lambda z}}{(1+|\Omega_0|^2)^2} + \frac{2\lambda}{i}\frac{\Omega_0 e^{i\lambda z}}{(1+|\Omega_0|^2)^2}\bigg|_{z=b} = \frac{1}{2}i\lambda\Omega_0 e^{i\lambda z} + \frac{\lambda}{2i}\Omega_0 e^{i\lambda z}\bigg|_{z=b} = \\
&= \frac{1}{2}i\lambda\Omega_0 e^{i\lambda z} - \frac{1}{2}i\lambda\Omega_0 e^{i\lambda z}\bigg|_{z=b} = 0
\end{aligned}$$

for any value of z. Therefore the natural boundary conditions are satisfied by the spiral solution $\Omega = \Omega_0 e^{i\lambda z}$ with $|\Omega_0| = 1$.
Next we will briefly consider the energies. Since the exchange energy is

$$E_{exch} = \frac{1}{2}\int_{\mathbb{R}} \partial_x \mathbf{m}\cdot\partial_x\mathbf{m}\,\mathrm{d}x \tag{17}$$

it can be clearly seen that $E_{exch} \geq 0$ since $\partial_x\mathbf{m}\cdot\partial_x\mathbf{m} \geq 0$.
Similarly, as the anisotropy energy is

$$E_{anis} = \frac{\kappa}{2}\int_{\mathbb{R}} (1 - m_3^2)\,\mathrm{d}x \tag{18}$$

and $|\mathbf{m}| = 1$, $1 - m_3^2 \geq 0$ it is clear that $E_{anis} \geq 0$.
On the other hand, let us compute the total energy of the spiral solution

$$\begin{aligned}
&E_{spiral} = 2\int_{\mathbb{R}} \frac{\Omega_z\overline{\Omega}_z}{(1+\Omega\overline{\Omega})^2}\,\mathrm{d}z + \frac{2\lambda}{i}\int_{\mathbb{R}} \frac{\overline{\Omega}_z\Omega - \Omega_z\overline{\Omega}}{(1+\Omega\overline{\Omega})^2}\,\mathrm{d}z \\
&= 2\int_{\mathbb{R}} \frac{i\lambda\Omega_0 e^{i\lambda z}(-i\lambda)\overline{\Omega_0}e^{-i\lambda z}}{(1+|\Omega_0|^2)^2}\,\mathrm{d}z + \frac{2\lambda}{i}\int_{\mathbb{R}} \frac{(-i\lambda)\overline{\Omega_0}e^{-i\lambda z}\Omega_0 e^{i\lambda z} - i\lambda\Omega_0 e^{i\lambda z}\overline{\Omega_0}e^{-i\lambda z}}{(1+|\Omega_0|^2)^2}\,\mathrm{d}z \\
&= -\frac{1}{2}\lambda^2\int_{\mathbb{R}}\mathrm{d}z < 0
\end{aligned}$$

where in the last step we have used that $|\Omega_0| = 1$.
Therefore we can conclude that since the energy of the domain wall solution is comprised of the exchange and anisotropy energies and the energy of the spiral solution is comprised of the exchange and Dzyaloshinskii-Moriya energies, the spiral has a lower energy than the domain wall.

Now we will move on to two dimensions. We will follow [1] and consider only the exchange energy. Let us show that finding an extremum of the exchange energy reduces to finding a solution to the differential equation

$$\partial_\mu\mathbf{m} = \pm\varepsilon_{\mu\nu}\mathbf{m}\times(\partial_\nu\mathbf{m}) \tag{19}$$

Motivated by this equation, we start from the inequality

$$\int [(\partial_\mu \mathbf{m} \pm \varepsilon_{\mu\nu}\mathbf{m} \times \partial_\nu \mathbf{m}) \cdot (\partial_\mu \mathbf{m} \pm \varepsilon_{\mu\sigma}\mathbf{m} \times \partial_\sigma \mathbf{m})] \,\mathrm{d}^2 x \geq 0 \tag{20}$$

which holds since it is an integral of something non-negative as the integrand is an inner product of something with itself.
We expand to get

$$\begin{aligned}&\int (\partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m}) + \varepsilon_{\mu\nu}(\mathbf{m} \times \partial_\nu \mathbf{m}) \cdot \varepsilon_{\mu\sigma}(\mathbf{m} \times \partial_\sigma \mathbf{m}) \\ &\pm \varepsilon_{\mu\nu}(\mathbf{m} \times (\partial_\nu \mathbf{m}) \cdot \partial_\mu \mathbf{m} \pm \varepsilon_{\mu\sigma}\partial_\mu \mathbf{m} \cdot (\mathbf{m} \times \partial_\sigma \mathbf{m}) \,\mathrm{d}^2 x \geq 0\end{aligned}$$

First we look at the following term:

$$\begin{aligned}&\varepsilon_{\mu\nu}\varepsilon_{\mu\sigma}(\mathbf{m} \times \partial_\nu \mathbf{m}) \cdot (\mathbf{m} \times \partial_\sigma \mathbf{m}) = \\ &= \delta_{\nu\sigma}\{[(\mathbf{m} \times \partial_\nu \mathbf{m}) \times \mathbf{m}] \cdot \partial_\sigma \mathbf{m}\} \\ &= \delta_{\nu\sigma}\{[\partial_\nu \mathbf{m}(\mathbf{m} \cdot \mathbf{m}) - \mathbf{m}(\mathbf{m} \cdot \partial_\nu \mathbf{m})] \cdot \partial_\sigma \mathbf{m}\} \\ &= \delta_{\nu\sigma}\{[(\mathbf{m} \cdot \mathbf{m})(\partial_\nu \mathbf{m} \cdot \partial_\sigma \mathbf{m}) - (\mathbf{m} \cdot \partial_\nu \mathbf{m})(\mathbf{m} \cdot \partial_\sigma \mathbf{m}]\} \\ &= \partial_\nu \mathbf{m} \cdot \partial_\nu \mathbf{m}\end{aligned}$$

where we have used the properties of the scalar and vector triple products as well as the fact that $\mathbf{m} \cdot \mathbf{m} = 1$ and consequently that $\mathbf{m} \cdot \partial_\nu \mathbf{m} = 0$.
Thus the inequality becomes

$$\int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \,\mathrm{d}^2 x \geq \pm \int \varepsilon_{\mu\nu}\mathbf{m} \cdot (\partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}) \,\mathrm{d}^2 x \tag{21}$$

where we just got rid of silent indices. But the exchange energy, which now we will call E, is

$$E = \int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \,\mathrm{d}^2 x \tag{22}$$

and the topological charge is

$$Q = \frac{1}{4\pi} \int \varepsilon_{\mu\nu}\mathbf{m} \cdot (\partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}) \,\mathrm{d}^2 x \tag{23}$$

Therefore,

$$E \geq 4\pi |Q| \tag{24}$$

This inequality holds in any given Q-sector. The energy minimization can be done in each Q-sector separately since any configuration cannot move from one sector to another under continuous deformation. Therefore the energy is minimized when the equality in Equation 24 is satisfied, which happens if and only if the equality in Equation 20 is satisfied, which leads to the field equation, Equation 19.

Now, in order to solve the field equation, we return to the stereographic projection and define the following complex variables:

$$\omega_1 = \frac{2m_1}{1-m_3}, \quad \omega_2 = \frac{2m_2}{1-m_3}$$
$$\omega \equiv \omega_1 + i\omega_2 = \frac{2(m_1 + im_2}{1-m_3} = \frac{2m}{1-m_3}$$

where $m \equiv m_1 + im_2$. The difference from Ω as defined and used above is a difference of north and south pole in the stereographic projection.
While our aim is to solve the field equation, the strategy is that we will work with various quantities and eventually express the field equation as a relation between the derivatives of ω which we will then recognize as the Cauchy-Riemann equations for an analytic complex function. To simplify the calculations, we will use a kind of "derivative":

$$x\ddot{\partial}_\mu y = x\partial_\mu y - y\partial_\mu x$$

which will prove useful for dealing with terms coming from the cross product. Now,

$$\begin{aligned}\partial_1\omega = \frac{\partial\omega}{\partial x_1} &= 2\frac{(1-m_3)\partial_1 m + m\partial_1 m_3}{(1-m_3)^2} = \\ = \frac{2}{(1-m_3)^2}&[\partial_1 m - m_3\partial_1 m + m\partial_1 m_3] = \\ = \frac{2}{(1-m_3)^2}&[\partial_1 m + m\ddot{\partial}_1 m_3]\end{aligned}$$

Next, we use the field equation, Equation 19, to find

$$\begin{aligned}&\partial_1 m_1 = \pm\varepsilon_{1\nu}\big[\mathbf{m}\times(\partial_\nu\mathbf{m})\big]_1 \\ &= \pm\varepsilon_{1\nu}(m_2\partial_\nu m_3 - m_3\partial_\nu m_2) \\ &= \pm\big[\varepsilon_{11}(m_2\partial_1 m_3 - m_3\partial_1 m_2) + \varepsilon_{12}(m_2\partial_2 m_3 - m_3\partial_2 m_2)\big] \\ &= \pm(m_2\partial_2 m_3 - m_3\partial_2 m_2) \\ &= \pm m_2\ddot{\partial}_2 m_3\end{aligned}$$

Similarly,

$$\partial_1 m_2 = \pm m_1\ddot{\partial}_2 m_3$$

Therefore,

$$\begin{aligned}
&\partial_1 m = \partial_1 m_1 + i\partial_1 m_2 = \pm m_2 \ddot{\partial}_2 m_3 \pm i m_1 \ddot{\partial}_2 m_3 \\
&= \mp i(m_1 + i m_2)\ddot{\partial}_2 m_3 = \mp i m \ddot{\partial}_2 m_3
\end{aligned}$$

Similarly,

$$\partial_2 m = \pm i m \ddot{\partial}_1 m_3$$

Therefore,

$$\begin{aligned}
\partial_1 \omega &= \frac{2}{(1-m_3)^2}[\mp i m \ddot{\partial}_2 m_3 + m \ddot{\partial}_1 m_3] \\
&= \frac{2}{(1-m_3)^2}[m(\ddot{\partial}_1 m_3 \mp i \ddot{\partial}_2 m_3)]
\end{aligned}$$

and

$$\begin{aligned}
\partial_2 \omega &= \frac{2}{(1-m_3)^2}[\pm i m \ddot{\partial}_1 m_3 + m \ddot{\partial}_2 m_3] \\
&= \frac{2}{(1-m_3)^2}[m(\pm i \ddot{\partial}_1 m_3 + \ddot{\partial}_2 m_3)] \\
&= \pm i \frac{2}{(1-m_3)^2}[m(\ddot{\partial}_1 m_3 \mp i \ddot{\partial}_2 m_3)]
\end{aligned}$$

Therefore,

$$\partial_2 \omega = \pm i \partial_1 \omega \Leftrightarrow i\partial_2 \omega = \mp \partial_1 \omega \Leftrightarrow \partial_1 \omega = \mp i \partial_2 \omega$$

In terms of ω_1 and ω_2, we have

$$\begin{aligned}
&\partial_1 \omega_1 + i\partial_1 \omega_2 = \mp i(\partial_2 \omega_1 + i\partial_2 \omega_2) \Leftrightarrow \\
&\partial_1 \omega_1 + i\partial_1 \omega_2 = \pm \partial_2 \omega_2 \mp i\partial_2 \omega_1 \Leftrightarrow \\
&\partial_1 \omega_1 = \pm \partial_2 \omega_2 \quad \text{and} \quad \partial_1 \omega_2 = \mp \partial_2 \omega_1
\end{aligned}$$

or

$$\frac{\partial \omega_1}{\partial x_1} = \pm \frac{\partial \omega_2}{\partial x_2} \quad \text{and} \quad \frac{\partial \omega_1}{\partial x_2} = \mp \frac{\partial \omega_2}{\partial x_1} \tag{25}$$

which we recognize as the Cauchy-Riemann equations for $\omega(z)$ or $\omega(z^*)$ to be analytic, with $z = x_1 + ix_2$. Therefore, any analytic function $\omega(z)$ is a solution of the field equation.

Magnetostatic Field

Let us start from the following two equations:

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{m} \tag{26}$$
$$\mathbf{H} = -\nabla \Psi \tag{27}$$

These equations can clearly be combined in the following way:

$$\begin{aligned} &\nabla \cdot (-\nabla \Psi) = -(\nabla \cdot \nabla)\Psi = -\Delta \Psi = -\nabla \cdot \mathbf{m} \Leftrightarrow \\ &\Delta \Psi = \nabla \cdot \mathbf{m} \end{aligned} \tag{28}$$

We will use Equation 28 to calculate the magnetic potential Ψ from the magnetization $\mathbf{m}$ and then use Equation 27 to calculate the magnetic field $\mathbf{H}$ from the magnetic potential Ψ.

Now let us show how we got these equations. We begin with one of Maxwell's equations:

$$\begin{aligned} &\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{m} \Rightarrow \\ &\nabla \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{m} \right) = \mathbf{J} \Rightarrow \\ &\nabla \times \mathbf{H} = \mathbf{J} \end{aligned} \tag{29}$$

where $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{m}$. Therefore, for $\mathbf{J} = 0$, $\nabla \times \mathbf{H} = 0$. But then, using Stokes' theorem,

$$\oint \mathbf{H} \cdot \mathrm{dl} = \iint_S \nabla \times \mathbf{H} = 0$$

therefore $\mathbf{H}$ is path-independent. But the converse of the gradient theorem [2] states that then $\mathbf{H}$ is the gradient of some scalar-valued function, as long as $\mathbf{H}$ is defined on simply-connected space. Therefore, we are led to Equation 27.

To get Equation 26, we start from another of Maxwell's equations and use the definition of $\mathbf{H}$:

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0 \Leftrightarrow \\ &\nabla \cdot (\mathbf{H} + \mathbf{m}) = 0 \Leftrightarrow \\ &\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{m} \end{aligned}$$

For the moment, we want to look at axially symmetric solutions, therefore we will use cylindrical coordinates and reduce the problem from three dimensions to two since the angle (ϕ) will not vary due to axial symmetry, i.e. the magnetization we are looking for will not be a function of ϕ. Thus our program will consist of the remaining dimensions, ρ and z.

The magnetization $\mathbf{m}$ will be expressed in cylindrical coordinates as

$$\mathbf{m} = m_\rho(\rho, z)\hat{\rho} + m_\phi(\rho, z)\hat{\phi} + m_z(\rho, z)\hat{z}$$

Let us first transform the Laplacian to cylindrical coordinates. We know that for a function f has the following Laplacian in cylindrial coordinates:

$$\Delta f = \partial_\rho^2 f + \frac{1}{\rho}\partial_\rho f + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$$

Using this, the chain rule and the fact that $\hat{z}$ stays the same, we get

$$\Delta\mathbf{m} = \Delta m_\rho\hat{\rho} + \Delta m_\phi\hat{\phi} + \Delta m_z\hat{z} + m_\rho\Delta\hat{\rho} + m_\phi\Delta\hat{\phi}$$

and we still have to figure out the Laplacians of the new unit vectors. To do this, we express them as functions of the Cartesian unit vectors

$$\begin{aligned}\hat{\rho} &= \cos\phi\hat{x} + \sin\phi\hat{y}\\ \hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y}\end{aligned}$$

and, after some simple algebra, we get

$$\begin{aligned}\Delta\hat{\rho} &= -\frac{\hat{\rho}}{\rho^2}\\ \Delta\hat{\phi} &= -\frac{\hat{\phi}}{\rho^2}\end{aligned}$$

Therefore, and from now on we will use the notation $\mathbf{m} = (m_\rho, m_\phi, m_z) = (m_1, m_2, m_3)$,

$$\Delta\mathbf{m} = (\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 - \frac{1}{\rho^2})m_1\hat{\rho} + (\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 - \frac{1}{\rho^2})m_2\hat{\phi} + (\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2)m_3\hat{z}$$

and similarly,

$$\nabla\times\mathbf{m} = -\partial_z m_2\hat{\rho} + (\partial_z m_1 - \partial_\rho m_3)\hat{\phi} + (\partial_\rho m_2 + \frac{1}{\rho}m_2)\hat{z}$$

We will further restrict our attention to solutions in which the Dzyaloshiinski-Moriya interaction is uniform in the z-direction therefore we will disregard the partial derivatives with respect to z in the cross product. Therefore we will use

$$\nabla\times\mathbf{m} = -\partial_\rho m_3\hat{\phi} + (\partial_\rho m_2 + \frac{1}{\rho}m_2)\hat{z}$$

We are now in a position to write the Landau-Lifshitz-Gilbert equation governing the dynamics:

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{f} + \alpha \mathbf{f} - \alpha(\mathbf{m} \cdot \mathbf{f})\mathbf{m} \tag{30}$$

where $\mathbf{f} = \Delta \mathbf{m} + 2\lambda \nabla \times \mathbf{m} + \kappa m_3 \hat{z} + H\hat{z}$ is the effective field and λ, α, κ are the Dzyaloshiinski-Moriya, damping, and anisotropy constants respectively.

We now turn back to the magnetostatic field. We need the divergence of $\mathbf{m}$ which is given by

$$\nabla \cdot \mathbf{m} = \frac{1}{\rho}\left(\frac{\partial(\rho m_1)}{\partial \rho}\right) + \frac{\partial m_3}{\partial z}$$

where we have used that we only look for solutions which are not functions of ϕ therefore $\partial_\phi \mathbf{m} = 0$.
For our program it will be easier to expand the divergence into

$$\nabla \cdot \mathbf{m} = \frac{1}{\rho} m_1 + \partial_\rho m_1 + \partial_z m_3$$

We see that for any magnetization configuration we can compute this divergence and then we will have to solve the Poisson equation (since $\Delta\Psi = \nabla \cdot \mathbf{m}$), or, more specifically, the Poisson equation in the region of the magnetic material, and the Laplace equation (since $\mathbf{m} = 0$) outside.
As we try to solve $\Delta\Psi = \nabla \cdot \mathbf{m}$, we notice that the source term does not depend on the angle ϕ. Therefore Ψ will not depend on ϕ either. Therefore we can write and discretize

$$\Delta\Psi = \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho} + \frac{\partial^2 \Psi}{\partial z^2} \tag{31}$$

First let's treat and discretize the singularity at $\rho = 0$. We have (to be shown):

$$\begin{aligned}
&\Delta\Psi(\rho = 0, z) \approx \\
&\approx \frac{4}{\Delta\rho^2}[\Psi(\Delta\rho, z) - \Psi(0, z)] + \frac{1}{\Delta z^2}[\Psi(0, z + \Delta z) - 2\Psi(0, z) + \Psi(0, z - \Delta z)] \\
&= (-\frac{4}{\Delta\rho^2} - \frac{2}{\Delta z^2})\Psi(0, z) + \frac{4}{\Delta\rho^2}\Psi(\Delta\rho, z) + \frac{1}{\Delta z^2}\Psi(0, z + \Delta z) + \frac{1}{\Delta z^2}\Psi(0, z - \Delta z)
\end{aligned}$$

For $\rho \neq 0$, we have

$$\Delta\Psi(\rho \neq 0, z) \approx$$
$$\approx \frac{1}{\Delta\rho^2}[\Psi(\rho+\Delta\rho, z) - 2\Psi(\rho, z) + \Psi(\rho-\Delta\rho, z)] + \frac{1}{\rho}\frac{1}{\Delta\rho}[\Psi(\rho+\Delta\rho, z) - \Psi(\rho, z)]$$
$$+\frac{1}{\Delta z^2}[\Psi(\rho, z+\Delta z) - 2\Psi(\rho, z) + \Psi(\rho, z-\Delta z)]$$
$$= (-\frac{2}{\Delta\rho^2} - \frac{1}{\rho\Delta\rho} - \frac{2}{\Delta z^2})\Psi(\rho, z) + (\frac{1}{\Delta\rho^2} + \frac{1}{\rho\Delta\rho})\Psi(\rho+\Delta\rho, z)$$
$$+\frac{1}{\Delta\rho^2}\Psi(\rho-\Delta\rho, z) + \frac{1}{\Delta z^2}\Psi(\rho, z+\Delta z) + \frac{1}{\Delta z^2}\Psi(\rho, z-\Delta z)$$

Let's discretize $\nabla \cdot \mathbf{m}$ in the same way.

$$\nabla \cdot \mathbf{m}(\rho = 0, z) \approx$$
$$\approx \frac{2m_1(0, z)}{\Delta\rho} + \frac{m_1(\Delta\rho, z) - m_1(0, z)}{\Delta\rho} + \frac{m_3(0, z+\Delta z) - m_3(0, z)}{\Delta z}$$
$$= \frac{1}{\Delta\rho}m_1(0, z) + \frac{1}{\Delta\rho}m_1(\Delta\rho, z) + \frac{1}{\Delta z}m_3(0, z+\Delta z) - \frac{1}{\Delta z}m_3(0, z)$$

And for $\rho \neq 0$:

$$\nabla \cdot \mathbf{m}(\rho \neq 0, z) \approx$$
$$\approx \frac{m_1(\rho, z)}{\rho} + \frac{m_1(\rho+\Delta\rho, z) - m_1(\rho, z)}{\Delta\rho} + \frac{m_3(\rho, z+\Delta z) - m_3(\rho, z)}{\Delta z}$$
$$= (\frac{1}{\rho} - \frac{1}{\Delta\rho})m_1(\rho, z) + \frac{1}{\Delta\rho}m_1(\rho+\Delta\rho, z) + \frac{1}{\Delta z}m_3(\rho, z+\Delta z) - \frac{1}{\Delta z}m_3(\rho, z)$$

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