Contents

V2 Notes

Anachthonic

27 February 2025

0 Prove it!

This is a brief digression to study proof tactics. Some commonly used proof tactics include:

- **Contradiction:** Suppose we are trying to prove p. If we prove that $\neg p \implies \perp$, then the only option is for p to be true.
- **Induction:** It is an axiom (schema) in the Peano axioms and provable in ZFC that if $P(0)$ and $P(n) \implies P(n+1)$ both hold, then $P(k)$ is true for any k. We won't delve into this too much, look no further than Wikipedia for a smoother introduction to this topic.
- **Pigeonhole Principle:** A "trivial" theorem that says there is no injective function from a set to another such that the first set has greater size. Again, look to other sources for more information.

Theorem 0.0.1

There are infinitely many primes.

Proof. Suppose for contradiction that we have a complete finite list of all primes, $\{p_1, p_2, \ldots, p_k\}$. Consider $X = p_1 p_2 \cdots p_k + 1$. Notice that, since all primes are greater than or equal to 2, none of them divide X. This means that the only possible divisors for X are X and 1. This means that X is prime, contradicting that our list is complete. □

Proposition 0.0.2

Let F_n be the Fibonacci sequence defined with $F_0 = 0$ and $F_1 = 1$. Prove that

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

Proof. We induct on n. Note that $F_3 = 2$, so this holds for F_1. Then suppose it holds for F_n. Consider the sum

$$F_1 + \cdots + F_{n+1} = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1.$$

So this holds for all n. □

1 Logarithms

Instead of doing a hefty introduction, which can be found elsewhere, we shall relay the properties of logarithms, given that log means an arbitrary logarithm, ln is the natural logarithm, and the base 10 logarithm is denoted $\log_{10}$.

- $\log b^n = n \log b$
- $\log b + \log c + \log bc$
- $(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$
- $\log_a b = \frac{\log b}{\log a}$.
- $\log_{a^n} b^n = \log_a b$.

Example 1.0.1

Let $x = \log_2 3$ and $y = \log_2 5$. Then:

- $\log_2 15 = x + y$
- $\log_2 7.5 = x + y - 1$
- $\log_3 2 = \frac{\log_2 2}{\log_2 3} = \frac{1}{x}$
- $\log_3 15 = \frac{\log_2 15}{\log_2 3} = \frac{x+y}{x}$
- $\log_4 9 = \log_2 3 = x$
- $\log_5 6 = \log_5 2 + \log_5 3 = \frac{x+1}{y}$

Exercise 1.0.2. Find all x such that $\log_6(x+2) + \log_6(x+3) = 1$

Solution. This is the same as solving $(x+2)(x+3) = 6$. Aside from an obvious solution at $x = 0$, we can expand to get $x^2 + 5x = 0$, which gives $x = -5$. However, we require arguments of logarithms to be positive, so $x = 0$. □

Exercise 1.0.3. Find the sum

$$\log \frac{1}{2} + \cdots + \log \frac{99}{100}$$

Solution.

$$\log \frac{1 \cdots 99}{2 \cdots 100} = \log(1/100) = -2$$

□

Problem 1.0.4. Evaluate $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)$

Solution. Use the "bouncing around" property of logarithm to get $3 \cdot 1 \cdots 1$ □

Problem 1.0.5. How many points do $y = 2\log x$ and $y = \log 2x$ intersect?

Solution. We must have $2\log x = \log 2x$ so that $2x = x^2$, solution from there. □

Problem 1.0.6. Find all solutions of

$$x^{\log x} = \frac{x^3}{100}$$

Solution. □

Problem 1.0.7.

Solution. □

Problem 1.0.8.

Solution. □

Problem 1.0.9.

Solution. □

Problem 1.0.10.

Solution. □

Problem 1.0.11.

Solution. □