Existence and uniqueness of solutions to dynamic models with occasionally binding constraints

Tom D. Holden, Deutsche Bundesbank*

Abstract: Policy makers would like to prevent self-fulfilling fluctuations. This requires determinacy conditions for models with occasionally binding constraints (OBCs) like the zero lower bound (ZLB). To this end, we derive existence and uniqueness conditions for otherwise linear models with OBCs. Our main result gives necessary and sufficient conditions for such models to have a unique perfect foresight solution returning to a given steady state, for any initial condition. We show that standard New Keynesian models have multiple perfect-foresight paths eventually escaping the ZLB, but price level targeting restores determinacy. In supplemental results, we derive equilibrium existence conditions under rational expectations.

Keywords: *occasionally binding constraints, zero lower bound, existence, uniqueness, price targeting*

JEL Classification**:** *C62, E3, E4, E5*

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* Address: Deutsche Bundesbank, Wilhelm-Epstein-Straße 14, 60431 Frankfurt am Main, Germany.

Email: thomas.holden@gmail.com. Website: http://www.tholden.org/. The views expressed in this paper are those of the author and do not represent the views of the Deutsche Bundesbank, the Eurosystem or its staff.

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1. Introduction

Consider an otherwise linear model with occasionally binding constraints (OBCs). This paper gives conditions under which the model always has a unique perfect foresight solution that eventually escapes the bound, whatever the initial state. This provides determinacy conditions for models with occasionally binding constraints.

For uniqueness, positive news shocks to the bounded variable must have a sufficiently positive impact on that variable. When this does not hold, there are some states in which there exist multiple transition paths back to the unconstrained steady state. For example, one path may hit the bound, while another does not.

To see how this multiplicity is possible, suppose the model's agents knew the economy would escape the bound next period. Then expectations of next period's outcomes would be linear in today's variables, as the model is linear apart from the OBC. However, substituting out these expectations does not leave a linear system in today's variables, due to the OBC. This non-linear system may have two solutions, with one featuring a slack constraint, and the other having a binding constraint. Alternatively, the non-linear system may have no solution at all, giving non-existence. Without the assumption that next period the economy is away from the bound, the scope for multiplicity is even greater, and there may be infinitely many solutions.

Our uniqueness condition is sufficient in all models. It is also necessary provided either that the model's state space is rich enough, or that we consider a broad enough set of perfect foresight exercises. A standard perfect foresight exercise calculates the return to steady state given an initial state and perhaps some initial shock. We can also allow for shocks that hit in future though. If we want uniqueness for all possible sequences of future shocks, then our uniqueness condition is necessary. We also give conditions under which an otherwise linear model with OBCs has at least one solution eventually escaping the bound.

To relate our perfect-foresight results to rational expectations, we prove supplemental results on existence under rational expectations for arbitrary non-linear models (not just otherwise linear ones with OBCs). Under mild assumptions, we show that for each solution under perfect foresight, there is a corresponding solution under rational expectations. This solution approximately follows the dynamics of the perfect foresight one until a "reset" shock hits, at which point it is as if time were set back to zero. Additionally, when there are multiple perfect foresight solutions, we show there are a continuum of rational expectations solutions that switch between them. Applied to models with OBCs, this gives existence conditions under rational expectations.

We apply our results to New Keynesian (NK) models with a zero lower bound (ZLB). Responding aggressively to inflation is generally insufficient to achieve determinacy in the presence of the ZLB. This contrasts with the case without the ZLB, where determinacy just requires the Taylor principle to be satisfied (Clarida, Galí & Gertler 1997; 2000), meaning interest rates respond more than one for one to inflation. We find NK models with a ZLB and an endogenous state variable usually have multiple solutions that eventually escape the ZLB, even with a monetary rule that satisfies the Taylor principle. However, a weak response to the price level in the monetary rule is sufficient to restore determinacy.

The next section presents simple examples of multiplicity and non-existence, and illustrates our main results. Section 3 provides the key equivalence result enabling us to examine models with OBCs via an associated linear complementarity problem. Section 4 then provides our main results on existence and uniqueness under perfect foresight, with applications to NK models. Section 5 gives additional results under rational expectations. Finally, Section 6 places our results in the context of the broader literature and discusses key assumptions.

2. Multiplicity in simple models

We start by presenting two simple models with multiple perfect foresight solutions. These will make clear why non-uniqueness is so common in models with OBCs. We also use these models to introduce our general results, and to illustrate the key ideas behind them. Next, we give a demonstration that price level targeting produces uniqueness, which is a robust conclusion of our results. We conclude the section by showing that the multiplicity we find under perfect foresight is also present under rational expectations.

We focus on New Keynesian examples due to the continued relevance of the ZLB. Other examples of multiplicity of transition paths in NK models are provided by Hebden, Lindé & Svensson (2011), Brendon, Paustian & Yates (2013; 2019), and our Appendix E. We will also examine the standard three equation NK model in Subsection 4.3.

2.1. A simple first example

We consider the simplest possible NK setting. Suppose the central bank follows the Taylor-type rule:

$$i_t = \max\{0, r_t + \phi\pi_t\},$$

where r_t is the real interest rate (not the natural rate), i_t is the nominal interest rate, π_t is inflation, and $\phi > 1$ to ensure the Taylor principle is satisfied. Suppose further that the Fisher equation holds:

$$i_t = r_t + \mathbb{E}_t\pi_{t+1}.$$

Away from the ZLB, combining the two equations implies $\mathbb{E}_t\pi_{t+1} = \phi\pi_t$, which has the unique, non-explosive solution, $\pi_t = 0$. Thus, $\pi = 0$ is one steady state of the model. The model has an additional steady state, in which $i = 0$ and $\pi = -r$, but we will focus on solutions returning to the "standard" steady state with $\pi = 0$. This is in line with the evidence of Gürkaynak, Levin & Swanson (2010) who find that under inflation targeting, agents expect a return to the non-deflationary steady state.

For simplicity, we consider an economy with exogenous real interest rates, as under flexible prices. In particular, suppose $r_t = r + \varepsilon_t$, with $r > 0$ and ε_t acting as a shock to real rates. We assume $\varepsilon_t = 0$ for $t > 1$, meaning the shock can only occur in period 1.

We seek to solve for π_t for $t = 1,2,\dots$. If inflation is to return to the standard steady state, then we must have that $\pi_t = 0$ for $t \geq 2$. To see this, suppose $i_t = 0$ for some $t \geq 2$. Then by the Fisher equation $\pi_{t+1} = -r$, implying $i_{t+1} = 0$ by the Taylor rule. By induction, $i_s = 0$ for all $s \geq t$, contradicting our assumption of a return to the standard steady state. Thus, $i_t > 0$ for all $t \geq 2$, so $\pi_t = 0$ for $t \geq 2$. Hence, from the period one Fisher equation and monetary rule:

$$r + \varepsilon_1 = i_1 = \max\{0, r + \varepsilon_1 + \phi\pi_1\},$$

so:

$$0 = \max\{-r - \varepsilon_1, \phi\pi_1\}. \tag{1}$$

This is a simple linear complementarity problem. It means that $-r - \varepsilon_1 \leq 0$, $\phi\pi_1 \leq 0$ and either $-r - \varepsilon_1 = 0$, or $\phi\pi_1 = 0$. These conditions are illustrated in Figure 1.

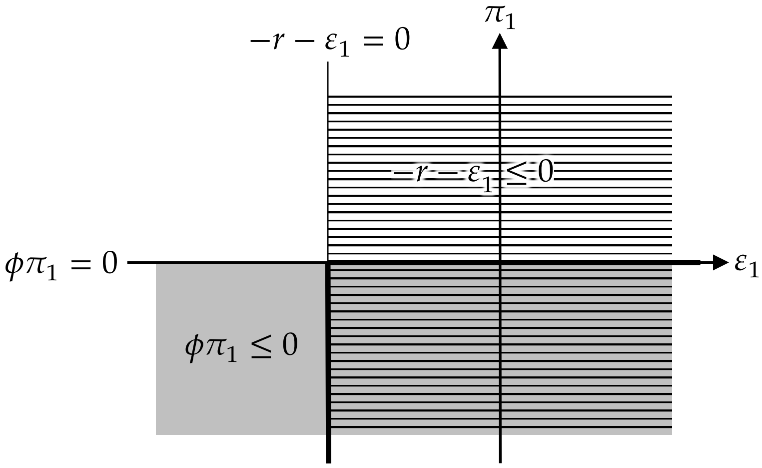

Figure 1: The solution to equation (1)**: Inflation in period one as a function of the shock.**

The hatched area shows where $-r - \varepsilon_1 \leq 0$. The shaded area shows where $\phi\pi_1 \leq 0$.

A value for π_1 is a solution if and only if it is within both of these areas, and on the border of one of them.

These points are marked with a thick (rotated L shaped) line.

If $\varepsilon_1 < -r$ then (1) has no solution, so the model has no solution returning to the standard steady state. In fact, there is no bounded solution in this case.[1] If $\varepsilon_1 = -r$, then any $\pi_1 \leq 0$ is consistent with (1): there is indeterminacy. This is the thick vertical line in Figure 1. Finally, if $\varepsilon_1 > -r$, then $\pi_1 = 0$ is the unique solution. This is the thick horizontal line in Figure 1. So, this model has either zero, one or infinitely many solutions returning to the standard steady state, depending on the value of the shock.

2.2. An example with robust multiplicity

While the last example only had multiplicity in a knife-edge case, multiplicity is more robust in richer models. For example, suppose the central bank responds to lagged as well as current inflation.[2] This is an easy way of generating some endogenous persistence, but almost any state variable would have a similar effect. Assuming r_t is now constant, the model becomes:

$$r + \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1}\},$$

where $\phi - \psi > 1$ and $\psi > 0$. These assumptions are sufficient for a determinate solution when there is no ZLB. The initial state, π_0, is given. To simplify presentation, we set $\phi := 2$, so we need $\psi < 1$. Our results are not specific to this case.

[1] Ensuring $i_1 \geq 0$ when $r + \varepsilon_1 < 0$ requires $\pi_2 > 0$ by the Fisher equation. This leads to explosive inflation as $\phi > 1$.

[2] Responding negatively to lagged inflation is optimal if firms index to past inflation (Giannoni & Woodford 2003). This is one possible justification for a response to lagged inflation.

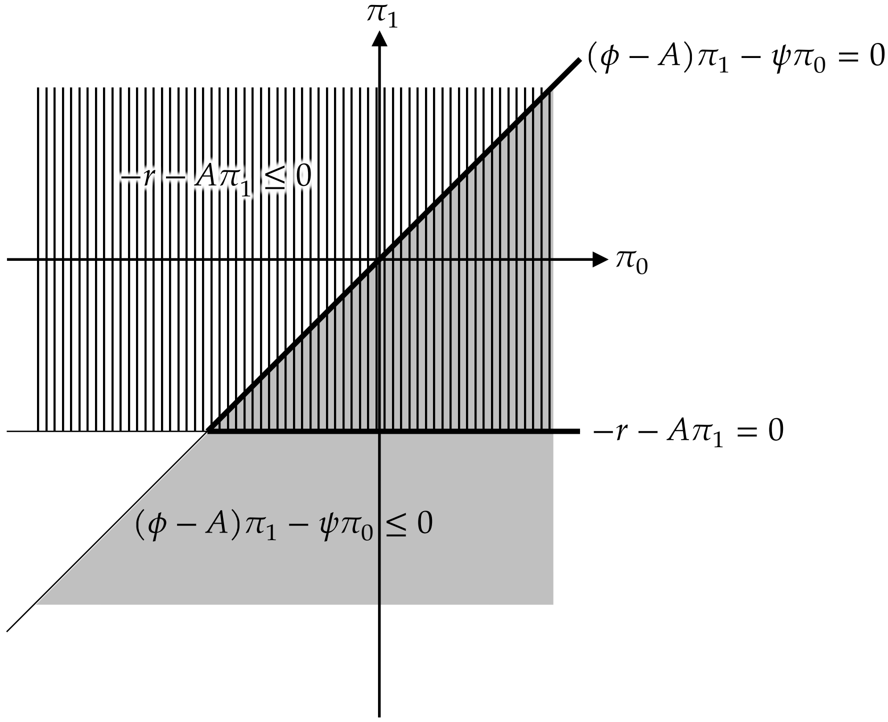

Figure 2: The solution to equation (2): Inflation in period one as a function of initial inflation.

The hatched area shows where $-r - A\pi_1 \leq 0$. The shaded area shows where $(\phi - A)\pi_1 - \psi\pi_0 \leq 0$.

A value for π_1 is a solution if and only if it is within both of these areas, and on the border of one of them.

These points are marked with a thick (wedge shaped) line.

Away from the ZLB, the model's solution takes the form $\pi_t = A\pi_{t-1}$, where $A^2 = \phi A - \psi$, so $A = 1 - \sqrt{1-\psi} \in (0,1)$. We first prove that the model cannot be at the ZLB for more than one period, if it is to ever escape the bound. Suppose that for some $t \geq 1$, $i_{t+1} = 0$ but $i_{t+2} > 0$, i.e., the economy is at the bound in $t+1$, but escapes in $t+2$. Since the economy is away from the bound in $t+2$, $\pi_{t+2} = A\pi_{t+1}$. Thus, by the Fisher equation, $0 = i_{t+1} = r + A\pi_{t+1}$, so $\pi_{t+1} = -\frac{r}{A}$. Hence, $i_t = r - \frac{r}{A} = r\left(\frac{A-1}{A}\right) < 0$ which is inconsistent with the monetary rule. This contradiction proves that if the economy eventually escapes the bound, then for $t \geq 1$, $i_{t+1} > 0$. In particular, the economy must be away from the ZLB in period two, so $\pi_2 = A\pi_1$ and:

$$r + A\pi_1 = i_1 = \max\{0, r + \phi\pi_1 - \psi\pi_0\}.$$

Much like before, this implies:

$$0 = \max\{-r - A\pi_1, (\phi - A)\pi_1 - \psi\pi_0\}, \tag{2}$$

which is another simple linear complementarity problem. Equation (2) means $-r - A\pi_1 \leq 0$, $(\phi - A)\pi_1 - \psi\pi_0 \leq 0$ and either $-r - A\pi_1 = 0$ or $(\phi - A)\pi_1 - \psi\pi_0 = 0$. We plot these conditions in Figure 2.

Figure 2 shows two solutions exist for large enough values of π_0. The upward sloping thick line captures the "fundamental" solution with $\pi_t = A^t \pi_0$ for $t \geq 0$. The horizontal thick line captures an alternative solution that jumps to the ZLB, with $\pi_t = -A^{t-2} r$ for $t > 0$ and $i_1 = 0$. This alternative solution is at the bound in the first period but escapes it in the next, with a gradual return to the standard steady state. Crucially, the alternative solution does not require agents to expect convergence to a different steady state. This is in contrast to the literature on the consequences of steady-state multiplicity (Benhabib, Schmitt-Grohé & Uribe 2001a; 2001b; Schmitt-Grohé & Uribe 2012; Mertens & Ravn 2014; Aruoba, Cuba-Borda & Schorfheide 2018).

The two solutions agree when $\pi_0 = -A^{-2} r$, giving a unique solution. For $\pi_0 < -A^{-2} r$, there is no solution returning to the standard steady state, as the fundamental solution violates the ZLB and the alternative solution violates the Taylor rule. Both solutions exist and are distinct when $\pi_0 > -A^{-2} r$.

As we approach the canonical model with $\psi \to 0^+$, the region of non-existence shrinks but the multiplicity region grows until it encompasses the entire state space.[3] The Fisher equation and Taylor rule are the core of all NK models, so it is unsurprising that this result generalizes. We have found robust multiplicity in all the NK models with endogenous state variables we have analysed. While we show in Subsection 4.1 that the standard three equation NK model has a unique solution, this is not a robust result. With a positive inflation target, price dispersion enters as a state variable, and this is sufficient to produce multiplicity. We show examples of this and other multiplicity in NK models in Subsection 4.3 and Appendix E.

2.3. The mechanics of our main results

Even in such simple models, deriving these multiplicity and non-existence results is cumbersome. Our theoretical results provide an easier alternative. To understand how they work, it is helpful to begin by looking at the impact of a monetary policy shock in the previous model. So, suppose:

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[3] With $\psi = 0$ and constant r, there is a unique solution returning to the standard steady state (as with $\psi = 0$, if $i_t = 0$ for some $t > 0$, then $\pi_{t+1} = -r$, so $i_{t+1} = 0$ as well). This no longer holds once a shock is introduced, as seen above.

$$r + \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + \nu_t\},$$

where ν_t is a monetary policy shock with $\nu_t = 0$ for $t > 1$, and where π_0 is given. The solution away from the ZLB must take the form $\pi_t = A\pi_{t-1} + F\nu_t$, with A as before and $F = -\frac{1}{\phi - A} < 0$. Thus, away from the ZLB, $i_1 = r + A^2\pi_0 + AF\nu_1$. Hence, since $AF < 0$, a positive monetary policy shock actually lowers nominal interest rates.

This solution would just touch the ZLB if $0 = i_1 = r + A^2\pi_0 + AF\nu_1$. This happens when $\nu_1 = \nu_1^* := -\frac{r + A^2\pi_0}{AF}$. In this case, $\pi_1 = A\pi_0 - F\nu_1^* = -\frac{r}{A}$, so $\pi_t = -A^{t-2}r$ for $t > 0$. Note that a monetary policy shock of this magnitude is a positive innovation (i.e., $\nu_1^* > 0$) if and only if $\pi_0 > -A^{-2}r$. Thus, when $\pi_0 > -A^{-2}r$:

$$\begin{aligned} r + \phi\pi_1 - \psi\pi_0 + \nu_1^* = 0 = i_1 &= \max\{0, r + \phi\pi_1 - \psi\pi_0 + \nu_1^*\} \\ &= \max\{0, r + \phi\pi_1 - \psi\pi_0 + 0\}. \end{aligned}$$

In other words, with $\nu_1 = \nu_1^* > 0$, there is no observable evidence that a shock has arrived at all, since the ZLB means that nominal interest rates should be zero even without such a shock. $\pi_1 = -\frac{r}{A}$ satisfies the monetary rule both with the shock $\nu_1 = \nu_1^*$ and also when $\nu_1 = 0$. Moreover, $\pi_t = -A^{t-2}r$ for $t > 0$ must be an equilibrium in either case.

This establishes that when $\pi_0 > -A^{-2}r$, $\pi_t = -A^{t-2}r$ for $t > 0$ is an equilibrium of the model without the shock, as we had already discovered in the previous subsection. We have learnt something extra though. The outcome is as if the monetary policy shock ν_1^* hit, whether or not it did in reality. This construction is valid as long as there is a positive shock that reduces nominal interest rates to the ZLB. The negative effect of the positive innovation permits such shocks to be "censored" away. This explains why the condition for ν_1^* to be positive ($\pi_0 > -A^{-2}r$) should be the same as the multiplicity condition we found in the previous subsection ($\pi_0 > -A^{-2}r$), and why the condition for a positive shock to have a negative effect ($\psi > 0$), should be the same as the condition for there to be multiplicity for some π_0 ($\psi > 0$).

This reveals a tight connection between multiplicity and positive shocks having negative effects. Indeed, our key uniqueness condition requires that positive shocks to the bounded variable have positive effects. The condition calls for strict positivity to ensure cases like $\psi = 0$ are correctly classified as having multiple solutions. Our uniqueness condition also requires that news today about a future positive shock to the bounded variable results in the bounded variable being higher in the period the shock arrives. This is the natural generalisation for models in which the bound may be hit in future periods. More than this, it requires that the impact of news shocks to the bounded variable at different horizons be "jointly" positive, in a sense to be made clear in the next subsection.

2.4. Examining our first example through the lens of our main results

We will now analyse our first simple example (from Subsection 2.1) using the general results presented in this paper. This illustrates the form of our main results.

A crucial object for these main results is the "M" matrix. In the current context, the first column of M gives the impulse response of i_t to a contemporaneous monetary policy shock, without the bound. The second column of M gives the impulse response of i_t to news today that next period there will be a monetary policy shock, again ignoring the bound. The third column gives the impulse response of i_t to news today about a shock in two periods, and so on. More generally, the first column will be the impulse response of the bounded variable to a contemporaneous shock to the equation defining that variable, ignoring the bound, and similarly for other columns. In practice, we will usually only consider T periods of IRFs, for news shocks out to horizon $T-1$, giving a $T \times T$ matrix M. Given this truncation, the M matrix is easy to calculate from a solution to the model without the bound.

To calculate the M matrix for our current model, we start by augmenting the model without bound or shocks to r_t by an exogenous forcing process, ν_t, giving:

$$r + \pi_{t+1} = i_t = r + \phi\pi_t + \nu_t.$$

We suppose that the entire path of ν_t is known in period one, to enable us to capture the effects of news. Thus, the solution must have the form $\pi_t = \sum_{j=0}^{\infty} F_j \nu_{t+j}$. Matching coefficients implies that $F_j = -\phi^{-(j+1)}$ for all $j \in \mathbb{N}$, so $i_t = r - \sum_{j=1}^{\infty} \phi^{-j} \nu_{t+j}$. From this, we can read off the columns of the M matrix. The first column is the path of $i_t - r$ when $\nu_1 = 1$ and $\nu_t = 0$ for $t \neq 1$, which is $0,0,\dots$. The second column is the path of $i_t - r$ when $\nu_2 = 1$ and $\nu_t = 0$ for $t \neq 2$, which is $\phi^{-1},0,0,\dots$. The third is $\phi^{-2},\phi^{-1},0,0,\dots$, and so on. Thus, for any matrix size T, the M matrix has a zero diagonal, a strictly negative upper triangle, and a zero lower triangle.

Applied to the current context, our general results give necessary and sufficient conditions for there to be a unique perfect foresight solution to the model:

$$r_t + \pi_{t+1} = i_t = \max\{0, r_t + \phi\pi_t\}$$

for any possible values of $r_1, r_2, r_3, \ldots$. I.e., we want to ensure there is a unique solution for any current and anticipated future shocks to real interest rates. Since without the bound the unique solution has $i_t = r_t$, this is equivalent to saying we want uniqueness for any path the bounded variable might take with the bound removed. This is the form taken by our general results.

The necessary and sufficient condition for this uniqueness is that the M matrix is a "P-matrix". A matrix is a P-matrix if and only if the determinants of all its principal sub-matrices are positive (where principal sub-matrices are sub-matrices formed by taking the same subset of rows as of columns). For the current model, since M is an upper triangular matrix with a zero diagonal, all the principal sub-matrices of M will also be upper triangular matrices with zero diagonals. The determinant of a triangular matrix is the product of its diagonal elements, so all of the determinants of M's principal sub-matrices are zero. Thus M is not a P-matrix, so this model does not have a unique solution for all possible sequences of shocks to real rates, as we already saw.

2.5. Uniqueness under price targeting

In applying our general results to NK models, a robust finding is that a response to the price level in the monetary rule is sufficient to produce uniqueness. Examples of this are given in Subsection 4.3 and Appendix E. To better understand this result, we examine price level targeting in the simple model used in the previous subsection (introduced in Subsection 2.1). We start by modifying the model to include a response to the price level, p_t, in the Taylor rule, so it becomes:

$$r_t + p_{t+1} - p_t = i_t = \max\{0, r_t + \phi(p_t - p_{t-1}) + \chi p_t\},$$

where $\chi > 0$ controls the strength of the response to the price level, and where $p_0 = 0$.

To find the M matrix for this new model, we need to solve the news shock model:

$$r + p_{t+1} - p_t = i_t = r + \phi(p_t - p_{t-1}) + \chi p_t + \nu_t,$$

where, as before, ν_t is an exogenous forcing process whose entire path is known in period one. This must have a solution of the form $p_t = \sum_{j=-\infty}^{\infty} G_j \nu_{t+j}$, where $\nu_t = 0$ for all $t \leq 0$. By matching coefficients, we can derive closed form expressions for G_j, given in Appendix H.1. Furthermore, we show there that for any matrix size T, there exists $\overline{\chi}_T \in (0, \infty]$ such that for all $\chi \in (0, \overline{\chi}_T)$, M (of size $T \times T$) is a P-matrix. Consequently, a weak but positive response to the price level restores determinacy in this model. Since all NK models have a Fisher equation and a Taylor rule, it is unsurprising that this result is robust across NK models.

To understand why price level rules robustly produce determinacy, suppose that some model includes a monetary rule of the form:

$$i_t = \max\{0, r + \pi + \phi(\pi_t - \pi) + \chi(p_t - \pi t) + \text{other terms}\},$$

where r is the steady-state real interest rate, π is the inflation target and ϕ & χ are non-negative. The rest of the model is unrestricted and could include all the usual "DSGE" frictions.

Assume that in period 0, the price level is on its target path, so $p_0 = 0$. Then from summing the Fisher equation over periods 1 to ∞, under perfect foresight we have:

$$\sum_{t=1}^{\infty}(i_t - r - \pi) = \sum_{t=1}^{\infty}(r_t + p_{t+1} - p_t - r - \pi) = \sum_{t=1}^{\infty}(r_t - r) - (p_1 - \pi) + \lim_{t\to\infty}(p_t - \pi t).$$

If the central bank is price level targeting, meaning $\chi \neq 0$, then the limit on the right-hand side is zero. Thus, if there were a self-fulfilling jump to the ZLB, making the left side negative, then either current and future real interest rates would have to fall by the same amount, or the current price level would have to increase. However, with $\phi > 0$ and/or $\chi > 0$, the only way for i_1 to hit zero is for there to be sharp decline in p_1 (assuming that any other terms in the monetary rule are positively correlated with π_t, as is usually the case). So, for there to be a self-fulfilling jump to the ZLB, current and future real rates would have to fall by significantly more than current and future nominal rates. But real rates are no way near that responsive to nominal rates in NK models (see e.g. Rupert & Šustek 2019). For example, with a standard log-linearized Euler equation, $r_t - r$ is proportional to $c_{t+1} - c_t$ (where c_t is log-consumption), so $\sum_{t=1}^{\infty}(r_t - r)$ is proportional to $-c_1$. Hence, for current and future real rates to fall, current consumption would have to rise, at a time when inflation and inflation expectations are low. This is ruled out by any standard Phillips curve specification.

Indeed, sticky prices or wages help price level targeting rules achieve uniqueness. For there to be a self-fulfilling jump to the ZLB, inflation must fall sharply. But under a price level rule, this means the central bank is committed to make-up inflation in future. Given this expected future inflation, firms and workers are less keen to cut prices and wages today, making it harder to produce the fall in inflation required for a self-fulfilling jump to the ZLB.

2.6. Multiplicity under rational expectations

While the bulk of our paper will focus on perfect foresight solutions, in Section 5 we give results under rational expectations. We show that if there are multiple solutions under perfect foresight, then there are usually a continuum of solutions under rational expectations. These rational expectations solutions switch between neighbourhoods of the various perfect foresight ones. Even if we only consider a single perfect foresight solution, there can still be a continuum of rational expectations solutions if that perfect foresight solution depends on time. We give an example of such a case below. A key difference to the prior literature on multiplicity under rational expectations is that jumps to the ZLB do not have to be persistent in these equilibria.

As in the example from Subsection 2.2, we examine the model:

$$r + \mathbb{E}_t \pi_{t+1} = i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1}\}$$

with $\phi := 2$ and $\psi \in (0,1)$. We look for a rational expectations solution of a similar form to the perfect foresight solution $\pi_t = -A^{t-2} r$ for $t > 0$ with $i_1 = 0$. Since we do not want ZLB episodes to be confined to period 1, we will effectively allow for the clock to be reset. This will happen with probability $\delta \in (0,1)$.

In particular, the model has a solution of the following form. Each period, with probability $1 - \delta$, $\pi_t = A_\delta \pi_{t-1} + B_\delta$ and $i_t = r + \phi\pi_t - \psi\pi_{t-1} > 0$ (away from the ZLB), while with probability δ, $\pi_t = C_\delta$ and $i_t = 0$ (at the ZLB), where $A_\delta \in (0,1)$, $B_\delta < 0$ and $C_\delta < 0$. This is proven in Appendix H.2, where closed form expressions for A_δ, B_δ and C_δ are derived. As $\delta \to 0$, these coefficients satisfy $A_\delta \to A$ (as defined in Subsection 2.2), $B_\delta \to 0$ and $C_\delta \to -A^{-1} r$. Thus, this solution converges to the desired perfect foresight solution as $\delta \to 0$. However, since $B_\delta < 0$ for $\delta > 0$, the rational expectations solution has a deflationary bias.

Unlike in the prior literature on sunspot equilibria in ZLB-models (see e.g. Nakata & Schmidt 2021), here there is no requirement that ZLB episodes are sufficiently persistent. With δ small, ZLB episodes will generally last only one period in this model. In richer models, ZLB episodes can be much longer. However, their length is determined by the model's persistence and dynamics, not by the parameters governing exogenous switches between solution regimes. This difference comes from the fact that the solutions we examine are not time invariant. For example, we can think of the model presented above as one with a single "regime". It is only at the ZLB if the regime "clock" says it is the model's first period in that regime. However, the regime clock is reset after the δ shock hits, in a kind of self-transition.

3. Equivalence result

We saw in the previous section that for simple models with occasionally binding constraints, solving the model under perfect foresight was equivalent to solving a linear complementarity problem (LCP). This section establishes that this is a general result. Solving a model with an OBC is always equivalent to solving an LCP. This equivalence is behind all our results. It enables us to leverage prior theorems on existence and uniqueness for LCPs.

For now, we assume that there is a single OBC of the form $i_t = \max\{0, \dots\}$, where i_t is the constrained variable (not necessarily interest rates). This covers all OBCs one encounters in practice, possibly via a transformation. For example, the Karush-Kuhn-Tucker (KKT) type constraints $i_t \geq 0$, $\lambda_t \geq 0$, $i_t\lambda_t = 0$ hold if and only if $0 = \min\{i_t, \lambda_t\}$ which in turn holds if and only if $i_t = \max\{0, i_t - \lambda_t\}$. It is also straightforward to generalize to multiple constraints, or to constraints that bind in steady state (see Appendix D.3).

We continue to look for perfect foresight solutions converging to a steady state with $i_t > 0$. We assume throughout that without the bound, the model would be determinate around a unique steady state. We take as given the period 0 value of the model's endogenous variables.

Without loss of generality, the equation containing the bound is of the form:

$$i_t = \max\{0, f(x_{t-1}, x_t, x_{t+1}, \varepsilon_t)\}, \tag{3}$$

and the model's other equations are of the form:

$$0 = g(x_{t-1}, x_t, x_{t+1}, \varepsilon_t).$$

The vector x_t contains the model's period t endogenous variables, including i_t. The vector ε_t gives exogenous "shocks", with the entire path $(\varepsilon_t)_{t=1}^{\infty}$ known in period one, as we are working under perfect foresight. For example, anticipated shocks to interest rates may reflect forward guidance. We assume $\varepsilon_t \to 0$ as $t \to \infty$, consistent with our assumption of an eventual return to steady state. f and g are some differentiable functions, later restricted to be linear.

Now define:

$$y_t := \max\{0, f(x_{t-1}, x_t, x_{t+1}, \varepsilon_t)\} - f(x_{t-1}, x_t, x_{t+1}, \varepsilon_t).$$

By construction, $y_t \geq 0$. Also note that:

$$i_t = f(x_{t-1}, x_t, x_{t+1}, \varepsilon_t) + y_t. \tag{4}$$

Despite its simplicity (we have just added and subtracted a term), this result is important. It states that the value of the bounded variable is given by its value with the constraint removed (but given other endogenous variables), plus an additional positive "forcing" term capturing the effect of the constraint. Furthermore, by construction, if $i_t > 0$, then $y_t = 0$ and if $y_t > 0$, then $i_t = 0$. Thus, for all t, the bounded variable i_t and the forcing term y_t satisfy the complementary slackness condition, $i_t y_t = 0$. For further intuition, note that when the constraint originally came from the KKT conditions $i_t \geq 0$, $\lambda_t \geq 0$, $i_t \lambda_t = 0$, so $i_t = \max\{0, i_t - \lambda_t\}$, then $y_t = \max\{0, i_t - \lambda_t\} - i_t + \lambda_t = \lambda_t$, meaning y_t recovers the original KKT multiplier. Finally, note there must be some period T such that for all $t > T$, $y_t = 0$, since we are assuming the model returns to a steady state where $i_t > 0$.

In the previous section, we analysed models with OBCs via companion "news shock" models which removed the OBC but added an exogenous forcing process to the equation defining i_t. We used this to calculate the M matrix for the simple models analysed there. We can do this for our general model by replacing equation (3) with equation (4), but where we now treat y_t as an exogenous forcing process. Since we are working under perfect foresight, we assume the entire path of y_t is known in period one. We also assume there exists some period T such that for $t > T$, $y_t = 0$, as this always holds when y_t arises endogenously from an OBC.

We now make the following key definitions:

Definition 1 ($\boldsymbol{y, q, M}$) Under the setup of the preceding text:

- $y := [y_1, \dots, y_T]'$ is a vector giving the path of the forcing variable.
- $i: \mathbb{R}^T \to \mathbb{R}^T$ is a function, where for all y, $i(y)$ is a vector containing the first T elements of the path of i_t, for the given path of the forcing variable y, as determined by equation (4).

- $q := i(0)$ is a vector giving the first T elements of the path of i_t when $y_t = 0$ for all t, i.e. q gives the path i_t would follow were there no bound or forcing process in the model.
- M is a $T \times T$ matrix where the first column equals $\left.\frac{\partial i(y)}{\partial y_1}\right|_{y=0}$, the second equals $\left.\frac{\partial i(y)}{\partial y_2}\right|_{y=0}$, and so on.

Then, by Taylor's theorem $i(y) = q + My + \mathrm{O}(y'y)$ for small y. Henceforth, we restrict f and g to be linear, in which case this approximation is exact and $i(y) = q + My$, with only q, not M, depending on the initial state. We prove this and establish expressions for the elements of M in Appendix D. The proof proceeds by backwards induction, starting from the known transition matrix in period $T + 1$ from which point on the economy is away from the bound.

Note that with f and g linear, the first column of M gives the impulse response to a contemporaneous shock to i_t, the second column of M gives the impulse response to a one period ahead news shock to i_t, and so on. [4] This is how we defined the M matrix in the previous section. The result $i(y) = q + My$ means the path of i_t is its path without the OBC or forcing process, plus a linear combination of impulse responses to the "news" contained in y.

When y arises endogenously from an OBC, $i(y) = q + My$ still holds (shown in Appendix D). In effect, the OBC provides "endogenous news" that in periods when the bound is hit, i_t will be higher than it would be without the bound. Given the complementary slackness conditions for y_t already established, and the positivity of the path of the bounded variable i_t, we have that $y \geq 0$, $q + My \geq 0$ and $y'(q + My) = 0$. These conditions mean that y solves the following linear complementarity problem:

Definition 2 (LCP) We say $y \in \mathbb{R}^T$ solves the **LCP** (q, M) if and only if $y \geq 0$, $q + My \geq 0$ and $y'(q + My) = 0$.

Recall that we needed to solve a scalar LCP to find the solution to the simple models in the last section. We now see that LCPs appear in the solution to models with OBCs more generally. In fact, the LCP (q, M) completely characterises the solution of the OBC model, as shown in

[4] The idea of imposing an OBC by adding news shocks is also present in Holden (2010), Hebden, Lindé & Svensson (2011), Holden & Paetz (2012) and Bodenstein, Guerrieri & Gust (2013). Laséen & Svensson (2011) use a similar technique to impose a path of nominal interest rates, in a non-ZLB context. None of these papers formally establish our equivalence result. News shocks were introduced by Beaudry & Portier (2006).

the following key theorem:

Theorem 1

1) Suppose x_t is a solution to the model without an OBC in which equation (3) is replaced with equation (4), with y_t exogenous. Suppose there is some $T \geq 0$ such that for all $t > T$, $y_t = 0$. Then x_t is also a solution to the original model with an OBC if and only if $y \in \mathbb{R}^T$ solves the LCP (q, M) and for all $t > T, f(x_{t-1}, x_t, x_{t+1}, \varepsilon_t) \geq 0$.
2) Suppose x_t is a solution to the model with an OBC which eventually escapes the bound. Then there exists $T \geq 0$ such that for all $t > T, f(x_{t-1}, x_t, x_{t+1}, \varepsilon_t) \geq 0$. Furthermore, there exists a unique vector $y \in \mathbb{R}^T$ solving the LCP (q, M), such that x_t is the unique solution to the model without an OBC in which equation (3) is replaced with equation (4), with y_t exogenous.

The proof (in Appendix D) again relies on backward induction arguments. This theorem shows that to solve a model with OBCs under perfect foresight, we just need to guess a sufficiently high T, then find a forcing process y solving the LCP (q, M).

LCPs have been extensively studied in mathematics. See Cottle (2009) for a brief introduction, and Cottle, Pang & Stone (2009a) for a definitive survey. LCPs can be solved via mixed-integer linear programming (MILP), for which optimised solvers exist. This approach is developed into a solution algorithm for models with OBCs in Holden (2016).

Note that if y solves the LCP (q, M), then for any $\kappa > 0$, κy solves the LCP $(\kappa q, M)$. Thus, the properties (existence, uniqueness, difficulty, etc.) of an LCP cannot depend on the magnitude of q. Recall that q_t gives the value of i_t without the bound or the forcing process y_t. Hence, for large t, q_t will be close to the steady-state level of i_t. For example, in models with a ZLB, raising the inflation target will tend to shift q upwards. Under Calvo-type models with full price and wage indexation, M is unaffected by the raised inflation target. Thus, the scale invariance of LCPs implies that the rise in q will not affect existence or uniqueness in this case.

Without full indexation, the higher inflation target will alter model dynamics and so change M, but these indirect effects are unlikely to increase the chance of uniqueness. For one, even without the ZLB, higher inflation targets increase the likelihood of indeterminacy (see e.g.

Coibion & Gorodnichenko 2011). Furthermore, while the basic three equation NK model with a zero inflation target has a unique perfect foresight solution (see Subsection 4.3), there is multiplicity under a positive inflation target (see Appendix E.2). Thus, raising the inflation target is unlikely to prevent self-fulfilling jumps to the ZLB.

4. Existence and uniqueness results

We now present our main results on the existence and uniqueness of perfect foresight solutions to models that are linear apart from an OBC. Our results exploit the bijection between solutions of the model with an OBC and solutions to the LCP. This permits us to import the conclusions of the LCP literature. The LCP results all rest on the properties of the M matrix. Here we will focus on just two: that of being a P-matrix and that of being an S-matrix. The former will be key for uniqueness, and the latter for existence. We apply the conditions we derive to New Keynesian models. Supplemental results are contained in Appendices C and G.

We want to establish conditions under which there is a unique solution for any possible initial state x_0 and shocks $(\varepsilon_t)_{t=1}^{\infty}$. This guarantees that we will always be able to find a unique solution to the generalized perfect foresight exercise of finding a path for the model's variables given an initial state and a known sequence of current and future shocks. This is a common exercise due to the interest in "news shocks" and anticipated policy changes. Without an OBC, the Blanchard & Kahn (1980) conditions are necessary and sufficient for there to be a unique solution to this generalized perfect foresight exercise.

Another reason to be interested in finding the perfect foresight path with anticipated shocks is that this gives one way to approximate the solution under rational expectations. This is the basis of the original stochastic extended path algorithm of Adjemian & Juillard (2013). This algorithm draws multiple samples of future shocks for periods $1, \dots, S$, calculates the perfect-foresight paths conditional on those future shocks, then averages over these realised paths.[5] This suggests that the conditions under which there is a (unique) perfect foresight solution for any possible sequence of future shocks are likely to be close to the conditions for existence (and

[5] This is not fully rational, as it is equivalent to assuming that agents act as if the uncertainty in all future periods would be resolved next period. However, this appears to be a close approximation to full rationality, as demonstrated by Holden (2016). The authors of the stochastic path method now have a version fully consistent with rationality (Adjemian & Juillard 2016).

uniqueness) under rational expectations. Indeed, our proof of existence under rational expectations requires existence of a perfect foresight solution for all sufficiently small anticipated forcing processes.

How do we relate existence or uniqueness for any possible initial state x_0 and shocks $(\varepsilon_t)_{t=1}^{\infty}$ to the prior LCP literature? Note that by linearity, for any $T \geq 1$, there exists a vector $q_0 \in \mathbb{R}^T$ and matrices $Q_x, Q_1, Q_2, \ldots$, each with T rows, such that for any x_0 and $(\varepsilon_t)_{t=1}^{\infty}$:

$$q = q_0 + Q_x x_0 + \sum_{t=1}^{\infty} Q_t \varepsilon_t,$$

where $q \in \mathbb{R}^T$ is as defined in Definition 1. If for some $S \geq 0$, the T-row matrix $[Q_x, Q_1, \ldots, Q_S]$ is rank T, then the vector q will be completely unrestricted: for any possible q, there will be some initial state x_0 and shocks $\varepsilon_1, \ldots, \varepsilon_S$ that will result in the given q. This fits perfectly with the prior LCP literature, which considers existence and uniqueness for any possible $q \in \mathbb{R}^T$.

To simplify the statements of our main results, we make the following definition:

Definition 3 (Sequential Radius) We say the **sequential radius** of the model is **at least *T*** if there exists $S \geq 0$ such that the T-row matrix $[Q_x, Q_1, \ldots, Q_S]$ is rank T. We say the **sequential radius** of the model is **infinite** if the sequential radius is at least T for any $T \geq 1$.

Some of our results will require the model's sequential radius to be sufficiently large. Others will assume an infinite sequential radius. Even this is an incredibly weak assumption, providing there is at least one shock. For the model of Subsection 2.1, this assumption holds with just a real rate shock (as without the bound $i_t = r_t$), or with just a monetary policy shock (shown in Subsection 2.4). In fact, the assumption always holds for a generic model with at least one shock. This means if you draw a model from an absolutely continuous distribution over the space of all n-dimensional linear models with at least one shock, then the model's sequential radius will be infinite. Informally, in almost all models, all shocks have at least some effect on all variables. Under the mild assumption of infinite sequential radius, q is completely unrestricted. Thus, existence and uniqueness results on LCPs that hold for all possible q will translate into results for OBCs that hold for all possible initial states x_0 and shocks $(\varepsilon_t)_{t=1}^{\infty}$.

If the reader is uninterested in results that hold for all possible sequences of future shocks, and instead only wishes to consider current shocks, then they should work with a modified "sequential radius at least T" definition, which only looks at the rank of the matrix $[Q_x, Q_1]$.

4.1. General uniqueness results

We now present our main uniqueness results. The principal definition follows:

Definition 4 (P-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is a **P-matrix** if and only if for all $z \in \mathbb{R}^{T\times 1}$ with $z \neq 0$, there exists $t \in \{1, \dots, T\}$, such that $z_t(Mz)_t > 0$. Equivalently, M is a **P-matrix** if and only if all of the principal sub-matrices of M have positive determinants. (Cottle, Pang & Stone 2009b)

Clearly, all symmetric positive definite matrices are P-matrices, so this definition captures a broader notion of positivity for an arbitrary matrix. Additionally, the diagonal of any P-matrix must be positive. In the context of models with a ZLB, this means that if M is a P-matrix then positive monetary policy shocks must increase nominal interest rates. Additionally, news about future positive monetary shocks must lead to higher nominal interest rates in the period the shock actually hits. Recall that in Subsection 2.3 we found that multiplicity was driven by positive monetary policy shocks having negative effects. Thus, it is unsurprising that some type of positivity of the responses of the bounded variable to shocks is key for uniqueness. In fact:

Theorem 2 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix. If M is not a P-matrix, then for some q the LCP (q, M) has multiple solutions.
(Samelson, Thrall & Wesler 1958; Cottle, Pang & Stone 2009b)

To see why being a P-matrix is the correct notion of positivity, suppose that y and $\tilde{y}$ both solved the LCP (q, M). Thus, for all $t \in \{1, \dots, T\}$, $0 = y_t(q + My)_t = \tilde{y}_t(q + M\tilde{y})_t$, so:

$$\begin{aligned}(y-\tilde{y})_t\big(M(y-\tilde{y})\big)_t &= (y-\tilde{y})_t\big((q+My)-(q+M\tilde{y})\big)_t \\ &= y_t(q+My)_t + \tilde{y}_t(q+M\tilde{y})_t - y_t(q+M\tilde{y})_t - \tilde{y}_t(q+My)_t \le 0\end{aligned}$$

as y_t, $\tilde{y}_t$, $q + My$ and $q + M\tilde{y}$ must all be non-negative. Hence, if we define $z = y - \tilde{y}$, then we have that for all $t \in \{1, \dots, T\}$, $z_t(Mz)_t \le 0$. If M is a P-matrix, this implies that $z = 0$ so $y = \tilde{y}$, meaning the solution is unique.[6] Informally, M being a P-matrix guarantees positive shocks to i_t increase i_t enough on average that one cannot have the kinds of self-fulfilling jumps

[6] This argument just follows that of Cottle, Pang & Stone (2009b).

to the bound we saw in Section 2.

The direct approach to assessing whether M is a P-matrix involves checking the positivity of the determinants of all M's 2^T principal sub-matrices. Since this is rather onerous, in Appendix C.1 we present both easier to verify necessary conditions, and easier to verify sufficient conditions. These give a fast answer one way or the other in most cases. See Appendix C.4 for a practical guide to checking the various conditions. Note that if M is not a P-matrix for some T, then M will not be a P-matrix for any larger T, so to show multiplicity it suffices to show that M is not a P-matrix for some small T.

Using Theorem 1, we can apply Theorem 2 to models with an OBC, giving:

Corollary 1 Consider an otherwise linear model with an OBC. Let $T > 0$. Then:

1) If M is a P-matrix, then for any x_0 and $(\varepsilon_t)_{t=1}^{\infty}$ there exists a unique path $(x_t)_{t=1}^{\infty}$ satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T+1$ on.
2) [Implied by 1.] If M is a P-matrix, and $(x_t)_{t=1}^{\infty}$ satisfies the model's equations, with $i_t > 0$ for $t > T$, then $(x_t)_{t=1}^{\infty}$ is the unique solution for which $i_t > 0$ for $t > T$.

Furthermore, suppose the model's sequential radius is at least T, then:

3) If M is not a P-matrix then there exists x_0 and $(\varepsilon_t)_{t=1}^{\infty}$ (with finitely many non-zero elements) such that there are multiple paths $(x_t)_{t=1}^{\infty}$ satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e. with the max removed) from period $T+1$ onwards.

This is our most important result. Parts (1) and (2) give sufficient conditions for uniqueness. These do not require us to be considering anticipated shocks, or for the model to have a sufficiently large sequential radius. Thus, they are universal conditions, both across models and across different types of perfect foresight exercises. The second part is particularly powerful, as having solved for a perfect foresight path, we know a T large enough such that $i_t > 0$ for $t > T$. To ensure that it is the unique such solution, we just need to check that the M matrix for that T is a P-matrix. The first part is also helpful, as with large T we expect the model to be permanently away from the bound by $T+1$, thanks to the model's mean reversion without the

OBC. Even if it is not, $T = 1000$ quarters may be practically equivalent to $T = \infty$, as it stretches the plausibility of rational expectations to suppose outcomes today depend on conditions in 250 years' time.

Part (3) of the corollary gives a necessary condition for uniqueness, under the assumption that the model's sequential radius is large enough, and that we want uniqueness for all possible sequences of future shocks. We have already argued that both of these assumptions are mild and reasonable. Given these assumptions, part (3) implies the existence of multiple perfect foresight paths not violating the bound for at least the first T periods, if M is not a P-matrix. For large enough T, this generally implies multiple solutions to the model with the bound. In any case, as before $T = 1000$ (say) may be equivalent to $T = \infty$ in practice.

An added reason for relying on finite T results is that technological changes are likely to make many OBCs obsolete. For example, a move to electronic cash would mean the ZLB is no longer a constraint. If agents believe this will happen within 250 years, then taking $T = 1000$ quarters would be appropriate.

4.2. Uniqueness in purely forward or backward looking models

We can derive stronger results for purely forward-looking or purely backward-looking models. A model is purely forward-looking if it has no state variables ($t - 1$ dated terms) other than the exogenous shock processes. For example, the basic three equation NK model is purely forward-looking. A model is purely backward-looking if it does not contain any future ($t + 1$) dated or expectational terms. Models in which agents have adaptive, not rational expectations, are purely backward-looking. Additionally, some indeterminate models may be transformed into determinate backward-looking models with an extra "sunspot" shock, via the method of Farmer, Khramov & Nicolò (2015). See Appendix E.4 for an example.

For purely forward-looking models, the M matrix will always be upper triangular: anticipated shocks to the bounded equation have effects even before they hit, but the period after they hit the economy is back to steady state. For purely backward-looking models, the M matrix will always be lower triangular: anticipated shocks have no effect until the period they hit but may continue having effects after this. Since the determinant of a triangular matrix is the product of its diagonal entries, and principal sub-matrices of triangular matrices are triangular, this simplifies checking whether M is a P-matrix.

Furthermore, both for purely forward-looking models and for purely backward-looking models, the diagonal of the M matrix is constant. Every element of M's diagonal just gives the contemporaneous response of the bounded variable, i_t, to a (hypothetical) shock to the equation that defines it. So, in a ZLB context, each element of the diagonal of the M matrix is equal to the contemporaneous response of nominal interest rates to a unit, i.i.d, monetary policy shock (ignoring the bound). This further simplification enables results that just depend on $M_{1,1}$:

Corollary 2 Consider a purely forward-looking otherwise linear model with an OBC. Then:

1) If $M_{1,1} > 0$, then for any x_0 and $(\varepsilon_t)_{t=1}^{\infty}$ with $\varepsilon_t \to 0$ as $t \to \infty$, there exists a unique path $(x_t)_{t=1}^{\infty}$ satisfying the model's equations and eventually escaping the bound.[7]

Furthermore, suppose the model has at least one t-dated shock with a non-zero impact on i_t (if the model has a shock to the bounded equation, then $M_{1,1} \neq 0$ is sufficient for this), then:

2) If $M_{1,1} \leq 0$, then for any x_0, there exists $(\varepsilon_t)_{t=1}^{\infty}$ with $\varepsilon_t = 0$ for $t > 1$ and with multiple paths $(x_t)_{t=1}^{\infty}$ satisfying the model's equations and eventually escaping the bound.

We give a weaker result for purely backward-looking models in Appendix E.4.

Corollary 2 gives an alternative proof that the model of Subsection 2.1 has multiple solutions. It is a model for which $M_{1,1} = 0$, since in the model, monetary policy shocks have no contemporaneous effect on interest rates when the ZLB is removed. With the real interest rate shock included, it then satisfies the conditions for the corollary's second result.

4.3. Uniqueness and multiplicity in New Keynesian models

The most important consequence of Corollary 2 is that it implies the three equation NK model always has a unique solution when the Taylor principle is satisfied. Including a monetary policy shock ν_t, but no other shocks, the model is given by:

$$\pi_t = \kappa y_t + \beta \pi_{t+1},$$

$$y_t = y_{t+1} - \sigma^{-1}(i_t - \pi_{t+1} + \log \beta), \tag{5}$$

[7] Existence of a path escaping the bound comes from the fact that if we only impose the bounds for T periods, then i_{T+1} is linear in the shock, and the shock is converging to 0, meaning i_{T+1} must be away from the bound for large enough T.

$$i_t = \max\{0, -\log\beta + \phi_\pi \pi_t + \phi_y y_t + \nu_t\},$$

where π_t is inflation, y_t is the output gap and i_t is the nominal rate, and where the parameters β, κ, σ, ϕ_π and ϕ_y are all finite and non-negative. See e.g. Woodford (2003) for further details on the model. We assume $\kappa(\phi_\pi - 1) + (1-\beta)\phi_y > 0$ to ensure determinacy without the ZLB (Bullard & Mitra 2002). To verify the conditions of Corollary 2 we just need to check the sign of $\frac{di_1}{d\nu_1}$, ignoring the ZLB, when $\nu_t = 0$ for all $t \neq 1$. Standard calculations give:

$$\frac{di_1}{d\nu_1} = \frac{\sigma}{\sigma + \kappa\phi_\pi + \phi_y} \in (0,1),$$

thus $M_{1,1} > 0$ for this model, implying uniqueness by Corollary 2.

Superficially, this may look like robust determinacy. M would remain a P-matrix even if we reduced all of the elements of its diagonal by some small amount. But recall that the determinant of a triangular matrix is the product of its diagonal entries. For this model, with an M matrix of size T, this gives $\det M = \left(\frac{\sigma}{\sigma+\kappa\phi_\pi+\phi_y}\right)^T$, which tends to 0 as $T \to \infty$. So, while M is a P-matrix no matter its size, as M gets larger, it gets arbitrarily close to not being a P-matrix. Thus, the determinacy here is still a knife-edge result: for large T, a small change in the elements of M can be enough to push M's determinant below zero.

In particular, we prove in Appendix H.3 that for any small $\varepsilon > 0$, for sufficiently large T, increasing or decreasing a single element of M by ε is sufficient to make the determinant of the resulting matrix negative. The proof relies on the following property of the model without the ZLB: if $\nu_s = 1$ and $\nu_t = 0$ for all $t \neq s$, then for sufficiently large s, $\sum_{t=1}^{\infty}(i_t + \log\beta) < 0$ and this inequality remains strict in the limit as $s \to \infty$. In other words: the sum of the IRF of interest rates to a distant, positive monetary "news shock" is actually negative and bounded away from zero. The cumulated endogenous response of the policy rate to the contraction caused by the bad news is actually larger than the shock that caused the contraction. This is another example of problems caused by positive shocks to the bounded equation having a negative effect.

How do things change if we include a response to the price level? By the determinant's continuity, for a small enough response to the price level, M must remain a P-matrix for any finite T. Now note the Euler equation (5) may be rewritten in terms of the price level p_t as:

$$i_t + \log\beta = (p_{t+1} - p_t) + \sigma(y_{t+1} - y_t).$$

Taking the sum of both sides over time, much as in Subsection 2.5, then gives:

$$\begin{aligned}\sum_{t=1}^{\infty}(i_t + \log\beta) &= \sum_{t=1}^{\infty}(p_{t+1} - p_t) + \sigma\sum_{t=1}^{\infty}(y_{t+1} - y_t)\\ &= -p_1 - \sigma y_1 + \lim_{t\to\infty} p_t + \lim_{t\to\infty} y_t = -p_1 - \sigma y_1.\end{aligned}$$

Hence, the only way we could have $\sum_{t=1}^{\infty}(i_t + \log\beta) < 0$ would be if $p_1 + \sigma y_1 > 0$. But as the period of the anticipated shock, $s \to \infty$, we must have that $p_1 \to 0$ and $y_1 \to 0$, since for determinate models, the current response to distant news decays to zero as the time to the shock's realisation goes to infinity (see Appendix H.4). Hence, with any response to the price level, $\sum_{t=1}^{\infty}(i_t + \log\beta) \to 0$ as the period of the anticipated shock, $s \to \infty$, unlike in the case with a standard monetary rule. Thus, a response to the price level produces more robust uniqueness than a standard monetary rule.

This is confirmed by our numerical findings from a variety of richer models in Appendix E. For example, if we augment the Smets & Wouters (2007) model with a ZLB on nominal interest rates,[8] and set its parameters to their estimated posterior-modes, then for $T \geq 9$, M is not a P-matrix, so the model will possess multiple solutions. However, with a monetary rule including a response to the price level,[9] M is a P-matrix even with $T = 1000$. Hence, there is a unique solution conditional on escaping the bound after at most 250 years.

As an example of multiplicity in the Smets & Wouters (2007) model, Figure 3 plots two different solutions following the combination of shocks that are most likely to produce negative interest rates for a year without the ZLB.[10] This combination is dominated by expansionary supply shocks, reducing prices (positive productivity and negative mark-up). For both solutions, the dashed line shows the response ignoring the ZLB, for reference.

These solutions have radically different consequences. The "good" solution remains close

[8] The monetary rule has the form $i_t = \max\{0, \rho_i i_{t-1} + (1-\rho_i)(\cdots) + \cdots\}$, where the ... are as in the original paper.

[9] We use the rule $i_t = \max\left\{0, \rho_i i_{t-1} + (1-\rho_i)\log\left(P_t \frac{Y_t}{Y_{t-1}}\right)\right\}$, where ρ_i is as in the original model, Y_t is real GDP and where the price level P_t evolves according to $\log P_t = \log P_{t-1} + \log\left(\frac{\Pi_t}{\Pi}\right)$.

[10] We find the vector w that minimises $w'w$ subject to $\bar{r} + Zw \leq 0$, where $\bar{r}$ is the steady state interest rate, and columns of Z give four periods of the IRF of interest rates to the given shocks. This gives: productivity, 3.56 s.d.; risk premium, −2.70 s.d.; government, −1.63 s.d.; investment, −4.43 s.d.; monetary, −2.81 s.d.; price mark-up, −3.19 s.d.; wage mark-up, −4.14 s.d..

to the path the economy would have taken without the ZLB. Given the dominance of expansionary shocks, output and consumption expand, at least after the initial impact. However, in the “bad” solution, despite the identical impulse, the economy is at the ZLB for much longer. With the economy at the ZLB, interest rates are higher than they would be according to the usual monetary rule. This acts as if there were a series of anticipated contractionary monetary policy shocks. Consequently, demand-type dynamics dominate, and output, consumption & inflation all fall together. The longer ZLB spell is sustainable as there is a combination of anticipated contractionary monetary policy shocks that jointly lower nominal interest rates, since M is not a P-matrix.

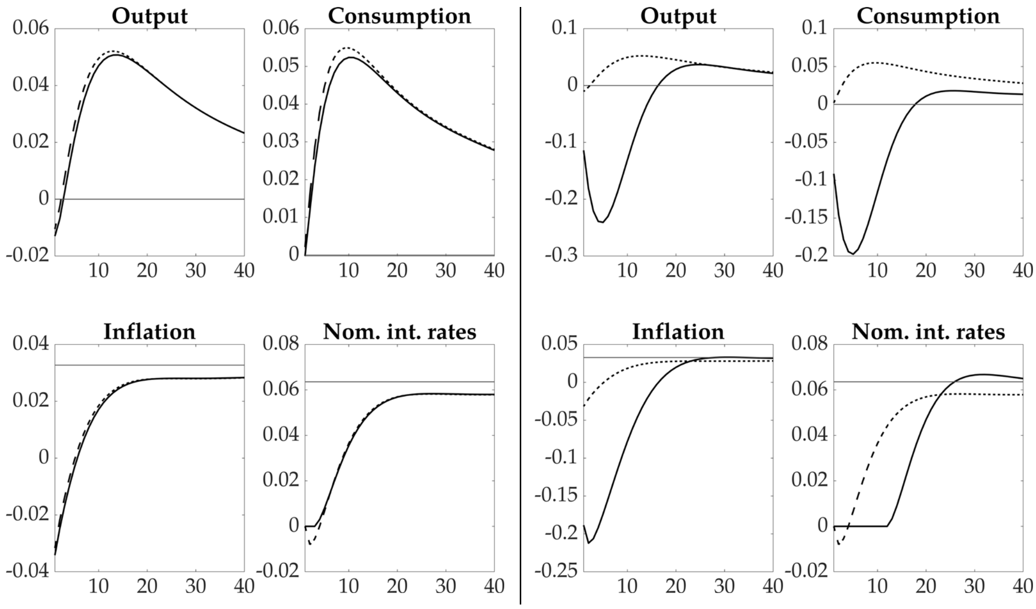

Figure 3: A “good” solution (left 4 panels) and a “bad” solution (right 4 panels), following a mixture of unexpected period-1 shocks to the Smets & Wouters (2007) model

All variables are in logarithms. Inflation and nominal interest rates are annualized. The precise combination of shocks is detailed in Footnote 10. In all plots, dashed lines show the path the economy would have followed without the ZLB.

4.4. Existence results

We conclude this section by deriving results on solution existence without also requiring uniqueness. In this case, the key property is being an S-matrix:

Definition 5 (S-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$. *Note: all P-matrices are S-matrices.*

Again, this captures a type of positivity of M. It is considerably weaker than the condition of being a P-matrix required for uniqueness. In a model with a ZLB it would be satisfied, for example, if raising rates today raised rates at all horizons thanks to the model's persistence. (This corresponds to taking $y = [1,0,0, \dots]'$.) We can check whether a matrix is an S-matrix in time proportional to $T^{2.37}$, by solving a linear programming problem (see Appendix B). This is identical to the computational complexity of matrix multiplication (up to a scaling factor).

The property of being an S-matrix is closely related to the feasibility of an LCP:

Definition 6 (Feasibility) We say $y \in \mathbb{R}^T$ is **feasible** for the LCP (q, M) if and only if $y \geq 0$ and $q + My \geq 0$. We say a path $(x_t)_{t=1}^{\infty}$ is **feasible** for a model with an OBC given initial state x_0 and shocks $(\varepsilon_t)_{t=1}^{\infty}$, if when equation (3) is replaced by equation (4), with y_t exogenous, there is some $(y_t)_{t=1}^{\infty}$ with $y_t \geq 0$ for all t, such that $(x_t)_{t=1}^{\infty}$ solves the model with equation (4), and $i_t \geq 0$ for all t.

By definition, if an LCP has a solution, then it is feasible. Likewise, if a model with an OBC has a solution, then it is feasible. If a monetary policy maker could make credible promises about (positive) future monetary policy shocks, then feasibility would be sufficient to allow the policy maker to ensure a solution.

If M is an S-matrix then feasibility is guaranteed:

Proposition 1 The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. If the LCP (q, M) has a solution for all $q \in \mathbb{R}^T$, M is an S-matrix. (Cottle, Pang & Stone 2009b)

Moreover, in most cases one encounters in practice, an LCP is solvable whenever it is feasible, i.e., whenever M is an S-matrix. This has immediate practical consequences: if M is an S-matrix for some T, then we are likely to be able to solve all the size T LCPs we encounter in simulating the model, whatever the model's path without the bound (q).

Additionally, from Theorem 1, we have:

Corollary 3 Let $T > 0$. Consider an otherwise linear model with sequential radius of at least T. Then if M is not an S-matrix, there exists x_0 and $(\varepsilon_t)_{t=1}^{\infty}$ such that:

1) There is no path $(x_t)_{t=1}^{\infty}$ with x_t satisfying the model's equations from period 1 to T and satisfying the model's equations without the OBC (i.e., with the max removed) from period $T + 1$ onwards.

2) [Implied by 1.] There is no path $(x_t)_{t=1}^{\infty}$ satisfying the model's equations which escapes the bound after at most T periods.

Since large T may be equivalent to $T = \infty$ for all practical purposes, this result is already a helpful guide to the non-existence of relevant solutions. For example, for the Smets & Wouters (2007) model considered in the previous subsection, M is not an S-matrix even with $T = 1000$, so even allowing for 250 years at the bound is not enough to guarantee existence. However, under the price targeting rule from Footnote 9, M is an S-matrix with $T = 1000$.

We can also directly obtain results on the existence or feasibility of solutions when the constraint is imposed for all periods (i.e., $T = \infty$). Proposition 1 implies that the infinite LCP (q, M) is feasible for all $q \in \mathbb{R}^{\mathbb{N}^+}$ if and only if $\varsigma := \sup_{y \in [0,1]^{\mathbb{N}^+}} \inf_{t \in \mathbb{N}^+} (My)_t > 0$. It turns out that we can bound this quantity, as in Appendix H.4 we prove:

Proposition 2 Given an otherwise linear model with an OBC, there exist easy to calculate, non-trivial bounds $\underline{\varsigma}$, $\overline{\varsigma}$, such that $\underline{\varsigma} \le \varsigma \le \overline{\varsigma}$.

This enables us to derive existence results for models with OBCs despite the infeasible infinite dimensional problem that defines ς. In particular:

Corollary 4 Suppose that $\underline{\varsigma} > 0$. Then for any x_0 and $(\varepsilon_t)_{t=1}^{\infty}$ the model with an OBC has a feasible path (a necessary condition for existence of a solution). Conversely, suppose $\overline{\varsigma} = 0$ and that the model's sequential radius is infinite. Then there is some x_0 and $(\varepsilon_t)_{t=1}^{\infty}$ with which the model has no solution.

Importantly, this result gives existence conditions without any dependence on T. It answers the question: are there states and anticipated shocks for which there is no solution that eventually escapes the bound? This is true for the Smets & Wouters (2007) model for example, for which we have $\overline{\varsigma} = 0$ to numerical precision. However, under the price targeting rule considered previously, $\underline{\varsigma} > 0.009$, so the model always has a feasible path. Thus, under this rule, if the central bank can commit to future positive monetary policy shocks, then the central bank can ensure a solution exists that eventually escapes the bound. This gives "infinite T" evidence on the performance of price targeting to supplement the finite T evidence of the last subsection.

5. Multiplicity under rational expectations

So far, we have concentrated on multiplicity under perfect foresight. Perfect foresight exercises allow us to analyse the model's response to probability zero ("MIT") shocks, assuming no other shocks arrive in future. If the initial shock is much larger than the regular shocks that hit the economy, then the relative error from ignoring future uncertainty may be moderate. Both the financial crisis and the Covid recession were large shocks that were considered very unlikely a priori, so perfect foresight analysis may be appropriate for them. Additionally, the extended path method of Fair & Taylor (1983) provides a way of approximately simulating a stochastic model by repeatedly solving perfect foresight exercises. A recent prominent example of its use is Christiano, Eichenbaum & Trabandt (2015).

However, we are also interested in model dynamics that properly account for uncertainty. This requires examining rational expectations solutions. In this section, we show that given multiple perfect foresight solutions, we can construct sunspot rational expectations solutions that shift between them. We focus on describing the general form of these solutions, leaving the details of existence conditions to the full treatment in Appendix F.

5.1. Construction of sunspot solutions

Our results here will apply to any non-linear dynamic model, not just otherwise linear models with occasionally binding constraints. Let x_t be a vector of the model's endogenous variables, with $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$. Similarly, let ε_t be a vector of the model's exogenous i.i.d. shocks, with $\varepsilon_t \in \mathcal{E} \subseteq \mathbb{R}^m$, where $0 \in \mathcal{E}$. We assume that with probability $1 - \sigma$, $\varepsilon_t = 0$, while with probability σ, ε_t is drawn from a probability distribution over $\mathcal{E}$ with measure p. This distribution may be either continuous or discrete. Thus, $\sigma = 0$ corresponds to the perfect foresight case, while when $\sigma = 1$, the distribution of ε_t is unrestricted. We assume t dated variables are known at t. There is no requirement that either x_t or ε_t be in any sense "minimal". For example, ε_t may contain non-fundamental shocks with no impact on the value of the model's equations, except perhaps through beliefs.

We assume that at any point in time, the economy can be in any one of a set K of "regimes". Both the policy functions, and the model's equations may differ across these regimes. Thus, these regimes can capture both switching sunspot solutions (with differing policy functions but identical model equations) and switching model properties (with the model equations switching). If the model equations do not vary over K, then in the limit as uncertainty disappears, these regimes will capture $|K|$ different perfect foresight solutions to the model. (K may be finite or countably infinite.) We denote the regime in period t by k_t. Within each regime, the policy functions and model equations may be a function of the length of time the economy has been in the current regime, denoted by s_t. $s_t = 1$ in the first period in a new regime, $s_t = 2$ in the second, and so on.

At the start of each period a binary "transition shock" is realised. With probability $1 - \delta$, the transition shock does not hit, and the economy will remain in the regime it was in last period. However, with probability δ, the economy is hit with the transition shock, and transitions to another regime according to the period t Markov transition matrix $\Omega_t := [\omega_{k,l}^{(t)}]_{k,l\in K}$. $\omega_{k,l}^{(t)} \in [0,1]$ gives the probability of transitioning from regime k to regime l at the start of period t, conditional on the transition shock hitting. Rows of Ω_t sum to 1. If $\omega_{k,k}^{(t)} \neq 0$ for some k, then if $k_{t-1} = k$, there is a $\delta\omega_{k,k}^{(t)}$ chance of remaining in regime k at t but with the "clock" reset, as if the economy had just arrived at regime k. We assume that for all $t \in \mathbb{Z}$, $k, l \in K$, $\omega_{k,l}^{(t)} = \omega_{k,l,s_t,\varepsilon_t}(x_{t-1})$ where $\omega_{k,l,s,e}: \mathcal{X} \to [0,1]$ for all $k, l \in K$, $s \in \mathbb{N}^+$ and $e \in \mathcal{E}$. This allows transition probabilities to be deterministic functions of the current state and shock.

We assume that the model's equations (first order conditions, laws of motion, etc.) are in the general form:

$$0 = \mathbb{E}_t \mathscr{f}_{k_t,s_t,\varepsilon_t}(x_{t-1}, x_t, x_{t+1}),$$

where $\mathscr{f}_{k,s,e}: \mathcal{X}^3 \to \mathbb{R}^n$ for all $k \in K$, $s \in \mathbb{N}^+$, $e \in \mathcal{E}$. We impose no stability requirement beyond $x_t \in \mathcal{X}$. The rational expectations solutions we find will be near to a corresponding perfect foresight one, so by limiting the perfect foresight equilibria considered, we can rule out explosive equilibria. Such equilibria could also be ruled out by bounding $\mathcal{X}$.

Given some σ and δ, we write $\mathscr{g}_{k,s,e}^{(\sigma,\delta)}: \mathcal{D}_{k,s} \to \mathcal{X}$ for the (unknown) policy function in the s^{th} period in regime k with shock e, meaning that for all t, $x_t = \mathscr{g}_{k_t,s_t,\varepsilon_t}^{(\sigma,\delta)}(x_{t-1})$. $\mathcal{D}_{k,s} \subseteq \mathcal{X}$ is the x-domain of definition of the policy functions, taken to be independent of σ and δ. This may be less than then entire space due to non-existence in some areas.

Our goal is to establish existence of the policy function for some $\sigma > 0$ and $\delta > 0$. To be a solution, for all $k \in K$, $s \in \mathbb{N}^+$, $e \in \mathcal{E}$ and $x \in \mathcal{D}_{k,s}$, these policy functions must satisfy:

$$\begin{aligned}
0 &= (1-\delta)(1-\sigma) f_{k,s,e}\left(x, g_{k,s,e}^{(\sigma,\delta)}(x), g_{k,s+1,0}^{(\sigma,\delta)}\left(g_{k,s,e}^{(\sigma,\delta)}(x)\right)\right) \\
&+ (1-\delta)\sigma \int_{\mathcal{E}} f_{k,s,e}\left(x, g_{k,s,e}^{(\sigma,\delta)}(x), g_{k,s+1,\varepsilon}^{(\sigma,\delta)}\left(g_{k,s,e}^{(\sigma,\delta)}(x)\right)\right) dp(\varepsilon) \\
&+ \delta(1-\sigma) \sum_{l \in K} \omega_{k,l,s,e}(x) f_{k,s,e}\left(x, g_{k,s,e}^{(\sigma,\delta)}(x), g_{l,1,0}^{(\sigma,\delta)}\left(g_{k,s,e}^{(\sigma,\delta)}(x)\right)\right) \\
&+ \delta\sigma \sum_{l \in K} \omega_{k,l,s,e}(x) \int_{\mathcal{E}} f_{k,s,e}\left(x, g_{k,s,e}^{(\sigma,\delta)}(x), g_{l,1,\varepsilon}^{(\sigma,\delta)}\left(g_{k,s,e}^{(\sigma,\delta)}(x)\right)\right) dp(\varepsilon). \qquad (6)
\end{aligned}$$

This equation just encodes the rules for transitioning between regimes already discussed.

When $\sigma = 0$ and $\delta = 0$, all future uncertainty disappears, and we are left with perfect foresight solutions. Setting $\sigma = 0$ and $\delta = 0$ in equation (6) gives:

$$0 = f_{k,s,e}\left(x, g_{k,s,e}^{(0,0)}(x), g_{k,s+1,0}^{(0,0)}\left(g_{k,s,e}^{(0,0)}(x)\right)\right). \qquad (7)$$

These are the standard equations defining perfect foresight policy functions. In this case, the regime never changes from its initial value, and so "clock time", s, gives actual time, t. The perfect foresight iteration $x_t = g_{k,t,\varepsilon_1 \mathbb{1}[t=1]}^{(0,0)}(x_{t-1})$ may converge to a different steady state in different regimes, or for different initial states x_0 and first period shocks ε_1. It may also cycle rather than converging. We assume these perfect foresight policy functions are known.

Under further technical conditions, outlined in Appendix F, we then have the following:

Theorem 3 Under the conditions outlined in the text above and in Appendix F, there exists $\gamma > 0$ and $\zeta \in (0,1)$ such that for all $\sigma < \zeta$ and $\delta < \zeta$, there exists a policy function $\left(g_{k,s,e}^{(\sigma,\delta)}\right)_{k \in K, s \in \mathbb{N}^+, e \in \mathcal{E}}$ that solves the model (equation (6)). Moreover:

$$\sup_{k \in K, s \in \mathbb{N}^+, e \in \mathcal{E}, x \in \mathcal{D}_{k,s}} \left\| g_{k,s,e}^{(\sigma,\delta)}(x) - g_{k,s,e}^{(0,0)}(x) \right\|_2 \leq \gamma \max\{|\sigma|, |\delta|\}.$$

We prove this in Appendix H.6. Note that the proof is constructive, so this could form the basis of an effective algorithm for computing global solutions to non-linear rational expectations models. Theorem 3 is a powerful tool for proving the existence of rational expectations equilibria for general non-linear models. It implies that if there are multiple solutions under perfect foresight (so $|K| > 1$), then there are generally a continuum of solutions under rational expectations, parameterized by the $\omega_{k,l,s,e}$ functions. Even if $|K| = 1$, then there can still be a continuum of solutions under rational expectations if the one perfect-foresight solution is not time invariant, as in the example from Subsection 2.6.

One immediate corollary of Theorem 3 is that if $\mathcal{E}$ is compact, $|K|$ is finite and for all $k,l \in K$, $s \in \mathbb{N}^+$ and $e \in \mathcal{E}$, $\mathfrak{f}_{k,s,e}$ is linear, independent of s and also linear in e, and $\omega_{k,l,s,e}$ is Lipschitz, then providing each regime has a non-explosive solution, there is a solution under rational expectations for small enough σ and δ. This gives existence for endogenous regime switching linear models under weaker assumptions than in e.g. Barthélemy & Marx (2017), though the stronger conditions in that paper are also sufficient for local uniqueness.

5.2. Application to otherwise linear models with an OBC

We now apply Theorem 3 to otherwise linear models with an OBC. We restrict attention to models that always have a unique solution to obtain clean results. However, Theorem 3 applies more broadly to models with OBCs under more involved conditions. For example, it applies to the model of Subsection 2.2, confirming the results of Subsection 2.6.

Suppose then that we have an otherwise linear model with an OBC, and that for any $T > 0$, the associated $T \times T$ M matrix is a P-matrix. Suppose we are given a closed and bounded set $\mathcal{E} \subseteq \mathbb{R}^m$ giving the support of the shock distribution, with $0 \in \mathcal{E}$, and a non-empty, closed and bounded set $\tilde{\mathcal{X}} \subseteq \mathbb{R}^n$ such that x_t should be supported at least on $\tilde{\mathcal{X}}$, but may have larger support. $\tilde{\mathcal{X}}$ could capture the space of economically relevant conditions. We suppose that for any initial state $x_0 \in \mathbb{R}^n$, and initial shock $\varepsilon_1 \in \mathcal{E}$, with $\varepsilon_t = 0$ for $t > 1$, there is a perfect foresight solution under those conditions that eventually escapes the bound and returns to the given steady state. Other than this restriction, the bounds may be arbitrarily large, so may not be overly restrictive in practice. Then we prove the following result in Appendix H.7:

Corollary 5 Under the conditions of the preceding text, there exists a compact set $\mathcal{X} \subseteq \mathbb{R}^n$ with $\tilde{\mathcal{X}} \subseteq \mathcal{X}$, and there exists $T^* \in \mathbb{N}$, such that for any initial state $x_0 \in \mathcal{X}$ and initial shock $\varepsilon_1 \in \mathcal{E}$, with $\varepsilon_t = 0$ for $t > 1$, there is a unique perfect foresight solution satisfying $i_t > 0$ for $t > T^*$, and this solution remains within $\mathcal{X}$. Furthermore, if the distribution of the shock has sufficient mass at 0, then the model also has a rational expectations solution that remains within the set $\mathcal{X}$. As the mass at 0 converges to 1, the rational expectations policy function converges to the perfect-foresight one.

Any purely forward looking otherwise linear model with an OBC and $M_{11} > 0$ satisfies the conditions of this corollary, by Corollary 2. Thus, Corollary 5 applies to the three equation New Keynesian model presented in Subsection 4.3. This proves the existence of a rational expectations solution to this model, providing the shocks are bounded with sufficient mass at 0. This equilibrium remains close to the perfect foresight one escaping the bound, so it does not get stuck in the deflationary steady state. While equilibria of this model have been exhibited computationally in prior work, it is reassuring to have a theoretical guarantee of their existence. It can be hard to distinguish numerically between non-existence and mere approximation error.

6. Further discussion

To see the broader relevance of our various results, in this section we further examine them in the context of the prior literature. We start by providing further justification for our imposition of a fixed terminal condition under perfect foresight. We then look at our assumption that the model is linear apart from the OBC and discuss our uniqueness and multiplicity results. We go on to provide additional context for our results on existence. We finish with a discussion of the benefits of price level targets.

6.1. Our terminal condition

Our perfect foresight results are conditional on the economy returning to a given steady state about which the economy is locally determinate. For ZLB models, this means the steady state with positive inflation, unless the model is augmented with a sunspot equation following Farmer, Khramov & Nicolò (2015) (see Appendix E.4). This is in contrast to the prior literature, beginning with Benhabib, Schmitt-Grohé & Uribe (2001a; 2001b), and developed by Schmitt-Grohé & Uribe (2012), Mertens & Ravn (2014) and Aruoba, Cuba-Borda & Schorfheide (2018), amongst others. In this literature, indeterminacy comes from the fact that agents place positive probability on the economy converging towards the deflationary steady state.

A priori, it is unclear whether agents should place positive probability on the economy converging to deflation. Firstly, the central banks of most major economies have announced (positive) inflation targets. Thus, convergence to a deflationary steady state would represent a spectacular failure to hit the target. As argued by Christiano and Eichenbaum (2012), a central bank may rule out the deflationary equilibria in practice by switching to a money growth rule following severe deflation, along the lines of Christiano & Rostagno (2001) and Christiano & Takahashi (2018). Furthermore, Richter & Throckmorton (2015) and Gavin et al. (2015) present evidence that the rational expectations deflationary equilibrium is unstable (under policy function iteration) if shocks are large enough, making it much harder for agents to coordinate upon it. Finally, a belief that inflation will eventually return to the vicinity of its target appears to be in line with the empirical evidence of Gürkaynak, Levin & Swanson (2010). It is thus an important question whether there are still multiple equilibria when agents believe the economy will eventually return to the standard steady state.

In addition, our results have important implications even without assuming a return to the standard steady state. Our examples in Subsection 4.4 and Appendix E show that for standard NK models with endogenous state variables, there is a positive probability of arriving in a state of the world from which there is no perfect foresight path returning to the non-deflationary steady state.[11] Hence, if we suppose that in the presence of risk, agents deal with uncertainty by integrating over the space of possible future shock sequences, as in the original stochastic extended path algorithm of Adjemian & Juillard (2013),[5] then such agents would likely place positive probability on tending to the "bad" steady state.[12] This rationalises the beliefs needed to sustain multiplicity in the prior literature.

As switching to a price level target would remove the non-existence problem, it could also help ensure beliefs about long-run inflation remain positive, removing this source of indeterminacy. Given a credible central bank, it seems natural that agents should expect a return to the standard steady state if it is possible, as it always is under a price level target. This suggests that a price level target will succeed in producing a unique outcome, despite the existence of a deflationary steady state.

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[11] If the LCP (q, M) is not feasible, then for any $\hat{q} \leq q$ the LCP $(\hat{q}, M)$ will also not be feasible. Consequently, if q is a draw from an absolutely continuous distribution, then if there are some q for which the model has no solution satisfying the terminal condition, then there is no solution with positive probability.

[12] The lack of a solution tending to the standard steady state does not imply the existence of a solution tending to the deflationary one. However, given the indeterminacy of the deflationary steady state, it is easier to find a solution returning there in general.

6.2. Other non-linearities, and our uniqueness and multiplicity results

A limitation of our results is that they only apply to otherwise linear models, excluding other non-linearities. We argue here for the importance of these results despite this limitation. We also discuss how the tools of this paper could be applied to non-linear models.

Bodenstein (2010) showed that linearization can exclude equilibria. Additionally, Boneva, Braun & Waki (2016) show that there may be multiple solutions to a non-linear NK model with ZLB, converging to the standard steady state, even though the linearized version of their model (with a ZLB) has a unique equilibrium. Thus, any multiplicity we find is strictly in addition to the type found by those authors. Moreover, note the multiplicity found in a simple linearized model in Brendon, Paustian & Yates (2013) is also found in the equivalent non-linear model in Brendon, Paustian & Yates (2019). This is suggestive evidence for the continued relevance of our results in the fully non-linear case.

In fact, the tools of this paper can be used to analyse the properties of perfect-foresight models with nonlinearities other than an occasionally binding constraint. Recall that we showed $i(y) = q + My + \mathrm{O}(y'y)$ as $y'y \to 0$, where M is defined in terms of partial derivatives of the path (see Definition 1). We did not need to impose linearity to derive the complementary slackness constraints on y. Thus, in a fully non-linear perfect foresight context, we can still use the tools we develop here to look at the (first order approximate) properties of perfect foresight problems in which y does not become too large in the solution (which usually means that q does not go too negative). In particular, we do not need to linearize before deriving q or M, so we can preserve accuracy even though only large shocks might drive us to the bound. In this fully non-linear case, M will be a function of the initial state.

Furthermore, studying multiplicity in otherwise linear models is an independently important exercise. Firstly, macroeconomists have long relied on existence and uniqueness results based on linearization of models without occasionally binding constraints, even though this may produce spurious uniqueness in some circumstances.[13] Secondly, it is nearly impossible to find all perfect foresight solutions in general non-linear models, as this is

[13] Perturbation solutions are only valid within some domain of convergence, so even the results of e.g. Lan & Meyer-Gohde (2013; 2014) do not mean that first order determinacy implies global determinacy.

equivalent to finding all the solutions to a huge system of non-linear equations. Even finding all the solutions to large systems of quadratic equations is computationally intractable. At least if we have the full set of solutions to the otherwise linear model, we may use homotopy continuation methods to map these solutions into solutions of the non-linear model. Furthermore, finding all solutions under uncertainty is at least as difficult in general, as the policy functions are also defined by a large system of non-linear equations. The proof of Theorem 3 gives one way to map perfect foresight policy functions into rational expectations ones, via certain fixed-point iterations. Thirdly, Christiano and Eichenbaum (2012) argue that the additional equilibria of Boneva, Braun & Waki (2016) may be mere "mathematical curiosities" due to their non-e-learnability. This suggests that the equilibria that exist in the linearized model are of independent interest, whatever one's view on this debate. Finally, our main results for NK models imply non-uniqueness, so concerns of spurious uniqueness under linearization will not be relevant in these cases.

Indeed, our choice to focus on otherwise-linear models under perfect-foresight, with fixed terminal conditions, has biased our results in favour of uniqueness for three distinct reasons. Firstly, because there are many more solutions under rational expectations than under perfect foresight, as we showed in Section 5 (under mild conditions). Secondly, because there are potentially other solutions returning to alternate steady states. Thirdly, because the original fully non-linear model may have yet more solutions. It is thus even more surprising that we still find multiplicity under perfect foresight in otherwise linear NK models with a ZLB.

However, we are certainly not the first to look at multiplicity in otherwise linear models with OBCs. Hebden, Lindé & Svensson (2011) propose a simple way to find multiplicity: hit the model with a large shock, and see if one can find more than one set of periods such that being at the bound during those periods is an equilibrium. In practice, this suggests first seeing if there is a solution that finally escapes the bound after one period, then seeing if there is one that finally escapes the bound after two periods, and so on.[14] This procedure may succeed in finding an example of multiplicity, and thus proving that the original model does not possess a

[14] This is tractable in our context, as it is easy to constrain the MILP representation of the LCP problem to be at the bound in the final period. The "DynareOBC" toolkit takes this approach. See Holden (2016) for further details.

unique solution. However, it cannot work completely generally as the multiplicity may only arise in very particular states, or may feature multiple spans at the bound.

Like us, Jones (2015) presents a uniqueness result for models with occasionally binding constraints. He shows that if one knows the set of periods in which the constraint binds, then under standard assumptions, there is a unique path in which the constraint binds in those periods. However, the multiplicity for models with OBCs stems from there being multiple sets of periods at which the model could be at the bound. Our results are not conditional on knowing in advance the periods at which the constraint binds.

Finally, uniqueness results have also been derived in the Markov switching literature. Examples include Davig & Leeper (2007), Farmer, Waggoner & Zha (2010; 2011) and Barthélemy & Marx (2019). These papers assume regime switching is exogenous. This prevents their application to OBCs, which generate endogenous regime switches. Determinacy results with endogenous switching were derived by Barthélemy & Marx (2017) assuming regime transition probabilities are a smooth function of the state. These results are not directly applicable to OBCs as OBCs produce jumps in regime transition probabilities.

6.3. Existence and non-existence

We also produced conditions for the existence of a perfect-foresight solution to an otherwise linear model with a terminal condition. These results provide new intuition for the prior literature on existence under rational expectations, which has found that NK models with a ZLB might have no solution at all if the variance of shocks is too high. For example, Mendes (2011) derived analytic results on existence as a function of the variance of a demand shock, and Basu & Bundick (2015) showed the quantitative relevance of such results. Existence conditions in a simple NK model with discretionary monetary policy and a two-state Markov shock were derived in close form by Nakata & Schmidt (2019). They show that the economy must spend a small amount of time in the low real interest rate state for the equilibrium to exist, which again links existence to variance. Our rational expectations existence results have a similar flavour, with the shock distribution required to have sufficient mass at 0.

While most of our results are not directly related to the variance of shocks, as we work under perfect foresight, they are nonetheless linked. We showed that the existence of a perfect foresight solution depends on the path taken by nominal rates without the bound (q). Many of our results assumed that this path was arbitrary thanks to the model's sequential radius being sufficiently large. However, in a model with a small number of bounded shocks, and no "news" shocks, not all paths are possible for nominal rates without the bound. The more shocks are added, and the wider their support, the larger will be the space of paths for nominal interest rates ignoring the ZLB. Hence, the more likely will be solution non-existence for a positive measure of such paths. This helps to explain the literature's prior results. Indeed, our proof of existence under rational expectations requires the existence of a perfect foresight solution for any sequence of sufficiently small, anticipated future shocks.

Prior work by Richter & Throckmorton (2015) and Gavin et al. (2015; Appendix B) relates a kind of eductive stability (the convergence of policy function iteration) to other properties of the model. Non-convergence of policy function iteration is suggestive of non-existence, though not definitive evidence.

It is also possible to establish existence by finding a solution to the model, perhaps conditional on the initial state. Under perfect foresight, the methods described in Holden (2010; 2016) are a possibility, and the method of Guerrieri & Iacoviello (2015) (extending Jung, Teranishi & Watanabe (2005)) is a prominent alternative. Under rational expectations, policy function iteration methods have been used by Fernández-Villaverde et al. (2015) and Richter & Throckmorton (2015), amongst others. However, solution algorithms cannot help us establish non-existence: non-convergence of a solution algorithm does not imply non-existence. [15] Furthermore, if the problem is solved globally, there could still be an area of non-existence outside of the grid on which the model was solved. If we wish to guide policy makers in how they should act to ensure existence in any state, then there is an essential role for results on global existence, like those presented in this paper.

6.4. Price level targeting

Our results suggest that given belief in an eventual return to inflation, the central bank can

[15] Holden (2016) is an exception. This algorithm always converges, either producing a solution, or a proof of non-existence.

produce a determinate equilibrium if it switches to targeting the price level, rather than the inflation rate. The welfare benefits of this could be substantial, given the severe recessions associated with prolonged ZLB episodes. See Appendix E.3 for some suggestive calculations.

There is of course a large literature advocating price level targeting already. Vestin (2006) made an important early contribution by showing that its history dependence mimics the optimal rule, a conclusion reinforced by Giannoni (2014). Eggertsson & Woodford (2003) showed the particular desirability of price level targeting in the presence of the ZLB, since it produces inflation after the bound is escaped. A later contribution by Nakov (2008) showed that this result survived taking a fully global solution, and Coibion, Gorodnichenko & Wieland (2012) showed that it still holds in a richer model. More recently, Basu & Bundick (2015) have argued that a response to the price level ensures equilibria exists even when shocks have large variances, avoiding the problems stressed by Mendes (2011). Our argument is distinct from these; we showed that in the presence of the ZLB, inflation targeting rules are indeterminate, even conditional on an eventual return to inflation, whereas price level targeting rules produce determinacy, in the sense of the existence of a unique perfect-foresight path returning to the standard steady state.

Our results are also distinct from those of Adão, Correia & Teles (2011) who showed that if the central bank is not constrained to respect the ZLB out of equilibrium (i.e. for non-market-clearing prices),[16] and if the central bank uses a rule that responds to the right hand side of the Euler equation, then a globally unique equilibrium may be produced, even without ruling out explosive beliefs about prices. Their rule has the flavour of a (future) price-targeting rule, due to the presence of future prices in the right-hand side of the Euler equation. We assume though that the central bank must satisfy the ZLB even out of equilibrium (i.e., for all prices), which makes it harder to produce uniqueness. However, in line with the bulk of the NK literature, we

[16] Bassetto (2004) gives a precise definition of this. The distinction is between constraints that hold for any prices (e.g., agent first order conditions), and constraints that hold only for the market clearing prices (e.g., market clearing conditions). The contention of Bassetto (2004) is that the ZLB is in the latter category—the central bank can promise negative nominal interest rates off the equilibrium path, which gives determinacy without negative rates actually being required. (Negative rates provide an infinite nominal transfer, entirely devaluing nominal wealth, so pushing up prices and preventing negative rates ever being called for.) Bassetto notes how dangerous it would be to rely on such infinite transfers given the possibility of misspecification.

maintain the standard assumption that explosive paths for inflation are ruled out,[17] an assumption which the rules of Adão, Correia & Teles (2011) do not require.

Somewhat contrary to our results, Armenter (2017) shows that in a simple otherwise linear NK model, if the central bank pursues Markov (discretionary) policy subject to an objective targeting inflation, nominal GDP or the price level, then the presence of a ZLB produces additional equilibria quite generally. This difference between our results and those of Armenter (2017) is chiefly driven by the fact that we rule out getting stuck in the neighbourhood of the deflationary steady state by assumption. We also assume commitment to a rule.

In other related work, Duarte (2016) considers how a central bank might ensure determinacy in a simple continuous time new Keynesian model. Like us, he finds that the Taylor principle is not sufficient in the presence of the ZLB. He shows that determinacy may be produced by using a rule that holds interest rates at zero for a history dependent amount of time, before switching to a $\max\{0, \dots\}$ Taylor rule. While we do not allow for such switches in central bank behaviour, we find a key role for history dependence, through price targeting.

7. Conclusion

Determinacy conditions are crucial for understanding the behaviour of the models we work with in macroeconomics. This paper provides the first general theoretical results on existence and uniqueness for otherwise linear models with occasionally binding constraints, given terminal conditions. Applying our results, we showed that multiplicity is the norm in New Keynesian models, but that a response to the price level can restore determinacy. Our conditions may be easily checked numerically using the "DynareOBC" toolkit we provide.[18]

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[17] Note that the unstable solutions under price level targeting feature exponential growth in the logarithm of the price level, which also implies explosions in inflation rates.

[18] Available from https://github.com/tholden/dynareOBC/releases. A guide to getting started with DynareOBC is contained in Appendix A.

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