Learning from Experience in the Stock Market*

Anton Nakov
ECB and CEPR

Galo Nuño
European Central Bank

November 26, 2014

Abstract

New evidence suggests that individuals "learn from experience," meaning they learn from events occurring during their lives as opposed to the entire history of events. Moreover, they weigh more heavily recent events compared to events occurring in the distant past. This paper analyzes the implications of such learning for stock pricing in a model with finitely-lived agents. Individuals learn about the rate of change of the stock price and of dividends using a weighted decreasing-gain algorithm. As a result of waves of optimism and pessimism, the stock price exhibits stochastic fluctuations around the rational expectations equilibrium. Conditional on the historical path of dividends, the model produces a price-dividend ratio which is in line with the evidence for the last century, except for the "dot-com" bubble in the 1990s.

Keywords: heterogeneous beliefs, constant-gain learning, OLG, asset pricing

JEL codes: G12, D83, D84

---

*We are grateful for comments and suggestions to Klaus Adam, George-Marios Angeletos, Gunter Coenen, Alistair Dieppe, Oliver de Groot, Cars Hommes, Miguel Angel Iraola, Christian Matthes, Albert Marcet, Alessio Moro, Giulio Nicoletti, Kristoffer Nimark, John Roberts, the Associate Editor, seminar participants at ECB, Bank of Spain, Central Bank of Ireland, University of Surrey, and University of Cagliari, and conference participants at the Society for Computational Economics 2011 and the Bank of Spain Workshop on Expectations in Macroeconomics 2013. Anton Nakov thanks the Bank of Spain and the Federal Reserve Board for their hospitality during earlier drafts of this paper. The views expressed here are those of the authors only and do not necessarily coincide with the views of the ECB or the Eurosystem. Corresponding author: Galo Nuño, European Central Bank, Kaiserstrasse 29, 60311 Frankfurt am Main, Germany. E-mail: galo.nuno@ecb.europa.eu

1 Introduction

The key role of expectations about the future is well understood in economics. The rational expectations hypothesis (REH) has been a major step forward, allowing rigorous formalization of the process of expectations formation. Yet it has often been criticized for endowing people with "too much" knowledge about their environment.[1] Empirical studies of individuals forming expectations about aggregate economic variables does not, in general, corroborate the REH. In particular, Malmendier and Nagel (2011, 2013) find evidence that, contrary to the REH, people "learn from experience," meaning that they are more influenced by observations from their own lifetimes than by earlier historical events. More specifically, Malmendier and Nagel (2011) find that individuals who experienced low stock market returns during their lives are more pessimistic about future stock returns and invest a lower fraction of their liquid assets in stocks. In addition, Malmendier and Nagel (2013) find that young individuals place more weight on recently experienced inflation than older individuals do. The by-product of this is that learning may take forever if history "gets lost" as new generations replace older ones.

In this paper, we explore how replacing the REH with "learning from experience" affects the dynamics of a simple general equilibrium model of the stock market. We are interested in the dynamics of heterogeneous beliefs and in the feedback loop that arises when individuals learn about variables which are the result of their collective actions given their beliefs.[2] To this end, we extend the basic Lucas-tree asset pricing model to a stochastic Blanchard-Yaari overlapping generations (OLG) setup in which individuals learn the parameters of the *endogenous* evolution of the stock price as well as the *exogenous* process for dividends.

Specifically, we assume that a small random fraction $1 - \phi$ (with $\phi \lesssim 1$) of individuals exit the stock market every period, and an equal measure of new individuals enter the market. As in Brown and Rogers (2009), each new entrant inherits the assets but does not inherit the accumulated knowledge of his parent about the economy. Instead, children learn from their own experience, updating their beliefs with information about stock prices and dividends which they observe during their own lifetimes.[3] As in Malmendier and Nagel (2013), agents use a decreasing-gain learning scheme with gain parameter θ. A value of $\theta = 1$ implies that individuals assign the same weight to all observations they witness. In contrast, if $\theta > 1$, as the evidence of Malmendier and Nagel suggests, then individuals weigh more heavily recent events compared to older events.

To analyze the model quantitatively, we propose a method that allows us to solve for the equilibrium with heterogeneous agents. In the model, the equilibrium stock price equals the reservation

[1] See, for example, Blume et. al. (1982), Arrow (1986), and Adam and Marcet (2011).

[2] See Eusepi and Preston (2011) who emphasize this type of self-referentiality.

[3] Thus, information in the model is dispersed across age cohorts, with older generations observing longer time series than younger ones. See Angeletos and La'O (2009, 2013) for the role of *geographically* dispersed information for macroeconomic dynamics.

price of the marginal stock holder. Individuals who are more optimistic than the marginal agent choose to hold the maximum possible amount of the stock, while those who are more pessimistic choose not to hold the stock. Since the asset holding decision of each individual depends on the current stock price (which is used to forecast future prices), the solution method involves finding a fixed-point for the market-clearing price given the non-linear pricing function.

The introduction of "learning from experience" has several novel implications. First, we find that, even if the retirement rate $1-\phi$ is quite low, so that in any given period only a small fraction of individuals are novice, the asset price dynamics are different from those in the rational expectations equilibrium (REE). Two forces create the oscillating dynamics. On the one hand, there is "momentum" rooted in the information loss due to the retirement of individuals from the market and to the stronger discounting of older information relative to the more recent one. As a result, beliefs about dividends and the stock price are biased towards extrapolation of the more recent past, and trading on these beliefs pushes the asset price further away from the fundamental.

On the other hand, there is a force of reversal toward the REE trend. Namely, when the stock price rises too far above the fundamental value, individual asset exposure constraints begin to bind. Because any given individual (including the optimistic types) can afford to buy less of the stock, the asset price must decline to the valuation of less optimistic individuals for the market to clear. The same reflecting force works also "from below", when the stock price falls far below the fundamental value. The combination of these two factors – momentum and trend reversal – results in boom-and-bust cycles, which are only loosely related to dividends and are mainly due to speculation about the future course of the stock price, in the spirit of Harrison and Kreps (1978).

We assess the empirical performance of the model by simulating the evolution of prices conditional on the historical series of US dividends. Our results show that the model generates stock market cycles similar to the ones observed in the data for the period 1920-1990. It is also able to match the main moments of the data for the period 1920-1990, although it fails to replicate the "dot-com" bubble. In contrast, the REE model produces a constant price-dividend ratio.

A second finding is that, although individual expectations in our framework are not model-consistent, the agents' forecasting performance is quite close to that under rational expectations. In particular, the mean forecasting error (averaged across cohorts) is similar to that in the rational expectations model. Likewise, the root mean square errors (again averaged across cohorts) are not too far from the ones obtained under rational expectations.

A third finding of our paper is that the dynamics of average beliefs in this heterogeneous-beliefs economy can be approximated reasonably well by a CGL scheme in which the social gain parameter is a nonlinear function of the survival rate ϕ and of the individual gain parameter θ. This implies that memories of the distant past are lost with the passage of time as a result of population turnover combined with "learning from experience."

Usually CGL is derived from the assumption that a representative agent uses the Kalman filter as in Ljung and Soderstrom (1983) or Sargent (1999, ch. 8). This type of learning has received much attention in the literature due to its ability to produce realistic model features, such as amplification of the persistence of macroeconomic variables in response to aggregate shocks.[4] Yet Malmendier and Nagel (2013) show empirically that even though individual learning follows a decreasing-gain scheme, the average learning-from-experience forecast can be approximated quite closely with a CGL algorithm. Complementary to their work, we propose a theoretical model of stock pricing which reproduces this feature and we provide an expression for the approximation error. We verify numerically that in our model the approximation error is relatively small. We further show that the social gain parameter is increasing in the individual gain parameter θ, and in the rate of generational turnover, $1-\phi$.

Finally, we compare the behavior of the price-dividend ratio in the heterogeneous-agents model with that in a representative-agent CGL model. We show that the representative-agent CGL produces completely different dynamics for prices, which bear very little resemblance to those in the data or in the OLG model.[5] The reason for such a discrepancy is the fact that in the CGL model the path of the stock price depends on the evolution of the average, rather than the marginal, belief of the population. This exercise should be taken as a warning that, despite the fact that CGL describes well the evolution of the average belief, assuming a representative agent with CGL is not equivalent to simulating a heterogeneous-agents economy with individuals who learn from experience.

Our paper is related to several strands of research. First, it relates to the emerging literature on learning with heterogeneous agents, such as Cogley, Sargent and Tsyrennikov (2012), Giannitsarou (2003), Branch and McGough (2004), Branch and Evans (2006), Honkapohja and Mitra (2006), and Graham (2011). In contrast to these papers, individuals in our economy use the same decreasing-gain learning scheme, have the same preferences, and observe the same public variables. The only source of heterogeneity in our model is in the individual information sets used to update beliefs, with younger cohorts focusing on a subset of the observations used by older generations.

Second, our work is related to the literature on bounded rationality with heterogeneous beliefs, following Brock and Hommes (1998). In this literature agents switch between heterogeneous expectations based on the short-run profitability of the investment strategies. These models have been estimated on different financial time series as in Boswijk, Hommes and Manzan (2007). [6]

Third, a related line of research analyzes the dynamics of asset prices under learning by a

[4]The value of the gain parameter typically is estimated or calibrated to yield the smallest mean-squared forecasting error. See, e.g., Milani (2007), Carceles-Poveda and Giannitsarou (2008), Branch and Evans (2011), Adam, Marcet, and Nicolini (2008).

[5]In fact, the heterogeneous-beliefs OLG model provides a better fit to the moments for price growth than the representative-agent CGL or the REE model.

[6]For a survey, see Hommes (2006).

representative agent. Timmermann (1994), Weitzman (2007), and Cogley and Sargent (2008), among others, explain some puzzling asset pricing phenomena based on rational learning by a representative agent. Unlike our setup, individuals in their models use all available past information and know *ex ante* the correct mapping between asset prices and fundamentals. Hence, they only need to learn about the latter in order to achieve convergence to the REE.

Finally, a recent line of research focuses on the role of higher-order expectations for asset prices. For example, Allen, Morris, and Shin (2006) analyze a linear model with asymmetric information. They find that, in the absence of common knowledge about higher-order beliefs, asset prices generally depart from the market consensus of the expected fundamental value, typically reacting more sluggishly to changes in fundamentals.

The rest of our paper is organized as follows. Section 2 presents the model. In section 3, we calibrate it and analyze the properties of "learning from experience." In section 4, we analyze to what extent the benchmark model can be approximated by a representative agent with CGL. Conclusions are presented in section 5.

2 The model

The economy is populated by N risk-neutral ex-ante identical dynasties, with N large. Members of each dynasty have stochastic lifetimes with death (or retirement) occurring with a constant exogenous probability, $1-\phi$. Thus, in each period, the number of dynasts of age $s \in \mathbb{N}_0$ is constant and equal to $f_s = N(1-\phi)\phi^s$. Upon retirement, a successor inherits the assets of the former dynast but not his accumulated knowledge about the processes governing the stock price and dividends. Instead, successors embark on their own learning experience "from scratch", starting with the identical initial belief that their predecessors had at birth, namely the belief consistent with REE.

The dynasts trade among themselves a single divisible stock which is in fixed supply, normalized to N. Each individual decides how much to invest in the asset based on inter-temporal arbitrage. Note, however, that the relevant arbitrage is not the one between selling the stock and holding it forever for its dividends. Instead, the condition that governs savings decisions is a one-period-ahead comparison between the value of the stock in the current period and the subjective expected payoff in the following trading period.

The equilibrium stock price in our model equals the *marginal* asset holder i's *subjectively* expected payoff from holding the stock for one period – that is, the present value of *his* expected dividend $\mathbb{E}_{it}(D_{t+1})$ plus *his* expected price $\mathbb{E}_{it}(P_{t+1})$ in the following period. Because expectations about future prices and dividends differ across individuals, the law of iterated expectations does not apply, and the pricing conditions of individuals do not aggregate to the familiar asset pricing formula with a representative agent.

2.1 Preferences and constraints

The dynast $i \in \{1, ..., N\}$ receives utility from consumption $u(C_{it}) = C_{it}$ per period. He discounts future consumption by factor $\beta\phi$, where $\beta < 1$ is a time preference parameter and $\phi < 1$ is a constant probability of survival. The expected value of lifetime utility for dynast i is thus

$$\mathbb{E}_{i0} \sum_{t=0}^{\infty} (\beta\phi)^t \, u(C_{it}), \tag{1}$$

where $\mathbb{E}_{i0}$ is individual i's expectation formed at time 0.

Individual i faces the period budget constraint

$$C_{it} + P_t S_{it} \leq (P_t + D_t) S_{it-1} + Y_{it}, \tag{2}$$

where S_{it} denotes his stock holdings, P_t is the asset price, D_t is the dividend, and Y_{it} is a per period income endowment. We assume for simplicity that $Y_{it} = \kappa D_t$, with κ a positive constant.

In addition, the individual faces constraints on the minimum and the maximum *asset exposure*, defined as the maximum value in terms of consumption that he stands to lose (or gain if short-selling) if the stock price falls to zero.

$$\underline{L}_t \leq P_t S_{it} \leq \bar{L}_t. \tag{3}$$

Constraints (3) imply that an individual investor cannot go arbitrarily short or long in the stock. In a more detailed model, these limitations can be derived from underlying credit constraints that prevent agents from borrowing unlimited amounts of resources. Instead, we will simply assume that $\underline{L}_t = 0$ and $\bar{L}_t = \lambda D_t > 0$, where parameter $\lambda > 0$ (which we loosely refer to as the permissible "leverage") is the maximum multiple of the current dividend that an individual can maintain invested in the risky stock. Our specification of the stock holding constraints puts effective bounds on the price-to-dividend ratio, without the need for a "projection facility" that mechanically constrains beliefs to a pre-specified neighborhood.

Dividends follow the exogenous stochastic process

$$\log\left(D_t / D_{t-1}\right) = \mu + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2), \tag{4}$$

where $\mu > 0$ and $\sigma^2 > 0$ are, respectively, the mean and the variance of the growth rate of dividends and where D_{-1} is known.

Given the information set available to individual i, his problem is to choose consumption and equity holdings so as to maximize lifetime utility (1), subject to the budget constraint (2). The first-order optimality conditions (FOC) of the individual's problem are:

$$\text{if } P_t < P_{it}, \text{ then } S_{it} = \bar{L}_t / P_t \tag{5a}$$

$$\text{if } P_t = P_{it}, \text{ then } S_{it} \in [\underline{L}_t / P_t, \bar{L}_t / P_t], \tag{5b}$$

$$\text{if } P_t > P_{it}, \text{ then } S_{it} = \underline{L}_t / P_t, \tag{5c}$$

$\forall t$, where

$$P_{it} = \beta\phi\mathbb{E}_{it}\left(P_{t+1} + D_{t+1}\right), \tag{6}$$

is individual i's "reservation price". Because the objective function is linear and the feasible set is closed, a maximum exists (and generally is a corner solution). We assume that κ is large enough so that the condition $C_{it} \geq 0$ is never binding.

The FOC can also be written as

$$P_t = \beta\phi\mathbb{E}_{it}\left(P_{t+1} + D_{t+1}\right) + \mu_{it}, \tag{7}$$

where $\mu_{it} \in \mathbb{R}$ is the Lagrange multiplier associated with the exposure constraints (3).

The market clearing condition is

$$\sum_{i=1}^{N} S_{it} = \sum_{s=0}^{\infty} f_s S_{st} = N. \tag{8}$$

If all individuals share the same model consistent expectations, $\mathbb{E}_{it}(\cdot) = \mathbb{E}_t(\cdot)$ and the transversality condition $\lim_{T\to\infty}(\beta\phi)^T \mathbb{E}_t(P_T) = 0$ is satisfied, the REE solution is

$$P_t^{REE} = \frac{\beta\phi e^{\mu+\sigma^2/2}}{1-\beta\phi e^{\mu+\sigma^2/2}} D_t. \tag{9}$$

Further imposing the parameter restrictions

$$\beta\phi e^{\mu+\sigma^2/2} < 1 \text{ and } \sigma^2/2 \approx 0$$

would imply that the stock price is finite, and that it does not depend on the variance of dividends: $P_t^{REE} = \frac{\beta\phi e^{\mu}}{1-\beta\phi e^{\mu}} D_t$.

2.2 Learning from experience

We depart from model-consistent expectations by assuming that individuals have only limited information about the world they live in. In particular, they do not know anything about other market participants' preferences or constraints. However, they do know their own objectives and constraints and have a prior belief about parameters μ and σ^2 governing the dividend process (4). In the absence of common knowledge, from an individual's perspective, *the price of the asset itself is a stochastic process* affecting optimal savings decisions much like dividends do. Hence individuals try to forecast both the dividend *and* the stock price, conditioning their forecasts on the history of past dividends and stock price realizations.

Individuals update their beliefs about the mean growth rate of the stock price and of dividends, μ. Given P_{t-1} and D_{t-1}, the perceived law of motion (PLM) is

$$x_t = m + \epsilon_t, \tag{10}$$

where

$$x_t = \begin{bmatrix} \log\left(P_t/P_{t-1}\right) \\ \log\left(D_t/D_{t-1}\right) \end{bmatrix}, \quad m = \begin{bmatrix} m^P \\ m^D \end{bmatrix}, \quad \epsilon_t \sim N(0,\Sigma), \quad \Sigma = \begin{bmatrix} \sigma_P^2 & \sigma_{PD}^2 \\ \sigma_{DP}^2 & \sigma_D^2 \end{bmatrix}. \tag{11}$$

This specification allows for beliefs about the growth rates in the share price and dividends to take on different values and their innovations to be imperfectly correlated.

Individuals are assumed to "learn from experience," that is, the information set x_s^t of an agent of age s consists of the realizations of stock prices and dividends observed during his lifetime:

$$x_s^t = \{\log(P_\tau/P_{\tau-1}), \log(D_\tau/D_{\tau-1})\}_{\tau=t-s}^{t}.$$

Individuals are assumed to be born with identical beliefs, centered on the REE outcome in which the asset price grows in lockstep with dividends, $\left(m_0^P, m_0^D\right) = (\mu,\mu)$.

As in Malmenadier and Nagel (2011, 2013), individuals estimate m recursively using past information x_s^t. Let $m_{s,t}$ be the estimator of m using information x_s^t, $m_{s,t} = \mathbb{E}[m|x_s^t]$:

$$m_{s,t} = m_{s-1,t-1} + \gamma_{s,t}\left(x_t - m_{s-1,t-1}\right), \quad \text{where } m_{0,t} = [\mu,\mu]'. \tag{12}$$

This is a particular case of the formulation employed in Malmenadier and Nagel (2013). The sequence of gains $\gamma_{s,t}$ determines the rate of updating of cohort s. With $\gamma_{s,t} = 1/t$ it corresponds to the special case of recursive least squares, which uses all available data up to time t with equal weights. Instead, with $\gamma_{s,t}$ set to a constant, it is a constant-gain learning algorithm, which weighs past data with exponentially decaying weights. In the case of "learning from experience" the gain depends on the age s of each cohort. As a result, individuals of distinct ages differ in their forecasts and adjust forecasts differently in response to changes in prices and dividends. The particular decreasing-gain specification which we adopt from Malmenadier and Nagel is

$$\gamma_{s,t} = \gamma_s \equiv \begin{cases} \frac{\theta}{s}, & \text{if } s \geq \theta, \\ 1, & \text{if } s < \theta, \end{cases} , \tag{13}$$

where θ is a constant parameter that determines the shape of the implied function of weights on past prices and dividends. For $\theta = 1$, all observations since birth are equally weighted. For $\theta < 1$ observations early in life receive more weight, and for $\theta > 1$ less weight, than more recent observations. According to Malmenadier and Nagel (2013), the "learning from experience" framework appears to be consistent with a variety of data and choice settings. For the case of $\theta = 1$ Nakov and Nuño (2011) show that a decreasing-gain scheme is the optimal Bayesian learning strategy for a PLM such as (10).

2.3 Numerical algorithm

The source of heterogeneity in this model is that different age groups form different beliefs. The key difficulty in solving the model lies in finding the stock price in each period given the current dividend realization and the entire distribution of beliefs about price and dividend growth. Updating dividend beliefs $m_{s,t}^{D}$ following (12) poses no major problem since dividends D_t are exogenous and known at time t. The problem lies in the computation of P_t, since it determines the beliefs $m_{s,t}^{P}$, which in turn are used to compute the reservation price of each agent

$$P_{s,t} = \beta\phi \left[e^{m_{s,t}^{P}(P_t)} P_t + e^{m_{s,t}^{D}(D_t)} D_t \right], \quad s \in \mathbb{N}^{+}, \tag{14}$$

and the amount of stock $S_{s,t}(P_t)$ demanded by agents given FOCs (5a) – (5c). The equilibrium stock price should clear the market as per equation (8).

We find the equilibrium stock price P_t by solving a fixed point problem. Appendix B provides a sketch of the algorithm. In order to satisfy the FOCs, the price that clears the market must be that of the marginal stock holder. The cohorts in which agents are more optimistic ($P_t < P_{it}$) face condition (5a), $S_{it} = \bar{L}_t/P_t = \lambda D_t/P_t$, whereas the cohorts in which agents are more pessimistic ($P_t > P_{it}$) face condition (5c), $S_{it} = \underline{L}_t/P_t = 0$. Given an initial price guess P_t^{guess}, we have a series of reservation prices $P_{s,t}(P_t^{guess})$ as per (14). We sort these reservation prices in decreasing order and index them by $j \in 0, ..., s-1$, where $j = 0$ corresponds to the highest reservation price, $j = 1$ to the second highest, and so on. Proceeding down the list from the highest reservation price, we find the reservation price P_t^* of the marginal cohort n, $P_t^* = P_{n,t}$. The marginal cohort is the one for which

$$\frac{1}{N}\sum_{j=0}^{n-1} f_j \bar{S}_t < 1, \text{ and } \frac{1}{N}\sum_{j=0}^{n} f_j \bar{S}_t \geq 1, \tag{15}$$

where $\bar{S}_t = \lambda \frac{D_t}{P_t^{guess}}$. This means that cohorts which are more optimistic than the marginal one ($j < n$) choose to hold the stock to the upper limit $\bar{S}_t$, while cohorts which are more pessimistic ($j > n$) choose not to hold the stock at all, satisfying the FOCs. The error in the guess is $P_t^{guess} - P_t^*(P_t^{guess})$. We use a standard numerical algorithm to find the fixed point $P_t = P_t^*(P_t)$ starting from an initial guess $P_t^{guess} = P_{t-1}$.

Figure 1 illustrates the equilibrium computation under the baseline calibration described below. The solid line in the upper panel represents the reservation prices of the different cohorts arranged in declining order for some arbitrarily chosen period t, $P_{j,t}$ (the stock prices have been normalized by the price under rational expectations, P_t^{REE}). The most optimistic cohort has a reservation price of 0.925 times P_t^{REE}, whereas the most pessimistic one has $0.887 P_t^{REE}$. The marginal cohort is in position $n = 391$ and its reservation price is $0.889 P_t^{REE}$, equal to the equilibrium stock market price, depicted by the dashed line. The lower panel displays the cumulative normalized asset demand $S(n) = \frac{1}{N}\sum_{j=0}^{n} f_j \bar{S}_t$. The equilibrium is achieved for $S(391) = 1$, where demand for the stock equals its supply.

3 A learning-from-experience view of the evolution of the US stock market in the last century

In this section, we explore the empirical implications of heterogeneity due to agents being born on different dates and focusing on data realizations from their own lifetimes, rather than on all historical data.

3.1 Calibration

The model's parameters are calibrated to match the U.S. stock market evidence as documented by Shiller (2005). We assume that each period in the model is a month.

Dynasts discount future consumption by the factor $\beta\phi$, where β is a time preference parameter and ϕ is the probability of survival. The survival rate is set equal to $\phi = 0.9979$, implying an "average life on the market" of about 40 years. We use Shiller's (2005) stock market dataset covering the S&P index from January 1871 to May 2014 to calibrate our model. In particular, consistent with Shiller's data, we set the mean growth rate of dividends to $\mu = 0.0012$ per quarter, and its standard deviation to $\sigma = 0.0145$. We set the time preference parameter to $\beta = 0.9979$, consistent with a price-to-dividend ratio in the REE case of 27, as in the data, and a real interest rate of 5 percent. The leverage ceiling parameter is set to $\lambda = 480$. Note that, by imposing a limit on each individual's investment in the stock, λ affects the measure of households who hold the asset. Setting $\lambda = 480$ is consistent with an average stock market participation rate of around 65 percent, which is the estimate reported by Poterba et. al. (1995) for U.S. households of income \$50,000-\$75,000 in 1992.[7] The value of $\theta = 1.0147$, which controls the rate of learning of each agent, comes from the empirical analysis of Malmendier and Nagel (2013), who find it to be 3.044 in quarterly data. For our numerical simulations, we truncate the maximum number of cohorts to $S = 960$ (80 years), which includes most of the mass of the distribution.

3.2 From 1920 to 1990

In order to evaluate the empirical performance of the model, we take the (exogenous) dividend series directly from the data by Shiller (2005) for the period January 1871 to May 2014 and then simulate the model to obtain the price-dividend ratio.[8] We discard the first 50 years of the simulation to ensure the propagation of beliefs though the entire demographic structure.

Results are displayed in Figure 2. The solid (blue) line reproduces the model simulated price-dividend ratio, the dashed (green) line is the ratio in the historical data and the (black) dotted line is the constant ratio predicted by the REE model. Our heterogeneous-agents overlapping-generations model (HA-OLG) broadly reproduces the main dynamics of the price-dividend ratio: it grows in the 1920s and collapses in the early 1930s, recovering later in the decade. It falls again during the WWII period but, in contrast to the data, the post-war recovery is delayed until the late 1940s. This is one of the main discrepancies between the model and the data, as the model generates a price-dividend (PD) ratio around 25 in the first half of the 1950s whereas it is below 15 in the actual data. The model and the data broadly coincide again during the 1960s' boom, with PD ratios around 30. The decline in the 1970s is more gradual in the model than in the data whereas the recovery in the 1980s is somewhat stronger in the data than in the model.

In Table 1 we compare the moments in the data and the model for this period. We show the first two moments of the growth rate of stock prices and of dividends, and the price-dividend ratio, in the data and in alternative model simulations. The first column reports the actual data, and the second and third columns show the REE model and the HA-OLG model, respectively. Results confirm that the HA-OLG model is able to approximate the main moments of the data, in contrast to the REE model.

[7] Including mutual fund and 401(k) plan participation. The number varies from 10 percent for households with income below \$15,000 to 79 percent for households with income above \$250,000.

[8] Notice that our test of the model is more demanding than the one in Adam, Beutel and Marcet (2013) who feed their model with the historical *capital gains* (both prices *and* dividends). In contrast, we only use the information on dividends.

The stochastic oscillations of the stock price in the model around the REE are related to the dynamics of learning. To see this, Figure 3 plots the evolution of the expected growth of prices across generations, $\mathbb{E}_{st}\left[\log(P_{t+1}/P_t)\right]$, where $\mathbb{E}_{st}(\cdot)$ is the expected value using the information set of cohort s at time t. We plot the expectations of the youngest ($s = 1$, thin blue line) and the oldest ($s = 960$, thick black line) cohorts.[9] Notice that individuals' beliefs regarding the rate of change of the stock price do not converge to the REE value.[10] Instead, they go through successive waves of optimism and pessimism.

Two elements of our model are responsible for the oscillating dynamics of the PD ratio. On the one hand, there is a force of *momentum*, which is rooted in the infrequent resetting of the learning of successive cohorts of individuals as well as in the fact that agents discount older data more heavily than more recent information. Thus, at any given date, a fraction of young individuals enters the market whose learning path initially is more strongly influenced by the more recent stock price and dividend realizations. Their forecasts inform their trading activities, and, through trade, affect the realized stock price, pulling the beliefs of older generations toward the more recent price change realizations.

On the other hand, there is a force of *trend-reversion*, emanating from the constraints on individual risky asset exposure. Namely, as the stock price rises far above the REE, the upper bound in (3) implies that optimistic investors can buy less shares for any given dividend realization. Because, in equilibrium, all shares must be held by someone, the stock price has to fall to the valuation of less optimistic investors. The same reflecting force operates "from below", when the stock price falls too far beneath the REE.[11] The combination of the two factors – momentum and trend reversion – results in boom-and-bust cycles that are only loosely related to dividends.

Indeed, similar to Harrison and Kreps (1978), asset price cycles in our model are partially the result of speculation about the future course of the asset price. Naturally, shocks to dividends do have an influence on the stock price, although the link is not nearly as direct as in the case of REE. Recall that in the REE model, the percentage change in the stock price tracks one-for-one the change in dividends, inheriting the persistence of dividend growth. In contrast, in the OLG model with "learning from experience," a sequence of positive dividend surprises has an escalating effect on asset price changes. This amplification occurs because, through trade, the overreaction to

[9] Recall that the belief of newborns (cohort $s = 0$) is set to the REE solution.

[10] To be precise, numerical simulations suggest that there is no convergence in probability, although we do not provide any theoretical proof of this. In contrast, we do not rule out the possibility of beliefs converging in distribution to the REE. We leave this analysis for future research. Convergence to REE has been studied in several papers such as Vives (1993), Marcet and Sargent (1989) or Ferrero (2007). Evans and Honkapohja (2003, ch. 15) establish that in recursive least squares learning for gain sequences of the form $t^{-\chi}$ the speed of convergence is asymptotically $t^{\chi/2}$. Nakov and Nuño (2011) discuss the convergence to REE of a particular case of this model with infinitely-lived agents.

[11] Note that trend reversal kicks in *before* the aggregate leverage constraint $P_t/D_t = \lambda$ becomes binding. Thus, the turning points of the stock price cycles are endogenous in the model.

more recent information affects the stock price and, progressively, the beliefs of other individuals, creating a non-linear feedback, which reinforces the effects of dividend shocks on the stock price.

3.3 From 1990 to 2014

We extend our analysis to cover the most recent period from 1990 to 2014. This period is characterized by the surge in the PD ratio during the "dot-com bubble"and its collapse, together with the fall in the PD ratio during the "Great Recession." Following the methodology described above, we compare the simulated outcome of the model to that in the data in Figure 4. The results show that the HA-OLG model fails to replicate the observed dynamics in the last quarter of the century. Instead, it produces some moderate oscillations around the REE solution.

The model fails to match the data due to the presence of an individual upper bound in (3). This bound limits the maximum PD ratio to the value of λ, which is calibrated (to replicate the stock market participation rate) to a value of 480, equivalent to a PD ratio of 40. In the period 1920-1990, this constraint is not binding at the aggregate level. However, during the period 1990-2014, the PD ratio rises above this level, so it is impossible for the model to replicate the data.

We next explore the possibility of relaxing the constraint by allowing λ to linearly increase from 480 to 1440 between January 1995 and June 2000, in line with the build-up of the bubble. We interpret this as an increase in the allowed leverage ratio (and market participation) of the agents in the stock market.

Figure 4 displays this case with a solid line with dots. The expansion in λ allows the model to replicate the 1990s boom and the crash in 2001. However, the model generates a counterfactual rise in the PD ratio again in 2004 which collapses abruptly in 2008-2009. The conclusion is that, even if the change in λ improves the fit of the model, it still offers an unsatisfactory evolution of the PD ratio in the 2000s.

In Table 2 we compare the moments in the data and the model for the full period 1920-2014. The first column reports the actual data, the second column shows the REE model, while columns three to four show the baseline and the alternative parameterization of our HA-OLG model. One clear advantage of the HA-OLG model is that it generates dynamics of the price-dividend ratio, unlike the REE model, due to the extra volatility of the growth rate of stock prices. The table shows that the HA-OLG provides a better fit of the moments in the data than the REE model.

To summarize these results, the baseline HA-OLG model reasonably reproduces the evolution of stock prices during the period 1920-1990, generating most of its boom-bust cycles, but fails to reproduce the last 25 years. In contrast, the REE model predicts a constant PD ratio with prices proportional to dividends. Taking into account that the model lacks many realistic features, such as risk-aversion, portfolio diversification between risky and riskless assets or financial frictions, the good fit in the first seven decades of the sample suggests that learning from experience may explain, at least partially, the evolution of stock prices. At the same time, the fact that the model fails to generate the large PD ratios observed in the late 1990s might indicate the important role of expectations about future prices in contrast to the mechanic updating of beliefs during the "dot-com" bubble.

3.4 Forecasting errors

Learning from experience implies that individuals' forecasts in our model are not fully rational because agents do not take into account all the available information. The important question however is to what extent the suboptimality of agents' forecasts is detectable from the data. Figure 5 plots the mean and the standard deviation of the 1-period ahead forecasting error in the HA-OLG model as a function of the cohort age,

$$e_{s,t} = \begin{bmatrix} e^P_{s,t} \\ e^D_{s,t} \end{bmatrix} = \begin{bmatrix} \log(P_t/P_{t-1}) - m^P_{s,t-1} \\ \log(D_t/D_{t-1}) - m^D_{s,t-1} \end{bmatrix},$$

alongside the forecasting error of the REE model. The two top panels show that in both models forecasts are approximately unbiased in the sense that mean forecasting errors for both prices and dividends are very close to zero. The two bottom panels show the root mean squared error, which, given unbiasedness, equals the standard deviation of the error. The standard deviations of the forecasting errors in the HA-OLG model for both dividends and stock prices are larger than in the REE model.

In the REE case the root mean squared errors for prices and dividends are the same, equal to σ. In contrast, in the HA-OLG model, the volatility of prices is considerably higher than that of dividends and hence the volatility of the forecasting errors for prices is higher. This occurs because the stock price depends on market expectations, creating self-referential dynamics as emphasized by Eusepi and Preston (2011). Conversely, in the REE model, uncertainty about prices and dividends is the same because agents coordinate ex-ante onto "the right model" for asset pricing.

Younger cohorts display more significant forecasting error variances in both prices and dividends than older cohorts. This is because older cohorts observe longer histories of data.

4 Comparison with a constant-gain learning representative-agent model

This section explores the comparison between the HA-OLG model and the more standard representative-agent constant-gain learning (RA-CGL) framework.

4.1 Constant-gain learning as an approximation to the average belief

We begin by approximating the aggregate dynamics of our economy without having to deal with the entire distribution of beliefs across agents. We show that the average belief can be approximated by a RA-CGL scheme. Let us define the average belief across age cohorts,

$$\bar{m}_t = \left[\bar{m}_t^P, \bar{m}_t^D\right]' = \frac{1}{N}\left[\sum_{s=0}^{\infty} f_s m_{s,t}^P(P_t), \sum_{s=0}^{\infty} f_s m_{s,t}^D(D_t)\right]', \tag{16}$$

and the maximum belief dispersion $\Sigma = \left[\Sigma^P, \Sigma^D\right]'$

$$\Sigma^i = \max_t \max_s \left\| m_{s,t}^i - \bar{m}_t^i \right\|, \quad i = P, D.$$

The evolution of the average belief is given by the following proposition:

Proposition 1 (Average market beliefs) *The average market belief is given by*

$$\bar{m}_t = \bar{m}_{t-1} + \bar{\gamma}\left(x_t - \bar{m}_{t-1}\right) + \xi_t, \tag{17}$$

where

$$\bar{\gamma}(\phi, \theta) \equiv \frac{1}{N}\sum_{s=1}^{\infty} f_s \gamma_s = (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor} \phi^s \left(1 - \frac{\theta}{s}\right) - \theta \log(1-\phi)\right], \tag{18}$$

is the average gain factor across age cohorts and the residual term ξ_t is bounded by

$$\left\|\xi_t - (1-\phi) m_0 + \left(\phi + \bar{\gamma} - 1\right)\bar{m}_{t-1}\right\| \le \bar{\gamma}\Sigma.$$

Proof. See Appendix A. ■

Notice that $\lfloor\theta\rfloor$ is the largest integer smaller or equal to θ and $m_0 = \mu$ (prior belief). Therefore, average beliefs about price and dividend growth are updated approximately according to a constant gain learning (CGL) scheme plus a residual term ξ_t. The approximation will be good provided that ξ_t is small compared to the rest of the terms in (17). The value of ξ_t depends on a time-varying term $(1-\phi)\left(m_0 - \bar{m}_{t-1}\right)$ plus a term proportional to the maximum belief dispersion Σ. CGL can thus be viewed as an approximate aggregation of the beliefs of individuals who learn from experience, using data realized in their lifetimes. Notice that the CGL algorithm differs from the actual learning scheme of any of the individual agents because individual learning happens with a decreasing gain, as shown in (13). The population as a whole, however, learns approximately with a constant gain. This has been empirically verified by Malmendier and Nagel (2013).

The social gain parameter $\bar{\gamma}$ is a non-linear function of the survival probability ϕ and the individual gain θ. The value of $\bar{\gamma}$ is increasing and concave in θ and $1-\phi$. Under our baseline calibration, the social gain is equal to 0.013, which in quarterly terms is around 0.039. These numbers are larger, but of the same order of magnitude as existing estimates of the constant-gain parameter both from macro time series data and from surveys. Milani (2007) estimates the gain-parameter in a representative agent model to be 0.018 in U.S. data, which is the same as the 0.018 estimated by Malmendier and Nagel (2013).

Proposition 1 states that the average beliefs follow a CGL updating scheme provided that ξ_t is small relative to $\bar{m}_{t-1}$ and $\bar{\gamma}\left(x_t - \bar{m}_{t-1}\right)$. We analyze whether this condition holds for the baseline calibration considered here. To do so, we simulate the baseline model (HA-OLG) and construct the vector x_t using the simulated prices and historical dividends. Then we run the CGL algorithm (17) and compute $\bar{m}_t$ and ξ_t. We call this "open-loop" as there is no feedback from the CGL beliefs to the model, that is, these are the results taking the series of prices and dividends as exogenous for the representative-agent CGL (RA-CGL) algorithm.

Figure 6 compares the HA-OLG model with the RA-CGL approximation. Both expectations about price and dividend growth are well approximated by the CGL algorithm. Notice that the values of $\Sigma' = \left[\Sigma^P, \Sigma^D\right]$ are $[1.68, 0.06]$ so that the error bounds $\bar{\gamma}\Sigma^i$ are 2.2 percent and 0.08 percent for prices and dividends respectively. The first one is one order of magnitude smaller than the volatility of the growth in prices, $53.76/\sqrt{12} = 15.52$ percent, according to Table 2. The second one is two orders of magnitude smaller than the volatility of the growth in dividends, $14.07/\sqrt{12} = 4.06$. In addition, the term $(\phi + \bar{\gamma} - 1)$ is 0.01. This confirms that the CGL approximation is a reasonable approximation to the average belief under this calibration.

4.2 A "closed-loop" RA-CGL model

We next analyze the "closed-loop" RA-CGL model. This amounts to analyzing an independent representative-agent economy with CGL. First we present the model in a similar fashion to the ones in Adam, Marcet and Nicolini (2008) or Adam, Beutel and Marcet (2013). A representative agent maximizes his utility (1) subject to the budget constraint (2) and the bounds (3). As in Adam, Beutel and Marcet (2013), the agent observes prices with one period delay, so that the Euler equation of the representative agent is

$$P_t = \beta\phi\tilde{\mathbb{E}}_{t-1}P_{t+1} + \beta\phi\tilde{\mathbb{E}}_t D_{t+1}, \tag{19}$$

where $\tilde{\mathbb{E}}_t\left(\cdot\right)$ is the expected value according to her PLM (10). Adam, Beutel and Marcet (2013) state that they choose this information delay in order to avoid the simultaneity of prices, which may give rise to multiple clearing prices and belief pairs.[12]

The agent updates her beliefs according to a CGL scheme similar to (17) with $\xi_t = 0$, $\forall t$. Therefore

$$P_t = \min\left\{\beta\phi\left[P_{t-1}e^{2*\bar{m}_{t-1}^P(P_{t-1})} + D_t e^{\bar{m}_t^D(D_t)}\right], \lambda D_t\right\},$$

is the equation that describes the evolution of prices given the beliefs.

How does the RA-CGL model relate to the HA-OLG described above? Taking into account the market clearing condition (8), we can compute the asset price in the HA-OLG model as a function of the average belief

$$\begin{aligned} P_t &= \beta\phi\frac{1}{N}\sum_{s=0}^{\infty} f_s \mathbb{E}_{st}\left(P_{t+1}+D_{t+1}\right) + \frac{1}{N}\sum_{s=0}^{\infty} f_s\mu_{it} \qquad (20)\\ &= \beta\phi\left\{P_t\sum_{s=0}^{\infty} f_s e^{m_{s,t}^P(P_t)} + D_t\sum_{s=0}^{\infty} f_s e^{m_{s,t}^D(D_t)}\right\} + \mu_t \\ &\approx \beta\phi\left\{P_t\exp\left[\sum_{s=0}^{\infty} f_s m_{s,t}^P(P_t)\right] + D_t\exp\left[\sum_{s=0}^{\infty} f_s m_{s,t}^D(D_t)\right]\right\} + \mu_t \\ &= \beta\phi\left[P_t e^{\bar{m}_t^P(P_t)} + D_t e^{\bar{m}_t^D(D_t)}\right] + \mu_t, \end{aligned}$$

where $\mu_t = \frac{1}{N}\sum_{s=0}^{\infty} f_s\mu_{st}$ is the average Lagrange multiplier across age cohorts. Equation (20) states that the market price is equal to the discounted average belief of next period payoff plus a term reflecting the deviation of the average from the marginal asset holder's belief. There are two differences between (19) and (20). The first one is the informational delay. The second one is the presence of μ_t, which is due to the fact that the price is not related to the beliefs of the *average* agent but to those of the *marginal* agent. This is a key difference between the two models. Going from the second to the third line we have also assumed that the growth rate of beliefs is sufficiently small. Notice that $\left[\bar{m}_t^P, \bar{m}_t^D\right]$ are a function of the current level of prices and dividends, respectively, and are updated according to the CGL rule (17) with $\xi_t \neq 0$.

To summarize this comparison, the main differences between RA-CGL model and HA-OLG is the fact that in the HA-OLG case the price depends on the beliefs of the marginal agent ($\mu_t \neq 0$) and that the aggregate CGL learning scheme includes a distortion term $\xi_t \neq 0$. In addition, the existence of multiple equilibria in the RA-CGL case, absent in the HA-OLG, introduce an extra informational assumption. As we see below, these differences have a major impact on the performance of the two models.

Figure 7 displays the comparison between the data, the HA-OLG model and the RA-CGL. We consider two alternative values for $\bar{\gamma}$: the baseline one of 0.013 and an alternative value of

[12] In the context of our model, not considering this informational delay generates explosive oscillating dynamics (not shown). This is the reason why we follow this literature on RA-CGL.

$0.018/3 = 0.006$, based on the empirical studies of Milani (2007) and Malmendier and Nagel (2013). The results show that in both cases the RA-CGL model produces a path for stock prices which is very different from the one in the HA-OLG case and from the one observed in the data. This suggests that the differences between the two models described above play a major role in explaining the alternative dynamics. In fact, in both CGL cases the PD ratios "hit" against the boundary λ, which does not happen in the HA-OLG model.[13]

The empirical performance of the RA-CGL is quite disappointing. The last two columns in Table 2 show that RA-CGL generates smaller mean stock price growth rates and volatilities than the HA-OLG model.[14] However, even if the rest of the moments are relatively similar to those in the data, the simulated price path is totally unrelated to the actual one and it is impossible to recognize any historical cycle.[15]

5 Conclusions

In order to coordinate *a priori* to a REE, individuals must be endowed with incredible amounts of information not only about the structure of the economy and the exogenous shocks but also about the higher-order beliefs of all other market participants. If individuals lack this information, the law of iterated expectations is no longer valid and "beauty contest" dynamics may emerge as individuals embark on speculative trading as in Harrison and Kreps (1978). In particular, empirical research by Malmendier and Nagel (2009, 2013) suggests that expectations are not "externally rational" in the sense of Adam and Marcet (2011); rather, they find evidence that people "learn from experience," giving more weight to recent events realized during their lives than to older ones.

We consider a Lucas-tree stochastic OLG setup and analyze the effects of "learning from experience." The fact that different generations of individuals hold different beliefs leads to boom-and-bust cycles of the stock price around the REE with statistical moments similar to those found in the data. In particular the model produces a sequence of stock price cycles similar to the actual ones in the US stock market for the period 1920-1990, although it fails to generate the "dot-com" boom-and-bust in the late 1990s. We show that, despite the fact that individuals learn with decreasing gain, aggregate learning by the population as a whole can be approximated by a constant gain. The social gain parameter is a nonlinear function of the survival rate and the individual gain parameters, reflecting both the fact that historical data is lost when older generations are replaced

[13] If the boundary is removed prices explode.

[14] These conclusions remain unchanged if we consider the subsample 1920-1990, as shown in the last column of Table 1.

[15] It should be stressed that the informational assumptions are not the same in the RA-CGL and the HA-OLG models. However, it seems unlikely that the disparity of results would be much reduced in the case of the same information timing.

by young ones and the higher weight that each individual gives to more recent events.

Our approach provides discipline by tying the gain parameter to the survival rate and to survey-based information on individual learning gains. The fact that the average learning can be approximated by a constant gain scheme does not mean that a standard representative-agent model with CGL produces similar results to the baseline model. In fact, we verify that it produces completely different outcomes. The main reason is that in the RA-CGL model the economy is driven by the average belief, whereas in the HA-OLG it is the belief of the marginal stock holder that matters.

References

[1] Adam, K., Beutel, J., and Marcet, A. 2013. Stock Price Booms and Expected Capital Gains. Unpublished manuscript, University of Mannheim.

[2] Adam, K. and A. Marcet. 2011. "Internal Rationality, Imperfect Market Knowledge and Asset Prices," *Journal of Economic Theory*, In Press.

[3] Adam, K., A. Marcet, and J. P. Nicolini. 2008. "Stock Market Volatility and Learning," European Central Bank Working Paper No. 862.

[4] Allen, F., S. Morris, and H. S. Shin. 2006. "Beauty Contests and Iterated Expectations in Asset Markets," *Review of Financial Studies*, vol. 19 (3), pp. 719–52.

[5] Angeletos, G. M., and J. La'O. 2010. Noisy business cycles. In NBER Macroeconomics Annual 2009, Volume 24 (pp. 319-378). University of Chicago Press.

[6] Angeletos, G. M. and J. La'O. 2011. "Decentralization, Communication, and the Origins of Fluctuations," Working paper, MIT

[7] Arrow, K.J. 1986. "Rationality of Self and Others in an Economic System," *The Journal of Business*, vol. 59 (4), S385–99.

[8] Blume, L. E., M. M. Bray, and D. Easley. 1982. "Introduction to the Stability of Rational Expectations Equilibrium," *Journal of Economic Theory*, vol. 26 (2), pp. 313–17.

[9] Boswijk, H. P., Hommes, C. H., & Manzan, S. 2007. Behavioral heterogeneity in stock prices. Journal of Economic Dynamics and Control, 31(6), 1938-1970.

[10] Branch, W. A., and G. W. Evans. 2006. "Intrinsic Heterogeneity in Expectation Formation," *Journal of Economic Theory*, vol. 127, pp. 264–95.

[11] Branch, W., and G. W. Evans. 2011. "Learning about Risk and Return: A Simple Model of Bubbles and Crashes," *American Economic Journal: Macroeconomics*, vol. 3 (3), pp. 159–91.

[12] Branch, W. and B. McGough. 2004. "Multiple Equilibria in Heterogeneous Expectations Models," *The B. E. Journal of Macroeconomics*, Berkeley Electronic Press, vol. 0 (1), pp. 12.

[13] Brock, William A., and Cars H. Hommes. 1998. "Heterogeneous beliefs and routes to chaos in a simple asset pricing model." Journal of Economic Dynamics and Control 22.8-9: 1235-1274.

[14] Brown, A., and L. Rogers. 2009. "Heterogeneous Beliefs with Finite-Lived Agents,“ University of Cambridge, Statistical Laboratory, Working Paper.

[15] Carceles-Poveda, E., and C. Giannitsarou. 2007. "Adaptive Learning in Practice," *Journal of Economic Dynamics and Control*, vol. 31 (8), pp. 2659-97.

[16] Cogley, T. and T. J. Sargent. 2008. "The Market Price of Risk and the Equity Premium: A Legacy of the Great Depression?," *Journal of Monetary Economics*, vol. 55 (3), pp. 454–76.

[17] Cogley, T., T. J. Sargent and V. Tsyrennikov. 2012. "Market Prices of Risk with Diverse Beliefs, Learning, and Catastrophes," *American Economic Review*, vol. 102 (3), pp. 141-46.

[18] DeGroot, M. H. 1970. *Optimal Statistical Decisions*. McGraw-Hill Press.

[19] Eusepi, S., and B. Preston. 2011. "Expectations, Learning and Business Cycle Fluctuations," *American Economic Review*, forthcoming.

[20] Evans, G. W. and S. Honkapohja. 2003. *Learning and Expectations in Macroeconomics*, Princeton University Press.

[21] Ferrero, G. 2007. "Monetary Policy, Learning and the Speed of Convergence," *Journal of Economic Dynamics and Control*, vol. 31 (9), pp. 3006–41.

[22] Giannitsarou, C. 2003. "Heterogeneous Learning," *Review of Economic Dynamics*, vol. 6 (4), pp. 885–906.

[23] Graham, L. 2011. "Learning, Information and Heterogeneity," Working paper.

[24] Harrison, J. M. and D. M. Kreps. 1978. "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quarterly Journal of Economics*, vol. 92 (2), pp. 323–36.

[25] Hommes, C. H. 2006. Heterogeneous agent models in economics and finance. Handbook of computational economics, 2, 1109-1186.

[26] Honkapohja, S., and K. Mitra. 2006. "Learning Stability in Economies with Heterogeneous Agents," *Review of Economic Dynamics*, vol. 9 (2), pp. 284–309.

[27] Ljung, L. and T. Soderstrom. 1983. *Theory and Practice of Recursive Identification*, MIT Press.

[28] Malmendier, U. and S. Nagel. 2011. "Depression Babies: Do Macroeconomic Experiences Affect Risk-Taking?," *Quarterly Journal of Economics*, vol. 126 (1), pp. 373–416.

[29] Malmendier, U. and S. Nagel. 2009. "Learning from Inflation Experiences," Unpublished manuscript, U.C. Berkley

[30] Marcet, A. and T. J. Sargent. 1989. "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, vol. 48 (2), pp. 337–68.

[31] Milani, F., 2007. "Expectations, Learning and Macroeconomic Persistence," J*ournal of Monetary Economics*, vol. 54, pp. 2065–82.

[32] Nakov, A. and G. Nuño. 2011. "Learning from experience in the stock market," Working Paper Series 1396, European Central Bank.

[33] Poterba, J. M., A. A. Samwick, A. Shleifer, and R. J. Shiller. 1995. "Stock Ownership Patterns, Stock Market Fluctuations, and Consumption," *Brookings Papers on Economic Activity*, vol. 1995 (2), pp. 295–372.

[34] Sargent, T. J. 1999. *The Conquest of American Inflation*. Princeton University Press.

[35] Shiller, R. 2005. *Irrational Exuberance*, 2nd edition, Princeton University Press.

[36] Timmerman, A. 1994. "Can Agents Learn to Form Rational Expectations? Some Results on Convergence and Stability of Learning in the UK Stock Market," *The Economic Journal*, vol. 104 (425), pp. 777–97.

[37] Vives, X. 1993. "How Fast do Rational Agents Learn?," *Review of Economic Studies*, vol. 60 (2), pp. 329–47.

[38] Weitzman, M. L. 2007. "Subjective Expectations and Asset-Return Puzzles," *American Economic Review*, vol. 97 (4), 1102–30.

Appendix A: Proof of Proposition 2

Proof. Let

$$\bar{m}_t = \frac{1}{N}\sum_{s=0}^{\infty} f_s m_{s,t} = \frac{f_0 m_{0,t}}{N} + \frac{1}{N}\sum_{s=1}^{\infty} f_s \left[m_{s-1,t-1} + \gamma_s \left(x_t - m_{s-1,t-1}\right)\right],$$

given the fact that

$$\frac{1}{N}\sum_{s=1}^{\infty} f_s m_{s-1,t-1} = \phi \bar{m}_{t-1},$$

and that

$$\begin{aligned}
\bar{\gamma} &= (1-\phi)\sum_{s=1}^{\infty}\phi^s\gamma_s = \frac{1}{N}\sum_{s=1}^{\lfloor\theta\rfloor} f_s + \frac{1}{N}\sum_{s=\lfloor\theta\rfloor+1}^{\infty} f_s\frac{\theta}{s} = (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor}\phi^s + \theta\sum_{s=\lfloor\theta\rfloor+1}^{\infty}\frac{\phi^s}{s}\right] \\
&= (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor}\phi^s + \theta\sum_{s=\lfloor\theta\rfloor+1}^{\infty}\int\phi^{s-1}d\phi\right] = (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor}\phi^s + \theta\int\left(\sum_{s=\lfloor\theta\rfloor+1}^{\infty}\phi^{s-1}\right)d\phi\right] \\
&= (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor}\phi^s + \theta\int\frac{\phi^{\lfloor\theta\rfloor}}{1-\phi}d\phi\right] = (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor}\phi^s - \theta\sum_{s=1}^{\lfloor\theta\rfloor}\frac{\phi^s}{s} - \theta\log(1-\phi)\right] \\
&= (1-\phi)\left[\sum_{s=1}^{\lfloor\theta\rfloor}\phi^s\left(1-\frac{\theta}{s}\right) - \theta\log(1-\phi)\right],
\end{aligned}$$

we have

$$\bar{m}_t = \frac{1}{N}\sum_{s=0}^{\infty} f_s m_{s,t} = (1-\phi)m_0 + \phi\bar{m}_{t-1} + \bar{\gamma}x_t - \frac{1}{N}\sum_{s=1}^{\infty} f_s\gamma_s m_{s-1,t-1},$$

and we define

$$\xi_t \equiv (1-\phi)m_0 + \left(\phi+\bar{\gamma}-1\right)\bar{m}_{t-1} - \frac{1}{N}\sum_{s=1}^{\infty} f_s\gamma_s m_{s-1,t-1}.$$

In order to find the bound (18), we calculate

$$\begin{aligned}
\left\|\xi_t - (1-\phi)m_0 + \left(\phi+\bar{\gamma}-1\right)\bar{m}_{t-1}\right\| &= \left\|(1-\phi)\sum_{s=1}^{\infty}\phi^s\gamma_s m_{s-1,t-1})\right\| \\
&\leq (1-\phi)\sum_{s=1}^{\infty}\phi^s\gamma_s\left\|m_{s-1,t-1}\right\| \\
&\leq \left[(1-\phi)\sum_{s=1}^{\infty}\phi^s\gamma_s\right]\Sigma \\
&= \bar{\gamma}\Sigma.
\end{aligned}$$

■

Appendix B: Simulation algorithm

We briefly sketch the algorithm used to find the equilibrium price of our heterogeneous-beliefs economy. The idea is to simulate the evolution of dividends while keeping track of each agent's stock holdings and beliefs. To compute the equilibrium price given equations (7) and (8), we employ a numerical routine to find a fixed-point for the price which is consistent with agents' beliefs and constraints and which guarantees that the market clears.

Here we describe a single Monte Carlo simulation of the model:

1. Generate an exogenous series for dividends D_t following (4) and assuming that $D_0 = 1$. Set $P_0 = P_0^{REE}$, where P_t^{REE} is given by (9).

2. Initialize the prior beliefs, $\left(m_{s,0}^P, m_{s,0}^D\right) = (\mu, \mu)$, for all cohorts, $s = 0, ..., S$.

3. **Main loop**. At each point in time $t = 1, ..., T$:

   (a) Compute the dividend beliefs across cohorts

   $$m_{s,t}^D = m_{s-1,t-1}^D + \gamma_s \left[\log(D_t/D_{t-1}) - m_{s-1,t-1}^D\right], s = 1, ..., S.$$

   (b) Compute the price P_t as a *fixed-point* given equations (7) and (8). The initial guess $P_t^{guess} = P_{t-1}$. Employ the following subroutine:

      i. Given the guess, compute the price beliefs across cohorts

      $$m_{s,t}^P = m_{s-1,t-1}^P + \gamma_s \left[\log(P_t^{guess}/P_{t-1}) - m_{s-1,t-1}^P\right], s = 1, ..., S.$$

      ii. Compute the reservation price across cohorts

      $$P_{s,t} = \beta\phi \left[e^{m_{s,t}^P} P_t^{guess} + e^{m_{s,t}^D} D_t\right].$$

      iii. Sort the reservation prices $P_{s,t}$ in decreasing order. Index the prices by j, where $j = 0$ corresponds to the highest $P_{s,t}$.

      iv. Proceeding from the highest reservation price, find the reservation price of the marginal cohort $P_t^* = P_{n,t}$. The marginal cohort n is such that

      $$\sum_{j=0}^{n-1} f_j \bar{S}_t < N, \text{ and } \sum_{j=0}^{n} f_j \bar{S}_t \geq N$$

where $\bar{S}_t = \lambda \frac{D_t}{P_t^{guess}}$ is the amount allocated to each agent given the leverage constraint (3) and f_j is the size of the cohort corresponding to index j.

v. The error in the guess is $\zeta_t = P_t^{guess} - P_t^*$.

vi. Repeat (i) to (v) until $|\zeta_t|$ is less than the error tolerance level.

(c) Given the price, compute the price beliefs across cohorts

$$m_{s,t}^P = m_{s-1,t-1}^P + \gamma_s \left[\log(P_t/P_{t-1}) - m_{s-1,t-1}^P\right], s = 1, ..., S.$$

4. Repeat the main loop (3) for periods $t = 1, ..., T$.

Appendix C: Tables and Figures

Table 1. Moments of prices and dividends 1920-1990

	Data	REE	HA-OLG	RA-CGL
$\log(P_t/P_{t-1})$				
mean (%)	2.47	1.66	2.79	3.14
st. dev. (%)	56.78	15.34	35.01	10.30
$\log(D_t/D_{t-1})$				
mean (%)	1.66	1.66	1.66	1.66
st. dev. (%)	15.34	15.34	15.34	15.34
P_t/D_t				
mean	23.21	26.25	21.75	25.33
st. dev.	6.52	0	5.17	11.31

Note: REE stands for "rational expectations equilibrium."

HA-OLG stands for "heterogeneous-agents overlapping generations."

RA-CGL stands for "representative-agent constant-gain learning."

Table 2. Moments of prices and dividends 1920-2014

	Data	REE	HA-OLG		RA-CGL	
			baseline	changing λ	baseline	$\gamma = 0.006$
$\log(P_t/P_{t-1})$						
mean (%)	3.06	1.82	2.94	3.92	1.58	1.16
st. dev. (%)	53.76	14.07	34.04	35.71	9.42	5.92
$\log(D_t/D_{t-1})$						
mean (%)	1.82	1.82	1.82	1.82	1.82	1.82
st. dev. (%)	14.07	14.07	14.07	14.07	14.07	14.07
P_t/D_t						
mean	30.71	26.25	22.83	31.47	26.21	24.69
st. dev.	15.92	0	5.28	21.00	10.82	12.90

Note: REE stands for "rational expectations equilibrium."

HA-OLG stands for "heterogeneous-agents overlapping generations."

RA-CGL stands for "representative-agent constant-gain learning."

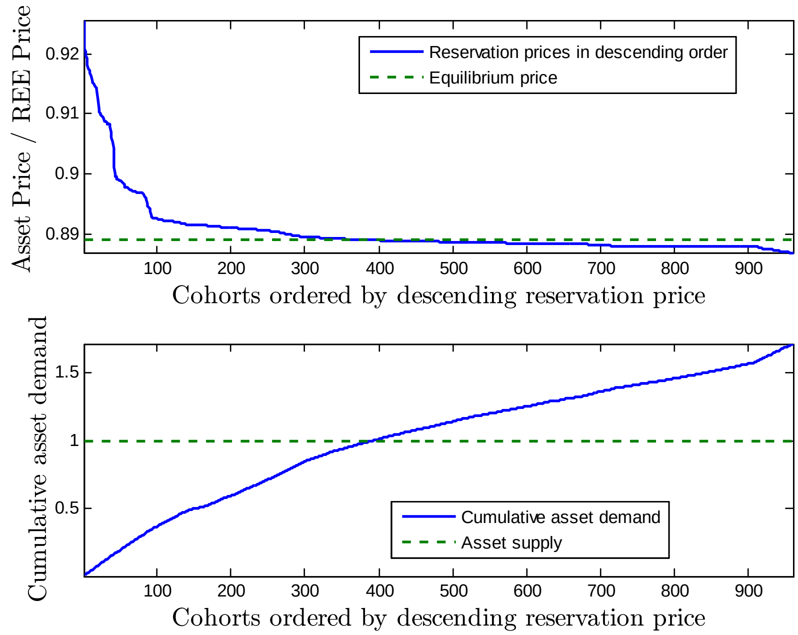

Figure 1: Distribution of reservation prices and asset demand across cohorts

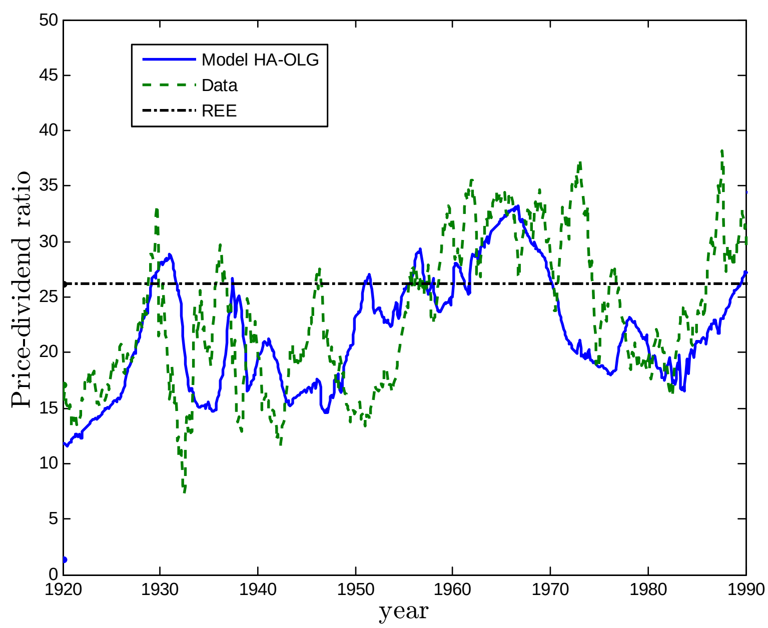

Figure 2: Evolution of the price-dividend ratio 1920-1990

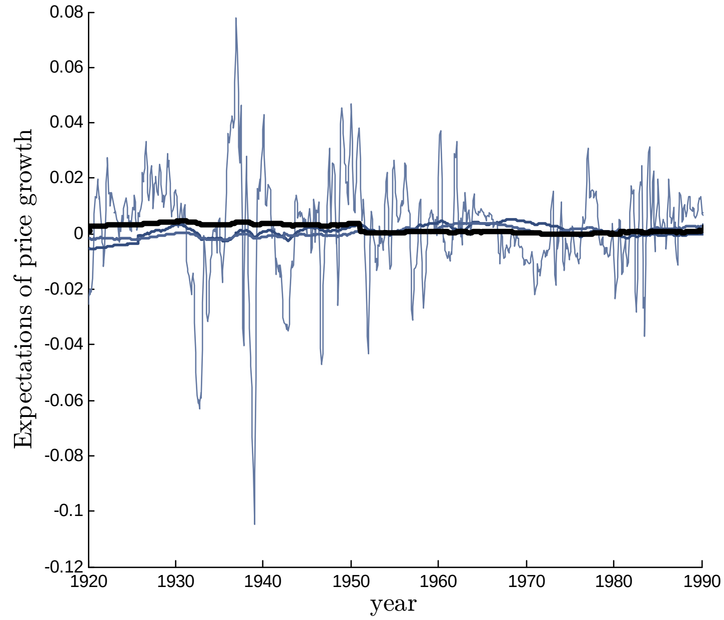

Figure 3: Expectations of price growth: thicker line indicates older cohort

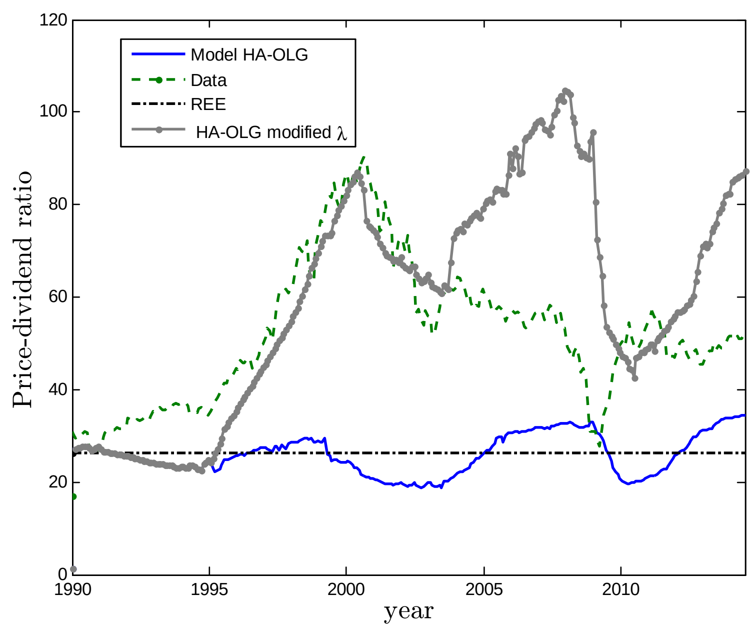

Figure 4: Evolution of the price-dividend ratio 1990-2014

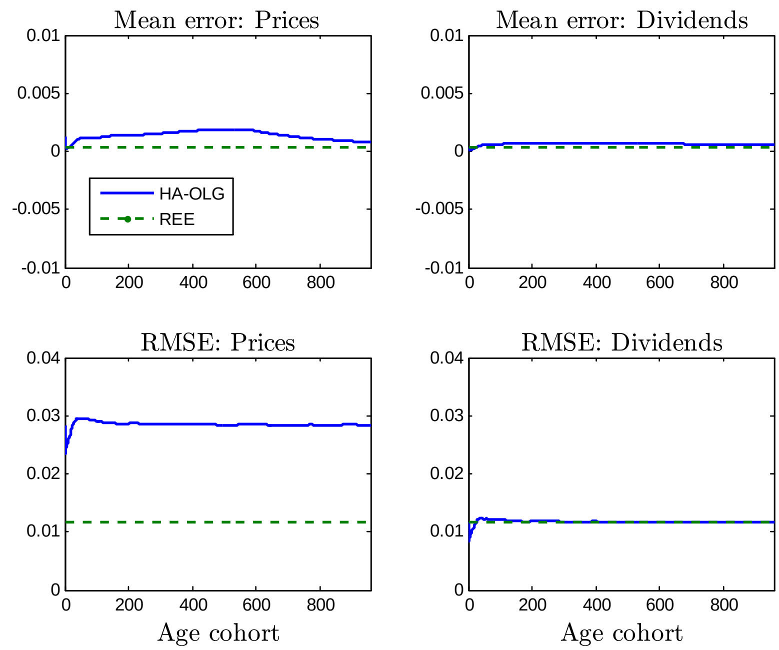

Figure 5: One-period-ahead mean and root mean squared forecasting errors for prices and dividends in the learning from experience model and the REE

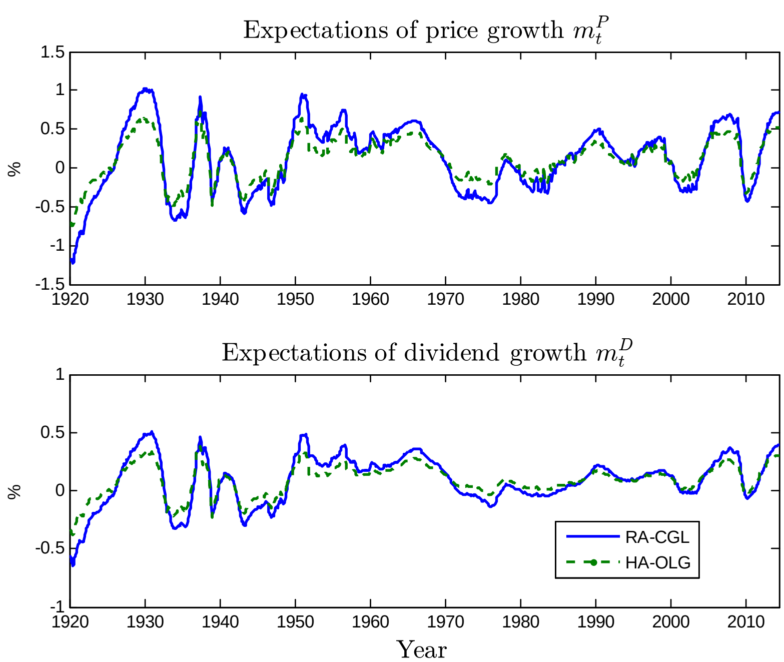

Figure 6: Price and dividend growth beliefs in the heterogeneous agents model (HA-OLG) and in the representative-agent CGL approximation (RA-CGL).

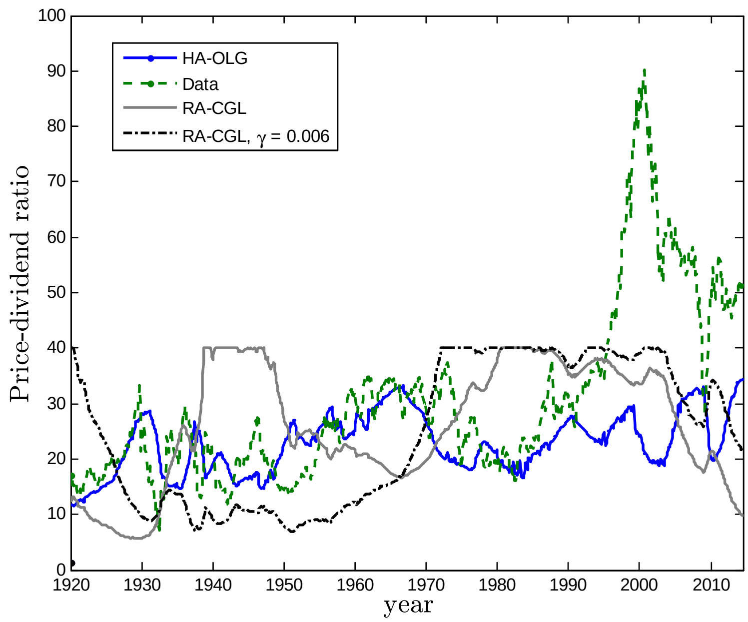

Figure 7: Evolution of the price-dividend ratio 1920-2014