Manifolds. Lecture Notes - Fall 2017

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Contents

Some words about this course

The goal of this course is to introduce and to study basic properties of *smooth manifolds*. The easiest way to picture smooth manifolds is as generalizations of smooth curves (1-dimensional) and surfaces (2-dimensional) in $\mathbb{R}^3$ to higher dimensions (and in $\mathbb{R}^n$, instead of $\mathbb{R}^3$). However, the fact that these spaces lie in $\mathbb{R}^3$ is merely a tool for us to picture them; the axioms of a smooth manifold do not refer to an ambient space (surprisingly, these axioms recover precisely the intuitive notion of a manifold in $\mathbb{R}^n$; this is the content of Whitney's Embedding Theorem, to be discussed in the lectures).

Manifolds are the mathematical objects that are used to model the abstract shapes of "physical spaces". A d-dimensional manifold is a topological space that locally looks like $\mathbb{R}^d$. For example, the surface of the Earth looks locally flat, like a piece of the plane, but globally its topology is that of a sphere. The Universe is modeled by a 3-dimensional manifold because locally it looks like a piece of $\mathbb{R}^3$, but its global topology might be more complicated. Space-time is a 4-dimensional manifold. The space of possible positions of a ball rolling on a plane is a 5-dimensional manifold.

Smooth manifolds are not just topological spaces; they are endowed with a so-called *differentiable structure*, which allows one to build classical multi-variable calculus on smooth manifolds. In particular, one defines smooth maps, derivatives, integration etc. This forces one to treat many standard constructions from calculus in a rather abstract way, which yields a more conceptual understanding of such operations. On the other hand, using manifolds to model physical systems, the differentiable structure allows one to write the laws of Physics, which are usually expressed as differential equations.

Smooth manifolds are used as the "background space" in many branches of modern geometry: Riemannian geometry = manifold + smooth distance, complex geometry = holomorphic manifolds, symplectic geometry = manifold + symplectic structure, Lie groups = manifold + smooth group multiplication etc.

So, let's begin!

Some typographical conventions

- When a section is colored in blue, it indicates that the section was not covered in the lectures.

- The symbol ♣ next to the statement of a theorem indicates that the proof will not be given in this course, mostly because it is too involved or too long.

1.1. Brief review of notions from Topology and Analysis

Did I use to know all these things? Was the summer so long?

We start by recalling some notions from Analysis and Topology, which will be used throughout the course.

Some Topology

Definition 1.1.1. *A **topology** on a set X is a set $\mathcal{T}$ of subsets of X, whose elements are called **open** subsets, satisfying the axioms:*

$$\emptyset, X \in \mathcal{T}, \qquad U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}, \qquad (U_i \in \mathcal{T}, \ \forall i \in I) \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}.$$

*The pair $(X, \mathcal{T})$ is called a **topological space**.*

Definition 1.1.2. *Let Y be a subset of a topological space $(X, \mathcal{T})$. The **induced topology** on Y, also called the **subspace topology**, is given by:*

$$\mathcal{T}|_Y := \{U \cap Y \ : \ U \in \mathcal{T}\}.$$

Definition 1.1.3. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function*

$$f : X \longrightarrow Y$$

*is called **continuous** if $f^{-1}(U)$ is open in X for every open set U in Y.*

*The function f is called a **homeomorphism** if it is continuous, bijective and its inverse $f^{-1} : Y \to X$ is also continuous.*

Definition 1.1.4. *A topological space $(X, \mathcal{T})$ is called **Hausdorff** if every two distinct points can be separated by two open sets; in other words, for every $x, y \in X$, with $x \neq y$, there exist $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Definition 1.1.5. *A topological space $(X, \mathcal{T})$ is called **second countable** if there exists a countable family of open sets $\{U_i\}_{i \in \mathbb{N}}$ with the following property: for every open set U there exists an index set $I \subset \mathbb{N}$ such that $U = \bigcup_{i \in I} U_i$. In other words, the topology has a countable basis.*

Next, we recall the definition of the Euclidean topology on

$$\mathbb{R}^n = \{x = (x^1, \ldots, x^n) \ : \ x^1, \ldots, x^n \in \mathbb{R}\}.$$

Definition 1.1.6. *(1) The **open ball** in $\mathbb{R}^n$ of radius $\varepsilon > 0$ and center $x_0 \in \mathbb{R}^n$ is the set*

$$B_\varepsilon(x_0) = \{x \in \mathbb{R}^n \ : \ |x - x_0| < \varepsilon\},$$

where $|\cdot|$ is the usual Euclidean length:

$$|x - y| := \sqrt{(x^1 - y^1)^2 + \ldots + (x^n - y^n)^2}.$$

*(2) A subset $U \subset \mathbb{R}^n$ is called **open** if for every $x_0 \in U$ there exists a positive number $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset U$. The collection of all such sets forms the so-called **Euclidean topology** on $\mathbb{R}^n$.*

Smooth functions

Let $U \subset \mathbb{R}^n$ be an open set. A function $f : U \to \mathbb{R}^m$ is called **differentiable** at a point $x \in U$, if there exists a linear map

$$\mathrm{d}_x f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

such that:

$$\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - \mathrm{d}_x f(h)|}{|h|} = 0.$$

The map $\mathrm{d}_x f$ is called the **differential** of f at x. If $\mathrm{d}_x f$ exists, then it is also uniquely determined, since it can be described as the directional derivative of f:

$$\mathrm{d}_x f(v) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon}, \quad v \in \mathbb{R}^n.$$

Denote the components of f by $f = (f^1, \ldots, f^m)$. The matrix of $\mathrm{d}_x f$ in the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by the **Jacobian matrix**:

$$\mathrm{d}_x f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}.$$

Recall also the chain rule for the differential:

Proposition 1.1.7 (The chain rule). *Consider two maps $f : U \to \mathbb{R}^m$, $g : V \to U$, where $U \subset \mathbb{R}^l$ and $V \subset \mathbb{R}^n$ are open sets. If g is differentiable at $x \in V$ and f is differentiable at $g(x) \in U$ then $f \circ g$ is differentiable at x, and its differential is given by*

$$\mathrm{d}_x(f \circ g) = (\mathrm{d}_{g(x)} f) \circ \mathrm{d}_x g : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

In terms of partial derivatives the chain rule takes the form:

$$\frac{\partial(f \circ g)}{\partial x^i}(x) = \sum_{j=1}^{l} \frac{\partial f}{\partial y^j}(g(x)) \frac{\partial g^j}{\partial x^i}(x), \quad \text{for all } 1 \leq i \leq n.$$

The first, and most general class of maps are continuous maps; these are called also functions of **class zero**, or just C^0**-functions**.

Denote the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$. Writing the linear maps in the standard bases, this space can be identified with the space of $n \times m$-matrices, hence with $\mathbb{R}^{n \times m}$. Using this identification, we endow $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ with the Euclidean topology on $\mathbb{R}^{n \times m}$.

Definition 1.1.8. *Let $U \subset \mathbb{R}^n$ be an open set. A map $f : U \to \mathbb{R}^m$ is said to be a C^1-**function** if f is differentiable at every $x \in U$, and its differential is a continuous function from U to $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$:*

$$\mathrm{d}f : U \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m), \quad x \mapsto \mathrm{d}_x f \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m).$$

*Functions of **class** C^k are defined inductively. We say that $f : U \to \mathbb{R}^m$ is a C^k-**function**, for $k \geq 2$, if it is differentiable and its differential is a C^{k-1}-**function**.*

Let us recall some standard properties of C^k-functions:

Proposition 1.1.9. *Let $U \subset \mathbb{R}^n$ be an open set, and let*

$$f = (f^1, \ldots, f^m) : U \longrightarrow \mathbb{R}^m$$

be a function.

(1) If f is a C^k-function then it is also a C^{k-1}-function.
(2) The function f is of class C^k if and only if its partial derivatives of order $\leq k$:

$$\frac{\partial^l f^i}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_l}} = \frac{\partial}{\partial x^{j_1}} \left(\frac{\partial}{\partial x^{j_2}} \left(\ldots \frac{\partial f^i}{\partial x^{j_l}} \ldots \right) \right) : U \to \mathbb{R},$$

$$\text{for} \ \ 1 \leq i \leq m, \quad 0 \leq l \leq k, \quad 1 \leq j_1, \ldots, j_l \leq n$$

exist and are continuous on U.
(3) If f is of class C^k then its partial derivatives "commute", meaning that:

$$\frac{\partial^l f^i}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_l}} = \frac{\partial^l f^i}{\partial x^{j_{\sigma(1)}} \partial x^{j_{\sigma(2)}} \ldots \partial x^{j_{\sigma(l)}}},$$

for $1 \leq i \leq m$, $1 \leq l \leq k$, $1 \leq j_1, \ldots, j_l \leq n$, and every permutation σ of the set $\{1, \ldots, l\}$.

We will mostly work with the following type of maps:

Definition 1.1.10. *A **smooth function**, also called a C^∞-**function**, is a function $f : U \to \mathbb{R}^m$, where U is an open subset of $\mathbb{R}^n$, which is of class C^k for all $k \geq 0$. Equivalently, f is smooth if and only if its partial derivatives of all orders exist and are continuous.*

The following lemma, which will be repeatedly used throughout this course, illustrates the richness of the class of smooth functions.

Lemma 1.1.11. *Fix a point $a \in \mathbb{R}^m$, and positive numbers $0 < \epsilon < \delta$. There exists a smooth map $\chi : \mathbb{R}^m \to \mathbb{R}$ such that*

- $0 \leq \chi(x) \leq 1$*, for all $x \in \mathbb{R}^m$,*
- $\chi(x) = 1$*, for all $x \in B_\epsilon(a)$,*
- $\chi(x) = 0$*, for all $x \notin B_\delta(a)$.*

The functions constructed in the lemma are usually called **bump-functions** (think about their graph in $\mathbb{R}^{m+1}$). In Exercise 1.2 we show how to construct such functions and how to prove the lemma.

Analytic functions and Borel's Lemma

For completeness, we recall also the class of **analytic** functions. Let $f : U \to \mathbb{R}^m$ be a smooth function defined on an open set $U \subset \mathbb{R}^n$. The **formal Taylor series** of f at a point $a \in U$ is given by:

$$P_f^a(T_1, \ldots, T_n) := \sum_{i_1, \ldots, i_n \geq 0} \frac{1}{i_1! \ldots i_n!} \frac{\partial^{i_1 + \ldots + i_n} f}{(\partial x^1)^{i_1} \ldots (\partial x^n)^{i_n}}(a) T_1^{i_1} \ldots T_n^{i_n}.$$

This expression belongs to the vector space of formal power series in the T_i's with coefficients in $\mathbb{R}^m$, denoted:

$$\mathbb{R}^m[[T_1, \ldots, T_n]] := \left\{ \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} T_1^{i_1} \ldots T_n^{i_n} \; : \; c_{i_1, \ldots, i_n} \in \mathbb{R}^m \right\}.$$

Definition 1.1.12. *The function f is called **analytic** around a if there exists $\epsilon > 0$ such that for all $x \in B_\epsilon(a)$, the Taylor series evaluated on $T = x - a$ converges absolutely to $f(x)$:*

$$f(x) = \sum_{i_1, \ldots, i_n \geq 0} \frac{1}{i_1! \ldots i_n!} \frac{\partial^{i_1 + \ldots + i_n} f}{(\partial x^1)^{i_1} \ldots (\partial x^n)^{i_n}}(a)(x^1 - a^1)^{i_1} \ldots (x^n - a^n)^{i_n}.$$

*Finally, f is called **analytic** on U, or of class C^ω, if it is analytic around every point in U.*

In this course we will develop the theory of smooth manifolds starting from the class of smooth functions. There are theories of manifolds which are developed based on other classes of functions. There are two main reasons why smooth functions are usually preferred: first, smooth functions are closed under several operations (derivatives of smooth functions are smooth, solutions to ODE's with smooth coefficients are smooth etc.), and second, the class of smooth function is very rich and very flexible (especially when compared to analytic functions; see Exercise 1.3).

Also to show the contrast with analytic functions, we state the following (for a proof, solve Exercises 1.4 and 1.5):

Theorem ♣ 1.1.13 (Borel's Lemma)**.** *For any formal power series*

$$Q(T_1, \ldots, T_n) := \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} T_1^{i_1} \ldots T_n^{i_n} \in \mathbb{R}[[T_1, \ldots, T_n]],$$

there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which has Q as its Taylor series at 0.

1.2. Manifolds

Topological manifolds

Definition 1.2.1. *An m-dimensional **topological manifold** is a Hausdorff, second countable topological space, for which every point has an open neighborhood homeomorphic to an open set in $\mathbb{R}^m$.*

The last condition means that a topological manifold looks locally like $\mathbb{R}^m$. The following terminology is used:

Definition 1.2.2. *Let M be a topological space. An m-dimensional **chart** (U, φ) on M consists of a homeomorphism $\varphi : U \to \widetilde{U}$ from an open set $U \subset M$ to an open set $\widetilde{U} \subset \mathbb{R}^m$. The components of φ are denoted by*

$$\varphi = (x_\varphi^1, \dots, x_\varphi^m), \quad x_\varphi^i : U \longrightarrow \mathbb{R}, \ \ for \ \ 1 \le i \le m,$$

*and are called the **local coordinates** on M corresponding to the chart (U, φ). The inverse map $\varphi^{-1} : \widetilde{U} \to U$ is called a **local parameterization** of M.*

Non-mathematically, an atlas is a book with charts. Here is an atlas of Earth:

Definition 1.2.3. *Let M be a topological space. An m-dimensional **topological atlas** on M (or C^0**-atlas** on M) consists of a collection of m-dimensional charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ covering M, i.e. $M = \cup_{\alpha \in I} U_\alpha$.*

With this terminology, note that a Hausdorff, second countable topological space is an m-dimensional topological manifold if and only if it admits an m-dimensional topological atlas.

Smooth manifolds

Let us recall:

Definition 1.2.4. *A map $f : U \to V$ between open sets $U, V \subset \mathbb{R}^m$ is called a **diffeomorphism** if f is bijective and both f and f^{-1} are smooth.*

Note that (as in the case of homeomorphisms) the condition that f^{-1} be smooth is not automatically satisfied. The standard example of a smooth bijection which is not a diffeomorphism is

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \ \ f(t) = t^3.$$

In a local chart a topological manifold is described as an open piece of $\mathbb{R}^m$. To develop analysis on manifolds, one needs to introduce derivatives and integration of functions, notions which, locally in charts, should coincide with those from multivariable calculus. However, a function that is differentiable in one chart might fail to be differentiable in different chart! To circumvent this, on a smooth manifold one only works with mutually *compatible charts*.

Definition 1.2.5. *Let M be a topological space. Let (U, φ) and (V, ψ) be two m-dimensional charts on M. The map*

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$$

*is a called the **change of coordinates map** or the **transition map** between the two charts. The two charts are said to be **compatible** if the transition map is a diffeomorphism.*

Definition 1.2.6. *An m-dimensional topological atlas $\mathcal{A}$ on an m-dimensional topological manifold M is said to be an m-dimensional **smooth atlas** (or C^∞-**atlas**) if every two charts in $\mathcal{A}$ are compatible.*

Here are some simple examples of C^∞-atlases:

Example 1.2.7. (1) On $\mathbb{R}^m$ there is a smooth atlas with only one chart

$$\mathcal{A} = \{(\mathbb{R}^m, \mathrm{id}_{\mathbb{R}^m})\}.$$

Another smooth atlas is the collection of all diffeomorphisms between open subsets of $\mathbb{R}^m$:

$\mathcal{B} = \{ (U, \varphi) \; : \; U, V \subset \mathbb{R}^m$ are open and $\varphi : U \to V$ is a diffeomorphism$\}.$

In fact, $\mathcal{B}$ consists of all charts compatible with $(\mathbb{R}^m, \mathrm{id}_{\mathbb{R}^m})$.

(2) Consider the m-dimensional sphere with the induced topology from $\mathbb{R}^{m+1}$:

$$S^m = \left\{(x^0, x^1, \ldots, x^m) : (x^0)^2 + (x^1)^2 + \ldots + (x^m)^2 = 1\right\} \subset \mathbb{R}^{m+1}.$$

We construct an atlas on S^m with only two charts. The "south pole" and the "north pole" of S^m are the points

$$s := (0, 0, \ldots, 0, -1) \in S^m, \quad \text{resp.} \quad n := (0, 0, \ldots, 0, 1) \in S^m.$$

The **stereographic projection** through the north pole is the map

$$\pi_n : S^m \backslash \{n\} \longrightarrow \mathbb{R}^m$$

which sends a point $p \in S^m$, $p \neq n$, to the point $q = \pi_n(p)$ which is the intersection of the plane $x^m = 0$ and the line through n and p.
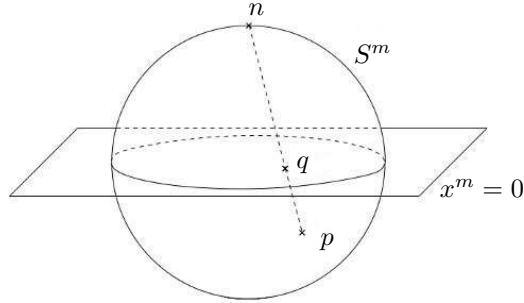
Explicitly, for $p = (x^0, \ldots, x^m)$, the point q must satisfy $q = (y, 0) = (1-t)n + tp$ for some $t \in \mathbb{R}$. The last coordinate gives $1 - t + tx^m = 0$, hence $t = \frac{1}{1-x^m}$, and so,

$$\pi_n(x^0, \ldots, x^m) = \frac{1}{1 - x^m}(x^0, \ldots, x^{m-1}).$$

Similarly, we have the stereographic projection through the south pole:

$$\pi_s : S^m \backslash \{s\} \to \mathbb{R}^m, \quad \pi_s(x^0, \ldots, x^m) = \frac{1}{1 + x^m}(x^0, \ldots, x^{m-1}).$$

We claim that the following is a smooth atlas on S^m:

$$\mathcal{A} = \{(S^m \backslash \{n\}, \pi_n), (S^m \backslash \{s\}, \pi_s)\}.$$

First we show that the stereographic projections are homeomorphisms. Note that the formula defining π_n makes sense on the open set $U = \mathbb{R}^{m+1} \backslash \{x^m = 1\}$, and that it defines a continuous map $U \to \mathbb{R}^m$. Thus π_n is the restriction to $U \cap S^m$ of a continuous map and, using the definition of the induced topology, this implies that π_n is continuous. The same argument implies that π_s is continuous. Finally, let us determine the inverse of π_n. We have that $p = \pi_n^{-1}(y)$ is the point at the intersection of S^m and the line through $q = (y, 0)$ and n, and which is not n. So $p = t(y, 0) + (1 - t)n$, with $t \in \mathbb{R}$. Writing $|p|^2 = 1$, we obtain: $t^2|y|^2 + (1 - t)^2 = 1$; or $t^2(|y|^2 + 1) = 2t$. The solution $t = 0$ corresponds to n; thus $t = \frac{2}{1+|y|^2}$. Using this, we obtain that the inverse of π_n is given by:

$$\pi_n^{-1} : \mathbb{R}^m \longrightarrow S^m \backslash \{n\},$$

$$\pi_n^{-1}(y^1, y^2, \ldots, y^m) = \left(\frac{2y^1}{|y|^2 + 1}, \frac{2y^2}{|y|^2 + 1}, \ldots, \frac{2y^m}{|y|^2 + 1}, \frac{|y|^2 - 1}{|y|^2 + 1} \right).$$

All functions appearing in this expression are continuous, and this proves that π_n is a homeomorphism. A similar calculation shows that also π_s is a homeomorphism.

Finally, we need to check that the charts are C^∞-compatible. Since $\pi_n(s) = \pi_s(n) = 0$, the transition map is defined as follows:

$$\pi_s \circ \pi_n^{-1} : \mathbb{R}^m \backslash \{0\} \longrightarrow \mathbb{R}^m \backslash \{0\},$$

and using the formulas above, we obtain that this map is given by:

$$\pi_s \circ \pi_n^{-1}(y) = \pi_s \left(\frac{2y}{|y|^2 + 1}, \frac{|y|^2 - 1}{|y|^2 + 1} \right) = \frac{1}{1 + \frac{|y|^2 - 1}{|y|^2 + 1}} \frac{2y}{|y|^2 + 1} = \frac{y}{|y|^2}.$$

This map is clearly smooth. Note that the map satisfies $(\pi_s \circ \pi_n^{-1})^2 = \mathrm{id}_{\mathbb{R}^m \backslash \{0\}}$. This shows that it equals its own inverse, hence it is a diffeomorphism.

We introduce a relation on atlases:

Definition 1.2.8. *Let M be an m-dimensional topological manifold. Consider the following relation on m-dimensional C^∞-atlases on M:*

$$\mathcal{A}_1 \sim \mathcal{A}_2 \quad \overset{def.}{\Longleftrightarrow} \quad \mathcal{A}_1 \cup \mathcal{A}_2 \text{ is a } C^\infty - atlas.$$

As expected:

Proposition 1.2.9. *The relation $\sim$ is an equivalence relation on the set of m-dimensional C^∞-atlases of M.*

Proof. Reflexivity and symmetry of the relation $\sim$ are obvious. We will check that transitivity holds. Consider three m-dimensional C^∞-atlases on M: $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ such that $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mathcal{A}_2 \sim \mathcal{A}_3$. To check that $\mathcal{A}_1 \sim \mathcal{A}_3$ we need to show that each pair of charts $(U_1, \varphi_1) \in \mathcal{A}_1$ and $(U_3, \varphi_3) \in \mathcal{A}_3$ are compatible, i.e. that the map

(1) $$\varphi_1 \circ \varphi_3^{-1} : \varphi_3(U_1 \cap U_3) \longrightarrow \varphi_1(U_1 \cap U_3)$$

is a diffeomorphism. This map is clearly a bijection, with inverse

(2) $$\varphi_3 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_3) \longrightarrow \varphi_3(U_1 \cap U_3).$$

So, it suffices to prove that the maps (1) and (2) are smooth. Let $p \in U_1 \cap U_3$. Since $\mathcal{A}_2$ is an atlas, there exists a chart $(U_2, \varphi_2) \in \mathcal{A}_2$ such that $p \in U_2$. Since $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mathcal{A}_2 \sim \mathcal{A}_3$ is follows that the following maps are diffeomorphisms:

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \longrightarrow \varphi_1(U_1 \cap U_2),$$
$$\varphi_2 \circ \varphi_3^{-1} : \varphi_3(U_2 \cap U_3) \longrightarrow \varphi_2(U_2 \cap U_3).$$

In particular, the restriction of their composition is a diffeomorphism:

$$\varphi_1 \circ \varphi_3^{-1} : \varphi_3(U_1 \cap U_2 \cap U_3) \longrightarrow \varphi_1(U_1 \cap U_2 \cap U_3).$$

Hence, the map (1) and its inverse (2) are smooth when restricted to the open neighborhoods $\varphi_3(U_1 \cap U_2 \cap U_3)$ and $\varphi_1(U_1 \cap U_2 \cap U_3)$, respectively, of $\varphi_3(p)$ and $\varphi_1(p)$, respectively. Since p was chosen arbitrary in $U_1 \cap U_3$, it follows that (1) and (2) are smooth everywhere. This finishes the proof. $\square$

Next, we define:

Definition 1.2.10. *An m-dimensional C^∞-atlas $\mathcal{A}$ on the topological manifold M is said to be* **maximal** *if*

$$\mathcal{A} \sim \mathcal{B} \quad \Longrightarrow \quad \mathcal{B} \subset \mathcal{A}$$

for any m-dimensional C^∞-atlas $\mathcal{B}$ on M.

We have that:

Proposition 1.2.11. *Any equivalence class of smooth atlases has a unique maximal representative. The maximal C^∞-atlas equivalent to the C^∞-atlas $\mathcal{A}$ is given by:*

$$\mathcal{A}_{\max} = \{(U, \varphi) \ : \ \mathcal{A} \cup \{(U, \varphi)\} \text{ is a } C^\infty\text{-atlas}\}.$$

Proof. Let $\mathcal{A}$ be an m-dimensional C^∞-atlas on M. It is clear that there exists at most one maximal smooth atlas which is equivalent to $\mathcal{A}$, because if there were two, $\mathcal{A}_{\max}^1$ and $\mathcal{A}_{\max}^2$, then by transitivity of the relation $\sim$ we would have that $\mathcal{A}_{\max}^1 \sim \mathcal{A}_{\max}^2$, and since both atlases are maximal, this implies that

$$\mathcal{A}_{\max}^1 \subset \mathcal{A}_{\max}^2 \subset \mathcal{A}_{\max}^1;$$

thus $\mathcal{A}_{\max}^1 = \mathcal{A}_{\max}^2$.

To prove existence of a maximal C^∞-atlas equivalent to $\mathcal{A}$, let $\mathcal{A}_{\max}$ be the set constructed in the statement. We check that $\mathcal{A}_{\max}$ is indeed a C^∞-atlas. First, since $\mathcal{A} \subset \mathcal{A}_{\max}$, it follows that the open sets in $\mathcal{A}_{\max}$ cover M. Second, we check that each pair of charts (U, φ) and (V, ψ) in $\mathcal{A}_{\max}$ are compatible. Clearly $\mathcal{A} \sim \mathcal{A} \cup \{(U, \varphi)\}$ and $\mathcal{A} \sim \mathcal{A} \cup \{(V, \psi)\}$, therefore, by transitivity of $\sim$, $\mathcal{A} \cup \{(U, \varphi)\} \sim \mathcal{A} \cup \{(V, \psi)\}$, hence $\mathcal{A} \cup \{(U, \varphi)\} \cup \{(V, \psi)\}$ is a C^∞-atlas. This implies that the charts (U, φ) and (V, ψ) are compatible, and proves that $\mathcal{A}_{\max}$ is indeed a C^∞-atlas.

Finally, we prove that $\mathcal{A}_{\max}$ is maximal. Note first that $\mathcal{A} \sim \mathcal{A}_{\max}$. Let $\mathcal{B}$ be a C^∞-atlas such that $\mathcal{B} \sim \mathcal{A}_{\max}$. Then, by transitivity of $\sim$, $\mathcal{B} \sim \mathcal{A}$, which implies that every chart in $\mathcal{B}$ is compatible with every chart in $\mathcal{A}$; hence $\mathcal{A} \cup \{(U, \varphi)\}$ is an atlas for every $(U, \varphi) \in \mathcal{B}$. The definition of $\mathcal{A}_{\max}$ implies now that $\mathcal{B} \subset \mathcal{A}_{\max}$. $\square$

Definition 1.2.12. *An m-dimensional* **differentiable structure** *(or* **smooth structure***) on a topological manifold M is an equivalence class of m-dimensional C^∞-atlases on M. Equivalently, by Proposition 1.2.11, a differentiable structure is the same as a maximal atlas on M.*

We are ready to give the general definition of a smooth manifold:

Definition 1.2.13. *An m-dimensional **smooth manifold** is a topological manifold endowed with an m-dimensional differentiable structure.*

Example 1.2.14. The topological space $\mathbb{R}^m$ and S^m are smooth manifolds when endowed with the differentiable structure corresponding to the C^∞-atlases constructed in Example 1.2.7. Note that $\mathcal{B}$ Example 1.2.7 (1) is the maximal atlas on $\mathbb{R}^m$. The obtained differentiable structures on $\mathbb{R}^m$ and S^m are called the standard differentiable structures. A differentiable structure which is not equivalent to the standard one is called an exotic differentiable structure. These do exist in certain dimensions; see Section 2.3 for more details.

Remark 1.2.15. The axiom of being second countable insures that manifolds are not "too big". Let us mention that there are examples of Hausdorff topological spaces, endowed with a smooth atlas, but which are not second countable. An easy example is the disjoint union of an uncountable collection of manifolds of the same dimension; e.g. an uncountable collection of points with the discrete topology is not second countable, and has a 0-dimensional atlas. There are also connected examples, but these are more difficult to describe; a famous one is the *Long Line* (see e.g. [**17, 18**]).

1.3. Exercises

Exercise 1.1. (a) Show that $\mathbb{R}^n$ is Hausdorff and second countable.
(b) Show that if X is a Hausdorff topological space and $Y \subset X$ is a subset, then the induced topology on Y is Hausdorff.
(c) Show that if X is a second countable topological space and $Y \subset X$ is a subset, then the induced topology on Y is second countable.

Exercise 1.2. (a) Prove that the following function is smooth but is not analytic

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} 0, & x \le 0; \\ e^{-1/x}, & x > 0. \end{cases}$$

(b) Show that $g : \mathbb{R} \to \mathbb{R}$, $g(x) := f(x)f(1-x)$ is a smooth function which is positive on $(0,1)$ and zero elsewhere. Show also that the function $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) := \frac{\int_0^x g(y)\mathrm{d}y}{\int_0^1 g(y)\mathrm{d}y}$$

is a smooth and it satisfies: $h(x) = 0$ for $x < 0$, $0 < h(x) < 1$ for $0 < x < 1$, and $h(x) = 1$ for $x > 1$.
(c) Use (a) and (b) to prove Lemma 1.1.11.

Exercise 1.3. Prove that there are no analytic functions $\chi : \mathbb{R}^m \to \mathbb{R}$ with the properties from Lemma 1.1.11.

The following exercise is used to prove Borel's Lemma:

Exercise 1.4. Let $C_b^k(\mathbb{R}^n)$ denote the set of C^k-function $f : \mathbb{R}^n \to \mathbb{R}$ which have all partial derivatives up to order k bounded. On this space we define the so-called C^k-norm:

$$\|f\|_k := \sup \left\{ \left| \frac{\partial^{i_1 + \ldots + i_n} f}{(\partial x^1)^{i_1} \ldots (\partial x^n)^{i_n}}(x) \right| \; : \; x \in \mathbb{R}^n, \; i_1 + \ldots + i_n \le k \right\}.$$

Let $C_b^\infty(\mathbb{R}^n)$ denote the set of smooth maps with **all** partial derivatives bounded.

(a) Prove that $(C_b^k(\mathbb{R}^n), \|\cdot\|_k)$ is a Banach space, i.e. prove that: every sequence of functions in $C_b^k(\mathbb{R}^n)$ which is Cauchy with respect to the norm $\|\cdot\|_k$ is in fact convergent to an element of $C_b^k(\mathbb{R}^n)$.

(b) Prove that $C_b^\infty(\mathbb{R}^n)$ endowed with the family of norms $\{\|\cdot\|_k\}_{k\geq 0}$ is a Fréchet space, i.e. prove that: every sequence of functions in $C_b^\infty(\mathbb{R}^n)$ which is Cauchy with respect to **all** the norms $\{\|\cdot\|_k\}_{k\geq 0}$ is in fact convergent with respect to all the norms $\{\|\cdot\|_k\}_{k\geq 0}$ to an element in $C_b^\infty(\mathbb{R}^n)$.

In the next exercise you are asked to prove Borel's Lemma; note that it uses the previous exercise.

Exercise 1.5. Consider a formal power series

$$Q(T_1,\ldots,T_n) := \sum_{i_1,\ldots,i_n \geq 0} c_{i_1,\ldots,i_n} T_1^{i_1} \ldots T_n^{i_n} \in \mathbb{R}[[T_1,\ldots,T_n]].$$

Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\chi(x) = 1$, for $x \in B_1(0)$, and $\chi(x) = 0$ for $x \notin B_2(0)$ (by Lemma 1.1.11, such functions exist).

For $k \geq 0$, define the following function depending on a positive number $\epsilon_k > 0$:

$$f_k(x) := \sum_{i_1+\ldots+i_n=k} \chi\left(\frac{x}{\epsilon_k}\right) \cdot c_{i_1,\ldots,i_n} (x^1)^{i_1} \ldots (x^n)^{i_n}.$$

(a) Prove that $f_k \in C_b^\infty(\mathbb{R}^n)$.

(b) For $k \geq 1$, prove that one can choose $\epsilon_k > 0$ such that

$$\|f_k\|_{k-1} < 2^{-k}.$$

(c) Let $\epsilon_k > 0$ be as in (b). Show that the series:

$$\sum_{k\geq 0} f_k$$

converges to a smooth function $f \in C_b^\infty(\mathbb{R}^n)$.

(d) Prove that the Taylor series of f at 0 is Q.

Exercise 1.6. Consider the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(t) = t^3.$$

(a) Show that f is a smooth homeomorphism of $\mathbb{R}$.

(b) Show that, there are no open neighborhoods U and V of 0 in $\mathbb{R}$ such that $f|_U : U \to V$ is a diffeomorphism.

(c) On the topological manifold $\mathbb{R}$ consider the atlases

$$\mathcal{A}_1 = \{(\mathbb{R}, \mathrm{id}_\mathbb{R})\} \quad \text{and} \quad \mathcal{A}_2 = \{(\mathbb{R}, f)\}.$$

Is $\mathcal{A}_1 \sim \mathcal{A}_2$?

Exercise 1.7. On the unit sphere $S^m \subset \mathbb{R}^{m+1}$ consider the following collection of $2(m+1)$ maps:

$$\mathcal{B} = \{(U_i^\epsilon, \varphi_i^\epsilon) \ : \ 0 \leq i \leq m, \ \epsilon = \pm 1\},$$
$$U_i^\epsilon = \{(x^0,\ldots,x^m) \in S^m \ : \ \epsilon \cdot x^i > 0\},$$
$$\varphi_i^\epsilon(x^0,\ldots,x^m) = (x^0,\ldots,\widehat{x^i},\ldots,x^m),$$

where the circumflex $\widehat{}$ indicates that the underlying term has been deleted.

(a) Prove that the pairs $(U_i^\epsilon, \varphi_i^\epsilon)$ are indeed charts, and prove that $\mathcal{B}$ is a smooth atlas on S^m.

(b) Prove that the atlas $\mathcal{A}$ from Example 1.2.7 (2) and the atlas $\mathcal{B}$ define the same differentiable structure on S^m.

LECTURE 2

2.1. Smooth maps

We introduce smooth maps between smooth manifolds. First, let us define:

Definition 2.1.1. *A **smooth chart** on a smooth manifold M is a chart (U, φ) which belongs to the maximal C^∞-atlas of M.*

Definition 2.1.2. *Let M and N be smooth manifolds of dimension m and n, respectively. A continuous map $f : M \to N$ is said to be **smooth** (or C^∞), if for every smooth chart (U, φ) on M and every smooth chart (V, ψ) on N, the map:*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \longrightarrow \psi(V)$$

*is smooth. The map $\psi \circ f \circ \varphi^{-1}$, which is defined between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ is called the **local representation** of f in the charts (U, φ) and (V, ψ).*

The space of smooth maps from M to N is denoted by $C^\infty(M, N)$. If $N = \mathbb{R}$ (as a smooth manifold) we simply write $C^\infty(M) := C^\infty(M, \mathbb{R})$.

The condition that f be continuous in the previous definition is necessary to insure that $U \cap f^{-1}(V)$ is open.

Remark 2.1.3. Smoothness can be checked on an open cover. Namley, a continuous map $f : M \to N$ is smooth iff for each $p \in M$ there exists a smooth chart (U, φ) on M, with $p \in U$, and a smooth chart (V, ψ) on N with $f(p) \in V$ such that the local representation of f in these charts is smooth. This follows because any two local representative around p and $f(p)$ are related by composing on both sides with local diffeomorphisms.

Composition of smooth maps is smooth:

Proposition 2.1.4. *Let $f : M \to N$ and $g : N \to P$ be smooth maps between smooth manifolds. Then $g \circ f : M \to P$ is a smooth map.*

Next, we extend also the notion of a diffeomorphism:

Definition 2.1.5. *A map $f : M \to N$ between smooth manifolds M and N is called a **diffeomorphism** if f is bijective and both f and f^{-1} are smooth. If there exists a diffeomorphism $f : M \to N$, then the manifolds M and N are said to be **diffeomorphic**.*

Let us mention here that there is a product operation on manifolds:

Proposition 2.1.6. *Let M and N be smooth manifolds of dimensions m, respectively n. Then the product $M \times N$ endowed with the product topology is a smooth $m + n$-dimensional manifold. A smooth atlas on $M \times N$ is*

$$\mathcal{A}_{M \times N} := \{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A}_M, \ (V, \psi) \in \mathcal{A}_N\},$$

where $\mathcal{A}_M$ is a smooth atlas on M and $\mathcal{A}_N$ is a smooth atlas on N. Moreover, the projection maps

$$\mathrm{pr}_1 : M \times N \to M, \ \ \mathrm{pr}_1(x, y) = x, \ \ \ \ \mathrm{pr}_2 : M \times N \to N, \ \ \mathrm{pr}_2(x, y) = y.$$

are smooth.

2.2. How many smooth structures?

This section is mostly for the general knowledge of the reader, and will not be used in further lectures. The results stated here are not proven in this course, as they are beyond the scope of these lectures; for more details, see the references in the Wikipedia article [**16**].

Let us state one of the problems which has been of central interest since the beginning of the theory of smooth manifolds:

Let M be a topological manifold. Can M be made into a smooth manifold? If so, in how many non-equivalent ways?

Let us be a bit more precise about the second part of the questions. Assume that M is a smooth manifold in two different ways, corresponding to the differentiable structures $\mathcal{D}_1$ and $\mathcal{D}_2$. We regard these differentiable structures are equivalent, if there exists a diffeomorphism relating the two structure:

$$\varphi : (M, \mathcal{D}_1) \longrightarrow (M, \mathcal{D}_2).$$

Thus, the second part of the questions asks how many non-equivalent such differentiable structures exist.

For small dimension, we have existence and uniqueness:

Theorem ♣ 2.2.1. *For $m \leq 3$, every m-dimensional topological manifold has a smooth structure, and any two such smooth structures are equivalent (for $m = 1, 2$ Radó, 1920s, for $m = 3$ Moise, 1953, [**9**]).*

However, in larger dimension:

Theorem ♣ 2.2.2. *There exist compact topological manifolds of dimension ≥ 4 which do not admit a differentiable structure (Kervaire 1960, [**6**] first example in dimension 10; nowadays many examples are known even of dimension 4).*

Finally, also uniqueness fails in dimension $m \geq 4$; there are topological manifolds which admit non-equivalent smooth structures. The non-standard ones are usually called *exotic*. The first examples of non-equivalent smooth structures on a topological manifold were Milnor's *exotic 7-spheres*:

Theorem ♣ 2.2.3. *The topological manifold S^7 admits exactly 28 non-equivalent smooth structures (Milnor 1956, [**7**]).*

However, it is not known whether S^4 has exotic smooth structures! Here is a table with the number of exotic structure for spheres of dim ≤ 20:

dimension	1	2	3	4	5	6	7	8	9	10
$C^\infty - \text{str.'s}$	1	1	1	?	1	1	28	2	8	6

dimension	11	12	13	14	15	16	17	18	19	20
$C^\infty - \text{str.'s}$	992	1	3	2	16256	2	16	16	523264	24

Even more surprisingly are the exotic $\mathbb{R}^4$'s:

Theorem ♣ 2.2.4. *For $m \neq 4$, the topological manifold $\mathbb{R}^m$ admits a unique smooth structure (Stallings, 1962 [12]). The topological manifold $\mathbb{R}^4$ admits uncountably many non-equivalent smooth structures (Taubes 1987, [13]).*

2.3. C^k-manifolds

Throughout this section we fix k which is either an integer $k \geq 1$, or $k \in \{\infty, \omega\}$. We briefly introduce the notion of C^k-manifolds.

All notions discussed so far can be extended directly to the C^k-setting, by simply replacing the requirement that a map be smooth with that of being C^k.

For example, a C^k-diffeomorphism is a map $f : U \to V$ between open sets $U, V \subset \mathbb{R}^m$ which is bijective and both f and f^{-1} are C^k-maps. Note the following:

Lemma 2.3.1. *If $f : U \to V$ is a C^1-diffeomorphism and a C^k-map, then f is a C^k-diffeomorphism.*

Proof. We need to check that the inverse of f, denoted by $g : V \to U$, is also of class C^k. For $k = \omega$, we omit the proof (one should write the power series expansion of the equation $f \circ g = \text{id}$, and determine a term-by-term formula for g, and then show that this is convergent). Let $k \neq \omega$. Differentiating the relation $f \circ g = \text{id}$, by the chain rule, we obtain that

$$\mathrm{d}_{g(x)}f \circ \mathrm{d}_x g = \text{id}.$$

This gives that the differential of g is the inverse of the differential of f:

$$\mathrm{d}_x g = (\mathrm{d}_{g(x)}f)^{-1}.$$

This relation shows that if g is C^l and f is C^{l+1} then g is C^{l+1} (the right hand side is a composition of C^l-maps, and the smooth (!) operation of taking the inverse of a matrix). An argument by induction, for $1 \leq l \leq k$, proves the statement. □

One introduces C^k-compatible charts, C^k-atlases, C^k-differentiable structures, and finally, C^k-manifolds, all in perfect analogy with the smooth setting.

Let $l, k \in \{0, 1, 2, \ldots, \infty, \omega\}$ $l < k$ (where $\infty < \omega$). A C^k-differentiable structure $\mathcal{D}^k$ on M is said to *extend* a given C^l-differentiable structure $\mathcal{D}^l$ on M if every C^k-atlas representing $\mathcal{D}^k$, when regarded as a C^l-atlas it represents $\mathcal{D}^l$. Two extensions $\mathcal{D}^k_1$ and $\mathcal{D}^k_2$ of $\mathcal{D}^l$ are said to be equivalent, if there exists a C^k-diffeomorphism relating them:

$$\varphi : (M, \mathcal{D}^k_1) \longrightarrow (M, \mathcal{D}^k_2).$$

The following theorem says that C^l-differentiable structures, with $0 < l < \infty$, are essentially the same as smooth structures (or even analytic structures).

Theorem ♣ 2.3.2. *If $1 \leq l < k$, with $l, k \in \{1, 2, \ldots, \infty, \omega\}$, then any C^l-differentiable structure can be extended to a C^k-differentiable structure, and any two extensions are equivalent (Whitney, 1936 [**15**]).*

Whitney's result also explains why the problem discussed in the previous section is only interesting for topological manifolds.

2.4. Embedded manifolds in $\mathbb{R}^n$

In this section we introduce *embedded manifolds in $\mathbb{R}^n$*. These are subsets of $\mathbb{R}^n$ which, locally, can be straighten out smoothly to look like an m-dimensional linear space. This gives a large class of examples of smooth manifolds, and in fact, Whitney's theorem (which will be discussed later on) states that any smooth manifold can be regarded as an embedded manifold in some $\mathbb{R}^n$.

Definition 2.4.1. *A subset $M \subset \mathbb{R}^n$ is called an m-dimensional **embedded manifold in** $\mathbb{R}^n$ if around every point in M there exists an open set $U \subset \mathbb{R}^n$ and there exists a diffeomorphism $f : U \to V$, where $V \subset \mathbb{R}^n$ is open, such that:*

$$f(M \cap U) = (\mathbb{R}^m \times \{0\}) \cap V.$$

*We will call a diffeomorphism (U, f) as above a **chart adapted to** M.*

Example 2.4.2. Let us check that the circle

$$S^1 \subset \mathbb{R}^2, \quad S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is a 1-dimensional embedded manifold in $\mathbb{R}^2$. Consider the map

$$f(x, y) := (x, x^2 + y^2 - 1).$$

Note that f is a diffeomorphism between the open sets

$$U := \{(x, y) \in \mathbb{R}^2 : y > 0\}, \quad V := \{(u, v) \in \mathbb{R}^2 : v > u^2 - 1\},$$

with inverse given by

$$f^{-1} : V \longrightarrow U, \quad f^{-1}(u, v) = (u, \sqrt{v + 1 - u^2}).$$

Note that $f(x, y) \in \mathbb{R} \times \{0\}$ if and only if $(x, y) \in S^1$; thus, f satisfies:

$$f(S^1 \cap U) = (\mathbb{R} \times \{0\}) \cap V.$$

The open set U covers the upper half of the circle. On the open sets $U_1 := -U$, $U_2 := \{(x, y) : x > 0\}$ and $U_3 := -U_2$, similar diffeomorphisms can be constructed:

$$f_1 := f|_{U_1}, \quad f_2 := g|_{U_2}, \quad f_3 := g|_{U_3}, \text{ where } g(x, y) := (y, x^2 + y^2 - 1).$$

We will use the following notations for the canonical projections and inclusions, respectively, between $\mathbb{R}^m$ and $\mathbb{R}^n$, with $m \leq n$:

$$\mathrm{pr}_m^n : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \mathrm{pr}_m^n(x^1, \ldots, x^n) := (x^1, \ldots, x^m)$$

$$\mathrm{i}_n^m : \mathbb{R}^m \longrightarrow \mathbb{R}^n, \quad \mathrm{i}_n^m(x^1, \ldots, x^m) := (x^1, \ldots, x^m, 0, \ldots, 0).$$

Proposition 2.4.3. *Let M be an m-dimensional embedded manifold in $\mathbb{R}^n$. Then M, endowed with the induced topology, is a smooth m-dimensional manifold with differentiable structure represented by the atlas*

$$\mathcal{A} = \{(M \cap U, \mathrm{pr}_m^n \circ f|_{M \cap U}) \ : \ (U, f) \text{ is a chart adapted to } M\}.$$

Proof. According to Exercise 1.1, the induced topology is Hausdorff and second countable. We need to check that $\mathcal{A}$ is a smooth atlas on M. By definition, M is covered by adapted charts. Let (U, f) be an adapted chart, and denote $V := f(U)$ and $\varphi := \mathrm{pr}^n_m \circ f|_{M \cap U}$. First, note that $M \cap U$ is open in the induced topology on M, and the map φ is continuous for the induced topology, because it is the restriction of a continuous map on U. We have that

$$\varphi(M \cap U) = \mathrm{pr}^n_m \circ f(M \cap U) = \mathrm{pr}^n_m \left((\mathbb{R}^m \times \{0\}) \cap V \right) \subset \mathbb{R}^m.$$

Since pr^n_m restricts to a homeomorphism between $\mathbb{R}^m \times \{0\}$ (with the induced topology) and $\mathbb{R}^m$, we have that $\varphi(M \cap U)$ is open in $\mathbb{R}^m$. Finally, the map φ is a homeomorphism onto its image, because its inverse is given by as a composition of continuous maps:

$$\varphi^{-1} : \varphi(M \cap U) \longrightarrow M \cap U, \quad \varphi^{-1} = f^{-1} \circ \mathrm{i}^m_n|_{\varphi(M \cap U)}.$$

This proves that $(M \cap U, \varphi)$ is an m-dimensional chart on M. Thus, M is a topological m-dimensional manifold.

To show that $\mathcal{A}$ is a smooth atlas, we need to check that any two charts are C^∞-compatible. Let $f : U \to V$ and $f' : U' \to V'$ be two adapted charts, inducing the following charts in $\mathcal{A}$:

$$(M \cap U, \varphi := \mathrm{pr}^n_m \circ f|_{M \cap U}), \quad (M \cap U', \varphi' := \mathrm{pr}^n_m \circ f'|_{M \cap U'}).$$

The corresponding transition map is smooth since it is given by:

$$\varphi' \circ \varphi^{-1} : \varphi(M \cap U \cap U') \longrightarrow \varphi'(M \cap U \cap U'), \quad \mathrm{pr}^n_m \circ f' \circ f^{-1} \circ \mathrm{i}^m_n.$$

By interchanging the roles of f and f' we see that its inverse is also smooth. Hence any two charts in $\mathcal{A}$ are C^∞-compatible. $\qquad\square$

The converse of this result also holds:

Theorem ♣ 2.4.4 (Whitney's Embedding Theorem). *Any m-dimensional smooth manifold, for $m \geq 1$, is diffeomorphic to an embedded manifold in $\mathbb{R}^{2m}$, which is a closed subset.*

2.5. Exercises

Exercise 2.1. Prove Proposition 2.1.6.

Exercise 2.2. Let M be an embedded manifold in $\mathbb{R}^n$. Show that the inclusion map $M \hookrightarrow \mathbb{R}^n$, $x \mapsto x$, is smooth.

Exercise 2.3. (a) Prove that S^m is an embedded manifold in $\mathbb{R}^{m+1}$ (as in Definition 2.4.1).
(b) Prove that the smooth structure on S^m resulting from Proposition 2.4.3 coincides with the smooth structure on S^m coming from the smooth atlas in Example 1.2.7.

<div align="right">

LECTURE 3

</div>

3.1. The topology is determined by the atlas

It is important to notice that an atlas determines the topology uniquely. This remark allows one to describe smooth manifolds by giving *a collection of charts on a set*. We state this in the following proposition.

Proposition 3.1.1. *Let M be a set and let $\mathcal{A}$ be a collection of maps*

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \; : \; \alpha \in I\}, \quad U_\alpha \subset M, \quad \varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^m,$$

which satisfy the following conditions:

(1) $\cup_{\alpha \in I} U_\alpha = M$;
(2) for each $\alpha \in I$, φ_α is injective;
(3) for each $\alpha, \beta \in I$, $\varphi_\beta(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^m$;
    in particular, for $\alpha = \beta$, $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^m$;
(4) for each $\alpha, \beta \in I$, the map $\varphi_\alpha \circ \varphi_\beta^{-1} \; : \; \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.

There exists a unique topology $\mathcal{T}_\mathcal{A}$ on M such that, for all $\alpha \in I$, the pair $(U_\alpha, \varphi_\alpha)$ is a chart on M. Moreover,

- *If I is at most countable, then $\mathcal{T}_\mathcal{A}$ is second countable.*
- *If for every two distinct points $p, q \in M$ there exist $\alpha, \beta \in I$ and open sets O_α and O_β in $\mathbb{R}^m$ such that*

$$p \in \varphi_\alpha^{-1}(O_\alpha), \quad q \in \varphi_\beta^{-1}(O_\beta), \quad \varphi_\alpha^{-1}(O_\alpha) \cap \varphi_\beta^{-1}(O_\beta) = \emptyset,$$

  then $\mathcal{T}_\mathcal{A}$ is Hausdorff.
- *If $\mathcal{T}_\mathcal{A}$ is Hausdorff and second countable, then M has an m-dimensional differentiable structure for which $\mathcal{A}$ is a C^∞-atlas.*

Proof. First, we show that there is at most one topology $\mathcal{T}_\mathcal{A}$ on M for which $\mathcal{A}$ is an atlas: since $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M w.r.t. $\mathcal{T}_\mathcal{A}$, we have that

$$V \in \mathcal{T}_\mathcal{A} \iff \big(\forall \, \alpha \in I \; : \; V \cap U_\alpha \in \mathcal{T}_\mathcal{A}\big);$$

and since $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism, we have that

$$V \in \mathcal{T}_\mathcal{A} \iff \big(\forall \, \alpha \in I \; : \; \varphi_\alpha(V \cap U_\alpha) \text{ is open in } \mathbb{R}^m\big).$$

The above defines $\mathcal{T}_\mathcal{A}$ uniquely. Next we show that $\mathcal{T}_\mathcal{A}$ is indeed a topology. Clearly, $\emptyset \in \mathcal{T}_\mathcal{A}$, and by (3), $M \in \mathcal{T}_\mathcal{A}$. Let $V_1, V_2 \in \mathcal{T}_\mathcal{A}$. For every $\alpha \in I$, we have that

$$\varphi_\alpha(U_\alpha \cap V_1 \cap V_2) = \varphi_\alpha\left((U_\alpha \cap V_1) \cap (U_\alpha \cap V_2)\right) \overset{(2)}{=} \varphi_\alpha(U_\alpha \cap V_1) \cap \varphi_\alpha(U_\alpha \cap V_2),$$

which shows that $\varphi_\alpha(U_\alpha \cap V_1 \cap V_2)$ is open. This implies that $V_1 \cap V_2 \in \mathcal{T}_\mathcal{A}$. Finally, consider an arbitrary family $\{V_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{T}_\mathcal{A}$. Then, for each $\alpha \in I$,

$$\varphi_\alpha\left(U_\alpha \cap (\cup_{\lambda \in \Lambda} V_\lambda)\right) = \varphi_\alpha\left(\cup_{\lambda \in \Lambda} U_\alpha \cap V_\lambda\right) = \cup_{\lambda \in \Lambda} \varphi_\alpha(U_\alpha \cap V_\lambda),$$

which shows that $\varphi_\alpha\left(U_\alpha \cap (\cup_{\lambda \in \Lambda} V_\lambda)\right)$ is open. Hence, $\cup_{\lambda \in \Lambda} V_\lambda \in \mathcal{T}_\mathcal{A}$. We conclude that $\mathcal{T}_\mathcal{A}$ is a topology.

Let us see that $(U_\alpha, \varphi_\alpha)$ is indeed a chart on M for the topology $\mathcal{T}_\mathcal{A}$. By (3), U_α is open. Next, we show that $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism. For any open set $O \subset \varphi_\alpha(U_\alpha)$, we have that $\varphi_\beta(U_\beta \cap \varphi_\alpha^{-1}(O)) = \varphi_\beta \circ \varphi_\alpha^{-1}(O)$ which is open because, by (4), $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a homeomorphism. Since β is arbitrary, $\varphi_\alpha^{-1}(O) \in \mathcal{T}_\mathcal{A}$. So, φ_α is continuous. By definition of $\mathcal{T}_\mathcal{A}$, we have that also $\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to M$ is continuous. Thus, the maps $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ are homeomorphisms, i.e. charts on M for the topology $\mathcal{T}_\mathcal{A}$.

Assuming that I is at most countable, we prove that $\mathcal{T}_\mathcal{A}$ is second countable. Let $\{O_k\}_{k \in \mathbb{N}}$ be a countable basis for the topology of $\mathbb{R}^m$, i.e. every open set $U \subset \mathbb{R}^m$ can be represented as $U = \cup_{k \in J_U} O_k$, for a subset $J_U \subset \mathbb{N}$. We claim that the family of open sets $\{\varphi_\alpha^{-1}(O_k)\}_{\alpha \in I, k \in \mathbb{N}}$ is a basis for $\mathcal{T}_\mathcal{A}$. Since I is at most countable, the index set $I \times \mathbb{N}$ is countable. Let $V \in \mathcal{T}_\mathcal{A}$ be an open set. First, decompose $V = \cup_{\alpha \in I}(V \cap U_\alpha)$. Since $\varphi_\alpha(V \cap U_\alpha)$ is open in $\mathbb{R}^m$, there exists a subset $J_\alpha \subset \mathbb{N}$ such that $\varphi_\alpha(V \cap U_\alpha) = \cup_{k \in J_\alpha} O_k$. Thus we have that

$$V = \cup_{\alpha \in I} V \cap U_\alpha = \cup_{\alpha \in I} \cup_{k \in J_\alpha} \varphi_\alpha^{-1}(O_k).$$

This shows that $\mathcal{T}_\mathcal{A}$ is second countable.

The criterion for being Hausdorff is clear: the points p and q are separated by the open sets $\varphi_\alpha^{-1}(O_\alpha)$ and $\varphi_\beta^{-1}(O_\beta)$.

By (4) the charts in $\mathcal{A}$ are pairwise compatible, which makes the final conclusion obvious. □

3.2. The real projective space

Proposition 3.1.1 allows us to describe the structure of a smooth manifold on a set in terms of a collection of local maps to $\mathbb{R}^m$ satisfying certain conditions. Here is an important example:

Example 3.2.1. The m-dimensional **real projective space** is defined as the set of all lines through the origin in $\mathbb{R}^{m+1}$:

$$\mathbb{P}^m(\mathbb{R}) := \{\, l \;:\; l \subset \mathbb{R}^{m+1} \text{ is a line with } 0 \in l\}.$$

Note first that a non-zero vector in $\mathbb{R}^{m+1}$ belongs to a unique line in $\mathbb{P}^m(\mathbb{R})$, and two vectors lie on the same line $\mathbb{P}^m(\mathbb{R})$ iff one is a scalar multiple of the other. This allows us to identify

$$\mathbb{P}^m(\mathbb{R}) = \left(\mathbb{R}^{m+1} \backslash \{0\}\right) / \sim,$$

where $\sim$ is the equivalence relation on $\mathbb{R}^{m+1} \backslash \{0\}$ given by

$$v \sim w \quad \Longleftrightarrow \quad v = tw, \text{ for some } 0 \neq t \in \mathbb{R}.$$

We denote the equivalence class of $v = (x^0, x^1, \ldots, x^m) \neq 0$ by:
$$[v] = [x^0 : x^1 : \ldots : x^m] = \left\{ (tx^0, tx^1, \ldots, tx^m) : 0 \neq t \in \mathbb{R} \right\}.$$
Consider the following $m + 1$ subsets of $\mathbb{P}^m(\mathbb{R})$:
$$U_i := \left\{ [x^0 : x^1 : \ldots : x^m] : x^i \neq 0 \right\} \subset \mathbb{P}^m(\mathbb{R}), \quad 0 \leq i \leq m.$$
and the maps:
$$\varphi_i : U_i \longrightarrow \mathbb{R}^m,$$
$$\varphi_i([x^0 : \ldots : x^{i-1} : x^i : x^{i+1} : \ldots : x^m]) := \left(\frac{x^0}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^m}{x^i} \right).$$
Note that the pair (U_i, φ_i) is well-defined: if $(x^0, \ldots, x^m) \sim (y^0, \ldots, y^m)$ then $y^j = tx^j$, for some $t \neq 0$. So, $x^i \neq 0$ iff $y^i \neq 0$, and $\frac{x^j}{x^i} = \frac{tx^j}{tx^i} = \frac{y^j}{y^i}$.

We claim that $\mathbb{P}^m(\mathbb{R})$ is a smooth m-dimensional manifold with C^∞-atlas
$$\mathcal{A} = \{(U_i, \varphi_i) : 0 \leq i \leq m\}.$$
We verify the condition (1)-(4) in Proposition 3.1.1. Since each point in $\mathbb{R}^{m+1}\backslash\{0\}$ has at least one non-zero coordinate, it follows that each point in $\mathbb{P}^m(\mathbb{R})$ lies in one of the sets U_i, hence condition (1) is satisfied
$$\mathbb{P}^m(\mathbb{R}) = \bigcup_{0 \leq i \leq m} U_i.$$
Condition (2) follows since $\varphi_i : U_i \to \mathbb{R}^m$ is a bijection with inverse
$$\varphi_i^{-1} : \mathbb{R}^m \longrightarrow U_i, \quad \varphi_i^{-1}(y^1, \ldots, y^m) = [y^1 : \ldots : y^i : 1 : y^{i+1} : \ldots : y^m].$$
For (3), note that
$$\varphi_i(U_i \cap U_j) = \begin{cases} \{y \in \mathbb{R}^m : y^{j+1} \neq 0\}, & \text{if } i > j \\ \{y \in \mathbb{R}^m : y^j \neq 0\}, & \text{if } i < j \\ \mathbb{R}^m, & \text{if } i = j \end{cases}$$
Finally, (4) follows by calculating the transition maps explicitly; for $i < j$:
$$\varphi_j \circ \varphi_i^{-1} : \{y \in \mathbb{R}^m : y^j \neq 0\} \longrightarrow \{y \in \mathbb{R}^m : y^{i+1} \neq 0\}$$
$$\varphi_j \circ \varphi_i^{-1}(y^1, \ldots, y^m) = \left(\frac{y^1}{y^j}, \ldots, \frac{y^i}{y^j}, \frac{1}{y^j}, \frac{y^{i+1}}{y^j}, \ldots, \frac{y^{j-1}}{y^j}, \frac{y^{j+1}}{y^j} \ldots, \frac{y^m}{y^j} \right),$$
and
$$\varphi_i \circ \varphi_j^{-1} : \{y \in \mathbb{R}^m : y^{i+1} \neq 0\} \longrightarrow \{y \in \mathbb{R}^m : y^j \neq 0\}$$
$$\varphi_i \circ \varphi_j^{-1}(y^1, \ldots, y^m) = \left(\frac{y^1}{y^{i+1}}, \ldots, \frac{y^i}{y^{i+1}}, \frac{y^{i+2}}{y^{i+1}}, \ldots, \frac{y^j}{y^{i+1}}, \frac{1}{y^{i+1}}, \frac{y^{j+1}}{y^{i+1}}, \ldots, \frac{y^m}{y^{i+1}} \right).$$

By Proposition 3.1.1, $\mathbb{P}^m(\mathbb{R})$ has a unique topology for which $\mathcal{A}$ is an m-dimensional atlas. Since the atlas is finite, this topology is second countable.

It remains to check that the topology is Hausdorff. Consider two distinct points $p = [p^0 : \ldots : p^m]$, $q = [q^0 : \ldots : q^m] \in \mathbb{P}^m(\mathbb{R})$. Assume first that there exists $0 \leq i \leq m$ such that $p, q \in U_i$. If O_p and O_q are two open sets in $\mathbb{R}^m$ separating $\varphi_i(p)$ and $\varphi_i(q)$, then p and q can be separated by the open sets $\varphi_i^{-1}(O_p)$ and $\varphi_i^{-1}(O_q)$. Assume now that there is no $0 \leq i \leq m$ such that U_i contains both p and q. By (1) there exist $0 \leq i, j \leq m$ such that $p \in U_i$ and $q \in U_j$. Since $p \notin U_j$ and $q \notin U_i$, we have that
$$p^i \neq 0, \quad p^j = 0, \quad q^i = 0, \quad q^j \neq 0.$$

We claim that the following two sets are open and separate p and q:

$$U(p) := \{[x^0 : \ldots : x^m] : |x^i| > |x^j|\}, \quad U(q) := \{[x^0 : \ldots : x^m] : |x^i| < |x^j|\}.$$

Note first that these sets are well-defined. Clearly, $p \in U(p)$, $q \in U(q)$ and $U(p) \cap U(q) = \emptyset$. Next, note that $U(p) \subset U_i$ and that $\varphi_i(U(p))$ is open in $\mathbb{R}^m$. Since φ_i is a homeomorphism, we conclude that $U(p)$ is open. Similarly, $U(q)$ is open. This proves that $\mathbb{P}^m(\mathbb{R})$ is indeed Hausdorff.

3.3. Smooth manifolds as quotients of group actions

New examples of manifolds can be constructed by taking the quotient of a manifold under a group action. We begin with some definitions.

Definition 3.3.1. *Consider an action of a group G on a smooth manifold M:*

$$a : G \times M \longrightarrow M, \quad a(g, x) = g \cdot x.$$

(1) *The action is called **free** if, for all $x \in M$, the equality $g \cdot x = x$ with $g \in G$ implies that $g = e$, where e denotes the identity element in G.*

(2) *The action is called **smooth** (resp. **continuous**) if, for each $g \in G$, the action by g is smooth (resp. continuous):*

$$a_g : M \longrightarrow M, \quad x \mapsto g \cdot x$$

(3) *A continuous action of G on M is called **proper** if every two points $x, y \in M$ have open neighborhoods U_x and U_y such that there are only finitely many $g \in G$ for which $U_x \cap g \cdot U_y \neq \emptyset$.*

(4) *The quotient space M/G is defined as the set of orbits*

$$M/G := \big\{[x] \ : \ x \in M\big\}, \quad [x] := G \cdot x.$$

This set comes with a canonical projection

$$\pi : M \longrightarrow M/G, \quad \pi(x) = [x].$$

(5) *The **quotient topology** on M/G is defined by*

$$U \subset M/G \ \text{ is open} \quad \Longleftrightarrow \quad \pi^{-1}(U) \subset M \ \text{ is open}.$$

In other words, the quotient topology is the largest topology (in the sense that it has the most open sets) for which the projection π is still continuous.

In the following result we construct a smooth manifold structure on the quotient space of a smooth, free and proper action.

Theorem 3.3.2. *Consider a smooth, free and proper action of group G on a smooth m-dimensional manifold M. Then M/G has a unique m-dimensional differentiable structure for which the canonical projection $\pi : M \to M/G$ is a local diffeomorphism.*

In the statement we have used the following concept:

Definition 3.3.3. *We say that a smooth map $f : M \to N$ is a **local diffeomorphism** if each $p \in M$ has an open neighborhood $U \subset M$ such that $f(U) \subset N$ is open and $f|_U : U \to f(U)$ is a diffeomorphism.*

Proof of Theorem 3.3.2. We consider the quotient topology on M/G. First note that π is an open map, i.e. $\pi(V)$ is open M/G for every open set $V \subset M$. This holds because $\pi^{-1}(\pi(V))$ is the union of the open sets $g \cdot V$, $g \in G$.

To see that M/G is second countable, let $\{U_k\}_{k \in \mathbb{N}}$ be a countable basis for the topology of M. Then the collection $\{\pi(U_k)\}_{k \in \mathbb{N}}$ is a basis for the topology of M/G (since π is an open map, these sets are indeed open). Let $V \subset M/G$ be any open set. Then $\pi^{-1}(V) = \cup_{k \in I} U_k$, for some subset $I \subset \mathbb{N}$, and therefore $V = \cup_{k \in I} \pi(U_k)$.

Next, we show that M/G is Hausdorff. Let $[x], [y] \in M/G$ with $[x] \neq [y]$. Since the action is proper, the points have neighborhoods U_x and U_y, respectively, such that U_x and $g \cdot U_y$ intersect only for a finite number of elements $g \in G$; call these elements $g_1, \ldots, g_k$. Since $[x] \neq [y]$ it follows that $g_i y \neq x$, for $1 \leq i \leq k$; thus, since M is Hausdorff, we can find open neighborhoods U_i of $g_i y$ and $U'_x \subset U_x$ of x such that $U_i \cap U'_x = \emptyset$. Let $U'_y := \left(\cap_{i=1}^k g_i^{-1} U_i \right) \cap U_y$. Then the open neighborhoods U'_x and U'_y of x and y, respectively, have the property that $U'_x \cap g U'_y = \emptyset$ for all $g \in G$. This implies that $g U'_x \cap h U'_y = \emptyset$ for all $g, h \in G$; or equivalently, $\pi(U'_x) \cap \pi(U'_y) = \emptyset$. Since π is an open map, the sets $\pi(U'_x)$ and $\pi(U'_y)$ are disjoint open neighborhoods of $[x]$, respectively of $[y]$. Thus M/G is Hausdorff.

Next, we construct charts on M/G. Consider $[x] \in M/G$. Since the action is proper, there is an open neighborhood U of x such that $g \cdot U$ intersects U only for a finite number of elements $g \in G$; call these elements $g_0 := e, g_1, \ldots, g_k$. Since the action is free, $g_i \cdot x \neq x$, for $1 \leq i \leq k$; thus we can find open neighborhoods U_i of $g_i \cdot x$ and $U' \subset U$ of x such that $U_i \cap U' = \emptyset$. Let $V := \left(\cap_{i=1}^k g_i^{-1} U_i \right) \cap U'$. This open neighborhood of x satisfies $V \cap g \cdot V = \emptyset$, for all $g \neq e$. This implies that $\pi|_V : V \to \pi(V)$ is injective. By shrinking V, we may assume that it is the domain of a chart (V, φ) on M. We define the following chart on M/G:

$$\varphi \circ (\pi|_V)^{-1} : \pi(V) \longrightarrow \mathbb{R}^m.$$

Since π is open, it follows that $\pi|_V : V \to \pi(V)$ is a homeomorphism; hence $(\pi(V), \varphi \circ (\pi|_V)^{-1})$ is indeed a chart on M/G. Consider the collection of all charts obtained as above:

$$\mathcal{A} := \left\{ (\pi(V), \varphi \circ (\pi|_V)^{-1}) : (V, \varphi) \text{ is a smooth chart on } M \text{ s.t. } \pi|_V \text{ is injective} \right\}.$$

Above we have constructed a chart around each $[x] \in M/G$; thus the open sets in $\mathcal{A}$ cover M/G. Finally, we need to prove that the transition maps are smooth. Let (V, φ) and (W, ψ) be two smooth charts on M such that $\pi|_V$ and $\pi|_W$ are injective. We need to show that the following map is a diffeomorphism:

$$\varphi \circ (\pi|_V)^{-1} \circ (\pi|_W) \circ \psi^{-1} : \psi(V \cap \pi^{-1}(\pi(W))) \longrightarrow \varphi(\pi^{-1}(\pi(V)) \cap W).$$

Let $p \in \pi(V) \cap \pi(W)$, and let $x \in V$ and $y \in W$ such that $[x] = p = [y]$. We show that the transition map is smooth in a neighborhood of $\psi(y)$. Since $[x] = [y]$, there exists $g \in G$ such that $x = g \cdot y$. Let $U := g^{-1} \cdot V \cap W$. On U, we claim that the following equality holds:

$$(\pi|_V)^{-1} \circ \pi|_U = a_g|_U : U \longrightarrow V.$$

By composing on both sides with $\pi|_V$, the equality clearly holds; and so the claim follows by injectivity of $\pi|_V$. Hence, on $\psi(U)$, the transition map is a composition of diffeomorphisms

$$\varphi \circ a_g \circ \psi^{-1} : \psi(U) \longrightarrow \varphi(g \cdot U);$$

therefore it is itself a diffeomorphism. This finishes the proof that $\mathcal{A}$ is a smooth atlas, and so M/G is a smooth m-dimensional manifold.

Finally, let us show that $\pi : M \to M/G$ is a local diffeomorphism. As we have seen above, around any point in M we can find a chart (V, φ) such that $\pi|_V$ is injective. The local representation of π in the charts (V, φ) and $(\pi(V), \varphi \circ (\pi|_V)^{-1})$ is simply given by the identity map of the set $\varphi(V)$. This implies that π is smooth, and $\pi|_V$ is a diffeomorphism; hence π is a local diffeomorphism.

Denote by $\mathcal{T}$ the quotient topology and by $\mathcal{D}$ the differentiable structure constructed above. Next, we show that $\mathcal{D}$ and $\mathcal{T}$ are unique such that π is a local diffeomorphism. Consider a second topology $\mathcal{T}'$ and a second differentiable structure $\mathcal{D}'$ on M/G such that π is a local diffeomorphism.

First we show that $\mathcal{T} = \mathcal{T}'$. Since $\pi : M \to M/G$ is continuous, it follows that $\mathcal{T}' \subset \mathcal{T}$. Since local diffeomorphisms are open maps, $\pi(U) \in \mathcal{T}'$ for any open set U in M. Thus, if $V \in \mathcal{T}$ then $\pi^{-1}(V)$ is open in M, hence $V = \pi(\pi^{-1}(V)) \in \mathcal{T}'$. This proves that $\mathcal{T} \subset \mathcal{T}'$; hence the two topologies coincide.

Since π is a local diffeomorphism for $\mathcal{D}'$, every point has an open neighborhood V such that $\pi(V)$ is open and $\pi|_V : V \to \pi(V)$ is a diffeomorphism. By shrinking V, we may assume that it is the domain of a smooth chart (V, φ) on M. Hence, $\varphi \circ (\pi|_V)^{-1} : \pi(V) \to \mathbb{R}^m$ is a diffeomorphism for $\mathcal{D}'$, and so $(\pi(V), \varphi \circ (\pi|_V)^{-1})$ is a smooth chart for $\mathcal{D}'$. The collection of such charts covers M/G, hence it is an atlas $\mathcal{A}'$ representing $\mathcal{D}'$. On the other hand, $\mathcal{A}' \subset \mathcal{A}$, and so $\mathcal{D} = \mathcal{D}'$. $\qquad\square$

3.4. Manifolds of dimensions 1 and 2

Without providing proofs, in this section we discuss some classification results for manifolds in dimension one and two.

Theorem ♣ 3.4.1. *Any connected 1-dimensional manifold is diffeomorphic to either $\mathbb{R}$ or S^1.*

Two-dimensional smooth manifolds, also called **surfaces**, are more complicated. For example every open subset U of $\mathbb{R}^2$ is a surface; just think about the cases $U = \mathbb{R}^2 \backslash \mathbb{Z}^2$, or U is the complement of the Cantor set.

Nevertheless, **compact** surfaces have a simple classification; we introduce these basic examples in a less-rigorous fashion.

The **genus-g surface**, denoted by Σ_g, is a surface which looks like the one in the picture, where $g \geq 0$ stands for the number of holes:

$$\Sigma_g$$
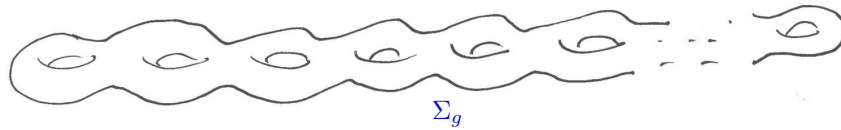
For $g = 0$, we obtain the 2-sphere S^2 and for $g = 1$ the 2-torus T^2:

$$\Sigma_0 \cong S^2 \qquad\qquad \Sigma_1 \cong T^2$$
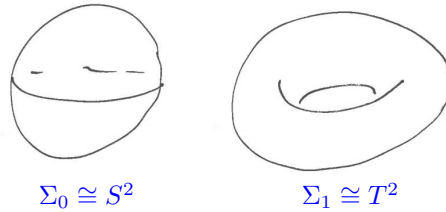
Place Σ_g in $\mathbb{R}^3$ such that it is symmetric with respect to the three coordinate planes; here is the picture for Σ_3:

Σ_3
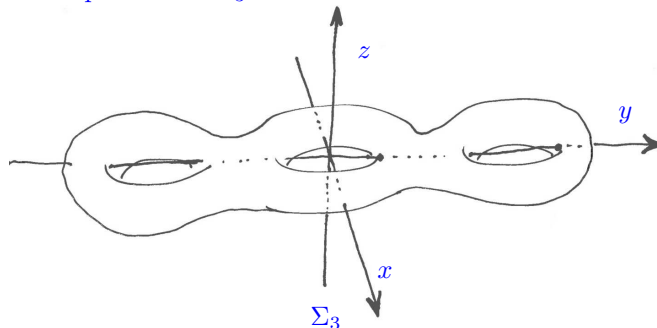
In this position, the group $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$ acts freely on Σ_g:

$$\hat{0} \cdot (x, y, z) = (x, y, z), \quad \hat{1} \cdot (x, y, z) = (-x, -y, -z).$$

The quotient space $\Sigma_g / \mathbb{Z}_2$ inherits therefore a smooth structure. For $g = 0$, $S^2 / \mathbb{Z}_2$ is the projective space (see Exercise 3.7), and for $g = 1$, $T^2 / \mathbb{Z}_2$ is the Klein bottle (described also in Exercise 3.9).

We are ready to state the classification theorem for compact surfaces:

Theorem ♣ 3.4.2. *Any compact connected 2-dimensional manifold is diffeomorphic to either Σ_g or $\Sigma_g / \mathbb{Z}_2$, for some $g \geq 0$.*

The manifolds in the list Σ_g, $\Sigma_g / \mathbb{Z}_2$, for $g \geq 0$, are non-diffeomorphic. Let us mention that, the surfaces Σ_g are the orientable ones, and the surfaces $\Sigma_g / \mathbb{Z}_2$ are the non-orientable ones; the notion of orientable manifold will be introduced in a later lecture.

3.5. Exercises

In the following exercise we construct a non-example of a manifold; a non-Hausdorff second countable topological space with a smooth 1-dimensional atlas.

Exercise 3.1 (Two intervals glued along an open subinterval). Consider the set:

$$M = (-1, 1) \cup [2, 3).$$

Let $U = (-1, 1)$ and $V = (-1, 0) \cup [2, 3)$. On these subsets of M define the maps

$$\varphi : U \to \mathbb{R}, \quad \varphi(t) = t, \quad \text{and}$$

$$\psi : V \to \mathbb{R}, \quad \psi(t) = \begin{cases} t, & t \in (-1, 0) \\ t - 2, & t \in [2, 3) \end{cases}$$

Show that $\mathcal{A} = \{(U, \varphi), (V, \psi)\}$ satisfies the conditions (1)-(4) of Proposition 3.1.1. Show that the topology $\mathcal{T}_{\mathcal{A}}$ on M is non-Hausdorff.

Exercise 3.2. Let L be the set of all lines in $\mathbb{R}^2$. For $a, b, c \in \mathbb{R}$, with $a \neq 0$ or $b \neq 0$, denote by $l(a, b, c)$ the line given by the equation

$$ax + by + c = 0.$$

Let $U \subset L$ be the set of lines $l(a, b, c)$ with $a \neq 0$, and let $V \subset L$ be the set of lines $l(a, b, c)$ with $b \neq 0$. Define the maps

$$\varphi : U \to \mathbb{R}^2, \quad \varphi(l(a, b, c)) = (b/a, c/a), \quad \psi : V \to \mathbb{R}^2, \quad \psi(l(a, b, c)) = (a/b, c/b).$$

Prove that L is a smooth 2-dimensional manifold with atlas $\mathcal{A} = \{(U, \varphi), (V, \psi)\}$.

Exercise 3.3. Let S^1 denote the unit circle in $\mathbb{C}$

$$S^1 := \{z = x + iy \ : \ |z| = \sqrt{x^2 + y^2} = 1\} \subset \mathbb{C}.$$

Let M be the collection of subsets of S^1 with 2 elements:

$$M := \{\{z, w\} \ : \ z, w \in S^1, \ z \neq w\}.$$

(Pay attention to the fact that $\{z, w\} = \{w, z\}$!!)

(a) Consider the set

$$V := \{(\alpha, \beta) \ : \ 0 < \alpha < \beta < 2\pi\} \subset \mathbb{R}^2,$$

and for $k \in \{0, 1, 2\}$ consider the map $\psi_k : V \to M$ given by

$$\psi_k(\alpha, \beta) := \{e^{i(\alpha + \frac{2\pi k}{3})}, e^{i(\beta + \frac{2\pi k}{3})}\}.$$

Prove that ψ_k is injective.

(b) Let $U_k := \psi_k(V)$, and let $\varphi_k := \psi_k^{-1} : U_k \to V$. Prove that M has the structure of a smooth 2-dimensional manifold, for which

$$\mathcal{A} := \{(U_k, \varphi_k) \ : \ k \in \{0, 1, 2\}\}$$

is a C^∞-atlas. *(Recall that $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$).*

Exercise 3.4. Show that the manifolds L and M constructed in Exercise 3.2 and 3.3, respectively, are diffeomorphic. *Note that both are a Möbius band:*
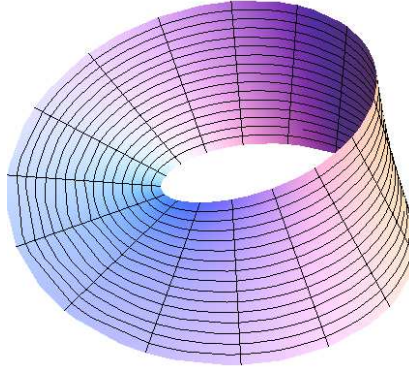
The following exercise generalizes the construction of the projective space.

Exercise 3.5. The **Grassmannian** $\mathrm{Gr}(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. We construct a smooth manifold structure on $\mathrm{Gr}(k, n)$ of dimension $k \cdot (n - k)$.

Let $\{e_i \ : \ 1 \leq i \leq n\}$ denote the standard basis of $\mathbb{R}^n$. Consider a subset I of $\{1, 2, \ldots, n\}$ with k elements. Denote the elements of I and of its complement increasingly as follows:

$$I = \{i_1 < i_2 < \ldots < i_k\}, \quad \{1, \ldots, n\}\backslash I := \{j_1 < j_2 < \ldots < j_{n-k}\}.$$

For each such subset I we construct the following map

$$\chi_I : \mathbb{R}^{k \times (n-k)} \longrightarrow \mathrm{Gr}(k, n)$$

$$\chi_I\left(A\right) = \mathrm{Span}\Big\langle e_{i_u} + \sum_{v=1}^{n-k} a_{u,v} e_{j_v} : 1 \leq u \leq k \Big\rangle, \quad \text{for } A = (a_{u,v})_{1 \leq u \leq k, 1 \leq v \leq n-k},$$

where Span denotes the linear span of a subset of $\mathbb{R}^n$.

(a) Prove that χ_I is injective.
(b) Let $U_I := \chi_I(\mathbb{R}^{k \times (n-k)})$ and let $\varphi_I := \chi_I^{-1} : U_I \to \mathbb{R}^{k \times (n-k)}$.
Prove that $\mathrm{Gr}(k,n)$ is a smooth $k \times (n-k)$-dimensional manifold with atlas:

$$\mathcal{A} = \{(U_I, \varphi_I) \; : \; I = \{1 \leq i_1 < i_2 < \ldots < i_k \leq n\}\}.$$

(c) Prove that the manifolds $\mathrm{Gr}(k,n)$ and $\mathrm{Gr}(n-k,n)$ are diffeomorphic.

The following example is from [**14**].

Exercise 3.6. A continuous action of a group G on a manifold M is called **wandering** if every point has a neighborhood U such that $gU \cap U = \emptyset$, for all $g \neq e$.
Consider the action of $(\mathbb{Z}, +)$ on $M := \mathbb{R}^2 \backslash \{0\}$ given by

$$n \cdot (x,y) = (2^n x, 2^{-n} y).$$

(a) Prove that this action is wandering, but not proper.
(b) Consider the quotient topology on $M/\mathbb{Z}$. Prove that $M/\mathbb{Z}$ is second countable, but not Hausdorff.
(c) Prove that $M/\mathbb{Z}$ has a smooth 2-dimensional atlas, i.e. a collection of charts $\mathcal{A}$ satisfying conditions (1)-(4) from Proposition 3.1.1.

In the following exercise we construct the projective plane as a quotient of the sphere.

Exercise 3.7. Consider the following action of $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$ on S^m:

$$\hat{0} \cdot x = x, \quad \hat{1} \cdot x = -x.$$

Prove that the action is smooth, proper and free. Prove that $S^m/\mathbb{Z}_2$ is diffeomorphic to $\mathbb{P}^m(\mathbb{R})$.

In the following exercise we construct the m-dimensional torus.

Exercise 3.8. Consider the action of the group $(\mathbb{Z}^m, +)$ on $\mathbb{R}^m$ given by:

$$(k^1, \ldots, k^m) \cdot (x^1, \ldots, x^m) := (k^1 + x^1, \ldots, k^m + x^m).$$

Prove that the action is smooth, proper and free. The quotient manifold

$$T^m := \mathbb{R}^m/\mathbb{Z}^m = \{x + \mathbb{Z}^m : x \in \mathbb{R}^m\}$$

is called the m-**dimensional torus**. Prove that T^m is diffeomorphic to the product of m-copies on S^1 (see Proposition 2.1.6 for the construction of the product).

By using actions on $\mathbb{R}^2$, in the following exercise we construct several 2-dimensional manifolds: the cylinder, the Möbius band, the torus and the Klein bottle.

Exercise 3.9. Consider the following two diffeomorphisms of $\mathbb{R}^2$:

$$a, b : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad a(x,y) = (x, y+1), \quad b(x,y) = (x + 1/2, -y).$$

Let $\mathrm{Diff}(\mathbb{R}^2)$ be the group of diffeomorphisms of $\mathbb{R}^2$. Let G_a, G_b, G_{a,b^2} and $G_{a,b}$, respectively, denote the subgroups of $\mathrm{Diff}(\mathbb{R}^2)$ generated by the sets $\{a\}$, $\{b\}$, $\{a, b^2\}$ and $\{a, b\}$, respectively.

(a) Prove that the following relation holds: $ba = a^{-1}b$. Using this relation, show that every element in $G_{a,b}$ can be represented as $a^n b^m$, for unique $n, m \in \mathbb{Z}$. Prove that the action of all four groups G_a, G_b, G_{a,b^2} and $G_{a,b}$ on $\mathbb{R}^2$ is smooth, free and proper.

(b) Prove that the quotient $\mathbb{R}^2/G_a$ is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

(c) Prove that $\mathbb{R}^2/G_{a,b^2}$ is diffeomorphic to the 2-torus T^2 (from Exercise 3.8).

(d) Prove that $\mathbb{R}^2/G_b$ is diffeomorphic to the manifold from Exercise 3.2. *(A manifold diffeomorphic to $\mathbb{R}^2/G_b$ is called a **Möbius band**; see also Exercise 3.4.)*

(e) Prove that G_{a,b^2} is an index two subgroup of $G_{a,b}$, i.e. $G_{a,b}/G_{a,b^2} \cong \mathbb{Z}_2$. Using this, define a smooth, free and proper action of $\mathbb{Z}_2$ on T^2 such that $T^2/\mathbb{Z}_2$ is diffeomorphic to $\mathbb{R}^2/G_{a,b}$. *(A manifold diffeomorphic to $\mathbb{R}^2/G_{a,b}$ is called a **Klein bottle**.)*

Exercise 3.10. Let $a : G \times M \to M$ be a proper smooth action, and assume that M is compact. Prove that G is finite.

LECTURE 4

4.1. Bump functions

Recall the following topological notion:

Definition 4.1.1. *Let X be a topological space, and let $\chi : X \to \mathbb{R}$ be a function. The **support** of χ is defined as the closure of the set where χ is not zero:*

$$\operatorname{supp}(\chi) := \overline{\{x \in X : \chi(x) \neq 0\}}.$$

Lemma 1.1.11 shows that around any point in $\mathbb{R}^n$ we can find a smooth **bump function**, i.e. a function with support in a fixed neighborhood of the point, and which is equal to the constant function 1 on a smaller neighborhood. This can be easily generalized to manifolds.

Lemma 4.1.2. *Let M be a smooth manifold, let $p \in M$ and let $U \subset M$ be an open neighborhood of p. Then there exists a smooth function $\chi : M \to [0, 1]$ such that $\operatorname{supp}(\chi) \subset U$ and $\chi|_V = 1$, where V is some neighborhood of p.*

Proof. Let (W, φ) be a chart with $p \in W \subset U$, and let $\epsilon > 0$ be such that the closed ball $\overline{B}_\epsilon$ around $\varphi(p)$ is included in $\varphi(W)$. By Lemma 1.1.11, there exists a smooth function $\theta : \mathbb{R}^m \to [0, 1]$ such that $\theta|_{B_{\epsilon/2}} = 1$ and $\operatorname{supp}(\theta) \subset \overline{B}_\epsilon$. Define

$$\chi : M \to [0, 1], \quad \chi|_W = \theta \circ \varphi, \quad \chi|_{M \setminus \varphi^{-1}(\overline{B}_\epsilon)} = 0.$$

Note that χ is well-defined: on the overlap of the sets W and $M \setminus \varphi^{-1}(\overline{B}_\epsilon)$, we have:

$$\theta \circ \varphi|_{W \setminus \varphi^{-1}(\overline{B}_\epsilon)} = 0.$$

Since χ is smooth on these two open sets that cover M, it follows that χ is smooth on M. The conclusion holds with $V := \varphi^{-1}(B_{\epsilon/2})$. $\qquad\square$

4.2. Partitions of unity

Partitions of unity play an important technical role in the theory of smooth manifolds, because they allow to build global objects by gluing together locally defined objects. In this section we prove the existence of partitions of unity.

We start by introducing the necessary terminology.

Definition 4.2.1. *Let X be a topological space.*

(1) An **open cover** of a subset $Y \subset X$ is a family of open subsets of X

$$\mathcal{U} = \{U_i\}_{i \in I}, \quad \text{such that} \quad Y \subset \cup_{i \in I} U_i,$$

where I is an index set.

(2) A **subcover** of an open cover $\mathcal{U}$ of Y is a subfamily $\mathcal{V} \subset \mathcal{U}$ which is still a cover of Y.

(3) A subset $Y \subset X$ is called **compact**, if any open cover of Y has a finite subcover.

(4) A **refinement** of an open cover $\mathcal{U}$ of Y is a second open cover $\mathcal{V}$ of Y such that for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$.

(5) A family $\mathcal{S} = \{S_i\}_{i \in I}$ of subsets of X is called **locally finite** if every $x \in X$ has an open neighborhood V which intersects only finitely many elements of $\mathcal{S}$.

(6) A family of functions $\{\rho_i : X \to \mathbb{R}\}_{i \in I}$ is called **locally finite** if the family of supports $\{\text{supp}(\rho_i)\}_{i \in I}$ is locally finite.

The following will be often used:

Lemma 4.2.2. *Let M be a smooth manifold, and consider a locally finite family of smooth functions on M, $\{\rho_i \in C^\infty(M)\}_{i \in I}$. Since at every point $x \in M$ there are only a finite number of $i \in I$ such that $\rho_i(x) \neq 0$, we can define:*

$$\rho = \sum_{i \in I} \rho_i.$$

The function ρ is smooth.

Proof. Let $x \in M$. Since the family is locally finite, x has an open neighborhood U which intersects only the supports of a finite number of functions, say $\rho_{i_1}, \ldots, \rho_{i_n}$. Therefore, on U we have that:

$$\rho|_U = \sum_{k=1}^{n} \rho_{i_k}|_U.$$

The sum of a finite number of smooth functions is smooth, hence $\rho|_U$ is smooth. So, every point has a neighborhood on which ρ is smooth; thus ρ is smooth on M. $\quad \square$

In order to define the function ρ in the lemma above, it suffices that at every point only a finite number of functions are non-zero. However, this condition does not insure smoothness of the resulting function; in Exercise 4.1, you are asked to provide an example.

The main notion of this section is the following:

Definition 4.2.3. *A **partition of unity** on a smooth manifold M is a locally finite family of smooth functions*

$$\{\rho_i\}_{i \in I}, \quad \rho_i \in C^\infty(M), \quad \text{for } i \in I,$$

such that, for all $x \in M$, $0 \leq \rho_i(x) \leq 1$ and

$$\sum_{i \in I} \rho_i = 1.$$

*The partition of unity $\{\rho_i\}_{i \in I}$ is said to be **subordinate** to an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ if $\text{supp}(\rho_i) \subset U_i$ for all $i \in I$.*

The following is the main result about partitions of unity:

Theorem 4.2.4. *Any open cover of a smooth manifold has a partition of unity subordinate to it.*

4.3. Proof of Theorem 4.2.4

The proof is divided into three main steps. First, we show that it suffices to prove the result for a refinement of the cover (Lemma 4.3.1); next, we show that open covers of manifolds admit locally finite refinements whose elements have compact closures (Lemma 4.3.4; this result implies that manifolds are **paracompact**, see Remark 4.3.5); and finally, we prove Theorem 4.2.4 for open covers with these properties.

Lemma 4.3.1. *Let $\mathcal{V}$ be a refinement of the open cover $\mathcal{U}$. If $\mathcal{V}$ has a partition of unity subordinate to it, then so does $\mathcal{U}$.*

Proof. Denote $\mathcal{V} = \{V_i\}_{i \in I}$ and $\mathcal{U} = \{U_j\}_{j \in J}$. The fact that $\mathcal{V}$ is a refinement of $\mathcal{U}$ implies that there exists a function $f : I \to J$ (axiom of choice) such that $V_i \subset U_{f(i)}$. Let $\{\rho_i\}_{i \in I}$ be a partition of unity subordinate to $\mathcal{V}$. Since the family of supports is locally finite, the following functions are smooth: $\chi_j := \sum_{i \in f^{-1}(j)} \rho_i$ (if $f^{-1}(j) = \emptyset$, we set $\chi_j = 0$). We claim that $\{\chi_j\}_{j \in J}$ is a partition of unity subordinate to $\mathcal{U}$. First, note that:

$$(*) \qquad \mathrm{supp}(\chi_j) \subset \bigcup_{i \in f^{-1}(j)} \mathrm{supp}(\rho_i) \subset \bigcup_{i \in f^{-1}(j)} V_i \subset U_j.$$

Next, we show next that the family $\{\chi_j\}_{j \in J}$ is locally finite. Let $p \in M$, and consider O an open neighborhood of p such that O intersects only a finite number of the sets $\mathrm{supp}(\rho_i)$; denote these indexes by $i_1, \ldots, i_n \in I$. If $q \in \mathrm{supp}(\chi_j)$, then by $(*)$, $q \in \mathrm{supp}(\rho_i)$ for some i satisfying $f(i) = j$. Therefore, O intersects only the sets $\mathrm{supp}(\chi_j)$, for $j = f(i_1), \ldots, f(i_k)$. Finally, it is clear that:

$$\sum_{j \in J} \chi_j = \sum_{j \in J} \sum_{i \in f^{-1}(j)} \rho_i = \sum_{i \in I} \rho_i = 1.$$

$\square$

The following two results are used in the proof of Lemma 4.3.4.

Lemma 4.3.2. *The topology of a manifold has a countable basis all of whose elements have compact closures.*

Proof. Let $\mathcal{B}$ be a countable basis for M, and let $\mathcal{B}_c$ be the collection of sets in $\mathcal{B}$ with compact closure. Clearly, $\mathcal{B}_c$ is countable. We prove that $\mathcal{B}_c$ is a basis. Let $O \subset M$ be an open set. For $p \in O$, let $K_p \subset O$ be a compact neighborhood (e.g. let K_p be the preimage by a chart around p of a small closed ball). Since $\mathcal{B}$ is a basis, there exists $U_p \in \mathcal{B}$ such that $p \in U_p \subset \mathrm{int}(K_p) \subset O_p$. In particular: $U_p \in \mathcal{B}_c$ and $U_p \subset O$. Thus, we can write $O = \cup_{p \in O} U_p$, which shows that $\mathcal{B}_c$ is a basis. $\square$

Lemma 4.3.3. *Any manifold M has an open cover $\{G_k\}_{k=1}^{\infty}$ such that, for $k \geq 1$, $\overline{G}_k$ is compact and $\overline{G}_k \subset G_{k+1}$.*

Proof. By Lemma 4.3.2, there exists a countable basis $\mathcal{B} = \{B_n\}_{n \geq 1}$ for the topology of M such that $\overline{B}_n$ is compact for all $n \geq 1$. Let $O_m := \cup_{n=1}^{m} B_n$. Note that $O_m \subset O_{m+1}$, $\overline{O}_m = \cup_{n=1}^{m} \overline{B}_n$ is compact, and that $\cup_{m \geq 1} O_m = M$. Therefore, for any $m \geq 1$ there is a smallest integer $f(m) > m$ such that $\overline{O}_m \subset O_{f(m)}$. Let $m_1 := 1$, and define inductively for $k \geq 2$, $m_k := f(m_{k-1})$. The sequence $G_k := O_{m_k}$ satisfies all requirements. $\square$

Lemma 4.3.4. *Any open cover $\mathcal{U}$ of a manifold M has a refinement $\mathcal{V}$ which is at most countable, locally finite, and whose elements have compact closures.*

Proof. Denote $\mathcal{U} = \{U_i\}_{i \in I}$. Consider a sequence of open sets $\{G_k\}_{k=1}^{\infty}$ as in Lemma 4.3.3, and denote $G_{-1} = G_0 = \emptyset$. For each $k \geq 1$, the family
$$\{U_i \cap (G_{k+1} \backslash \overline{G}_{k-2})\}_{i \in I}$$
is an open cover of the compact set $\overline{G}_k \backslash G_{k-1}$, therefore it has a finite subcover $\mathcal{V}_k$. Let $\mathcal{V} := \cup_{k \geq 1} \mathcal{V}_k$. We prove that $\mathcal{V}$ has the required properties. Clearly, $\mathcal{V}$ is an open cover because it covers $\overline{G}_k \backslash G_{k-1}$, and it is a refinement of $\mathcal{U}$ because each element is contained in some U_i. Since, for each $k \geq 1$, $\mathcal{V}_k$ contains only a finite number of open sets, $\mathcal{V}$ is at most countable. Every set in $\mathcal{V}_k$ has compact closure because it is contained in the compact $\overline{G}_{k+1}$. Finally, let $p \in M$ and $l \geq 1$ be such that $p \in G_l$. Then G_l intersects at most elements in $\mathcal{V}_k$ with $k \leq l + 1$. This shows that $\mathcal{V}$ is locally finite. $\qquad\square$

Let us make a general remark about what we have proven so far.

Remark 4.3.5. A simple analysis of the proofs of the previous three Lemmas (4.3.2,4.3.3,4.3.4) shows that we have used only the following topological properties of a manifold:

- second countable;
- Hausdorff (in particular, compact sets are closed);
- locally compact (i.e. every point has a compact neighborhood).

A topological space for which every open cover has a locally finite refinement is called **paracompact**. Thus we have proven that: *Any topological space which is second countable, Hausdorff and locally compact is paracompact.*

Finally, we show that open covers which have the properties from Lemma 4.3.4 admit partitions of unity, and therefore we complete the proof of Theorem 4.2.4.

Lemma 4.3.6. *Let $\mathcal{V}$ be a locally finite open cover of M whose elements have compact closures. Then there exists a partition of unity subordinate to $\mathcal{V}$.*

Proof. Let $V \in \mathcal{V}$. By Lemma 4.1.2, for each $p \in V$ there is a smooth function $\psi_p^V : M \to [0,1]$ such that $\psi_p^V(p) = 1$ and $\text{supp}(\psi_p^V) \subset V$. Denote $W_p^V := \{x \in M : \psi_p^V(x) > 0\}$. Then the collection $\mathcal{W} := \{W_p^V : p \in V \in \mathcal{V}\}$ is an open cover of M which is a refinement of $\mathcal{V}$. By Lemma 4.3.4 $\mathcal{W}$ has a locally finite refinement $\mathcal{O}$. So, for $O \in \mathcal{O}$ there is a smooth function $\psi^O : M \to [0,1]$ and $V \in \mathcal{V}$ such that
$$\psi^O|_O > 0 \ \text{ and } \ \text{supp}(\psi^O) \subset V.$$

We show that an element $V \in \mathcal{V}$ intersects only finitely many elements in $\mathcal{O}$. Since $\mathcal{O}$ is locally finite, $\overline{V}$ can be covered by open sets which intersect only finitely many elements in $\mathcal{O}$; and by compactness of $\overline{V}$, we may extract a finite subcover. Let U be the union of the elements of this subcover. Then $\overline{V} \subset U$ and U intersects only finitely many elements in $\mathcal{O}$.

In particular, for any $V \in \mathcal{V}$ the set
$$\mathcal{O}^V := \{O \in \mathcal{O} \ : \ \text{supp}(\psi^O) \subset V\}$$
is finite. Consider the family of smooth maps
$$\psi^V := \sum_{O \in \mathcal{O}^V} \psi^O : M \to [0, \infty), \quad V \in \mathcal{V}.$$

Because it is subordinate to $\mathcal{V}$, it follows that the family is locally finite; and by the properties of $\mathcal{O}$, at every point in M at least one of these functions is positive. Therefore $\psi := \sum_{V \in \mathcal{V}} \psi^V > 0$. The family of smooth functions $\rho_V := \psi_V / \psi$, $V \in \mathcal{V}$ is a partition of unity subordinate to $\mathcal{V}$. $\qquad\square$

4.4. Corollaries of Theorem 4.2.4

For covers with two elements, Theorem 4.2.4 gives a smooth *Urysohn's lemma*:

Corollary 4.4.1. *Let $A, B \subset M$ be two disjoint closed sets. There exists a smooth function $f : M \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.*
    In particular, M is a normal topological space.

Proof. Consider the open cover $\{M \backslash A, M \backslash B\}$. If $\{f, g\}$ is a partition of unity subordinated to it, then f satisfies the condition from the statement. Finally, M is normal, because A and B can be separated by the open sets $\{x \ : \ f(x) < 1/2\}$ and $\{x \ : \ f(x) > 1/2\}$, respectively. $\qquad\square$

Theorem 4.2.4 can be used to extend smooth functions defined on closed embedded submanifolds in $\mathbb{R}^n$.

Corollary 4.4.2. *Let $M \subset \mathbb{R}^n$ be an embedded manifold in $\mathbb{R}^n$ which is closed as a subset. For every $f \in C^\infty(M)$ there exists $\widetilde{f} \in C^\infty(\mathbb{R}^n)$ such that $\widetilde{f}|_M = f$.*

Proof. Recall that around every point in M there is a diffeomorphism $\varphi : U \to V$, with $V, U \subset \mathbb{R}^n$ open, such that

$$\varphi(M \cap U) = (\mathbb{R}^m \times \{0\}) \cap V.$$

We cover M by a family of such charts $\{(U_i, \varphi_i)\}_{i \in I}$. Consider the following open cover: $\{\mathbb{R}^n \backslash M\} \cup \{U_i\}_{i \in I}$, and let $\{\rho\} \cup \{\rho_i\}_{i \in I}$ be a subordinate partition of unity. For each $i \in I$, consider the following local extension of f:

$$\widetilde{f}_i := f \circ (\varphi_i^{-1} \circ \mathrm{pr}_m^n \circ \varphi_i) \in C^\infty(U_i),$$

and consider also the function:

$$g_i \in C^\infty(M), \quad g_i(x) := \begin{cases} \rho_i(x)\widetilde{f}_i, & x \in U_i \\ 0, & x \notin \mathrm{supp}(\rho_i) \end{cases}$$

The family $\{g_i\}_{i \in I}$ is locally finite, and it is easy to see that the sum $\widetilde{f} := \sum_{i \in I} g_i$ satisfies $\widetilde{f}|_M = f$. $\qquad\square$

Recall the following notion:

Definition 4.4.3. *A continuous function $f : X \to Y$ between topological spaces X and Y is called **proper**, if for any compact subset $K \subset Y$, we have that $f^{-1}(K) \subset X$ is compact.*

Corollary 4.4.4. *On any manifold M there exists a proper smooth function $h : M \to [0, \infty)$.*

Proof. Let $\{\rho_k\}_{k \geq 1}$ be a partition of unity subordinated to a locally finite, at most countable cover $\{V_k\}_{k \geq 1}$, with $\overline{V}_k$ compact (as in Lemma 4.3.4). Then the function $h := \sum_{k \geq 1} k\rho_k$ is a positive proper function. $\qquad\square$

4.5. Exercises

Exercise 4.1. Construct a sequence of functions $\rho_k \in C^\infty(\mathbb{R})$, $k = 1, 2, \ldots$, such that at every $x \in \mathbb{R}$, $\rho_k(x) \neq 0$ only for a finite number of k's, but such that the function $\sum_{k \geq 1} \rho_k$ is not smooth.

Exercise 4.2. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be two open covers of M. Show that $\mathcal{W} := \{U_i \cap V_j\}_{(i,j) \in I \times J}$ is an open cover of M. If $\{\rho_i\}_{i \in I}$ is a partition of unity subordinate to $\mathcal{U}$ and $\{\chi_j\}_{j \in J}$ is a partition of unity subordinate to $\mathcal{V}$, prove that $\{\rho_i \chi_j\}_{(i,j) \in I \times J}$ is a partition of unity subordinate to $\mathcal{W}$.

The locally finite cover found in Lemma 4.3.4 is at most countable. However, this fact is automatic.

Exercise 4.3. Let $\mathcal{U}$ be a locally finite open cover of a manifold M.

(a) If M is compact, prove that $\mathcal{U}$ has a finite number of elements.
(b) In general, prove that $\mathcal{U}$ is at most countable.
   Hint: choose a second cover $\mathcal{V}$, as in Lemma 4.3.4, and show that every element in $\mathcal{V}$ hits only a finite number of elements of $\mathcal{U}$.

Exercise 4.4. Let $\{\rho_i\}_{i \in I}$ be a locally finite family of smooth functions on M.

(a) If M is compact, prove that only a finite number of functions are not identically zero.
(b) In general, prove that the set of functions which are not identically zero is at most countable.

LECTURE 5

Remark. For simplicity, from now on we will usually suppress the adjective *smooth*, by implicitly assuming smoothness of the objects involved: by *manifold* we mean *smooth manifold*, by *chart* we mean *smooth chart*, etc.

5.1. The tangent space

Intuitively, the tangent space of a smooth manifold M at a point p, is a vector space denoted by T_pM of dimension $m = \dim(M)$, which represents all velocities of curves passing through p.

Example 5.1.1. For example, the tangent space to the 2-sphere S^2 at a point $p \in S^2$ can be identified with the set of vectors perpendicular to p: $T_pS^2 \cong \{v \in \mathbb{R}^3 \ : \ v \cdot p = 0\}$. Namely, any smooth curve $\gamma : (-\epsilon, \epsilon) \to S^2$ (with $\gamma(0) = p$) satisfies $\gamma(t) \cdot \gamma(t) = 1$, hence taking the derivative, $\gamma'(t) \cdot \gamma(t) = 0$; in particular $v \cdot p = 0$, where $v = \gamma'(0)$. On the other hand, given a local parameterization $f : O \subset \mathbb{R}^2 \to S^2$ (i.e. the inverse of a local chart) with $f(q) = p$, we can describe the tangent space as the image of the differential of f:

$$T_pS^2 \cong (\mathrm{d}_q f)(\mathbb{R}^2).$$

Example 5.1.2. The tangent space of $\mathbb{R}^m$ at $p \in \mathbb{R}^m$ can be canonically identified $T_p\mathbb{R}^m \cong \mathbb{R}^m$: any smooth curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^m$ with $\gamma(0) = p$ induces the tangent vector $\gamma'(0) \in \mathbb{R}^m$. Conversely, any $v \in \mathbb{R}^m$ is the speed of some curve through p; for example, $\gamma(t) = p + tv$. We think about $T_p\mathbb{R}^m$ as being "vectors based at p". Let us see what happens when we change coordinates on $\mathbb{R}^m$, i.e. consider a diffeomorphism $\chi : U \xrightarrow{\sim} V$, between open sets $U, V \subset \mathbb{R}^m$. Then any smooth curve $\gamma : (-\epsilon, \epsilon) \to U$, with $\gamma(0) = p$, can be transported to a smooth curve $\chi \circ \gamma : (-\epsilon, \epsilon) \to V$, and their speeds are related by the chain rule:

$$(\chi \circ \gamma)'(0) = (\mathrm{d}_p\chi)\gamma'(0).$$

Thus, the differential of the change of coordinate matrix gives a natural identification between the tangent spaces.

On a general manifold, a fixed chart induces a model for the tangent space (i.e. just $\mathbb{R}^m$), and we use the differential of the change of coordinate maps to identify these various models.

Definition 5.1.3. *Let M be a smooth manifold of dimension m. A **tangent vector** at $p \in M$ is a map v which associates to each chart (U, φ) around p an element $v_\varphi \in \mathbb{R}^m$ satisfying the following rule*

$$v_\psi = \mathrm{d}_{\chi(p)}(\psi \circ \chi^{-1})(v_\chi),$$

*for any two charts (O, χ) and (V, ψ) around p. The set of all tangent vectors at p is called the **tangent space** of M at p and is denoted by $T_p M$.*

Since the differential of the change of coordinates map is a linear map, it follows that $T_p M$ is a vector space with operations

$$(\lambda v)_\varphi = \lambda v_\varphi, \quad (v + w)_\varphi = v_\varphi + w_\varphi, \ \forall v, w \in T_p M, \ \lambda \in \mathbb{R}.$$

We have that:

Lemma 5.1.4. *The tangent space of M at p is a vector space of dimension $m = \dim(M)$. Moreover, for any chart (U, φ) around p, the map*

$$T_p M \to \mathbb{R}^m, \quad v \mapsto v_\varphi$$

is a linear isomorphism.

Proof. That the map is linear is clear. The kernel of the map is zero: if $v_\varphi = 0$ then, by linearity of the differential, $v_\psi = \mathrm{d}_{\varphi(p)}(\psi \circ \varphi^{-1})(v_\varphi) = 0$ for any other chart (V, ψ), thus $v = 0$. Thus the map is injective. To show surjectivity, let $u \in \mathbb{R}^m$, and define v so that

$$v_\psi := \mathrm{d}_{\varphi(p)}(\psi \circ \varphi^{-1})(u),$$

for a chart (V, ψ). That v is indeed a tangent vector follows from the chain rule: for any two charts (V, ψ) and (O, χ), we have

$$v_\psi = \mathrm{d}_{\varphi(p)}(\psi \circ \varphi^{-1})(u) = \mathrm{d}_{\varphi(p)}(\psi \circ \chi^{-1} \circ \chi \circ \varphi^{-1})(u) =$$
$$= \mathrm{d}_{\chi(p)}(\psi \circ \chi^{-1}) \circ \mathrm{d}_{\varphi(p)}(\chi \circ \varphi^{-1})(u) = \mathrm{d}_{\chi(p)}(\psi \circ \chi^{-1})(v_\chi).$$

Clearly, $v_\varphi = u$; thus the assignment is indeed onto. $\square$

Using the standard chart $(\mathbb{R}^m, \mathrm{id})$ on $\mathbb{R}^m$, the lemma above allows us to identify

$$(*) \qquad\qquad T_p \mathbb{R}^m \to \mathbb{R}^m, \quad v \mapsto v_{\mathrm{id}}.$$

5.1.5. The differential of a smooth map

Next, we extend the notion of the differential of a smooth map. We will use the same notation as for maps between open sets in Euclidean space; the standard notion is obtained by using the identification $(*)$ above.

Definition 5.1.6. *Let $f : M \to N$ be a smooth map. The **differential** of f at $p \in M$ is the linear map*

$$\mathrm{d}_p f : T_p M \to T_{f(p)} N$$
$$\left(\mathrm{d}_p f(v)\right)_\psi = \mathrm{d}_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})(v_\varphi),$$

where (U, φ) is a chart around p and (V, ψ) is a chart around $f(p)$.

To show that the above assignment $\psi \mapsto (\mathrm{d}_p f(v))_\psi$ yields indeed a tangent vector, one needs to show that it transforms accordingly to the rule from Definition 5.1.3; however, this is a straightforward application of the chain rule, which we omit (as in the proof of Lemma 5.1.4).

With the canonical identification of the tangent space of $\mathbb{R}^m$, note that the differential of a chart (U, φ) around $p \in M$ becomes

$$\mathrm{d}_p\varphi : T_pM \to T_{\varphi(p)}\mathbb{R}^m, \quad \mathrm{d}_p\varphi(v) = v_\varphi.$$

The chain rule also extends to the setting of smooth manifolds, and its proof is again a straightforward application of the 'classical' chain rule:

Lemma 5.1.7. *Let $f : M \to N$ and $g : N \to P$ be smooth maps between smooth manifolds. The differentials satisfy the **chain rule**:*

$$\mathrm{d}_p(g \circ f) = \mathrm{d}_{f(p)}g \circ \mathrm{d}_pf, \quad \forall\, p \in M.$$

5.2. Notations for tangent vectors

Let M be a smooth manifold and consider a chart (U, φ) around $p \in M$ inducing local coordinates

$$x_\varphi^i : U \to \mathbb{R}, \quad \varphi = (x_\varphi^1, \ldots, x_\varphi^m).$$

Let $e^1, \ldots, e^m$ be the standard basis of $\mathbb{R}^m$. The corresponding basis of T_pM will be denoted by

$$\left.\frac{\partial}{\partial x_\varphi^1}\right|_p, \quad \left.\frac{\partial}{\partial x_\varphi^2}\right|_p, \quad \cdots, \quad \left.\frac{\partial}{\partial x_\varphi^m}\right|_p \in T_pM,$$

$$\left(\left.\frac{\partial}{\partial x_\varphi^i}\right|_p\right)_\varphi = \mathrm{d}_p\varphi\left(\left.\frac{\partial}{\partial x_\varphi^i}\right|_p\right) = e^i, \quad 1 \leq i \leq m.$$

To explain the notation, let $f \in C^\infty(M)$ be a smooth function on M. Then its differential is given in this standard basis by:

$$\mathrm{d}_pf\left(\left.\frac{\partial}{\partial x_\varphi^i}\right|_p\right) = \mathrm{d}_{\varphi(p)}(f \circ \varphi^{-1})(e^i) = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(p));$$

in other words the differential of f on the standard basis is given by the partial derivatives of the local expression $f \circ \varphi^{-1}$ of f.

More generally, let $f : M \to N$ be a smooth map, and consider a smooth chart (V, ψ) around $q = f(p)$, with local coordinates $\psi = (y_\psi^1, \ldots, y_\psi^n)$. The matrix of d_pf in the bases

$$\left\{\left.\frac{\partial}{\partial x_\varphi^i}\right|_p \; : \; 1 \leq i \leq m\right\} \quad \text{and} \quad \left\{\left.\frac{\partial}{\partial y_\psi^j}\right|_q \; : \; 1 \leq j \leq n\right\}$$

of T_pM and T_qN, respectively, is the Jacobian matrix of the local representation

$$\psi \circ f \circ \varphi^{-1}(x) = (f^1(x), \ldots, f^n(x))$$

of f in these charts:

$$[\mathrm{d}_pf] = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^m} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f^n}{\partial x^1} & \frac{\partial f^n}{\partial x^2} & \cdots & \frac{\partial f^n}{\partial x^m} \end{pmatrix}(\varphi(p));$$

$$\mathrm{d}_pf\left(\left.\frac{\partial}{\partial x_\varphi^i}\right|_p\right) = \sum_{j=1}^n \frac{\partial f^j}{\partial x^i}(\varphi(p))\left.\frac{\partial}{\partial y_\psi^j}\right|_q.$$

5.3. The commutative algebra $C^\infty(M)$

The "cleanest" way to introduce the tangent space is by using the commutative $\mathbb{R}$-algebra of smooth functions:

$$(C^\infty(M), +, \cdot), \quad (f + g)(x) := f(x) + g(x), \quad (f \cdot g)(x) := f(x) \cdot g(x).$$

Note that smooth maps induce algebra homomorphisms:

Proposition 5.3.1. *Any smooth map $\varphi : M \to N$ induces a homomorphism of algebras, called the **pullback** along φ:*

$$\varphi^* : C^\infty(N) \to C^\infty(M), \quad \varphi^*(f) = f \circ \varphi.$$

The more surprising result is the converse of this Proposition, which implies that the commutative algebra $C^\infty(M)$ encodes algebraically the *entire* information about the manifold M:

Theorem ♣ 5.3.2. *Any algebra homomorphism $s : C^\infty(N) \to C^\infty(M)$ is of the form $s = \varphi^*$ for a unique smooth map $\varphi : M \to N$.*

In particular, this result implies that two manifolds are diffeomorphic if and only if their commutative algebras of smooth functions are isomorphic. A proof of Theorem 5.3.2 is outlined in the exercises at the end of this lecture.

5.4. Tangent vectors as derivations

Recall that on $\mathbb{R}^m$ the differential of a function $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is defined using directional derivatives:

$$d_p f(v) = \frac{d}{dt} f(p + tv)\big|_{t=0}.$$

We introduce the analog terminology on manifolds:

Definition 5.4.1. *Let M be a smooth manifold. The **derivative** along $v \in T_p M$ is the map*

$$D_v : C^\infty(U) \to \mathbb{R}, \quad D_v(f) := d_p f(v).$$

The following holds:

Lemma 5.4.2. *For all $f, g \in C^\infty(M)$, and $v \in T_p M$ we have that:*

$$D_v(f \cdot g) = f(p) D_v(g) + D_v(f) g(p).$$

Proof. Using a chart, it suffices to prove this on open sets in R^m. Here it follows from the usual Leibniz rule $(\alpha \cdot \beta)' = \alpha \cdot \beta' + \alpha' \cdot \beta$ applied to the functions $\alpha(t) = f(p + tv)$ and $\beta(t) = g(p + tv)$ at $t = 0$. $\qquad \square$

Definition 5.4.3. *A **derivation** of $C^\infty(M)$ at $p \in M$ is a linear map*

$$D : C^\infty(M) \longrightarrow \mathbb{R}$$

satisfying the derivation rule:

$$D(f \cdot g) = f(p) D(g) + D(f) g(p), \quad \forall \ f, g \in C^\infty(M)$$

The space of derivations of $C^\infty(M)$ at p will be denoted by $\mathrm{Der}_p(C^\infty(M))$.

Note that the space of derivations is a vector space: for all $\lambda, \mu \in \mathbb{R}$ and $D_1, D_2 \in \mathrm{Der}_p(C^\infty(M))$,

$$\lambda D_1 + \mu D_2 : C^\infty(M) \to \mathbb{R}, \quad (\lambda D_1 + \mu D_2)(f) := \lambda D_1(f) + \mu D_2(f)$$

is also derivation of $C^\infty(M)$ at p.

The following result shows that all derivations come from derivatives along tangent vectors.

Theorem 5.4.4. *Let M be a smooth manifold, and let $p \in M$. The map*

$$T_p M \to \mathrm{Der}_p(C^\infty(M)), \quad v \mapsto D_v$$

is a linear isomorphism.

To prove the theorem, we first show that derivations of $C^\infty(M)$ at p are local, in the sense that they depend only on the behavior of the functions around p. The following terminology will be therefore rather useful:

Definition 5.4.5. *Let M be a manifold and $p \in M$. Two functions $f, g \in C^\infty(M)$ are said to have the same **germ** at p, if there is an open neighborhood W of p such that $f|_W = g|_W$.*

Lemma 5.4.6. *Let $f, g \in C^\infty(M)$ have the same germ at $p \in M$, then for any $D \in \mathrm{Der}_p(C^\infty(M))$, we have that $D(f) = D(g)$.*

Proof. Let $h := f - g$; then $h|_W = 0$, for some neighborhood W of p. We show that $D(h) = 0$. Consider a bump function $\chi \in C^\infty(M)$ such that $\chi(p) = 1$ and $\mathrm{supp}(\chi) \subset W$. Then, $\chi h = 0$. Therefore:

$$0 = D(0) = D(\chi h) = h(p)D(\chi) + \chi(p)D(h) = D(h). \quad \square$$

$\square$

Lemma 5.4.7. *Let $U \subset M$ be an open neighborhood of p. Then the map*

$$I : \mathrm{Der}_p(C^\infty(U)) \to \mathrm{Der}_p(C^\infty(M)), \quad I(D)(g) := D(g|_U)$$

is a linear isomorphism.

Proof. Let $\chi \in C^\infty(M)$ be a bump function such that $\chi|_W = 1$, on some neighborhood $W \subset U$ of p and such that $\mathrm{supp}(\chi) \subset U$. We define the map

$$E : C^\infty(U) \to C^\infty(M), \quad E(f)|_U := \chi f, \quad E(f)|_{M \backslash \mathrm{supp}(\chi)} := 0.$$

Note that f and $E(f)|_U = \chi f$ have the same germ at p. Consider the map

$$R : \mathrm{Der}_p(C^\infty(M)) \to \mathrm{Der}_p(C^\infty(U)), \quad R(D')(f) := D'(E(f)).$$

Since $E(fg)$ and $E(f)E(g)$ have the same germ at p, $R(D')$ is indeed a derivation:

$$\begin{aligned} R(D')(fg) =& D'(E(fg)) = D'(E(f)E(g)) = E(f)(p)D'(E(g)) \\ &+ E(g)(p)D'(E(f)) = f(p)R(D')(g) + g(p)R(D')(f). \end{aligned}$$

Note that $I \circ R(D')(f) = D'(E(f)|_U) = D'(f)$, hence $I \circ R(D') = D'$; and similarly, that $R \circ I(D)(g) = D(E(g|U)) = D(g)$, hence $R \circ I(D) = D$. We obtain that I is invertible with $I^{-1} = R$. $\square$

For the proof, we will need the following version of Taylor's theorem:

Lemma 5.4.8. *Let $B \subset \mathbb{R}^m$ be a convex open neighborhood of 0. For any $f \in C^\infty(B)$ there exist smooth functions $g_i \in C^\infty(B)$, $1 \le i \le m$, such that*

$$f(x) = f(0) + \sum_{i=1}^m g_i(x)x^i,$$

and moreover, $g_i(0) = \frac{\partial f}{\partial x^i}(0)$.

Proof. Fix $x \in B$, and consider the function $h : [0,1] \to \mathbb{R}$, $h(t) = f(tx)$. Clearly, h is smooth, therefore the fundamental theorem of calculus applies:

$$h(1) - h(0) = \int_0^1 h'(t)\mathrm{d}t.$$

This equation gives:

$$f(x) - f(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}f(tx^1, \ldots, tx^m)\mathrm{d}t = \int_0^1 \sum_{i=1}^m \frac{\partial f}{\partial x^i}(tx)x^i\mathrm{d}t = \sum_{i=1}^m g_i(x)x^i,$$

where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\mathrm{d}t$. Clearly, $g_i \in C^\infty(B)$ and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$. $\qquad\square$

Proof of Theorem 5.4.4. Consider a chart (U, φ) on M around p, with coordinates $\varphi = (x_\varphi^1, \ldots, x_\varphi^m)$ such that $\varphi(p) = 0$ and $\varphi(U) = B$ is convex.

By Lemma 5.4.7, it suffices to show that the map

$$T_pM \to \mathrm{Der}_p(C^\infty(U)), \quad v \mapsto D_v$$

is an isomorphism.

Let $v \in T_pM$ and denote $v_\varphi = (v^1, \ldots, v^m)$. By the formulas in Section 5.2,

$$D_v(x_\varphi^i) = \mathrm{d}_p x_\varphi^i(v) = \sum_{j=1}^m v^j \frac{\partial x^i}{\partial x^j}(0) = v^i;$$

thus, the map is injective.

Let $D \in \mathrm{Der}_p(C^\infty(U))$. We show that D vanishes on constant functions. Since

$$D(1) = D(1 \cdot 1) = 1 \cdot D(1) + D(1) \cdot 1 = 2 \cdot D(1),$$

we obtain that $D(1) = 0$. By linearity, $D(c) = D(c \cdot 1) = c \cdot D(1) = 0$, for all $c \in \mathbb{R}$.

Let $v \in T_pM$ be the vector with $v_\varphi = (v^1, \ldots, v^m) \in \mathbb{R}^m$, where $v^i := D(x_\varphi^i)$. We show that $D = D_v$. Let $f \in C^\infty(U)$. Applying Lemma 5.4.8, we find functions $g_i \in C^\infty(U)$, for $1 \le i \le m$, such that

$$f = f(p) + \sum_{i=1}^m x_\varphi^i g_i, \quad g_i(p) = \mathrm{d}_p f\Big(\frac{\partial}{\partial x_\varphi^i}\Big|_p\Big).$$

Applying D, we obtain:

$$D(f) = D(f(p)) + \sum_{i=1}^m D(x_\varphi^i g_i) =$$

$$= 0 + \sum_{i=1}^m \big(0 \cdot D(g_i) + D(x_\varphi^i) \cdot g_i(p)\big) =$$

$$= \sum_{i=1}^m v_i \mathrm{d}_p f\Big(\frac{\partial}{\partial x_\varphi^i}\Big|_p\Big) = \mathrm{d}_p f(v) = D_v(f).$$

Thus, $D = D_v$. $\qquad\square$

5.5. Exercises

Exercise 5.1. Let $M \subset \mathbb{R}^n$ be a manifold embedded in $\mathbb{R}^n$, as in Definition 2.4.1, and let $p \in M$. Let $V_p \subset \mathbb{R}^n$ consist of all vectors $\gamma'(0) \in \mathbb{R}^n$, where $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve such that $\gamma(0) = p$. Show that V_p is a linear subspace of $\mathbb{R}^n$, and that there is a linear isomorphism $V_p \xrightarrow{\sim} T_pM$, such that $\gamma'(0) \in V_p$ corresponds to $\mathrm{d}_0\gamma(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}) \in T_pM$.

The following exercise gives an alternative definition of the tangent space:

Exercise 5.2. Let M be a manifold and let $p \in M$. Denote by $I_p \subset C^\infty(M)$ the ideal of functions vanishing at p:

$$I_p = \{f \in C^\infty(M) \ : \ f(p) = 0\}.$$

Denote by I_p^2 the square of this ideal, i.e. I_p^2 consists of finite sums of the form:

$$f = \sum_{j=1}^{k} g_j h_j, \quad g_j, h_j \in I_p, \quad 1 \le j \le k.$$

(a) Prove that I_p/I_p^2 has dimension $m = \dim(M)$.
  Hint: use Lemma 5.4.8 and Lemma 4.1.2.
(b) Show that $\mathrm{d}_p f(v) = 0$ for all $f \in I_p^2$ and $v \in T_pM$.
(c) Prove that the following map is a linear isomorphism:

$$E : T_pM \to (I_p/I_p^2)^*, \quad E(v)(f + I_p^2) := \mathrm{d}_p f(v), \quad f \in I_p.$$

We introduce some terminology for the following exercises.

Definition 5.5.1. *An **algebra** over $\mathbb{R}$ is a real vector space A endowed with an associative, bilinear multiplication $\cdot : A \times A \to A$, and with a unit $1_A \in A$; i.e.*

$$(f \cdot g) \cdot h = f \cdot (g \cdot h), \qquad 1_A \cdot f = f = f \cdot 1_A,$$

$$(\lambda f + \mu g) \cdot h = \lambda(f \cdot h) + \mu(g \cdot h), \qquad f \cdot (\lambda g + \mu h) = \lambda(f \cdot g) + \mu(f \cdot h),$$

*for all $f, g, h \in A$ and all $\lambda, \mu \in \mathbb{R}$. The algebra A is called **commutative** if $f \cdot g = g \cdot f$ for all $f, g \in A$.*

*A **homomorphism** between the $\mathbb{R}$-algebras A and B is a linear map $s : A \to B$, which preserves multiplication and units:*

$$s(f \cdot g) = s(f) \cdot s(g) \quad and \quad s(1_A) = 1_B.$$

Definition 5.5.2. *Let A and R be commutative algebras over $\mathbb{R}$, and let $s : A \to R$ be an algebra homomorphism. A **derivation at** s of A is a linear map*

$$D : A \longrightarrow R \quad s.t. \quad D(u \cdot v) = s(u)D(v) + D(u)s(v), \quad for \ all \ \ u, v \in A.$$

We denote the space of all derivations at s by:

$$\mathrm{Der}_s(A, R).$$

Definition 5.5.3. *Let A be a commutative algebra over $\mathbb{R}$. A **character** of A is an algebra homomorphism $\chi : A \to \mathbb{R}$.*

Definition 5.5.4. *For a commutative algebra R, let $R[X]$ denote the polynomial algebra with coefficients in R, i.e. each element in $p \in R[X]$ can be written as a finite sum:*

$$p = u_0 + u_1 X + \ldots + u_k X^k,$$

for unique $u_0, \ldots, u_k \in R$; the algebraic operations are defined as usually for polynomials. Consider the following algebra:

$$R[\epsilon] := R[X]/(X^2),$$

where $(X^2) \subset R[X]$ is the ideal generated by X^2, i.e. each element $p \in R[\epsilon]$ can be uniquely written as

$$p = u_0 + u_1\epsilon,$$

where $\epsilon := X + (X^2) \in R[X]/(X^2)$; and the multiplication is such that $\epsilon^2 = 0$:

$$(u_0 + u_1\epsilon) \cdot (v_0 + v_1\epsilon) = u_0v_0 + (u_0v_1 + u_1v_0)\epsilon.$$

Exercise 5.3. (a) Let $s : A \to R$ be an algebra homomorphism, and let $D \in \mathrm{Der}_s(A, R)$. Prove that the map

$$s + \epsilon D : A \longrightarrow R[\epsilon], \quad (s + \epsilon D)(u) := s(u) + \epsilon D(u)$$

is an algebra homomorphism.
(b) Prove that any algebra homomorphism $\sigma : A \to R[\epsilon]$ is of the form described at (a).

Exercise 5.4. Let $s : A \to R$ be a homomorphism of algebras.

(a) Prove that a homomorphism of algebras $t : B \to A$ induces a linear map between the derivation spaces, defined as follows:

$$t_s^* : \mathrm{Der}_s(A, R) \longrightarrow \mathrm{Der}_{s \circ t}(B, R), \quad t_s^*(D) := D \circ t.$$

(b) If t is surjective, prove that t_s^* is injective.
(c) Prove that the following "chain rule" holds: if $t : B \to A$ and $r : C \to B$ are algebra homomorphisms, then

$$(t \circ r)_s^* = r_{s \circ t}^* \circ t_s^* : \mathrm{Der}_s(A, R) \longrightarrow \mathrm{Der}_{s \circ t \circ r}(C, R).$$

Exercise 5.5. Let X be a topological space, and let $C(X)$ be the algebra of continuous maps $X \to \mathbb{R}$. Consider a subalgebra $A \subset C(X)$, with unit 1_A the constant map $1_A(p) = 1$, and which satisfies the condition:

(*) $(f \in A, \ f(x) \neq 0, \ \forall\, x \in X) \quad \Longrightarrow \quad 1/f \in A.$

We denote by $A^\vee$ the set of characters. For $\chi \in A^\vee$, let I_χ denote the kernel of χ:

$$I_\chi := \{f \in A \ : \ \chi(f) = 0\}.$$

(a) Let $p \in X$. Prove that the following map gives a character of A:

$$\chi_p : A \longrightarrow \mathbb{R}, \quad \chi_p(f) := f(p).$$

We denote $I_p := I_{\chi_p}$.
(b) For $\chi \in A^\vee$, and $p \in X$, prove that $I_p \subset I_\chi$, implies $\chi = \chi_p$.
(c) Prove that if $f \in I_\chi$, then there exists $p \in X$ such that $f(p) = 0$.
(d) Let $\chi \in A^\vee$ be a general character, and assume that $\chi \neq \chi_p$ for every $p \in X$. Prove that for any point $p \in X$ there exists $f \in I_\chi$ such that $f(p) > 0$.
(e) Prove that if X is compact, then every $\chi \in A^\vee$ is of the form $\chi = \chi_p$ for some $p \in X$.
(f) Assume that there exists $f \in A$ such that $f^{-1}(\lambda) \subset X$ is compact for any $\lambda \in \mathbb{R}$. Show that every $\chi \in A^\vee$ is of the form $\chi = \chi_p$ for some $p \in X$.

Exercise 5.6. Prove that for any algebra homomorphism $\chi : C^\infty(M) \to \mathbb{R}$ (i.e. any character of $C^\infty(M)$) there exists a unique point $p \in M$ such that:

$$\chi(f) = f(p), \quad \text{for all} \quad f \in C^\infty(M).$$

Hint: use Exercise 5.5 and Corollary 4.4.4.

Exercise 5.7. Let M be a manifold. Show that there is a one-to-one correspondence between tangent vectors on M and algebra homomorphisms $C^\infty(M) \to \mathbb{R}[\epsilon]$.
Hint: use Exercises 5.3 and 5.6.

The following exercise proves Theorem 5.3.2.

Exercise 5.8. Let $s : C^\infty(N) \to C^\infty(M)$ be an algebra homomorphism.
(a) Prove that there exists a unique function $\varphi : M \to N$ such that $s(f) = f \circ \varphi$ for all $f \in C^\infty(N)$. *Hint: use Exercise 5.6.*
(b) Prove that for any smooth function $f \in C^\infty(N)$, the set

$$U_f := \{p \in N \ : \ f(p) \neq 0\}$$

is open, and that these sets forms a basis for the topology of N.
Hint: use Lemma 4.1.2.
(c) Prove that φ is continuous.
(d) Prove that φ is smooth.

LECTURE 6

6.1. The inverse function theorem

The inverse function theorem gives a very simple condition to check that a map is locally a diffeomorphism, namely, it suffices that its differential be invertible. For a proof of this classical result, see for example [**11**]. This can be directly extended to smooth manifolds:

Theorem 6.1.1 (The inverse function theorem)**.** *Let $f : M \to N$ be a smooth map and let $p \in M$. If $\mathrm{d}_p f : T_p M \to T_{f(p)} N$ is a linear isomorphism, then there exists an open neighborhood $O \subset M$ of p, such that $f(O)$ is open and f restricts to a diffeomorphism:*

$$f|_O : O \xrightarrow{\sim} f(O).$$

Proof. Let (U, φ) be a chart around p and (V, ψ) be a chart around $f(p)$. By shrinking U, we may assume that $f(U) \subset V$. By the chain rule, we have that

$$\mathrm{d}_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) = \mathrm{d}_{f(p)}\psi \circ \mathrm{d}_p f \circ (\mathrm{d}_p\varphi)^{-1}.$$

Since the charts are diffeomorphisms, their differentials are linear isomorphisms; therefore, the right hand side is the composition of linear isomorphisms. We obtain that also $\mathrm{d}_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})$ is a linear isomorphism. The map $\psi \circ f \circ \varphi^{-1}$ is a smooth map between open sets in $\mathbb{R}^m$, and its Jacobian matrix at $\varphi(p)$ coincides with its differential, thus it is invertible. By the standard inverse function theorem (see e.g. [**11**]), there exists an open neighborhood W of $\varphi(p)$ such that $\psi \circ f \circ \varphi^{-1}(W)$ is open and $\psi \circ f \circ \varphi^{-1}|_W : W \xrightarrow{\sim} \psi \circ f \circ \varphi^{-1}(W)$ is a diffeomorphism. Since the charts are diffeomorphisms, this implies that conclusion with $O := \varphi^{-1}(W)$. $\square$

6.2. Immersions and submersions

The following two classes of maps play an important role:

Definition 6.2.1. *Let $f : M \to N$ be a smooth map, and let $p \in M$.*

*(1) f is called an **immersion at** p if $\mathrm{d}_p f : T_p M \to T_{f(p)} N$ is injective. If this holds at all points in M, then f is called an **immersion**.*

*(2) f is called a **submersion at** p if $\mathrm{d}_p f : T_p M \to T_{f(p)} N$ is surjective. If this holds at all points in M, then f is called a **submersion**.*

The most basic example of an immersion is the linear inclusion:

$$\mathrm{i}_l^k : \mathbb{R}^k \longrightarrow \mathbb{R}^l, \quad \mathrm{i}_l^k(x^1, \ldots, x^k) := (x^1, \ldots, x^k, 0, \ldots, 0), \text{ for } k \leq l,$$

and the most basic example of a submersion is the linear projection

$$\mathrm{pr}_k^l : \mathbb{R}^l \longrightarrow \mathbb{R}^k, \quad \mathrm{pr}_k^l(x^1, \ldots, x^l) := (x^1, \ldots, x^k), \text{ for } k \leq l.$$

The following two theorems say that locally, up to a change of coordinates, these are the only examples:

Theorem 6.2.2 (The local immersion theorem). *Let $f : M \to N$ be a smooth map. If f is an immersion at $p \in M$, then there are charts (U, φ) and (V, ψ) around p and $f(p)$, respectively, such that $f(U) \subset V$ and, in these coordinates, f becomes the map i_n^m; i.e.*

$$\psi \circ f \circ \varphi^{-1} = \mathrm{i}_n^m|_{\varphi(U)} : \varphi(U) \longrightarrow \mathbb{R}^n,$$
$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

Theorem 6.2.3 (The local submersion theorem). *Let $f : M \to N$ be a smooth map. If f is a submersion at $p \in M$, then there are charts (U, φ) and (V, ψ) around p and $f(p)$, respectively, such that $f(U) \subset V$ and, in these coordinates, f becomes the map pr_n^m; i.e.*

$$\psi \circ f \circ \varphi^{-1} = \mathrm{pr}_n^m|_{\varphi(U)} : \varphi(U) \longrightarrow \mathbb{R}^n,$$

$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n).$$

The proofs of these two results are completely analogous, therefore, we only give the details for the second result; the proof of the local immersion theorem is left as an exercise.

Proof of the local submersion theorem. Let $(\widetilde{U}, \widetilde{\varphi})$ and (V, ψ) be two charts around p and $f(p)$, respectively, such that $f(\widetilde{U}) \subset V$. Denote the local expression of f by $h := \psi \circ f \circ \widetilde{\varphi}^{-1}$. We are going to "complete h to a diffeomorphism". By applying Exercise 6.1 to the linear map $A = \mathrm{d}_{\widetilde{\varphi}(p)} h$ (which, by the chain rule, is surjective), we obtain a linear map $B : \mathbb{R}^m \to \mathbb{R}^{m-n}$ such that $(\mathrm{d}_{\widetilde{\varphi}(p)} h, B) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear isomorphism. This map is precisely the differential at $\widetilde{\varphi}(p)$ of the map $(h, B) : \widetilde{\varphi}(\widetilde{U}) \to \mathbb{R}^m$. Therefore, by the inverse function theorem, there is a neighborhood $O \subset \widetilde{\varphi}(\widetilde{U})$ of $\widetilde{\varphi}(p)$ such that $W := (h, B)(O)$ is open and $(h, B) : O \xrightarrow{\sim} W$ is a diffeomorphism; denote its inverse by $g := (h, B)^{-1} : W \xrightarrow{\sim} O$. Then we have that $(h(g(x)), B(g(x))) = x$ for all $x \in W$, hence $h \circ g = \mathrm{pr}_n^m|_W$. Equivalently, $\psi \circ f \circ \varphi^{-1} = \mathrm{pr}_n^m|_W$, where

$$\varphi := g^{-1} \circ \widetilde{\varphi} = (h, B) \circ \widetilde{\varphi} : U \longrightarrow \mathbb{R}^m, \quad U := \widetilde{\varphi}^{-1}(O);$$

hence, the charts (U, φ) and (V, ψ) satisfy the conditions from the statement. $\square$

Remark 6.2.4. In the local submersion theorem, the chart on the codomain (V, ψ) can be chosen arbitrary; this follows from the proof. Similarly, in the local immersion theorem the chart (U, φ) on the domain can be chosen arbitrary.

6.3. Embedded submanifolds

The following generalizes the notion of *embedded manifolds in* $\mathbb{R}^n$ from Section 2.4:

Definition 6.3.1. *A subset $S \subset M$ of a smooth m-dimensional manifold M is called a k-**dimensional embedded submanifold** if for every point $p \in S$ there exists a chart (U, φ) around p, such that*

(*) $$\varphi(S \cap U) = \left(\mathbb{R}^k \times \{0\}\right) \cap \varphi(U) \subset \mathbb{R}^m.$$

*A chart (U, φ) with these property is called a chart **adapted to** S. The **codimension** of S is the number $m - k = \dim(M) - \dim(S)$.*

Next, we show that embedded submanifolds are indeed smooth manifolds:

Theorem 6.3.2. *Let $S \subset M$ be a k-dimensional embedded submanifold. Then S, endowed with the subset topology, has a k-dimensional differentiable structure, with atlas obtained by restricting adapted charts:*

$$\mathcal{A}_S = \{(U \cap S, \mathrm{pr}_k^m \circ \varphi|_{U \cap S}) \ : \ (U, \varphi) \text{ is a chart adapted to } S\}.$$

Moreover, the inclusion map $i : S \hookrightarrow M$, $i(x) = x$ is a smooth immersion.

Proof. The proof that S, endowed with the subset topology, has a k-dimensional differentiable structure for which $\mathcal{A}_S$ is a smooth atlas can be proven exactly as in the case of embedded manifolds in $\mathbb{R}^n$ (see Proposition 2.4.3).

Next, since S has the induced topology, it follows that i is continuous. Let (U, φ) be an adapted chart around $p \in S$. Then, note that in the pair of charts $(U \cap S, \mathrm{pr}_k^m \circ \varphi|_{U \cap S})$ and (U, φ) the local expression of i is the canonical immersion $\mathrm{i}_m^k : \mathbb{R}^k \to \mathbb{R}^m$. This proves that i is a smooth immersion. $\qquad\square$

In Definition 6.3.1 above, denote that the coordinates by $\varphi = (x_\varphi^1, \ldots, x_\varphi^m)$. Then condition (*) means that S is locally described by the $l = m - k$ equations:

$$S \cap U = \{q \in U \ : \ x_\varphi^{k+1}(q) = 0, \ x_\varphi^{k+2}(q) = 0, \ \ldots \ , \ x_\varphi^m(q) = 0\}.$$

The converse also holds, and turns out to be a very useful criterion: any subset which is locally described as the zeroes of by l *independent equations* is an embedded submanifold of codimension l (here, *independent* means that the equations form a submersion):

Proposition 6.3.3. *A subset $S \subset M$ is an embedded submanifold of codimension l if and only if for every $p \in S$ there exists an open set $p \in U \subset M$ and smooth map $h : U \to \mathbb{R}^l$ which is a submersion at p, such that $S \cap U = h^{-1}(0)$.*

Proof. Assume first that S is an embedded submanifold of codimension l. Let $p \in S$ and let (U, φ) be a chart adapted to S with $p \in U$. Define

$$h : U \longrightarrow \mathbb{R}^l, \quad h := \overline{\mathrm{pr}}_l^m \circ \varphi,$$

where $\overline{\mathrm{pr}}_l^m : \mathbb{R}^m \to \mathbb{R}^l$ denotes the projection onto the last l coordinates. Then h is a submersion and $S \cap U = h^{-1}(0)$.

Conversely, assume that $S \subset M$ is a subset such that, around any $p \in S$, we can find an open set U and a smooth map $h : U \to \mathbb{R}^l$ which is a submersion at p and satisfies $U \cap S = h^{-1}(0)$. We apply the local submersion theorem to h. By Remark 6.2.4, we do not need to change the coordinates on $\mathbb{R}^l$. Thus, after shrinking U, we may assume that there is a chart (U, φ) on M such that

$$h \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^{k+1}, \ldots, x^m), \quad \text{where } k = m - l;$$

or equivalently, in the usual notation for coordinates $\varphi = (x_\varphi^1, \ldots, x_\varphi^m)$, we have:

$$h(q) = (x_\varphi^{k+1}(q), \ldots, x_\varphi^m(q)), \quad \text{for } q \in U.$$

Thus, for $q \in U$: $h(q) = 0$ iff $\varphi(q) \in \mathbb{R}^k \times \{0\}$, and since $U \cap S = h^{-1}(0)$, we obtain: $\varphi(S \cap U) = (\mathbb{R}^k \times \{0\}) \cap \varphi(U)$. This concludes the proof. $\qquad \square$

Remark 6.3.4. Consider an embedded submanifold $S \subset M$. By Theorem 6.3.2 the inclusion map $i : S \hookrightarrow M$ is a smooth immersion. Therefore if $p \in S$ the map $d_p i : T_p S \to T_p M$ is injective, and so it sends $T_p S$ isomorphically onto a linear subspace of $T_p M$. We will identify these spaces:

$$T_p S = d_p i(T_p S) \subset T_p M.$$

Assume now that $h : U \to \mathbb{R}^l$ is a local submersion around p such that $S = h^{-1}(0)$. We claim that

(*) $$T_p S = \ker(d_p h) = \{v \in T_p M \;:\; d_p h(v) = 0\}.$$

Since $h \circ i(U \cap S) = 0$, it follows that $d_p h \circ d_p i(T_p S) = 0$. Hence $T_p S \subset \ker(d_p h)$. The other inclusion follows because $d_p h$ is surjective, and so these vector spaces have the same dimension:

$$\dim \ker(d_p h) = \dim T_p M - \dim \operatorname{im}(d_p h) = \dim M - \operatorname{codim} S = \dim S = \dim T_p S.$$

Let us introduce some standard terminology:

Definition 6.3.5. A **_regular value_** of a smooth map $f : M \to N$ is a point $q \in N$ such that f is a submersion at every point in $f^{-1}(q)$. A point $q \in N$ which is not a regular value is called a **_critical value_** for f.

Corollary 6.3.6 (Preimage Theorem)**.** _Let $f : M \to N$ be a smooth map. If $q \in N$ is a regular value f, then $f^{-1}(q)$ is an embedded submanifold of M with_

$$\operatorname{codim}(f^{-1}(q)) = \dim(N), \quad T_p(f^{-1}(q)) = \ker(d_p f), \quad \text{for } p \in f^{-1}(q).$$

Proof. Take a chart (U, φ) around p with $\varphi(q) = 0$. Then $h := \varphi \circ f : V \to \mathbb{R}^n$ is a submersion, and $f^{-1}(q) = h^{-1}(0)$. Proposition 6.3.3 implies the result. $\qquad \square$

Let us also mention a deep result of mathematical analysis, which states that most points are regular values (note that points that are not in the image are always regular values). Without introducing all notions involved, we state this result below (for details, the reader is encouraged to consult [**8**] or [**4**]).

Theorem ♣ 6.3.7 (Sard's Theorem)**.** _The set of critical values of a smooth map $f : M \to N$ has Lebesgue measure zero. In particular, the set of regular values of f is dense in N._

Remark 6.3.8. If $\dim M < \dim N$, then every point in M is a critical point of f. In this situation, Sard's Theorem says that $f(M)$ has measure zero in N; and therefore $N \backslash f(M)$ is dense in N. In particular, one cannot have a surjective map smooth $f : M \to N$ if $\dim M < \dim N$.

For this property, smoothness is essential. In the continuous setting, there are the co-called space-filling curves (e.g. Peano's curve [**21**]). For example, one can prove that any connected, nonempty manifold is the image of a continuous curve!!

6.4. Preimages of submanifolds

The following extends the notion of a regular value to submanifolds.

Definition 6.4.1. *A smooth map $f : M \to N$ is said to be **transverse** to an embedded submanifold $Q \subset N$, if*

$$T_{f(p)}N = T_{f(p)}Q + \mathrm{d}_p f(T_p M), \quad \forall \; p \in f^{-1}(Q).$$

Remark 6.4.2. Note that the sum in the decomposition is not assumed to be a direct sum, i.e. the vector spaces $T_{f(p)}Q$ and $\mathrm{d}_p f(T_p M)$ may have a nontrivial intersection. For example, a submersion is transverse to *any* embedded submanifold of the codomain.

Theorem 6.4.3. *Let $f : M \to N$ be a smooth map that is transverse to the embedded submanifold $Q \subset N$. Then $P := f^{-1}(Q)$ is an embedded submanifold of M with* $\mathrm{codim}\, P = \mathrm{codim}\, Q$ *and*

$$T_p P = (\mathrm{d}_p f)^{-1} \left(T_{f(p)} f^{-1}(Q) \right), \quad \forall p \in P.$$

Before giving the proof, let us consider an example.

Example 6.4.4. Consider the orthogonal projection from the 2-sphere to $\mathbb{R}^2$,

$$f : S^2 \longrightarrow \mathbb{R}^2, \quad f(x, y, z) := (x, y).$$

Note that f is a submersion outside of the equatorial circle, which we denote by Z. Consider two embedded submanifold Q_1 and Q_2 in $\mathbb{R}^2$ which are indicated on the picture below:
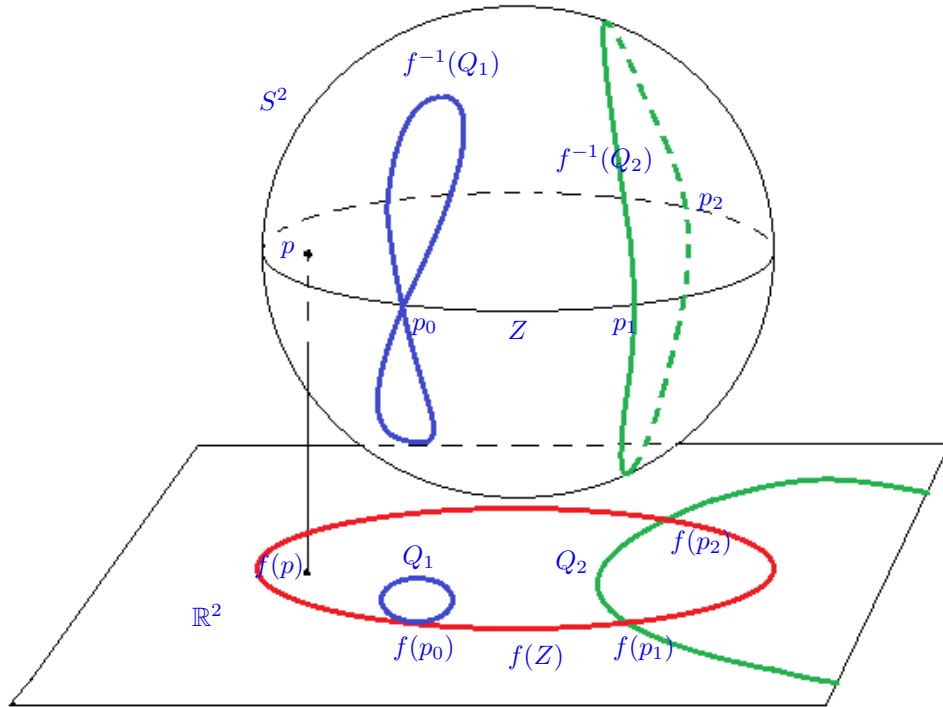
Note that f is not transverse to Q_1, because at the point of intersection $f(p_0)$ of $f(Z)$ and Q_1, we have that $T_{f(p_0)}Q_1 = \mathrm{d}_{p_0} f(T_{p_0} S^2)$. Note also that $f^{-1}(Q_1)$ is not

an embedded submanifold of S^2, because it has a singularity which looks like the intersection of two lines.

On the other hand, f is transverse to Q_2; outside of Z this is clear because f is a submersion, and at the two points $p_1, p_2 \in Z \cap f^{-1}(Q_2)$ note that

$$\mathrm{d}_{p_i} f(T_{p_i} S^2) + T_{f(p_i)}(Q_2) = T_{f(p_i)} \mathbb{R}^2, \quad i = 1, 2.$$

As predicted by the Theorem 6.4.3, $f^{-1}(Q_2)$ is an embedded submanifold of S^2; in fact it is diffeomorphic to a circle.

Proof of Theorem 6.4.3. We check that P satisfies the condition in Proposition 6.3.3. Consider $p \in P$, and denote $q := f(p) \in Q$. Since Q is an embedded submanifold there exists an open neighborhood U of q and a smooth map $h : U \to \mathbb{R}^l$, where $l = \mathrm{codim}Q$, such that h is a submersion at q and $h^{-1}(0) = U \cap Q$. On the open neighborhood $f^{-1}(U)$ of p, consider the smooth map $h \circ f : f^{-1}(U) \to \mathbb{R}^l$. We show that this map satisfies the conditions from Proposition 6.3.3. Note that

$$(h \circ f)^{-1}(0) = f^{-1}(h^{-1}(0)) = f^{-1}(U \cap Q) = f^{-1}(U) \cap f^{-1}(Q) = f^{-1}(U) \cap P.$$

By Remark 6.3.4, $T_q Q = \ker \mathrm{d}_q h$; hence $\mathrm{d}_q h(T_q Q) = 0$. Using this, that h is a submersion at q, that f is transverse to Q, and the chain rule, we obtain:

$$T_0 \mathbb{R}^l = \mathrm{d}_q h\,(T_q N) = \mathrm{d}_q h\,(T_q Q + \mathrm{d}_p f(T_p M)) = \mathrm{d}_q h \circ \mathrm{d}_p f(T_p M) = \mathrm{d}_p(h \circ f)(T_p M).$$

Thus $h \circ f$ is a submersion at p. So Proposition 6.3.3 implies that P is an embedded manifold of codim $P = l = \mathrm{codim}\ Q$. Finally, again by (*), we have that

$$T_p P = \ker \mathrm{d}_p(h \circ f) = \ker\,(\mathrm{d}_q h \circ \mathrm{d}_p f) = (\mathrm{d}_p f)^{-1}(\ker \mathrm{d}_q h) = (\mathrm{d}_p f^{-1})(T_q Q).$$

$\square$

Remark 6.4.5. Note that the empty set $\emptyset$ is a smooth manifold of *any dimension* $d \geq 0$: it has a (unique) topology; it is locally homeomorphic to the empty set in $\mathbb{R}^d$ (which is open); and the only transition map is the $\mathrm{id}_\emptyset$, which is smooth, when we regard $\emptyset \subset \mathbb{R}^d$ (prove this "by contradiction"!). In the setting of Theorem 6.4.3, it could happen that $f(M) \cap Q = \emptyset$; in this case f is transverse to Q and $f^{-1}(Q) = \emptyset$.

Theorem 6.4.3 can be used to intersect submanifolds. First, we need the following definition:

Definition 6.4.6. *Two embedded submanifold $R, S \subset M$ are said to intersect* **transversely**, *if for every $p \in R \cap S$ we have that*

$$T_p M = T_p R + T_p S.$$

This condition is equivalent to the inclusion $i : R \hookrightarrow M$ being transverse to S. Note also that

$$i^{-1}(S) = R \cap S, \quad (\mathrm{d}_p i)^{-1}(T_p R) = T_p R \cap T_p S, \quad p \in R \cap S.$$

Hence, Theorem 6.4.3 implies the following:

Corollary 6.4.7. *The intersection $R \cap S$ of two embedded submanifolds $R, S \subset M$ which intersect transversally is an embedded submanifold with*

$$\mathrm{codim}\ R \cap S = \mathrm{codim}\ R + \mathrm{codim}\ S;$$

moreover, for each $p \in R \cap S$ we have that

$$T_p(R \cap S) = T_p R \cap T_p S.$$

6.5. Exercises

Exercise 6.1. Let $k \leq n$, and let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map, and let $C : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear map.

(a) Prove that there exists a linear map $B : \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that the map

$$(A, B) : \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n, \quad (A, B)(x) := (A(x), B(x)),$$

is a linear isomorphism.

(b) Prove that there exists a linear isomorphism $G : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$A \circ G = \mathrm{pr}_k^n.$$

(c) Prove that there exists a linear map $D : \mathbb{R}^{n-k} \to \mathbb{R}^n$ such that the map

$$C + D : \mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n \to \mathbb{R}^n, \quad (C + D)(y, z) := C(y) + D(z),$$

is a linear isomorphism.

(d) Prove that there exists a linear isomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$H \circ C = \mathrm{i}_n^k.$$

Exercise 6.2. Let $a \in \mathbb{R}$. Show that the following map is an immersion:

$$f_a : \mathbb{R} \longrightarrow S^1 \times S^1, \quad f_a(t) := (e^{it}, e^{ita}).$$

Make a sketch of $f_a(\mathbb{R})$ for $a = 0, 1/2, 1, \sqrt{2}, 3$. For which $a \in \mathbb{R}$ is f_a injective?

Exercise 6.3. Let $f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}^3$ be the map defined by

$$f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$

Show that f is smooth and show that it only fails to be an immersion at 6 points. Make a sketch of the image of f.

Being an immersion/submersion is an open condition:

Exercise 6.4. If a smooth map $f : M \to N$ is an immersion (resp. a submersion) at $p \in M$ prove that there is an open neighborhood $V \subset M$ of p such that $f|_V$ is an immersion (resp. a submersion).
Do not use the local immersion/submersion theorem!

Exercise 6.5. Prove the local immersion theorem.

Exercise 6.6 (Alternative definition of immersions and submersions)**.** Let $f : M \to N$ be a smooth map, and let $p \in M$. Prove the following:

(a) f is an immersion at p iff it has a local left inverse, i.e. there exist open neighborhoods $U \subset M$ of p and $V \subset N$ of $f(p)$ and a smooth map $\sigma : V \to U$ such that $f(U) \subset V$ and

$$\sigma(f(x)) = x, \qquad \text{for all } x \in U.$$

(b) f is a submersion at p iff it has a local right inverse, i.e. there exist open neighborhoods $U \subset M$ of p and $V \subset N$ of $f(p)$ and a smooth map $j : V \to U$ such that $j(f(p)) = p$ and

$$f(j(y)) = y, \qquad \text{for all } y \in V.$$

Exercise 6.7. Let $f : M \to N$ be a submersion. Prove that f is an open map (i.e. for every open set $U \subset M$, $f(U)$ is open in N).

Exercise 6.8. Let $f : M \to N$ be a smooth map, and let $K \subset M$ be a compact subset. If $f|_K : K \to N$ is injective, and $d_p f : T_p M \to T_{f(p)} N$ is a linear isomorphism for every $p \in K$, prove that there is an open set U such that $K \subset U$, $f(U)$ is open in N and $f : U \to f(U)$ is a diffeomorphism.

Exercise 6.9. Use the Preimage Theorem (Corollary 6.3.6) to prove that S^n is an embedded submanifold of $\mathbb{R}^{n+1}$.

Exercise 6.10. Let $S_b^1 \subset \mathbb{R}^3$ be the circle in the plane $z = 0$ of radius $b > 0$ and with center at the origin. The 2-dimensional torus of radii $0 < a < b$, denoted by $T_{a,b}^2$, is the set of points in $\mathbb{R}^3$ at distance a from S_b^1. Use the Preimage Theorem (Corollary 6.3.6) to prove that $T_{a,b}^2$ is a 2-dimensional manifold (*Find an explicit equation describing it!*)

Exercise 6.11. Let $M(n)$ denote the space of all real $n \times n$ matrices. Let $S(n)$ denote the space of symmetric matrices:

$$S(n) := \{B \in M(n) \ : \ B = B^t\}.$$

Prove that I is a regular value of the map

$$f : M(n) \longrightarrow S(n), \quad f(A) = AA^t.$$

Using the Preimage Theorem (Corollary 6.3.6), prove that the set of orthogonal matrices

$$O(n) := \{A \in M(n) \ : \ AA^t = I\},$$

is a compact embedded submanifold of $M(n)$ of dimension $n(n-1)/2$.

Hint: For compactness, show that $O(n) \subset S^{n^2-1}$.

7.1. Immersed submanifolds

There are two ways to describe submanifolds: either by giving (local) equations that define them, or by giving a parameterization. As discussed in the previous lecture, the first class corresponds to embedded submanifolds. The second class, called immersed submanifolds, is more general; it incorporates several natural "submanifolds" which are not embedded, but play an important role in various geometric settings: Lie subgroups of Lie groups, orbits of Lie group actions, leaves of foliations, etc.

Definition 7.1.1. *Let M be a smooth manifold. An **immersed k-dimensional submanifold** of M is a subset $S \subset M$ endowed with a topology and a k-dimensional differentiable structure such that the inclusion map $i : S \hookrightarrow M$, $i(x) = x$ is a smooth immersion.*

Here are some obvious examples:

Example 7.1.2. (1) An embedded submanifold $S \subset M$ with the differentiable structure from Theorem 6.3.2 is also an immersed submanifold.
(2) If $f : N \to M$ is an injective immersion then $S := f(N)$ is an immersed submanifold, with differentiable structure and topology such that $f : N \to S$ is a diffeomorphism (this condition defines a unique topology and differentiable structure on S!). In fact, every immersed submanifold S is of this type: just take $S = N$, $f = i$.

Remark 7.1.3. It is important to note that:
- The topology of an immersed submanifold is not necessarily the induced topology (!)
- There can be several immersed submanifolds with the same underlying set $S \subset M$, i.e. there can exist several differentiable structures on the set S, such that S is an immersed submanifold (!)

Here are some examples illustrating these remarks:

Example 7.1.4. Denote the six curves represented below by $C_a, C_b, \ldots, C_f$:

a. Topologist's sine curve   b. Figure 8   c. Shade of a trefoil knot

d. Dutch figure 8   e. Alien creature   f. Non-analytic bifurcation
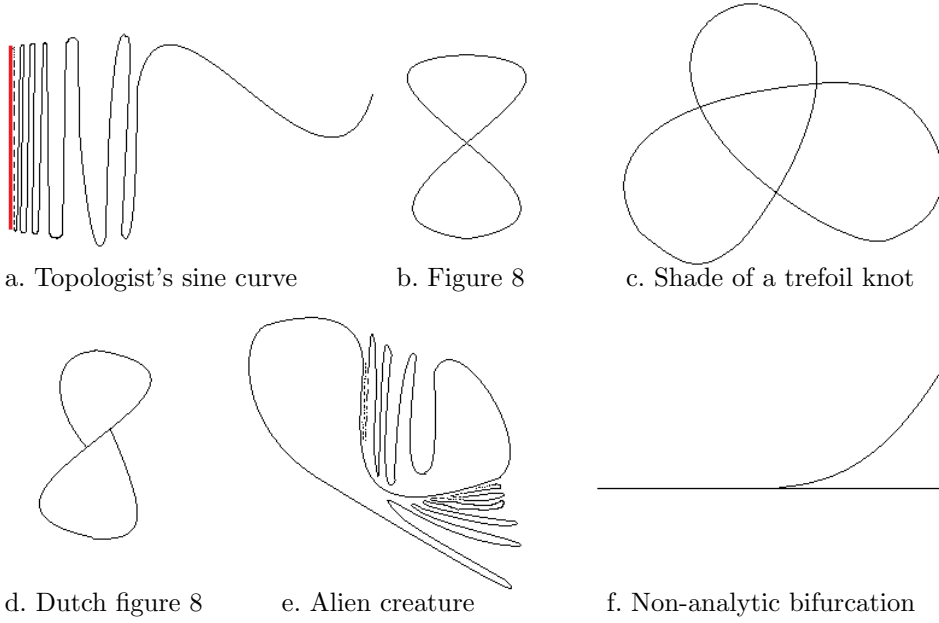
All these curves are immersed submanifolds in $\mathbb{R}^2$! Here are some of their properties:

a. The red line segment that C_a approaches is not part of C_a. An injective immersion that parameterizes C_a (or a curve resembling C_a) is:

$$j : (0, \infty) \longrightarrow \mathbb{R}^2, \quad j(t) = (t, \sin(1/t)).$$

Note that C_a is in fact an embedded submanifold of $\mathbb{R}^2$: this follows by Proposition 6.3.3, because C_a is the 0-set of the submersion:

$$h : (0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}, \quad h(x, y) = y - \sin(1/x).$$

The differentiable structure on C_a coming from j coincides with the differentiable structure of C_a regarded as an embedded submanifold; in fact, we will prove that, in general, on an embedded submanifold there is a unique differentiable structure which makes it into an immersed submanifold; namely, the one constructed in Theorem 6.3.2. Note that C_a is the only embedded submanifold among the 6 examples.

b. The curve C_b is the image of the immersion:

$$j : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad j(t) = (\sin(2t), \sin(t)).$$

Note that j is injective on either of the two intervals $I_1 := (-\pi, \pi)$ and $I_2 := (0, 2\pi)$; and that $j(I_1) = C_b = j(I_2)$. Therefore, each of the maps

$$j_1 := j|_{I_1} : I_1 \longrightarrow C_b, \text{ and } j_2 := j|_{I_2} : I_2 \longrightarrow C_b$$

endows C_b with the structure of an immersed submanifold. Note that the resulting topologies are different! In fact, these are the only two differentiable structures on C_b making C_b an immersed submanifold. Note that $(0, 0) = j_1(0)$

and $(0,0) = j_2(\pi)$. At this point, the tangent spaces corresponding to the different smooth structures are different:

$$T_{(0,0)}C_b = \mathrm{d}_0 j_1(T_0\mathbb{R}) = \left\langle 2\frac{\partial}{\partial x}\big|_{(0,0)} + \frac{\partial}{\partial y}\big|_{(0,0)} \right\rangle$$

$$T_{(0,0)}C_b = \mathrm{d}_\pi j_2(T_\pi\mathbb{R}) = \left\langle 2\frac{\partial}{\partial x}\big|_{(0,0)} - \frac{\partial}{\partial y}\big|_{(0,0)} \right\rangle.$$

c. The curve C_c has $2^3 = 8$ differentiable structures; at each intersection point there are 2 options how to separate the curve. Also, the 8 topologies are different!

d. The curve C_d has a unique differentiable structure as an immersed submanifold; but note that it is not an embedded submanifold!

e. The curve C_e is a closed version of the curve C_a; the behavior at the two "ends" is similar to that of the *Topologist's sine curve*. This curve has a unique differentiable structure as an immersed submanifold, but note that it is not an embedded submanifold!

f. The curve C_f represents two curves that are "infinitely tangent". There are two differentiable structure on C_f which make it into an immersed submanifold. For example, as a model one can take the following two decompositions of C_f:

$$C_f = \{(x,0) \ : \ x \in \mathbb{R}\} \cup \{(x, e^{-1/x}) \ : \ x > 0\};$$

$$C_f = \{(x,0) \ : \ x > 0\} \cup \{(x, f(x)) \ : \ x \in \mathbb{R}\},$$

where f is the smooth function: $f(x) = 0$, if $x \le 0$ and $f(x) = e^{-1/x}$, if $x > 0$, which give different decompositions of C_f into connected components. Note that both smooth structures have the same tangent space at $(0,0)$:

$$T_{(0,0)}C_f = \left\langle \frac{\partial}{\partial x}\big|_{(0,0)} \right\rangle.$$

The following is the main tool for comparing different immersed submanifolds with the same underlying subset.

Lemma 7.1.5. *Let $f : N \to M$ be an injective immersion. Consider a smooth map $\varphi : X \to M$ such that $\varphi(X) \subset f(N)$, and denote by $\widetilde{\varphi} : X \to N$ the unique map satisfying $f \circ \widetilde{\varphi} = \varphi$. If $\widetilde{\varphi}$ is continuous then it is smooth.*

$$\begin{array}{ccc} & X & \\ \widetilde{\varphi}\big\downarrow & & \searrow^{\varphi} \\ N & \xrightarrow{\ f\ } & M \end{array}$$

Proof. Let $x \in X$, and denote $y := \widetilde{\varphi}(x) \in N$ and $z := f(y) = \varphi(x) \in M$. Since f is an immersion, by Exercise 6.6, there exists a smooth map $\sigma : V \to U$, where $U \subset N$ is an open neighborhood of y and $V \subset M$ is an open neighborhood of z such that $f(U) \subset V$ and $\sigma \circ f|_U = \mathrm{id}_U$. Since $\widetilde{\varphi}$ is continuous, we have that $W := \widetilde{\varphi}^{-1}(U) \subset X$ is an open neighborhood of x.

$$\begin{array}{ccc} & x \in W & \\ \widetilde{\varphi}\big\downarrow & & \searrow^{\varphi} \\ y \in U & \underset{\sigma}{\overset{f}{\rightleftarrows}} & V \ni z \end{array}$$

Note that on W, the map $\widetilde{\varphi}$ is a composition of smooth functions:

$$\widetilde{\varphi}|_W = \sigma \circ \varphi|_W,$$

hence $\widetilde{\varphi}$ is smooth on W. So, around every point $x \in X$ we have found an open neighborhood on which $\widetilde{\varphi}$ is smooth; this implies that $\widetilde{\varphi}$ is smooth. $\square$

Corollary 7.1.6. *Let $f_i : N_i \to M$, $i = 1, 2$, be two injective immersions from manifolds of the same dimension $\dim(N_1) = \dim(N_2)$ which have the same image $f_1(N_1) = f_2(N_2) = S$. If the map $f_2^{-1} \circ f_1 : N_1 \to N_2$ is continuous, then it is a diffeomorphism. Hence f_1 and f_2 induce the same structure of a k-dimensional immersed submanifold on S (recall Example 7.1.2 (2)).*

Proof. By Lemma 7.1.5, $\chi := f_2^{-1} \circ f_1$ is smooth. Since $f_1 = f_2 \circ \chi$, it follows that χ is an immersion between manifolds of the same dimension, hence it is a local diffeomorphism. Since χ is a bijection, it is a diffeomorphism. $\square$

Remark 7.1.7. The condition $\dim(N_1) = \dim(N_2)$ in Corollary 7.1.6 superfluous. Namely, since $\chi : N_1 \to N_2$ is onto, it follows by Sard's Theorem (see Remark 6.3.8) that $\dim(N_1) \geq \dim(N_2)$, and the equality follows because χ is an immersion.

As a consequence of the previous result, we obtain:

Corollary 7.1.8. *The differentiable structure on an embedded k-dimensional submanifold $S \subset M$ constructed in Theorem 6.3.2 is the unique k-dimensional differentiable structure on S for which the inclusion map is an immersion.*

Proof. Let $f : N \to M$ be an injective immersion with $f(N) = S$, which induces a second k-dimensional differentiable structure on S. As an embedded submanifold, S has the induced topology from M; hence $f : N \to S$ is continuous, and so by Corollary 7.1.6, f is a diffeomorphism. Thus, the two differentiable structure coincide. $\square$

Let us introduce the following notion:

Definition 7.1.9. *(1) Let X and Y be topological spaces. An injective continuous map $f : X \to Y$ is called a **topological embedding** if $f : X \to f(X)$ is a homeomorphism, where $f(X)$ is endowed with the induced topology from Y.*
*(2) Let N and M be smooth manifolds. A smooth map $f : N \to M$ is called a **smooth embedding** if f is an injective immersion, and $f : N \to f(N)$ is a homeomorphism, where $f(N)$ is endowed with the induced topology.*

Embedded submanifolds are precisely the image of smooth embeddings:

Proposition 7.1.10. *Let $S \subset M$ be a subset. The following are equivalent:*
(1) S is an embedded k-dimensional submanifold.
(2) S is an immersed k-dimensional submanifold, whose topology is the induced topology from M (see Definition 1.1.2).
(3) There exists a k-dimensional manifold N and a smooth embedding $f : N \to M$ such that $S = f(N)$.

Proof. $(1) \Rightarrow (2)$ follows because the topology on an embedded submanifold is the induced topology. $(2) \Rightarrow (3)$ follows with $N := S$, $f := i : S \to M$. Finally, we prove $(3) \Rightarrow (1)$. Let $f : N \to M$ be a smooth embedding. Note the following consequence of the Local Immersion Theorem 6.2.2: every point in N has an open neighborhood $U \subset N$ such that $f(U)$ is an embedded k-dimensional submanifold. Since f is a topological embedding, $f(U) = V \cap f(N)$, for some open set $V \subset M$. We conclude that every point in $S := f(N)$ has an open neighborhood $V \subset M$

such that $S \cap V$ is a k-dimensional embedded submanifold. This implies that S is a k-dimensional embedded submanifold. □

Next, recall that a continuous map is called proper if the preimages of compact subsets are compact (see Definition 4.4.3). We note the following topological result:

Lemma 7.1.11. *Let $f : X \to Y$ be a continuous map. Assume that f is proper and injective and that Y is Hausdorff and locally compact (i.e. every point has a compact neighborhood). Then f is a topological embedding and $f(X)$ is closed.*

Proof. We show that f is a closed map, i.e. the image of any closed set is closed. Let $C \subset X$ be a closed set, and let $y \in Y \backslash f(C)$. Since Y is locally compact, y has an open neighborhood V with $\overline{V}$ compact. Since f is proper, $E := f^{-1}(\overline{V})$ is compact in X. Thus, $f(E)$ is compact. Since Y is Hausdorff, $f(E)$ is closed. Let $W = V \backslash f(E)$. One easily checks that W is a neighborhood of y disjoint from $f(C)$, as desired. In particular, $f(X) \subset Y$ is closed. Therefore, $f : X \to f(X)$ is a closed continuous bijection; which implies that it is a homeomorphism. □

Finally, Lemma 7.1.11 and Proposition 7.1.10 give:

Corollary 7.1.12. *The image of a proper injective immersion is a closed embedded submanifold. In particular, the image of an injective immersion from a compact manifold is an embedded submanifold.*

7.2. A weak version of Whitney's Embedding Theorem

As an application of the existence of partitions of unity, and of Corollary 7.1.12, we prove of a weak version of Whitney's Embedding Theorem 2.4.4.

Theorem 7.2.1. *A manifold which admits a finite atlas can be embedded in $\mathbb{R}^n$, for large enough n. In particular, a compact manifold can be embedded in some $\mathbb{R}^n$.*

Proof. Consider a finite atlas $\{(U_i, \varphi_i)\}_{i=1}^k$ on a manifold M. Let $\{\rho_i\}_{i=1}^k$ be a partition of unity subordinated to the cover $\{U_i\}_{i=1}^k$. For each i, define the function

$$\widetilde{\varphi}_i : M \to \mathbb{R}^m, \quad \widetilde{\varphi}_i(x) = \begin{cases} 0, & x \notin U_i; \\ \rho_i(x)\varphi_i(x), & x \in U_i. \end{cases}$$

Since $\widetilde{\varphi}_i$ is smooth on the open sets U_i and $M \backslash \mathrm{supp}(\rho_i)$, it is smooth on M. Let

$$f : M \to \mathbb{R}^{m(k+1)}, \quad f(x) = (\rho_1(x), \ldots, \rho_k(x), \widetilde{\varphi}_1(x), \ldots, \widetilde{\varphi}_k(x)).$$

We prove that f is an injective immersion. Let $x, y \in M$ such that $f(x) = f(y)$. Since $\sum_i \rho_i(x) = 1$, there is i such that $\rho_i(x) > 0$. Since $f(x) = f(y)$, we have that $\rho_i(x) = \rho_i(y) > 0$ and $\widetilde{\varphi}_i(x) = \widetilde{\varphi}_i(y)$. In particular, $x, y \in U_i$, and therefore:

$$\varphi_i(x) = \frac{\widetilde{\varphi}_i(x)}{\rho_i(x)} = \frac{\widetilde{\varphi}_i(y)}{\rho_i(y)} = \varphi_i(y).$$

Since φ_i is injective, we conclude that $x = y$. To see that f is an immersion, consider $x \in M$ and $v \in T_x M$ such that $\mathrm{d}_x f(v) = 0$. This implies that $\mathrm{d}_x \rho_i(v) = 0$ and $\mathrm{d}_x \widetilde{\varphi}_i(v) = 0$. Let i be such that $\rho_i(x) > 0$. Then $x \in U_i$, and note that

$$\mathrm{d}_x \widetilde{\varphi}_i(v) = \mathrm{d}_x(\rho_i \varphi_i)(v) = \mathrm{d}_x \rho_i(v)\varphi_i(x) + \rho_i(x)\mathrm{d}_x \varphi_i(v).$$

Thus, we obtain that $\mathrm{d}_x \varphi_i(v) = 0$. Since φ_i is a diffeomorphism, $v = 0$.

By Corollary 7.1.12, if M is compact then f is an embedding. In the general case consider a proper smooth function $h : M \to \mathbb{R}$, which exists by Corollary 4.4.4.

Then the map $(f, h) : M \to \mathbb{R}^{m(k+1)+1}$ is a proper injective immersion, thus, by Corollary 7.1.12 an embedding. $\square$

7.3. Exercises

Exercise 7.1. If X is compact and Z is Hausdorff, prove that any continuous bijection $f : X \to Z$ is a homeomorphism. *Do not use Lemma 7.1.11!*

Definition 7.3.1. *An immersed submanifold $S \subset M$ is called an **initial submanifold**, if for every smooth map $\varphi : X \to M$, such that $\varphi(X) \subset S$, we have that the induced map $\widetilde{\varphi} : X \to S$, $\widetilde{\varphi}(x) = \varphi(x)$ is smooth.*

$$
\begin{array}{ccc}
 & X & \\
\widetilde{\varphi} \downarrow & & \searrow \varphi \\
S & \xrightarrow{\ i\ } & M
\end{array}
$$

Exercise 7.2. Prove that embedded submanifolds are initial. *Hint: Use Lemma 7.1.5.*

Exercise 7.3. Prove that an initial submanifold has a unique differentiable structure such that the inclusion map is an immersion. *Hint: Use Corollary 7.1.6.*

Exercise 7.4. (a) Construct a smooth map $\chi : (-\epsilon, 1 + \epsilon) \to [0, 1]$ such that

$$\chi|_{(-\epsilon, 0]} = 0, \quad \chi|_{[1, 1+\epsilon)} = 1, \quad \chi(0, 1) = (0, 1),$$

and $\chi : (0, 1) \to (0, 1)$ is a diffeomorphism.
(b) Let M be a manifold and consider two smooth curves $\gamma_1, \gamma_2 : [0, 1] \to M$ such that $\gamma_1(1) = \gamma_2(0)$. Show that the following curve is smooth:

$$\gamma : [0, 2] \longrightarrow M, \quad \gamma(t) = \begin{cases} \gamma_1(\chi(t)), & 0 \le t \le 1 \\ \gamma_2(\chi(t-1)), & 1 \le t \le 2 \end{cases}$$

Recall that a map from a closed interval $f : [a, b] \to M$ is smooth iff there is a smooth map $\widetilde{f} : (a - \epsilon, b + \epsilon) \to M$, for some $\epsilon > 0$, such that $\widetilde{f}|_{[a,b]} = f$.
(c) Consider a subset in $\mathbb{R}^2$ which looks like the letter T:

$$S = \{(0, t) \ : \ t \in \mathbb{R}\} \cup \{(t, 0) \ : \ t \in [0, \infty)\}.$$

Prove that S is an immersed submanifold of $\mathbb{R}^2$.
(d) Using item (b) prove that S is not an initial submanifold of $\mathbb{R}^2$.

Exercise 7.5. Consider the six curves from Example 7.1.4. Show that C_b, C_c, C_d, and C_f are not initial submanifold. Explain why C_a and C_e are initial manifolds (for C_e, you do not need to give a rigorous proof). Show that C_e is not an embedded submanifold (note that C_e is a compact subset).

Exercise 7.6. Let M be a smooth manifold, and let $X \subset M$ be a subset which is at most countable. Prove that X is an initial submanifold of M. Prove that X is an embedded submanifold iff it is discrete, i.e. for every point $x \in X$ has an open neighborhood $U \subset M$ such that $X \cap U = \{x\}$.

Exercise 7.7. For $a \in \mathbb{R} \backslash \mathbb{Q}$, consider the map from Exercise 6.2:

$$f_a : \mathbb{R} \longrightarrow S^1 \times S^1, \quad f_a(t) := (e^{it}, e^{ita}).$$

Show that $f_a(\mathbb{R})$ is an initial submanifold of the 2-torus $S^1 \times S^1$.

For the following exercises, we consider another class of submanifolds. This notion is not standard; however, it seems useful because it is less general than initial submanifold, but it includes the interesting examples of leaves of foliations, Lie subgroups and orbits of smooth actions of Lie groups.

Definition 7.3.2. *A subset $S \subset M$ is called a **leaf-like submanifold** of codimension l, if for every $p \in S$ there exists a submersion $f : U \to \mathbb{R}^l$, where U is an open neighborhood of p in M, and there exists a subset $\Lambda \subset \mathbb{R}^l$ which is at most countable such that*

$$U \cap S = f^{-1}(\Lambda).$$

Exercise 7.8. Prove that the initial submanifold from Exercise 7.7 is a leaf-like submanifold of codimension one.

Exercise 7.9. Prove that an embedded submanifold $S \subset M$ of codimension l is a leaf-like submanifold of codimension l.

Exercise 7.10. Show that the initial submanifold C_e from Example 7.1.4 is not a leaf-like submanifold.

Exercise 7.11. Prove that any leaf-like submanifold is an initial submanifold. *Don't forget second countability!*

Exercise 7.12. Prove that a connected leaf-like submanifold $S \subset M$ which is closed (as a subset of M) is also embedded.

The following exercise represents the converse of Corollary 4.4.2, and gives an algebraic description of closed embedded submanifolds (recall also Theorem 5.3.2).

Exercise 7.13. Let $\varphi : S \to M$ be a smooth map such that the pullback map

$$\varphi^* : C^\infty(M) \longrightarrow C^\infty(S), \quad \varphi^*(f) = f \circ \varphi,$$

is surjective. The goal of this exercise is to show that φ is a closed embedding, and so, by Proposition 7.1.10 and Corollary 7.1.12, $\varphi(S) \subset M$ is a closed embedded submanifold and $\varphi : S \to \varphi(S)$ is a diffeomorphism. Prove this by showing that the following hold:

(a) φ is injective.
(b) φ is an immersion (argue "by contradiction": if it is not an immersion prove that there is $x \in S$ and a vector $v \in T_x S$ such that $v \neq 0$ but $v(g) = 0$ for all $g \in C^\infty(S)$).
(c) φ is an embedding (show first that open sets of the form $f^{-1}(0, \infty)$, $f \in C^\infty(S)$ form a basis for the topology of S; then show that the image via φ of such open sets is open in $\varphi(S)$).
(d) $\varphi(S)$ is closed.
(e) Show that $\varphi(S)$ is a closed embedded submanifold (apply Corollary 7.1.12).

8.1. The tangent bundle

Definition 8.1.1. *The **tangent bundle** of a smooth manifold M is the set of all tangent vectors:*
$$TM := \bigsqcup_{p \in M} T_p M.$$
*The **canonical projection** is the map*
$$\pi : TM \longrightarrow M, \qquad v \in T_p M \implies \pi(v) = p.$$

Theorem 8.1.2. *Let M be a smooth manifold of dimension m. The tangent bundle TM has a unique $2m$-dimensional differentiable structure such that, for any chart (U, φ) on M, the pair $(TU, \mathrm{d}\varphi)$ is a smooth chart on TM, where*
$$\mathrm{d}\varphi : TU \longrightarrow T\varphi(U) = \varphi(U) \times \mathbb{R}^m, \quad v \in T_p M \mapsto (\varphi(p), \mathrm{d}_p\varphi(v)).$$
Moreover, the canonical projection $\pi : TM \to M$ is a surjective submersion.

Proof. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be a maximal atlas on M. We will apply Proposition 3.1.1 to the collection $\widetilde{\mathcal{A}} := \{(TU_\alpha, \mathrm{d}\varphi_\alpha)\}_{\alpha \in I}$. Note that: (1) the collection $\{TU_\alpha\}_{\alpha \in I}$ covers TM; (2) the maps $\mathrm{d}\varphi_\alpha : TU_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^m$ are bijections; (3)
$$\mathrm{d}\varphi_\alpha(TU_\alpha \cap TU_\beta) = \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m$$
are open sets; (4) the transition maps of $\widetilde{\mathcal{A}}$ are smooth because they are the differentials of the transition maps of $\mathcal{A}$:
$$\mathrm{d}\varphi_\beta \circ (\mathrm{d}\varphi_\alpha)^{-1} = \mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1}) : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^m.$$
So the conditions (1)-(4) from Proposition 3.1.1 are satisfied, therefore TM has a unique topology for which the maps $(TU_\alpha, \mathrm{d}\varphi_\alpha)$ are (topological) charts. For this topology, the map $\pi : TM \to M$ is continuous: for any open set $V \subset M$, we have that $\pi^{-1}(V)$ is open:
$$\pi^{-1}(V) = \bigcup_{\alpha \in I} \pi^{-1}(V \cap U_\alpha) = \bigcup_{\alpha \in I} (\mathrm{d}\varphi_\alpha)^{-1}\left(\varphi_\alpha(U_\alpha \cap V) \times \mathbb{R}^m\right).$$

To show that the topology is second countable, by Lemma 4.3.4, we can consider a countable refinement $\{V_k\}_{k \geq 1}$ of the cover $\{U_\alpha\}_{\alpha \in I}$, i.e. $V_k \subset U_{i_k}$ for some $i_k \in I$. If $\{B_l\}_{l \geq 0}$ is a countable basis of $\mathbb{R}^m \times \mathbb{R}^m$, then $\{\pi^{-1}(V_k) \cap (\mathrm{d}\varphi_{i_k})^{-1}(B_l)\}_{k,l \geq 1}$ is a countable basis for the topology on TM.

Next, we show that the topology is Hausdorff. Let $v, w \in TM$, with $v \neq w$. If $\pi(v) = \pi(w)$, consider a chart $(U_\alpha, \varphi_\alpha)$ such that $\pi(v) \in U_\alpha$. Then $v, w \in TU_\alpha$. Since $\mathrm{d}\varphi_\alpha : TU_\alpha \rightarrow \varphi_\alpha(U) \times \mathbb{R}^m$ is a homeomorphism, and the second space is Hausdorff, v and w can be separated by open sets in TU_α. If $\pi(v) \neq \pi(w)$, let $V, W \subset M$ be open sets separating $\pi(v)$ and $\pi(w)$. Since π is continuous, $\pi^{-1}(V)$ and $\pi^{-1}(W)$ are open sets separating v and w.

Proposition 3.1.1 implies now that TM has a unique $2m$-dimensional differentiable structure such that $\widetilde{\mathcal{A}}$ is a smooth atlas. Finally, we check that $\pi : TM \rightarrow M$ is a submersion. Note that in the pair of charts $(TU_\alpha, \mathrm{d}\varphi_\alpha)$ and $(U_\alpha, \varphi_\alpha)$, π becomes the canonical projection:

$$\varphi_\alpha \circ \pi \circ (\mathrm{d}\varphi_\alpha)^{-1} = \mathrm{pr}_m^{2m} : \varphi_\alpha(U) \times \mathbb{R}^m \longrightarrow \varphi_\alpha(U).$$

Thus π is locally a submersion; hence it is a submersion. $\qquad\qquad\square$

8.2. Vector bundles

Note that the tangent bundle TM of a manifold M has more structure: it is a collection of vector spaces T_pM *depending smoothly* on $p \in M$. Such structures arise in many natural constructions, therefore we introduce the following concept:

Definition 8.2.1. *A smooth **vector bundle of rank** r consists of a surjective submersion $\pi : E \rightarrow M$ satisfying*

- *For each $p \in M$, the **fibre** of π over p, denoted by*

$$E_p := \pi^{-1}(p),$$

  is endowed with the structure of a real vector space of dimension r.
- *Around every point in M, there is an open set U and a diffeomorphism*

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^r,$$

  such that the following diagram commutes:

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \ \Phi\ \ } & U \times \mathbb{R}^r \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
U & \xrightarrow[\ \ \mathrm{id}_U\ \ ]{} & U
\end{array}
\qquad \mathrm{pr}_1 \circ \Phi = \pi,
$$

  where $\mathrm{pr}_1(x, v) = x$, and such that for every $p \in U$, the restriction

$$\mathrm{pr}_2 \circ \Phi|_{E_p} : E_p \longrightarrow \mathbb{R}^r$$

  is a linear isomorphism.

Here are some examples:

Example 8.2.2. The **trivial** vector bundle of rank r over a manifold M is

$$\mathrm{pr}_1 : M \times \mathbb{R}^r \longrightarrow M,$$

where the vector space structure on $\{p\} \times \mathbb{R}^r$ is the obvious one.

Example 8.2.3. The tangent bundle TM of a manifold M is a vector bundle of $\mathrm{rank}(TM) = \dim(M)$.

Example 8.2.4. The group $(\{\pm 1\}, \cdot)$ acts on $S^1 \times \mathbb{R}$ via

$$\pm 1 \cdot (z, x) = (\pm z, \pm x).$$

The action is free, and the quotient $E := (S^1 \times \mathbb{R})/\{\pm 1\}$ is diffeomorphic to the Möbius band. Then E is a vector bundle over S^1 with surjective submersion:

$$\pi : E \longrightarrow S^1, \quad \pi([z, x]) = z^2.$$

For $w \in S^1$, fix $z \in S^1$ one of the two solutions of the equation $z^2 = w$. The vector space structure on the fibre $E_w = \{[z, x] \ : \ x \in \mathbb{R}\}$ is given by:

$$a[z, x] + b[z, y] = [z, ax + by], \quad a, \ b \in \mathbb{R}, \quad [z, x], \ [z, y] \in E_w.$$

Next, we introduce vector bundle morphisms:

Definition 8.2.5. *A **morphism** between two vector bundles $\pi_E : E \to M$ and $\pi_F : F \to N$ is a pair of smooth maps $\Phi : E \to F$ and $\varphi : M \to N$ such that for all $p \in M$ we have $\Phi(E_p) \subset F_{\varphi(p)}$ and that the induced map*

$$\Phi|_{E_p} : E_p \longrightarrow F_{\varphi(p)}$$

*is linear. If Φ and φ are diffeomorphisms, then (Φ, φ) is said to be a **vector bundle isomorphism**, and the two bundles are said to be **isomorphic**.*

Remark 8.2.6. Note that the condition $\Phi(E_p) \subset F_{\varphi(p)}$ in the definition above is equivalent to the commutativity of the following:

$$
\begin{array}{ccc}
E & \xrightarrow{\ \Phi\ } & F \\
{\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\
M & \xrightarrow{\ \varphi\ } & N
\end{array}
\qquad \pi_F \circ \Phi = \varphi \circ \pi_E
$$

Example 8.2.7. If $\varphi : M \to N$ is a smooth map, then $(\mathrm{d}\varphi, \varphi)$, with $\mathrm{d}f : TM \to TN$, is a vector bundle morphism.

Definition 8.2.8. *The **restriction** of a vector bundle $\pi : E \to M$ to an open set $U \subset M$ is the vector bundle over U*

$$E|_U := \pi^{-1}(U), \quad \pi|_{E|_U} : E|_U \longrightarrow U.$$

Definition 8.2.9. *A vector bundle isomorphic to a trivial vector bundle is called **trivializable**.*

Remark 8.2.10. Every vector bundle $\pi : E \to M$ is locally trivializable. This holds by definition, since M can be covered by oven subsets U such that there exists a vector bundle isomorphism $\Phi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^r$ covering id_U. Such a local isomorphism to the trivial bundle is called a **local trivialization**.

8.3. Sections of vector bundles

Definition 8.3.1. *A **smooth section** of a vector bundle $\pi : E \to M$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$; in other words: $\sigma(p) \in E_p$, for all $p \in M$.*
We denote the space of all sections of $\pi : E \to M$ by $\Gamma(E)$.

Note that every vector bundle $\pi : E \to M$ has the **zero-section** $0 \in \Gamma(E)$ which sends a point $p \in M$ to the origin of the vector space E_p: $p \mapsto 0_p \in E_p$.

The space of sections $\Gamma(E)$ forms a **module** over the commutative algebra of smooth functions $C^\infty(M)$, with the point-wise defined operations:

$$+ : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E), \quad (\sigma_1 + \sigma_2)(p) := \sigma_1(p) + \sigma_2(p) \in E_p,$$
$$\cdot : C^\infty(M) \times \Gamma(E) \longrightarrow \Gamma(E), \quad (f \cdot \sigma)(p) := f(p)\sigma(p) \in E_p,$$

where, on the right hand side, the vector space operations on E_p are used. The zero-section plays the role of the zero-element of $\Gamma(E)$.

Consider the case of a trivial bundle $M \times \mathbb{R}^r \to M$. Its sections are smooth maps of the form:

$$\sigma : M \longrightarrow M \times \mathbb{R}^r, \quad \sigma(p) = (p, (f_1(p), \ldots, f_r(p))) \in M \times \mathbb{R}^r.$$

Thus, we have a one-to-one correspondence between $\Gamma(M \times \mathbb{R}^r)$ and $C^\infty(M; \mathbb{R}^r)$. Using the structure of a $C^\infty(M)$-module of $\Gamma(M \times \mathbb{R}^r)$, we can decompose any section σ uniquely as

$$\sigma = f_1\sigma_1 + \ldots + f_r\sigma_r,$$

where, for $1 \leq i \leq r$, $\sigma_i \in \Gamma(M \times \mathbb{R}^r)$ represents the section

$$\sigma_i(p) := (p, (0, \ldots, 1, \ldots, 0)) \in M \times \mathbb{R}^r,$$

with 1 is on the i-th position. In other words:

$$\Gamma(M \times \mathbb{R}^r) \text{ is a free } C^\infty(M)\text{-module with basis } \sigma_1, \ldots, \sigma_r.$$

Since a general vector bundle $\pi : E \to M$ is locally trivializable, there is a cover of M by open sets U such that $\Gamma(E|_U)$ is a free $C^\infty(U)$-module of rank r.

8.4. Frames

Definition 8.4.1. *Let $\pi : E \to M$ be a vector bundle. A **local frame** over $U \subset M$ is a smooth map*

$$\gamma : E|_U \longrightarrow \mathbb{R}^r,$$

such that, for all $p \in U$, $\gamma|_{E_p} : E_p \to \mathbb{R}^r$ is a linear isomorphism.

Lemma 8.4.2. *Let $\pi : E \to M$ be a vector bundle, and let $U \subset M$ be an open set. There is a one-to-one correspondence between the following objects:*

(1) local trivializations $\Phi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^r$;
(2) local frames $\gamma : E|_U \to \mathbb{R}^r$;
(3) sections $\sigma_1, \ldots, \sigma_r \in \Gamma(E|_U)$ such that, for every $p \in U$, $\sigma_1(p), \ldots, \sigma_r(p)$ forms a basis of E_p.

This correspondence is such that, given a local frame γ, the corresponding local trivialization is $\Phi := \pi \times \gamma : E|_U \to U \times \mathbb{R}^r$, and the corresponding set of sections $\sigma_1, \ldots, \sigma_r$ is given by $\sigma_i(p) := \gamma_p^{-1}(e_i)$, where $e_1, \ldots, e_r$ is the standard basis of $\mathbb{R}^r$.

Proof. By passing to the restriction $E|_U$, we may assume that $U = M$.

First, we explain the equivalence between (a) and (b). Consider a trivialization $\Phi : E \xrightarrow{\sim} M \times \mathbb{R}^r$. Then, by the definition of Φ, we have that $\Phi = \pi \times \gamma$, where $\gamma : E \to \mathbb{R}^r$ is a frame. Conversely, let $\gamma : E \to \mathbb{R}^r$ be a frame on E over M. We need to show that $\Phi = \pi \times \gamma : E \to M \times \mathbb{R}^r$ is a diffeomorphism. Since Φ is a bijection, and the manifolds have the same dimension, invoking the inverse function theorem, it suffices to show that $\pi \times \gamma$ is an immersion. Let $v \in T_e E$ be such that

$$0 = \mathrm{d}_e\,(\pi \times \gamma)\,(v) = (\mathrm{d}_e\pi(v), \mathrm{d}_e\gamma(v)).$$

By Corollary 6.3.6, $\ker(\mathrm{d}_e\pi) = T_e(E_{\pi(e)})$. Therefore, $v \in T_e(E_{\pi(e)})$, and so $0 = \mathrm{d}_e\gamma(v) = \mathrm{d}_e\big(\gamma|_{E_{\pi(e)}}\big)(v)$. Since $\gamma|_{E_{\pi(e)}}$ is a diffeomorphism, we have that $v = 0$.

We explain now the equivalence between (b) and (c). Let $\gamma : E \to \mathbb{R}^r$ be a frame on E. By the first part, $\Phi = \pi \times \gamma : E \to M \times \mathbb{R}^r$ is a diffeomorphism, therefore the maps:

$$\sigma_i : M \to E, \quad \sigma_i(p) := \Phi^{-1}(p, (0, \ldots, 1, \ldots, 0))$$

are smooth, and therefore $\sigma_i \in \Gamma(E)$. Now, for all $p \in M$, $\gamma_p(\sigma_i(p)) = e_i$, therefore $\sigma_1(p)$, ..., $\sigma_r(p)$ form a basis of E_p. Conversely, consider σ_1, ..., $\sigma_r \in \Gamma(E)$ which form a point-wise basis of E. Consider the map

$$\Psi : M \times \mathbb{R}^r \longrightarrow E, \quad \Psi(p, (a^1, \ldots, a^r)) := a^1\sigma_1(p) + \ldots + a^r\sigma_r(p).$$

Clearly, Ψ is a vector bundle morphism. An argument similar to the proof of the first part shows that Ψ is also a diffeomorphism; hence Ψ is a vector bundle isomorphism. The corresponding frame is the map $\gamma : E \to \mathbb{R}^r$ determined by $\Psi^{-1} = \pi \times \gamma$. $\qquad\square$

The Lemma implies that:

Corollary 8.4.3. *For a vector bundle $\pi : E \to M$, the following are equivalent:*

(1) E is trivializable;
(2) E admits a global frame $\gamma : E \to \mathbb{R}^r$;
(3) there are smooth sections $\sigma_1, \ldots, \sigma_r \in \Gamma(E)$ such that, for all $p \in M$, their values at p form a basis of E_p.

Definition 8.4.4. *Let $\pi : E \to M$ be a vector bundle of rank r. Let $\gamma : E|_U \to \mathbb{R}^r$ be a local frame over the open set $U \subset M$, and let $\sigma_1, \ldots, \sigma_r \in \Gamma(E|_U)$ denote the corresponding sections which, at every $p \in U$, form a basis $\sigma_1(p), \ldots, \sigma_r(p)$ of E_p.*

*The **local representation** of a section $\sigma \in \Gamma(E)$ in this local frame is the decomposition of $\sigma|_U \in \Gamma(E|_U)$ as a linear combination*

$$\sigma|_U = f_1\sigma_1 + \ldots + f_r\sigma_r,$$

with coefficients $f_i \in C^\infty(U)$. The coefficients are determined by

$$(f_1, \ldots, f_r) = \gamma \circ \sigma|_U : U \longrightarrow \mathbb{R}^r.$$

Let us mention that vector bundles are determined by their module of smooth sections; in fact, one can characterize algebraically which modules arise from vector bundles. Without introducing the necessary terminology, we state the following:

Theorem ♣ 8.4.5 (Serre-Swan)**.** *Let M be a smooth manifold. The correspondence $E \mapsto \Gamma(E)$ is a one-to-one correspondence between vector bundles over M (up to isomorphism) and finitely generated projective modules over $C^\infty(M)$ (up to isomorphism).*

8.5. Vector fields as sections of the tangent bundle

Definition 8.5.1. *A **vector field** on a smooth manifold M is a smooth section of the tangent bundle $\pi : TM \to M$; in other words it is a smooth map*

$$X : M \longrightarrow TM \quad \text{such that } X_p \in T_pM, \text{ for all } p \in M.$$

We use the notation X_p rather than $X(p)$ for the value of X at $p \in M$.

The space of all smooth vector fields is denoted by

$$\mathfrak{X}(M) := \Gamma(TM).$$

Let (U, φ) be a chart on M inducing coordinates $\varphi = (x^1_\varphi, \dots, x^m_\varphi)$, with $x^i_\varphi \in C^\infty(U)$. The chart induces a canonical frame on TM over U:

$$\frac{\partial}{\partial x^1_\varphi}, \quad \dots, \quad \frac{\partial}{\partial x^m_\varphi} \in \mathfrak{X}(U).$$

The local representation of a vector field $X \in \mathfrak{X}(M)$ is given by:

$$X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i_\varphi}, \quad X^i \in C^\infty(U),$$

where the coefficients are smooth functions on U.

The notion of trivializable specializes to:

Definition 8.5.2. *If the tangent bundle of a manifold is trivializable, then the manifold is said to be* **parallelizable**.

Example 8.5.3. Clearly, $\mathbb{R}^m$ is parallelizable. A global frame on $T\mathbb{R}^m$ is given by the vector fields $\frac{\partial}{\partial x^i}$, with $1 \le i \le m$.

Example 8.5.4. The circle S^1 is also parallelizable: if $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ denotes the "angle coordinate" on S^1, then a global frame is given by the vector field

$$\frac{\partial}{\partial \theta} \in \mathfrak{X}(S^1).$$

Even though the θ makes sense as a coordinate only on sets of the form $S^1 \backslash \{e^{i\beta}\}$, the vector field $\frac{\partial}{\partial \theta}$ makes sense globally, as it can be defined as the derivative of rotation:

$$\frac{\partial}{\partial \theta}\big|_{e^{i\alpha}} := \frac{\mathrm{d}}{\mathrm{d}\theta} e^{i\theta}\big|_{\theta=\alpha}.$$

Example 8.5.5. Products of parallelizable manifolds are parallelizable. In particular the m-dimensional torus $T^m = (S^1)^m$ is parallelizable. Using the angle coordinates $(\theta^1, \dots, \theta^m) \in (\mathbb{R}/(2\pi\mathbb{Z}))^m$, we obtain the global frame

$$\frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^m} \in \mathfrak{X}(T^m).$$

The 2-dimensional sphere S^2 is not parallelizable. This is a consequence of the following result (see Corollary 8.4.3):

Theorem ♣ 8.5.6 (The Hairy Ball Theorem). *Every smooth vector field $X \in \mathfrak{X}(S^2)$ has at least one zero, i.e. a point $p \in S^2$ such that $X_p = 0$.*

Here is an interesting application of this result. Thinking about the surface of the Earth as a manifold diffeomorphic to S^2, and assuming that the velocity of the wind can be represented by a smooth vector field $X \in \mathfrak{X}(S^2)$, this result implies that there always exists a point on Earth where the wind is not blowing (most likely not in this part of the World, but you can check it out here: [**22**]).

It is known precisely which spheres are parallelizable:

Theorem ♣ 8.5.7. *The only parallelizable spheres are S^0, S^1, S^3 and S^7 (i.e. S^{2^c-1}, with $c = 0, 1, 2, 3$).*

You can check that this is a consequence of the following classical theorem:

Theorem ♣ 8.5.8. *On the sphere S^n the maximum number $k = k(n)$ of vector fields $X_1, \ldots, X_k$ such that at every $p \in S^n$ the vectors $X_{1,p}, \ldots, X_{k,p} \in T_p S^n$ are linearly independent is given as follows: write $n + 1 = (2a + 1)2^{4b+c}$, with $a, b \in \mathbb{N}$ and $c \in \{0, 1, 2, 3\}$, then $k(n) = 8b + 2^c - 1$.*

If n is even then $k(n) = 0$, so every vector field on S^n has a zero. In general, note that $k(n)$ is quite small compared to n; in particular, $k(n) \leq 2 \log_2(n+1) + 1$.

8.6. Exercises

Exercise 8.1. Prove that TS^1 is isomorphic to the trivial vector bundle $S^1 \times \mathbb{R}$.

Exercise 8.2. Let $f : M \to N$ be a smooth map. Prove that $\mathrm{d}f : TM \to TN$ is a smooth vector bundle morphism.

Exercise 8.3. Let M be a smooth manifold. Prove that there exists a vector field V on the manifold TM (i.e. $V \in \mathfrak{X}(TM) = \Gamma\big(T(TM)\big)$) satisfying:

$$\mathrm{d}_v \pi(V_v) = v, \quad \forall\, v \in TM.$$

Hint: construct V locally, and use a partition of unity.

Exercise 8.4. Recall that the m-dimensional real projective space is the space of all lines through the origin in $\mathbb{R}^{m+1}$, with the manifold structure from Example 3.2.1:

$$\mathbb{P}^m(\mathbb{R}) := \{l \ : \ l \subset \mathbb{R}^{m+1} \text{ is a line with } 0 \in l\}.$$

The **tautological line bundle** over $\mathbb{P}^m(\mathbb{R})$ is the rank 1 vector bundle $\pi : L(m) \to \mathbb{P}^m(\mathbb{R})$ whose fiber over a line l is the line l:

$$L(m) := \{(l, v) : l \in \mathbb{P}^m(\mathbb{R}), \ \ v \in l\}.$$

(a) Sketch a picture of $L(1)$.
(b) Show that $\pi : L(m) \to \mathbb{P}^m(\mathbb{R})$ is indeed a vector bundle of rank one, such that the following map is morphism of vector bundles:

$$
\begin{array}{ccc}
L(m) & \xrightarrow{\ \Phi\ } & \mathbb{P}^m(\mathbb{R}) \times \mathbb{R}^{m+1} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
\mathbb{P}^m(\mathbb{R}) & \xrightarrow{\ \mathrm{id}\ } & \mathbb{P}^m(\mathbb{R})
\end{array}
, \quad \Phi(l, v) := (l, v) \in \mathbb{P}^m(\mathbb{R}) \times \mathbb{R}^{m+1}.
$$

(c) Sketch a picture of the maps above for $m = 1$.
(d) Prove that $L(1)$ is not trivializable (use Corollary 8.4.3).
(e) Prove that $L(m)$ is not trivializable, by using the morphism of vector bundles:

$$
\begin{array}{ccc}
L(1) & \xrightarrow{\ \Psi\ } & L(m) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\
\mathbb{P}^1(\mathbb{R}) & \longrightarrow & \mathbb{P}^m(\mathbb{R})
\end{array}
,
$$

$$\Psi(l, v) := (l \times \{0\}, (v, 0)) \in \mathbb{P}^m(\mathbb{R}) \times \mathbb{R}^{m+1}.$$

The following exercise gives a proof of the Hairy Ball Theorem 8.5.6. Any such proof requires some knowledge of algebraic topology. We will give a proof using only the winding number of a curve, which should be familiar from a standard course in one-variable complex analysis.

Definition 8.6.1. *Let $\gamma : S^1 \to \mathbb{C}^*$ be a smooth closed curve, where $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. The **winding number** of γ is defined by the formula:*

$$W(\gamma) := \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z} \in \mathbb{Z}.$$

This number is an integer, and counts how many of times γ goes around $0 \in \mathbb{C}$ in the positive direction, minus the number of times it goes around 0 in the negative direction.

Note that the winding number is a **homotopy invariant**: if $\gamma_t : S^1 \to \mathbb{C}^*$ is a family of closed curves depending continuously on $t \in [0,1]$, then $W(\gamma_0) = W(\gamma_1)$. This is because $W(\gamma_t)$ has integer values, and depends continuously on t; thus $W(\gamma_t)$ must be constant.

Exercise 8.5. For a vector field $V = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$, we denote by $\gamma_V : S^1 \to \mathbb{C}$ the closed curve

$$\gamma_V(\cos(\theta), \sin(\theta)) := a(\cos(\theta), \sin(\theta)) + ib(\cos(\theta), \sin(\theta)) \in \mathbb{C}.$$

(a) Let $V \in \mathfrak{X}(\mathbb{R}^2)$ be a vector field which is nowhere zero, i.e. $V_{(x,y)} \neq 0$ for all $(x,y) \in \mathbb{R}^2$. Construct a continuous family of closed curves $\gamma_t : S^1 \to \mathbb{C}^*$, with $t \in [0,1]$, such that $\gamma_1 = \gamma_V$ and $\gamma_0 = \lambda$, where $\lambda \in \mathbb{C}^*$ is a constant. Conclude that $W(\gamma_V) = 0$ (*Hint: look at the vector field V on circles of radius t*).

(b) Consider the stereographic atlas on S^2 with two charts from Example 1.2.7. Let $X \in \mathfrak{X}(S^2)$ be a vector field, and consider its representatives in the these two charts:

$$V^s = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2),$$

$$V^n = c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2).$$

Show that, at $(x,y) \neq (0,0)$, these local representatives are related by:

$$c(x,y) = (y^2 - x^2)a\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) - 2xy \cdot b\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right),$$

$$d(x,y) = -2xy \cdot a\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) + (x^2 - y^2)b\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

Conclude the following relation between the corresponding curves:

$$\gamma_{V^n} = e^{2i\theta}(-\overline{\gamma}_{V^s}).$$

(c) Prove that every vector field $X \in \mathfrak{X}(S^2)$ has at least one zero. *Hint: assume that a nowhere vanishing vector field X exists. By applying (a) to V^s, and then (b), show that γ_{V^n} is homotopic to the curve $e^{2i\theta}$. Deduce that $W(\gamma_{V^n}) = 2$, which contradicts (a).*

LECTURE 9

9.1. Vector fields as derivations

Vector fields have a natural action on smooth functions:

Definition 9.1.1. *The **derivative** along a vector field $X \in \mathfrak{X}(M)$ of a smooth function $f \in C^\infty(M)$ is the function $X(f) \in C^\infty(M)$, $M \ni p \mapsto X_p(f)$.*

The fact that $X(f)$ is indeed a smooth function can be checked locally. Represent X in a local chart $(U, \varphi = (x_\varphi^1, \ldots, x_\varphi^m))$ on M,

$$X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x_\varphi^i}, \quad X^i \in C^\infty(U),$$

then we have the explicit formula:

$$X(f)|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x_\varphi^i}(f|_U) = \sum_{i=1}^m X^i \frac{\partial f \circ \varphi^{-1}}{\partial x^i} \circ \varphi$$

which shows that $X(f)|_U$ is smooth; and since (U, φ) is arbitrary, $X(f)$ is smooth.

The resulting operation $f \mapsto X(f)$ on $C^\infty(M)$ has the following property:

Definition 9.1.2. *A **derivation** of $C^\infty(M)$ is a linear map*

$$D : C^\infty(M) \longrightarrow C^\infty(M)$$

satisfying the rule

$$D(f \cdot g) = f \cdot D(g) + D(f) \cdot g, \quad \text{for all } f, g \in C^\infty(M).$$

Proposition 9.1.3. *Every derivation D of $C^\infty(M)$ is the derivative along a unique vector field $X \in \mathfrak{X}(M)$.*

Proof. Let $X \in \mathfrak{X}(M)$. Because X_p is a derivation for all $p \in M$:

$$X_p(f \cdot g) = f(p)X_p(g) + X_p(f)g(p),$$

we obtain that the derivative along X is indeed a derivation of $C^\infty(M)$:

$$X(f \cdot g) = f \cdot X(g) + X(f) \cdot g.$$

Assume that $X, Y \in \mathfrak{X}(M)$ induces the same derivation, i.e. $X(f) = Y(f)$ for all $f \in C^\infty(M)$. Hence $X_p(f) = Y_p(f)$, for all $f \in C^\infty(M)$ and $p \in M$, and so $X_p = Y_p$ for all $p \in M$. Thus $X = Y$, which shows the uniqueness assertion.

Finally, consider a derivation D of $C^\infty(M)$. Evaluating the equality $D(fg) = fD(g) + D(f)g$ at a point $p \in M$, we obtain that the linear map

$$X_p : C^\infty(M) \to \mathbb{R}, \quad X_p(f) := D(f)(p)$$

is a derivation at p. Thus, we obtain a map $X : M \to TM$, with $X_p \in T_pM$, which satisfies that, for all $f \in C^\infty(M)$, the map $M \ni p \mapsto X_p(f)$ is smooth (because this map is $D(f)$). We show that this implies smoothness of X. Consider a chart $(U, \varphi = (x_\varphi^1, \dots, x_\varphi^m))$ around a point $p \in M$. Note that, even if X is not smooth, we can still decompose $X|_U$ in the corresponding frame:

$$X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x_\varphi^i},$$

just that the coefficients $X^i : U \to \mathbb{R}$ might not be smooth. For $1 \le i \le m$, let $\widetilde{x}_\varphi^i \in C^\infty(M)$ have the same germ as x_φ^i at p, i.e. $\widetilde{x}_\varphi^i|_V = x_\varphi^i|_V$ for some neighborhood $V \subset U$ of p. Then

$$X(\widetilde{x}_\varphi^i)|_V = X|_V(x_\varphi^i|_V) = X^i|_V.$$

Since $X(\widetilde{x}_\varphi^i) \in C^\infty(M)$, we have shown that $X^i|_V$ is smooth, hence $X|_V : V \to TM$ is smooth. So X is smooth because it is smooth around any $p \in M$. This concludes the proof, because D and X act the same on $C^\infty(M)$. $\qquad\square$

From now on, we will identify the space of vector fields with the space of derivations of $C^\infty(M)$.

9.2. The Lie bracket

Let $X, Y \in \mathfrak{X}(M)$ be two vector fields, which we regard as derivations:

$$X, Y : C^\infty(M) \longrightarrow C^\infty(M).$$

In general, their composition is not a derivation. The failure for this to hold is

$$(X \circ Y)(f \cdot g) - (X \circ Y)(f) \cdot g - f \cdot (X \circ Y)(g) = X(f) \cdot Y(g) + Y(f) \cdot X(g).$$

Note that the right hand side is symmetric in X and Y. Therefore, if we subtract from this the same equation with X and Y interchanged, we obtain:

$$(X \circ Y - Y \circ X)(f \cdot g) = (X \circ Y - Y \circ X)(f) \cdot g + f \cdot (X \circ Y - Y \circ X)(g).$$

Thus, $X \circ Y - Y \circ X$ is again a derivation of $C^\infty(M)$, and so, by Proposition 9.1.3, again a vector field.

Definition 9.2.1. *The **Lie bracket** is the following operation on vector fields:*

$$[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad [X, Y] := X \circ Y - Y \circ X.$$

*The Lie bracket $[X, Y]$ is also called the **commutator** of X and Y (because it measures whether the derivations X and Y commute, i.e. whether $X \circ Y = Y \circ X$).*

The Lie bracket has the following properties (the proof is left to the reader):

Proposition 9.2.2. *The Lie bracket*

*(1) is **skew-symmetric**:*

$$[X, Y] = -[Y, X];$$

*(2) is $\mathbb{R}$-**bilinear**:*

$$[X, aY + bZ] = a[X, Y] + b[X, Z];$$

*(3) satisfies the **Jacobi identity**:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0;$$

*(4) satisfies the **Leibniz rule**:*

$$[X, fY] = f[X, Y] + X(f)Y;$$

for all $X, Y, Z \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$ and $f \in C^\infty(M)$.

Remark 9.2.3. A vector space V endowed with an operation $[\cdot, \cdot] : V \times V \to V$ satisfying (1), (2) and (3) is called a **Lie algebra**. This notion will be discussed in Lecture 11.

Let us write the Lie bracket $[X, Y]$ of two vector fields $X, Y \in \mathfrak{X}(M)$ in a local chart $(U, \varphi = (x_\varphi^1, \ldots, x_\varphi^m))$ on M. Denote

$$X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x_\varphi^i}, \qquad Y|_U = \sum_{i=1}^m Y^i \frac{\partial}{\partial x_\varphi^i}.$$

Then, for any $f \in C^\infty(U)$ we have that:

$$X \circ Y(f)|_U = X\left(\sum_{j=1}^m Y^j \frac{\partial f}{\partial x_\varphi^j} \right) = \sum_{i=1}^m \sum_{j=1}^m X^i \frac{\partial}{\partial x_\varphi^i}\left(Y^j \frac{\partial f}{\partial x_\varphi^j} \right) =$$

$$= \left(\sum_{i=1}^m \sum_{j=1}^m X^i \frac{\partial Y^j}{\partial x_\varphi^i} \frac{\partial}{\partial x_\varphi^j} \right)(f) + \sum_{i=1}^m \sum_{j=1}^m X^i Y^j \frac{\partial^2 f}{\partial x_\varphi^i \partial x_\varphi^j}.$$

Note that there are second order derivatives of f appearing, and this is the reason why $X \circ Y$ is not a vector field. However, since $\frac{\partial^2}{\partial x_\varphi^i \partial x_\varphi^j} = \frac{\partial^2}{\partial x_\varphi^j \partial x_\varphi^i}$, these terms cancel in the commutator $[X, Y]|_U$. We obtain the local expression of the Lie bracket:

$$[X, Y]|_U = \sum_{j=1}^m \left(\sum_{i=1}^m X^i \frac{\partial Y^j}{\partial x_\varphi^i} - Y^i \frac{\partial X^j}{\partial x_\varphi^i} \right) \frac{\partial}{\partial x_\varphi^j}.$$

9.3. Related vector fields

Vector fields cannot be transported via smooth maps (only via diffeomorphisms). However, the following relation makes sense in general:

Definition 9.3.1. *Let $\varphi : M \to N$ be a smooth map. A vector field $X \in \mathfrak{X}(M)$ is said to be φ-**related** to a vector field $Y \in \mathfrak{X}(N)$ if, for all $p \in M$,*

$$\mathrm{d}_p\varphi(X_p) = Y_{\varphi(p)}.$$

Recall that a smooth map $\varphi : M \to N$ induces a **pullback map** on smooth functions, denoted:

$$\varphi^* : C^\infty(N) \to C^\infty(M), \quad \varphi^*(f) := f \circ \varphi.$$

Here is an alternative description of related vector fields in terms of the pullback:

Lemma 9.3.2. *Let $\varphi : M \to N$ be a smooth map. Then $X \in \mathfrak{X}(M)$ if φ-related to $Y \in \mathfrak{X}(N)$ iff*

$$X(\varphi^*(f)) = \varphi^*(Y(f)), \quad \forall\, f \in C^\infty(N).$$

Proof. At $p \in M$, the left-hand-side reads:

$$X(\varphi^*(f))(p) = X_p(f \circ \varphi) = (\mathrm{d}_p\varphi(X_p))(f),$$

and the right-hand-side:

$$\varphi^*(Y(f))(p) = Y_{\varphi(p)}(f).$$

The conclusion follows by noting that two vectors in $T_{\varphi(p)}N$ are equal iff they act the same on smooth functions. $\qquad\square$

The following is a very useful result:

Lemma 9.3.3. *Let $\varphi : M \to N$ be a smooth map, and let $X^1, X^2 \in \mathfrak{X}(M)$ and $Y^1, Y^2 \in \mathfrak{X}(N)$ be four vector fields. If X^1 is φ-related to Y^1 and X^2 is φ-related to Y^2, then also $[X^1, X^2] \in \mathfrak{X}(M)$ is φ-related to $[Y^1, Y^2] \in \mathfrak{X}(N)$.*

Proof. Using Lemma 9.3.2, this result is straightforward:

$$\begin{aligned}
\varphi^*([Y^1, Y^2](f)) &= \varphi^*\left(Y^1 \circ Y^2(f)\right) - \varphi^*\left(Y^2 \circ Y^1(f)\right) = \\
&= X^1\left(\varphi^*(Y^2(f))\right) - X^2\left(\varphi^*(Y^1(f))\right) = \\
&= X^1 \circ X^2(\varphi^*f) - X^2 \circ X^1(\varphi^*f) = \\
&= [X^1, X^2](\varphi^*f).
\end{aligned}$$

$\qquad\square$

Vector fields can always be transported along diffeomorphisms.

Definition 9.3.4. *The **pushforward** of a vector field $X \in \mathfrak{X}(M)$ via a diffeomorphism $\varphi : M \xrightarrow{\sim} N$ is the vector field*

$$\varphi_*X = \mathrm{d}\varphi \circ X \circ \varphi^{-1} \in \mathfrak{X}(N),$$

more explicitly:

$$(\varphi_*X)_q := \mathrm{d}_p\varphi(X_p), \quad \text{where } p = \varphi^{-1}(q).$$

Note that φ_*X is the unique vector field on N such that X and φ_*X are φ-related.

*The **pullback** of a vector field $Y \in \mathfrak{X}(N)$ via the diffeomorphism $\varphi : M \xrightarrow{\sim} N$ is the vector field*

$$\varphi^*Y := (\varphi^{-1})_*Y = (\mathrm{d}\varphi)^{-1} \circ Y \circ \varphi \in \mathfrak{X}(M).$$

Note that φ^*Y is the unique vector field on M such that φ^*Y and Y are φ-related.

Example 9.3.5. Let $\iota : M \hookrightarrow N$ denote the inclusion of the embedded submanifold M of N. Then $X \in \mathfrak{X}(M)$ is ι-related to $Y \in \mathfrak{X}(N)$ iff

$$Y_p = X_p \in T_pM \subset T_pN, \quad \forall\, p \in M$$

For example, consider $\iota : S^1 \hookrightarrow \mathbb{R}^2$, and the following vector field on $\mathbb{R}^2$

$$Y := x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

First, note that Y is tangent to S^1 because, for $(u, v) \in S^1$, we have that

$$T_{(u,v)}S^1 := \ker(\mathrm{d}_{(u,v)}(x^2 + y^2 - 1)) = \left\{a\frac{\partial}{\partial x}\Big|_{(u,v)} + b\frac{\partial}{\partial y}\Big|_{(u,v)} \;:\; ua + bv = 0\right\}.$$

Consider the "angle coordinate" $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ on S^1 (see Example 8.5.4),

$$\theta \mapsto (x(\theta), y(\theta)) := (\cos(\theta), \sin(\theta)).$$

We claim that Y is ι-related to $X := \frac{\partial}{\partial \theta}$, i.e. $Y|_{S^1} = \frac{\partial}{\partial \theta}$. We will use Lemma 9.3.2. For any $f \in C^\infty(\mathbb{R}^2)$, we have that

$$Y(f) = x \frac{\partial f}{\partial y}(x, y) - y \frac{\partial f}{\partial x}(x, y),$$

and so

$$\iota^*(Y(f)) = \cos(\theta) \frac{\partial f}{\partial y}(\cos(\theta), \sin(\theta)) - \sin(\theta) \frac{\partial f}{\partial x}(\cos(\theta), \sin(\theta)).$$

On the other hand, by the chain rule, we have that:

$$X(\iota^* f) = \frac{\partial}{\partial \theta}\left(f(\cos(\theta), \sin(\theta))\right) =$$

$$= \frac{\partial f}{\partial x}(\cos(\theta), \sin(\theta)) \frac{\partial \cos(\theta)}{\partial \theta} + \frac{\partial f}{\partial y}(\cos(\theta), \sin(\theta)) \frac{\partial \sin(\theta)}{\partial \theta} =$$

$$= \cos(\theta) \frac{\partial f}{\partial y}(\cos(\theta), \sin(\theta)) - \sin(\theta) \frac{\partial f}{\partial x}(\cos(\theta), \sin(\theta)).$$

9.4. Change of coordinates rule for vector fields

Let X be a smooth vector field on M, and let $(U, \varphi = (x_\varphi^1, \ldots, x_\varphi^m))$ be a chart. Denote the local expression of X by:

$$X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x_\varphi^i}, \quad X^i \in C^\infty(U).$$

Consider a second chart $(V, \psi = (y_\psi^1, \ldots, y_\psi^m))$ on M, and denote the corresponding local expression of X by

$$X|_V = \sum_{i=1}^m Y^i \frac{\partial}{\partial y_\psi^i}, \quad Y^i \in C^\infty(V).$$

Thus, on $U \cap V$, we have that:

$$\sum_{i=1}^m X^i \frac{\partial}{\partial x_\varphi^i} = \sum_{i=1}^m Y^i \frac{\partial}{\partial y_\psi^i}.$$

Applying this equality to the coordinate function y_ψ^j, we obtain

$$Y^j = \sum_{i=1}^m X^i \frac{\partial y_\psi^j}{\partial x_\varphi^i}.$$

This expression represents the rule of change of coordinates for vector fields, i.e. it gives the local expression of X in the chart (V, ψ) in terms of the local expression of X in the chart (U, φ).

9.5. Integral curves of a vector field

Let $I \subset \mathbb{R}$ be an open interval, and consider a smooth curve $\gamma : I \to M$. Recall that the **derivative** of γ is the following curve in TM:

$$\frac{d\gamma}{dt}(t) := d_t\gamma\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M.$$

We will consider curves whose derivatives are given by a fixed vector field:

Definition 9.5.1. *Let $X \in \mathfrak{X}(M)$ be a vector field. An **integral curve** of X is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval, which satisfies:*

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = X_{\gamma(t)}, \quad \text{for all } t \in I.$$

*The integral curve γ is said to **start at** a point $p \in M$, if $0 \in I$ and $\gamma(0) = p$.*

Let $(U, \varphi = (x_\varphi^1, \ldots, x_\varphi^m))$ be a chart on M. Denote the local expression of a vector field $X \in \mathfrak{X}(M)$ in this chart by:

$$X|_U = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x_\varphi^i}, \quad X^i \in C^\infty(U).$$

We will write locally what it means for a curve $\gamma : I \to U$ to be an integral curve of X. Denote the local expression of γ in this chart by:

$$\varphi \circ \gamma(t) = (\gamma_\varphi^1(t), \ldots, \gamma_\varphi^m(t)), \quad t \in I.$$

Then, as explained in Subsection 5.2, we have that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \mathrm{d}_t \gamma \left(\frac{\partial}{\partial t} \right) = \sum_{i=1}^{m} \frac{\mathrm{d}\gamma_\varphi^i}{\mathrm{d}t}(t) \frac{\partial}{\partial x_\varphi^i} \Big|_{\gamma(t)}.$$

Therefore, the condition that γ be an integral curve of X is equivalent to the following system of ordinary differential equations (ODE's)

$$\frac{\mathrm{d}\gamma_\varphi^i}{\mathrm{d}t}(t) = X^i \circ \varphi^{-1}(\gamma_\varphi^1(t), \ldots, \gamma_\varphi^m(t)), \quad \text{for} \quad 1 \le i \le m,$$

on curves $(\gamma_\varphi^1, \ldots, \gamma_\varphi^m) : I \to \varphi(U)$. We write this system in a more compact way:

$$\frac{\mathrm{d}z}{\mathrm{d}t}(t) = F(z(t)), \quad z : I \to V,$$

where $V = \varphi(U) \subset \mathbb{R}^m$ and

$$z = (\gamma_\varphi^1, \ldots, \gamma_\varphi^m) : I \to V, \quad F = (X^1, \ldots, X^m) \circ \varphi^{-1} : V \to \mathbb{R}^m.$$

Such systems are studied using standard techniques from the theory of ODE's. We will use the following result (e.g. [**1**, Appendix C]):

Theorem 9.5.2. *Let $F : V \to \mathbb{R}^m$ be a smooth map, where $V \subset \mathbb{R}^m$ is an open set. For some $p \in V$, consider the ordinary differential equation*

$$\frac{\mathrm{d}z}{\mathrm{d}t}(t) = F(z(t)), \quad z(0) = p.$$

Every point in V has an open neighborhood W and there exists a smooth map

$$y : (-\epsilon, \epsilon) \times W \longrightarrow V,$$

for some $\epsilon > 0$, such that the curve $t \mapsto y(t, p)$ is a solution of the ODE, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t, p) = F(y(t, p)), \quad y(0, p) = p.$$

Moreover, the solution is unique, in the sense that if $p \in W$ and $t \mapsto \widetilde{y}(t, p)$ is a second solution defined for t around 0, then $\widetilde{y}(t, p) = y(t, p)$ for t around 0.

9.6. Exercises

Exercise 9.1. Consider the polar coordinates (U, r, θ) on $\mathbb{R}^2$, where

$$U = \mathbb{R}^2 \backslash ((-\infty, 0] \times \{0\}), \quad x = r\cos(\theta), \ y = r\sin(\theta), \quad (r, \theta) \in (0, \infty) \times (-\pi, \pi).$$

Write in polar coordinates the expression of an arbitrary vector field on $\mathbb{R}^2$:

$$X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2).$$

Exercise 9.2. Let $\varphi : M \to N$ be a local diffeomorphism (recall Definition 3.3.3). Define the pullback $\varphi^* X$ of a vector field $X \in \mathfrak{X}(N)$ along φ (your definition should generalize Definition 9.3.4). Show that this operation preserves the Lie bracket:

$$\varphi^*[X, Y] = [\varphi^* X, \varphi^* Y], \quad \text{for all} \ \ X, Y \in \mathfrak{X}(N).$$

Exercise 9.3. Calculate the Lie bracket of the vector fields:

$$X = \sin(x + y)\frac{\partial}{\partial x} + e^{xy}\frac{\partial}{\partial y}, \quad Y = y^2\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}.$$

Exercise 9.4. Consider the map

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \varphi(x, y) := (e^x \cos(y), e^x \sin(y)),$$

i.e. φ is the complex exponential written in real coordinates.

(a) Find two vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^2)$ such that X is φ-related to $\frac{\partial}{\partial x}$ and Y is φ-related to $\frac{\partial}{\partial y}$. Are these vector fields uniquely determined by this condition? Calculate $[X, Y]$.
(b) Find two vector fields $U, V \in \mathfrak{X}(\mathbb{R}^2)$ such that $\frac{\partial}{\partial x}$ is φ-related to U and $\frac{\partial}{\partial y}$ is φ-related to V. Are these vector fields uniquely determined by this condition? Calculate $[U, V]$.

Exercise 9.5. Let $\varphi : M \to N$ and $\psi : N \to P$ be smooth maps. If $X \in \mathfrak{X}(M)$ is φ-related to $Y \in \mathfrak{X}(N)$ and if Y is ψ-related to $Z \in \mathfrak{X}(P)$ show that X is $\psi \circ \varphi$-related to Z.

Exercise 9.6. Let $M \subset N$ be an embedded submanifold. A vector field $X \in \mathfrak{X}(N)$ is said to be tangent to M, if $X_p \in T_p M$ for all $p \in M$. If $X \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(N)$ are tangent to M show that also $[X, Y] \in \mathfrak{X}(N)$ is tangent to M.

Exercise 9.7. (a) Let $X \in \mathfrak{X}(M)$ be a vector field. Show that a smooth curve $\gamma : I \to M$ is an integral curve of X iff $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ is γ-related to X.
(b) Let $\psi : M \to N$ be a smooth map, and assume that $X \in \mathfrak{X}(M)$ is ψ-related to $Y \in \mathfrak{X}(N)$. If $\gamma : I \to M$ is an integral curve of X, prove that $\psi \circ \gamma : I \to N$ is an integral curve of Y.

Exercise 9.8. Calculate the integral curves of the following vector fields on $\mathbb{R}^2$

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad Z = x^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

LECTURE 10

10.1. The flow of a vector field

Let us fix a vector field X on a smooth manifold M. In this section we will describe the integral curves of X. Most results discussed here are in essence consequences of Theorem 9.5.2. First let us remark:

Lemma 10.1.1. *If $\gamma : (a, b) \to M$ is an integral curve of X then, for any $s \in \mathbb{R}$, also $\sigma : (a - s, b - s) \to M$, $\sigma(t) := \gamma(t + s)$ is an integral curve of X.*

Proof. Let $\tau_s(t) := t + s$. Then $\sigma = \gamma \circ \tau_s$, and

$$\frac{\mathrm{d}(\gamma \circ \tau_s)}{\mathrm{d}t}(t) = \mathrm{d}_t(\gamma \circ \tau_s)\Big(\frac{\partial}{\partial t}\big|_t\Big) = (\mathrm{d}_{t+s}\gamma) \circ \mathrm{d}_t(\tau_s)\Big(\frac{\partial}{\partial t}\big|_t\Big) =$$

$$= (\mathrm{d}_{t+s}\gamma) \circ \Big(\frac{\partial}{\partial t}\big|_{t+s}\Big) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t + s) = X_{\gamma \circ \tau_s(t)}.$$

$\square$

Lemma 10.1.2. *For any $p \in M$ there exists an integral curve*

$$\gamma_p : I_p \to M$$

of X starting at p which is maximal in the sense that, any other integral curve $\gamma : I \to M$ of X starting at p satisfies

$$I \subset I_p \quad and \quad \gamma = \gamma_p|_I.$$

Proof. Consider two integral curves starting at p, denoted $\gamma^i : I^i \to M$, $i = 1, 2$. By continuity of the curves, the set

$$J := \{t \in I^1 \cap I^2 \ : \ \gamma^1(t) = \gamma^2(t)\}$$

is closed in $I^1 \cap I^2$. Let $s \in J$. Note that $\sigma^i(t) := \gamma^i(t + s)$, $i = 1, 2$, are two integral curves which start at $q := \gamma^i(s)$; hence, in a chart around q, they are solutions to the same ODE with the same initial condition. Thus, by the uniqueness assertion in Theorem 9.5.2, we have that $\sigma^1 = \sigma^2$ in a neighborhood of 0; in other words J contains a neighborhood of s. We have shown that J is both open and closed in the open, non-empty inteval open $I^1 \cap I^2$, and since $0 \in J$, this implies that $J = I^1 \cap I^2$. In other words any two integral curves starting at p coincide on the intersection of their domains.

By Theorem 9.5.2, there is at least one integral curve $\gamma : I \to M$ starting at p. Let I_p be the union of all intervals I for which there is an integral curve $\gamma : I \to M$ starting at p. By the above, for any $t \in I_p$ all integral curves starting at p and defined at $t \in I_p$ take the same value at t; we define $\gamma_p(t)$ to be this common value. Clearly, $\gamma_p : I_p \to M$ satisfies the maximality property. $\qquad\square$

Note that the maximality property of the integral curve $\gamma_p : I_p \to M$ implies its uniqueness. This allows us to define:

Definition 10.1.3. *Let X be a vector field on a manifold M. The **maximal domain** (or just the **domain**) of the flow of X is the set*

$$\mathcal{D}(X) := \{(t,p) \ : \ p \in M, \ t \in I_p\} \ \subset \ \mathbb{R} \times M,$$

*and the **maximal flow** (or just the **flow**) of X is the map*

$$\phi_X : \mathcal{D}(X) \to M, \quad (t,p) \mapsto \phi_X^t(p) := \gamma_p(t).$$

*For some $t \in \mathbb{R}$, the **flow at time** t is the map*

$$\phi_X^t : \mathcal{D}_t(X) \to M, \quad p \mapsto \phi_X^t(p),$$

where its domain is given by

$$\mathcal{D}_t(X) := \{p \in M \ : \ t \in I_p\} \ \subset \ M.$$

Lemma 10.1.4. *For $s \in I_p$, we have that $t \in I_{\phi_X^s(p)}$ iff $t + s \in I_p$; and moreover,*

$$\phi_X^t(\phi_X^s(p)) = \phi_X^{t+s}(p) \quad \forall \, t \in I_{\phi_X^s(p)}.$$

Proof. Denote $q := \phi_X^s(p)$ and $\tau_s(t) := t + s$. Then $\gamma_p \circ \tau_s : \tau_{-s}(I_p) \to M$ is an integral curve starting at q, therefore $\tau_{-s}(I_p) \subset I_q$ and $\gamma_p \circ \tau_s = \gamma_q|_{\tau_{-s}(I_p)}$; i.e.

$$\phi_X^{t+s}(p) = \phi_X^t(\phi_X^s(p)), \quad \forall \, t \in \tau_{-s}(I_p).$$

Similarly, $\gamma_q \circ \tau_{-s} : \tau_s(I_q) \to M$ is an integral curve starting at p, hence $\tau_s(I_q) \subset I_p$. We obtain that $I_q = \tau_{-s}(I_p)$, and this concludes the proof. $\qquad\square$

We have that

Theorem 10.1.5. *Let X be a vector field on M. Then the domain $\mathcal{D}(X)$ of X is open in $\mathbb{R} \times M$ and the flow of X is a smooth map*

$$\phi_X : \mathcal{D}(X) \to M, \quad (t,p) \mapsto \phi_X^t(p).$$

Proof. Theorem 9.5.2 implies that for every point $p \in M$ there exists an open neighborhood $W \subset M$ and there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times W \subset \mathcal{D}(X)$, and ϕ_X is smooth on $(-\epsilon, \epsilon) \times W$.

Let $(t,p) \in \mathcal{D}(X)$. Denote the image of the integral curve from p and $\phi_X^t(p)$ by

$$C := \{\phi_X^s(p) \ : \ s \in [0,t] \text{ (or } s \in [t,0])\}.$$

Since C is compact, the argument above implies that there is some open set $C \subset W \subset M$ and some $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times W \subset \mathcal{D}(X)$ and ϕ_X is smooth on $(-\epsilon, \epsilon) \times W$. Let $n > 0$ be such that $t/n \in (-\epsilon, \epsilon)$; then $\phi_X^{t/n} : W \to M$ is smooth. Consider the open sets $\{W_i\}_{i=0}^n$ defined inductively by

$$W_n := W, \quad W_i := \left(\phi_X^{t/n}|_W\right)^{-1}(W_{i+1}), \quad 0 \leq i \leq n - 1.$$

Since $C \subset W$, it follows, by induction on $j = 0, \ldots, n$ and by Lemma 10.1.4 that

$$\phi_X^{t\frac{n-j}{n}}(p) \in W_{n-j};$$

therefore, W_0 is not empty, because it contains p. Also, directly by the definition of the sets W_i and by Lemma 10.1.4, it follows that

$$\phi_X^{t\frac{i}{n}}(W_0) \subset W_i.$$

In particular, we obtain that $\{t\} \times W_0 \subset \mathcal{D}(X)$. Moreover, ϕ_X^t is smooth on W_0, because it is given by the composition of smooth maps:

$$\phi_X^t|_{W_0} = (\phi_X^{t/n}|_{W_n}) \circ \ldots \circ (\phi_X^{t/n}|_{W_0}) : W_0 \longrightarrow M.$$

On the other hand, since ϕ_X is smooth on $(-\epsilon, \epsilon) \times W$ (in particular continuous), there is an open set $p \in V \subset W$ and there is $0 < \delta < \epsilon$ such that $\phi_X((-\delta, \delta) \times V) \subset W_0$. Hence, $(t - \delta, t + \delta) \times V \subset \mathcal{D}(X)$ is an open neighborhood of (t, p) on which ϕ_X is smooth, since it is given by:

$$\phi_X^s|_V = (\phi_X^t|_{W_0}) \circ (\phi_X^{s-t}|_V), \quad s \in (t - \delta, t + \delta).$$

This finishes the proof. $\qquad\square$

The following is a rather direct consequence of Theorem 10.1.5 and Lemma 10.1.4; the proof is left to the reader.

Corollary 10.1.6. *For $t \in \mathbb{R}$, $\mathcal{D}_t(X)$ is open in M and ϕ_X^t is a diffeomorphism between $\phi_X^t : \mathcal{D}_t(X) \xrightarrow{\sim} \mathcal{D}_{-t}(X)$ with inverse $\phi_X^{-t} : \mathcal{D}_{-t}(X) \xrightarrow{\sim} \mathcal{D}_t(X)$.*

10.2. Complete vector fields

Definition 10.2.1. *A vector field $X \in \mathfrak{X}(M)$ is called **complete**, if its integral curves are defined at all times; in other words, if $\mathcal{D}(X) = \mathbb{R} \times M$.*

Given a complete vector field $X \in \mathfrak{X}(M)$, for each $t \in \mathbb{R}$, ϕ_X^t is a diffeomorphism of M. Moreover, this assignment satisfies

$$\phi_X^s \circ \phi_X^t = \phi_X^{s+t};$$

in other words, we obtain a group homomorphism:

$$(\mathbb{R}, +) \longrightarrow (\mathrm{Diff}(M), \circ), \quad t \mapsto \phi_X^t.$$

Equivalently, one can think about the flow of a complete vector field as a **smooth action** of the group $(\mathbb{R}, +)$ on M:

$$\phi_X : \mathbb{R} \times M \longrightarrow M, \quad (t, p) \mapsto \phi_X^t(p).$$

The family $\{\phi_X^t\}_{t \in \mathbb{R}}$ is also called a **one-parameter group of diffeomorphisms**, in the sense that it is a one-dimensional subgroup of the infinite-dimensional group of diffeomorphisms.

The following result is useful for checking completeness:

Lemma 10.2.2. *Let $X \in \mathfrak{X}(M)$ be a vector field. If there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times M \subset \mathcal{D}(X)$, then X is complete.*

Proof. Let $p \in M$, and let $t \in I_p$. Then $(-\epsilon, \epsilon) \subset I_{\phi_X^t(p)}$ and so, by Lemma 10.1.4, $(t - \epsilon, t + \epsilon) \subset I_p$. Hence, $I_p = \mathbb{R}$. $\qquad\square$

A similar argument implies the following behavior of integral curves:

Lemma 10.2.3. *Let $X \in \mathfrak{X}(M)$ be a vector field, and let $p \in M$. If there is a compact set $K \subset M$ such that $\phi_X^t(p) \in K$ for all $t \in I_p$, then $I_p = \mathbb{R}$ (in other words, an integral curve is either complete, or it leaves every compact subset).*

Proof. Since K is compact, there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times K \subset \mathcal{D}(X)$. So, for $q \in K$, $(-\epsilon, \epsilon) \subset I_q$. Thus if $t \in I_p$, then $(-\epsilon, \epsilon) \subset I_{\phi_X^t(p)}$, which by Lemma 10.1.4 implies that $(t - \epsilon, t + \epsilon) \subset I_p$. We conclude that $I_p = \mathbb{R}$. $\square$

Either of the two lemmas, implies the following:

Corollary 10.2.4. *Any vector field on a compact manifold is complete. More generally, any compactly supported vector field on a manifold is complete.*

10.3. The flow as the exponential of the Lie derivative

Definition 10.3.1. *Let $X \in \mathfrak{X}(M)$ be a vector field. The **Lie derivative** along X of a function $f \in C^\infty(M)$ is the smooth function*

$$\mathscr{L}_X f := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\phi_X^t)^* f \in C^\infty(M).$$

*The **Lie derivative** along X of a vector field $Y \in \mathfrak{X}(M)$ is the vector field*

$$\mathscr{L}_X Y := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\phi_X^t)^* Y \in \mathfrak{X}(M).$$

Remark 10.3.2. Let us be more precise about the definitions above. First, recall that the flow of X at time t is defined as a map $\phi_X^t : \mathcal{D}_t(X) \to M$ and that $\mathcal{D}_t(X)$ is an open subset of M. Thus, for $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$, we have

$$(\phi_X^t)^* f = f \circ \phi_X^t \in C^\infty(\mathcal{D}_t(X)) \quad \text{and} \quad (\phi_X^t)^* Y = (\mathrm{d}\phi_X^t)^{-1}(Y_{\phi_X^t}) \in \mathfrak{X}(\mathcal{D}_t(X)).$$

Let $p \in M$. The maps $I_p \ni t \mapsto ((\phi_X^t)^* f)(p)$ and $I_p \ni t \mapsto ((\phi_X^t)^* Y)_p$ are smooth maps into the vector spaces $\mathbb{R}$ and $T_p M$, respectively. Therefore their derivatives at $t = 0 \in I_p$ are well-defined, and these give $(\mathscr{L}_X f)(p)$ and $(\mathscr{L}_X Y)_p$, respectively.

The following shows that the operators $(\phi_X^t)^*$ play the role of the exponential of the operators $t\mathscr{L}_X$.

Proposition 10.3.3. *Let $X \in \mathfrak{X}(M)$. For $f \in C^\infty(M)$ and $Y \in \mathfrak{X}(M)$, we have:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^* f = (\phi_X^t)^* \circ X(f) = X \circ (\phi_X^t)^* f \in C^\infty(\mathcal{D}_t(X)),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^* Y = (\phi_X^t)^*[X, Y] = [X, (\phi_X^t)^* Y] \in \mathfrak{X}(\mathcal{D}_t(X)).$$

In particular,

$$\mathscr{L}_X(f) = X(f), \qquad \mathscr{L}_X Y = [X, Y].$$

Proof. The first equality is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\phi_X^t(p)) = X_{\phi_X^t(p)}(f), \quad \text{for all } p \in \mathcal{D}_t(X),$$

which holds because $\phi_X^t(p)$ is an integral curve of X. Using that $\phi_X^{t+s} = \phi_X^t \circ \phi_X^s$, and applying the first equality for $s = 0$, we obtain the second equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^* f = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\phi_X^{t+s})^* f = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\phi_X^s)^*((\phi_X^t)^* f) = X((\phi_X^t)^* f).$$

Next, note that, for any $g \in C^\infty(M)$ and $Z \in \mathfrak{X}(M)$, we have that:

$$(*) \quad ((\phi_X^t)^* Z)g = \left((\mathrm{d}\phi_X^{-t}) \circ Z_{\phi_X^t}\right)g = Z_{\phi_X^t}(g \circ \phi_X^{-t}) = (\phi_X^t)^* \circ Z \circ (\phi_X^{-t})^* g.$$

Taking the derivative of this equality, for $Z = Y$, and using the first part, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}((\phi_X^t)^* Y)g = \frac{\mathrm{d}}{\mathrm{d}t}\left((\phi_X^t)^* \circ Y \circ (\phi_X^{-t})^* g\right) =$$
$$= (\phi_X^t)^* \circ (X \circ Y - Y \circ X) \circ (\phi_X^{-t})^* g =$$
$$= (\phi_X^t)^* \circ [X, Y] \circ (\phi_X^{-t})^* g = ((\phi_X^t)^* [X, Y])g,$$

where in the last step we have used again $(*)$ for $Z = [X, Y]$. Since, g is arbitrary, we obtain the third equality $\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^* Y = (\phi_X^t)^* [X, Y]$. Since $[X, X] = 0$, this implies that $(\phi_X^t)^* X = X$. Next, note that the vector fields $(\phi_X^t)^* X = X$, $(\phi_X^t)^* Y$ and $(\phi_X^t)^* [X, Y]$ are ϕ_X^t-related to the vector fields X, Y and $[X, Y]$, respectively. Thus, by Lemma 9.3.3, and by the fact that ϕ_X^t is a local diffeomorphism, $(\phi_X^t)^* [X, Y] = [X, (\phi_X^t)^* Y]$. This implies also the last equality. $\square$

10.4. The Lie bracket as infinitesimal commutator of flows

The Lie bracket $[X, Y]$ of two vector fields $X, Y \in \mathfrak{X}(M)$ can be regarded as the infinitesimal obstruction for the flows of the vector fields to commute. Around every point in M we can find an open set W and $\epsilon > 0$ such that the expression:

$$\sigma_p(t, s) := \phi_Y^{-s} \circ \phi_X^{-t} \circ \phi_Y^s \circ \phi_X^t(p) \in M,$$

is defined for $(t, s, p) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times W$. If X and Y are complete, then $\sigma(t, s) \in \mathrm{Diff}(M)$ is the group-theoretic commutator of the diffeomorphisms ϕ_X^t and ϕ_Y^s. Geometrically, $\sigma_p(t, s)$ is obtained as follows: try to construct a "curved parallelogram" starting at p with one pair of sides pq and nm in the direction of X and of "length" t, and the other pair of sides qm and np' parallel to Y and of "length" s, as in the picture below:
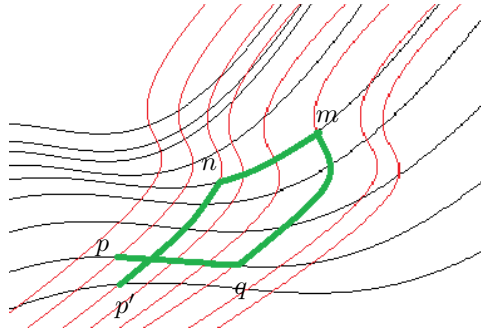
The black and red curves represent the flow lines of X and Y, respectively, and

$$q = \phi_X^t(p), \quad m = \phi_X^s(q), \quad n = \phi_X^{-t}(m), \quad p' = \phi_X^{-s}(n) = \sigma_p(t, s).$$

In general, the curved parallelogram does not close up, i.e. $p' = \sigma_p(t, s) \neq p$.

On the other hand, since $\sigma(t,0) = \mathrm{id}_M$, the derivative $t \mapsto \frac{\partial}{\partial s}\sigma_p(t,0)$ is a smooth vector field on M, which is given by:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\sigma_p(t,s)\big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\phi_Y^{-s} \circ \phi_X^{-t} \circ \phi_Y^{s} \circ \phi_X^{t}(p)\big|_{s=0} = \\
&= -Y_p + (\mathrm{d}\phi_X^{-t})(Y_{\phi_X^t(p)}) = \\
&= \big((\phi_X^t)^* Y - Y\big)_p.
\end{aligned}
$$

Applying Proposition 10.3.3, we obtain the following description of the Lie bracket:

Proposition 10.4.1. *With the notation from above, we have that:*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma(t,s) = [X,Y].
$$

10.5. Exercises

Exercise 10.1. Let $\psi : M \to N$ be a smooth map. Consider $X \in \mathfrak{X}(M)$ which is ψ-related to $Y \in \mathfrak{X}(N)$. Prove that their flows are ψ-related in the following sense: for every $p \in M$, $I_p \subset I_{\psi(p)}$, and for all $t \in I_p$ we have that

$$
\psi(\phi_X^t(p)) = \phi_Y^t(\psi(p)).
$$

In other words $(\mathrm{id}_{\mathbb{R}} \times \psi)(\mathcal{D}(X)) \subset \mathcal{D}(Y)$, and the following diagram commutes:

$$
\begin{CD}
\mathcal{D}(X) @>{\mathrm{id}_{\mathbb{R}} \times \psi}>> \mathcal{D}(Y) \\
@V{\phi_X}VV @VV{\phi_Y}V \\
M @>{\psi}>> N
\end{CD}
$$

Exercise 10.2. Let $X \in \mathfrak{X}(M)$. For $\lambda \in \mathbb{R}$, prove that the flow of λX is given by:

$$
\phi_{\lambda X}^t(p) = \phi_X^{\lambda t}(p),
$$

and moreover that

$$
(t,p) \in \mathcal{D}(\lambda X) \iff (\lambda t, p) \in \mathcal{D}(X).
$$

Exercise 10.3. Let X be a vector field on M. Let $p \in M$ be such that $X_p = 0$. Prove that $I_p = \mathbb{R}$ and that $\phi_X^t(p) = p$ for all $t \in \mathbb{R}$.

Exercise 10.4. On the m-dimensional sphere

$$
S^m = \{(x^1,\ldots,x^{m+1}) \in \mathbb{R}^{m+1} \;:\; \sum_{i=1}^{m+1}(x^i)^2 = 1\},
$$

consider the stereographical atlas:

$$
\mathcal{A} = \{(U_-,\varphi_-),(U_+,\varphi_-)\},
$$

where $U_\pm = S^m\backslash\{(0,\ldots,0,\pm 1)\}$ and $\varphi_\pm : U_\pm \xrightarrow{\sim} \mathbb{R}^m$ induce the coordinates $\varphi_\pm = (y_\pm^1,\ldots,y_\pm^m)$:

$$
y_\pm^i := \frac{x^i}{1 \mp x^{m+1}}, \quad 1 \leq i \leq m.
$$

(a) Show that there exists a smooth vector field X on S^m which in the coordinates y_+^i is given by

$$X|_{U_+} = \sum_{i=1}^m y_+^i \frac{\partial}{\partial y_+^i} \in \mathfrak{X}(\mathbb{R}^m),$$

and write the expression of X in the coordinates y_-^i.
(b) Determine the flow of X in both charts.
(c) Prove that there exists a vector field $\widetilde{X} \in \mathfrak{X}(\mathbb{R}^{m+1})$ satisfying

$$\widetilde{X}_p = X_p, \quad \forall \, p \in S^m.$$

Exercise 10.5. (a) For any vector field $X = f(x)\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R})$, prove that the vector field $\sin(x)f(x)\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R})$ is complete (*Hint: use Lemma 10.2.3*).
(b) Prove that any vector field on $\mathbb{R}$ can be written as the sum of two complete vector fields on $\mathbb{R}$ (*Hint: use part (a)*).

Exercise 10.6. Let $X = \sum_{i=1}^m X^i(x)\frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^m)$ be a bounded vector field, i.e. there is a constant $C > 0$ such that $|X^i(x)| < C$ for all $x \in \mathbb{R}^m$ and all $1 \le i \le m$. Prove that X is complete (*Hint: first, show that its flow at time t can displace a point only by a distance of $C\sqrt{m}|t|$; second, use an argument as in Lemma 10.2.3*).

Exercise 10.7. For $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, prove that the Taylor series of $(\phi_X^t)^* f$ at $t = 0$ is given by:

$$\exp(T\mathscr{L}_X)f := \sum_{n \ge 0} \frac{T^n}{n!} \mathscr{L}_X^n(f) \in C^\infty(M)[[T]],$$

where $C^\infty(M)[[T]]$ denotes the ring of formal power series in T with coefficients in $C^\infty(M)$.

Exercise 10.8. Let A be a real vector space endowed with a bilinear map

$$* : A \times A \longrightarrow A.$$

We are not assuming any other axiom for the operation $*$ (like associativity, commutativity, Jacobi identity etc.).

An **automorphism** of $(A, *)$ is a linear isomorphism $g : A \to A$ satisfying

$$g(a * b) = g(a) * g(b).$$

Denote the space of all automorphisms by $\mathrm{Aut}(A, *)$.

A **derivation** of $(A, *)$ is a linear map $D : A \to A$ satisfying:

$$D(a * b) = D(a) * b + a * D(b), \quad \text{for all} \ \ a, b \in A.$$

Denote the space of all derivations by $\mathrm{Der}(A, *)$.

(a) Show that $(\mathrm{Aut}(A, *), \circ)$ is a group.
(b) Show that, if $D_1, D_2 \in \mathrm{Der}(A, *)$, then

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1 \in \mathrm{Der}(A, *).$$

(c) Prove that the operation $[\cdot, \cdot]$ on $\mathrm{Der}(A, *)$ satisfies the condition (1),(2) and (3) of Proposition 9.2.1; i.e. $(\mathrm{Der}(A, *), [\cdot, \cdot])$ is a Lie algebra.

For (d) and (e), assume A to be **finite dimensional**[1] Hence, we can talk about smooth curves $g : (-\epsilon, \epsilon) \to \mathrm{Lin}_{\mathbb{R}}(A, A)$, where $\mathrm{Lin}_{\mathbb{R}}(A, A)$ is the (finite dimensional) vector space of linear maps from A to A. Identifying $T_{g(t)}\mathrm{Lin}_{\mathbb{R}}(A, A) \cong \mathrm{Lin}_{\mathbb{R}}(A, A)$, we regard $\frac{\mathrm{d}g}{\mathrm{d}t}(t) \in \mathrm{Lin}_{\mathbb{R}}(A, A)$.

(d) Consider a smooth curve $g : (-\epsilon, \epsilon) \to \mathrm{Lin}_{\mathbb{R}}(A, A)$, such that $g(0) = \mathrm{id}_A$ and

$$g(t) \in \mathrm{Aut}(A, *), \text{ for all } t \in (-\epsilon, \epsilon).$$

Prove that $\frac{\mathrm{d}g}{\mathrm{d}t}(0) \in \mathrm{Der}(A, *)$.

(e) Let $D \in \mathrm{Der}(A, *)$. Show that the following series converges absolutely to an automorphism of $(A, *)$

$$\exp(D) = \sum_{n \geq 0} \frac{1}{n!} D^n \in \mathrm{Aut}(A, *).$$

For (f) and (g) assume $(A, *)$ to be **associative and commutative**, however, A can be infinite dimensional.

(f) For $a \in A$ and $D \in \mathrm{Der}(A, *)$, prove that $a * D \in \mathrm{Der}(A, *)$, where $a * D$ is defined in the obvious way:

$$(a * D)(b) := a * (D(b)).$$

(g) Prove that the following Leibniz rule holds:

$$[D_1, a * D_2] = D_1(a) * D_2 + a * [D_1, D_2].$$

---

[1]This assumption simplifies the discussion; however, the results hold also in an infinite dimensional setting, e.g. for bounded operators on a Banach space.

LECTURE 11

11.1. Lie groups

Definition 11.1.1. *A **Lie group** is a manifold G endowed with a smooth group structure, i.e. the multiplication and the inversion are smooth maps:*

$$\mu : G \times G \longrightarrow G, \quad (g,h) \mapsto g \cdot h,$$

$$\iota : G \longrightarrow G, \quad g \mapsto g^{-1}.$$

*A **Lie subgroup** of G is a subgroup $H \subset G$ which is an immersed submanifold.*

Examples 11.1.2. (1) Every at most countable group is a 0-dimensional Lie group.
(2) The additive group $(\mathbb{R}^n, +)$ is an n-dimensional Lie group.
(3) The multiplicative group $(\mathbb{C}^*, \cdot)$ is a 2-dimensional Lie group. The following are 1-dimensional subgroups $\mathbb{R}^* \subset \mathbb{C}^*$ and $S^1 \subset \mathbb{C}^*$.
(4) **The general linear group**. The group $GL(n)$ of linear automorphisms of $\mathbb{R}^n$ is an n^2-dimensional Lie group. Its smooth structure comes by regarding $GL(n)$ as an open subset in the n^2-dimensional vector space $M(n)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.
(5) **The special linear group**. The group $SL(n)$ is the subgroup of $GL(n)$ consists of linear automorphisms of $\mathbb{R}^n$ which have determinant 1.
(6) **The orthogonal group**. The group $O(n)$ of orthogonal matrices (i.e. $A \in M(n)$ such that $AA^t = I$) is a compact Lie group of dimension $n(n-1)/2$ (see Exercise 6.11).
(7) There are complex versions of the groups above: $GL(n, \mathbb{C})$ the group of invertible complex matrices, with subgroup $SL(n, \mathbb{C})$ consisting of matices with determinant 1; $O(n, \mathbb{C})$ the group of complex matrices satisfying $AA^t = I$.
(8) The product $G_1 \times \ldots \times G_k$ of Lie groups is a Lie group. In particular, the k-torus $T^k := S^1 \times \ldots \times S^1$ is a k-dimensional compact Lie group.
(9) If G is any Lie group, then the connected component of the identity, denoted by $G^\circ \subset G$, is a Lie subgroup (which is both open and closed). In fact, G° is a normal subgroup.
(10) The group $GL(n)^\circ$ consists of matrices with positive determinant.
(11) **The special orthogonal group** is the Lie group $SO(n) := O(n)^\circ$.
(12) If G is a Lie group, then TG is a Lie group. Its structure maps are the differentials of the structure maps of G.

Definition 11.1.3. *The* **left translation** *by $g \in G$ is the diffeomorphism of G*
$$\lambda_g : G \longrightarrow G, \quad \lambda_g(h) := gh.$$
The **right translation** *by $g \in G$ is the diffeomorphism of G*
$$\rho_g : G \longrightarrow G, \quad \rho_g(h) := hg.$$

Using left translations one obtains the following:

Proposition 11.1.4. *Every Lie group is a parallelizable manifold.*

Proof. Using left translations, we construct a global frame on TG:
$$\theta^L : TG \longrightarrow T_eG, \quad \theta^L(X) := d_g\lambda_{g^{-1}}(X), \quad X \in T_gG.$$
By Corollary 8.4.3, the existence of θ^{L}[1] implies that G is parallelizable. $\qquad \square$

Definition 11.1.5. *Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is called* **left invariant** *if X is λ_g-related to X, for all $g \in G$. We denote the space of left invariant vector fields by $\mathfrak{X}_L(G)$. Similarly, one defines the space of* **right invariant** *vector fields $\mathfrak{X}_R(G)$.*

Explicitly, the condition that $X \in \mathfrak{X}(G)$ is left invariant can be written as:
$$d_h\lambda_g(X_h) = X_{gh}, \quad \text{for all} \ \ g, h \in G.$$
Letting $h = e$, this formula implies that X is determined by its value at the identity:
$$X_g = d_e\lambda_g(X_e).$$

Conversely, the **left invariant extension** of $v \in T_eG$ is the vector field:
$$v^L \in \mathfrak{X}(G), \quad v_g^L := d_e\lambda_g(v).$$
That v^L is left invariant follows by the chain rule:
$$d_h\lambda_g(v_h^L) = d_h\lambda_g \circ d_e\lambda_h(v) = d_e(\lambda_g \circ \lambda_h)(v) = d_e\lambda_{gh}(v) = v_{gh}^L.$$
We conclude with the following:

Proposition 11.1.6. *(1) The following map is a linear isomorphism:*
$$\mathfrak{X}_L(G) \xrightarrow{\sim} T_eG, \quad X \mapsto X_e, \quad \text{with inverse } v \mapsto v^L.$$
(2) If $X, Y \in \mathfrak{X}_L(G)$, then $[X, Y] \in \mathfrak{X}_L(G)$.

Proof. Part (1) was discussed above; part (2) follows from Lemma 9.3.3. $\qquad \square$

Definition 11.1.7. *A* **Lie algebra** *is a vector space $\mathfrak{g}$ endowed with an operation:*
$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$
called the **Lie bracket***, which*
(1) is **skew-symmetric***:*
$$[u, v] = -[v, u], \quad u, v \in \mathfrak{g},$$
*(2) is $\mathbb{R}$-**bilinear***:*
$$[u, av + bw] = a[u, v] + b[u, w], \quad a, b \in \mathbb{R}, \ u, v, w \in \mathfrak{g},$$
(3) and satisfies the **Jacobi identity***:*
$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \quad u, v, w \in \mathfrak{g}.$$

---

[1]The map θ^L is called the **left Maurer-Cartan form** of G.

Definition 11.1.8. *The **Lie algebra** of a Lie group G is the vector space*

$$\mathfrak{g} := T_e G$$

with Lie bracket:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad [v, w] := [v^L, w^L]_e.$$

By Proposition 9.2.1, $(\mathfrak{g}, [\cdot, \cdot])$ satisfies indeed the axioms of a Lie algebra.

Remarks 11.1.9. (1) The Lie algebra $\mathfrak{g}$ of a Lie group G plays the role of a "linearized version" of G; it encodes infinitesimal information about G. In fact, if G is simply connected, it can be fully recovered as a Lie group from the structure of $\mathfrak{g}$ (see for example [**2**]).

(2) In general Lie groups are denoted by capital Latin letters: G, H, K, L, P, Q, etc., while the corresponding Lie algebras are denoted by the corresponding small Fraktur letters (also called Gothic letters): $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{l}$, $\mathfrak{p}$, $\mathfrak{q}$ etc. (in LaTeX use $\backslash mathfrak\{..\}$).

(3) One can define the Lie algebra of G also using right invariant vector fields; nevertheless, one obtains an isomorphic object. The push-forward along the inversion map gives an isomorphism between left invariant and right invariant vector fields. Note however that, if both Lie algebras are identified with $T_e G$ (by evaluating vector fields at e), one obtains opposite Lie brackets on $T_e G$, because the differential of the inversion map at e is $-\mathrm{Id}_{T_e G}$.

Proposition 11.1.10. *Left invariant vector fields are complete. Moreover,*

$$\phi_X^t(gh) = g\phi_X^t(h), \quad \text{for all } X \in \mathfrak{X}_L(G), \quad g, h \in G, \quad t \in \mathbb{R}.$$

Proof. Let $X \in \mathfrak{X}_L(G)$ and let $g, h \in G$. As in Lecture 10, let I_h denote the maximal interval on which the flow line starting at h is defined. For $t \in I_h$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(g\phi_X^t(h)\big) = \mathrm{d}_{\phi_X^t(h)}\lambda_g\Big(\frac{\mathrm{d}}{\mathrm{d}t}\phi_X^t(h)\Big) = \mathrm{d}_{\phi_X^t(h)}\lambda_g\big(X_{\phi_X^t(h)}\big) = X_{g\phi_X^t(h)},$$

where we have used that X is left invariant. This shows that

$$I_h \ni t \mapsto g\phi_X^t(h)$$

is an integral curve of X. Since the curve starts at gh, by Lemma 10.1.2, we conclude that $I_h \subset I_{gh}$ and that $g\phi_X^t(h) = \phi_X^t(gh)$. In particular, $I_e \subset I_g$ for all $g \in G$, and so, by Lemma 10.2.2, X is complete. $\square$

The Proposition above allows us to define:

Definition 11.1.11. *The **exponential map** of a Lie group G with Lie algebra $\mathfrak{g}$ is the map:*

$$\exp : \mathfrak{g} \longrightarrow G, \quad v \mapsto \phi_{v^L}^1(e).$$

Exercise 11.1 guides you through a possible proof that the exponential map is indeed smooth. In fact, the exponential map is a local diffeomorphism around 0.

Proposition 11.1.12. *The exponential map $\exp : \mathfrak{g} \to G$ satisfies:*

$$\mathrm{d}_0 \exp = \mathrm{Id}_{\mathfrak{g}},$$

where we identify $T_0\mathfrak{g} = \mathfrak{g} = T_e G$. Hence, $\exp$ is a diffeomorphism between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of e in G.

Proof. The result follows from the formula $\phi^t_{sv^L}(g) = \phi^{st}_{v^L}(g)$ of Exercise 10.2:

$$d_0 \exp(v) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \exp(\epsilon v) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \phi^1_{\epsilon v^L}(e) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \phi^{\epsilon}_{v^L}(e) = v^L_e = v.$$

$\square$

Next, we discuss maps between Lie groups and Lie algebras

Definition 11.1.13. *(1) A **Lie group homomorphism** is a smooth group homomorphism*

$$F : (G, \cdot) \longrightarrow (H, \cdot).$$

*(2) A **homomorphism** of Lie algebras is a linear map*

$$f : (\mathfrak{g}, [\cdot, \cdot]) \longrightarrow (\mathfrak{h}, [\cdot, \cdot])$$

satisfying

$$f([u, v]) = [f(u), f(v)], \quad \text{for all } u, v \in \mathfrak{g}.$$

Lie group homomorphisms induce Lie algebra homomorphisms:

Proposition 11.1.14. *Let $(G, \cdot)$ and $(H, \cdot)$ be two Lie groups with corresponding Lie algebras $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{h}, [\cdot, \cdot])$. Consider a Lie group homomorphism:*

$$F : (G, \cdot) \longrightarrow (H, \cdot).$$

Since $F(e) = e$, define:

$$f := d_e F : \mathfrak{g} = T_e G \longrightarrow \mathfrak{h} = T_e H.$$

Then f is a homomorphism of Lie algebras, and the following diagram commutes:

$$
\begin{array}{ccc}
G & \xrightarrow{\ F\ } & H \\
{\scriptstyle\exp}\big\uparrow & & \big\uparrow{\scriptstyle\exp} \\
\mathfrak{g} & \xrightarrow{\ f\ } & \mathfrak{h}
\end{array}
$$

i.e. $F(\exp(v)) = \exp(f(v))$ for all $v \in \mathfrak{g}$.

Proof. For $v \in \mathfrak{g}$, we show that v^L and $f(v)^L$ are F-related. Since F is a Lie group homomorphism, we have $F \circ \lambda_g = \lambda_{F(g)} \circ F$. Therefore,

$$d_g F(v^L_g) = d_g F \circ d_e \lambda_g(v) = d_e(F \circ \lambda_g)(v) = d_e(\lambda_{F(g)} \circ F)(v) =$$
$$= d_e \lambda_{F(g)} \circ d_e F(v) = d_e \lambda_{F(g)}(f(v)) = f(v)^L_{F(g)}.$$

So, for any $w \in \mathfrak{g}$, w^L and $f(w)^L$ are also F-related, and so, by Lemma 9.3.3, $[v^L, w^L]$ is F-related to $[f(v)^L, f(w)^L]$. Evaluating this identity at the unit, implies that f is a Lie algebra homomorphism:

$$[f(v), f(w)] = [f(v)^L, f(w)^L]_e = d_e F([v^L, w^L]_e) = f([v, w]).$$

Finally, using again that v^L is F-related to $f(v)^L$, and Exercise 10.1, we obtain the commutativity of the diagram:

$$F(\exp(v)) = F(\phi^1_{v^L}(e)) = \phi^1_{f(v)^L}(F(e)) = \exp(f(v)).$$

$\square$

11.2. Exercises

Exercise 11.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote by $v^L \in \mathfrak{X}_L(G)$ the left invariant extension of a vector $v \in \mathfrak{g}$.

(a) Show that the following formula defines a smooth vector field on $G \times \mathfrak{g}$:

$$\xi \in \mathfrak{X}(G \times \mathfrak{g}), \quad \xi_{(g,v)} := (v^L_g, 0) \in T_g G \oplus T_v \mathfrak{g} \cong T_{(g,v)}(G \times \mathfrak{g}).$$

(b) Prove that the flow of ξ is given by

$$\phi^t_\xi(g, v) = (\phi^t_{v^L}(g), v),$$

and conclude that ξ is complete.

(c) Use (b) to prove that the exponential map $\exp : \mathfrak{g} \to G$ is smooth.

Exercise 11.2. Prove that left invariant vector fields commute with right invariant vector fields:

$$X \in \mathfrak{X}_L(G), \quad Y \in \mathfrak{X}_R(G) \quad \Longrightarrow \quad [X, Y] = 0.$$

Exercise 11.3. For $k \geq 1$, denote by T^k the k-dimensional torus, i.e.

$$T^k = (S^1)^k = \{(z_1, \ldots, z_k) \in \mathbb{C}^k \; : \; |z_1| = \ldots = |z_k| = 1\}.$$

(a) Prove that left invariant vector fields on the k-torus are the same as right invariant vector fields:

$$\mathfrak{X}_L(T^k) = \mathfrak{X}_R(T^k).$$

Moreover, if θ_i, $1 \leq i \leq k$ denote the "angle coordinates" on T^k (see Example 8.5.5), prove that the following is a basis of $\mathfrak{X}_L(T^k)$:

$$\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_k} \in \mathfrak{X}(T^k).$$

(b) Calculate the exponential map:

$$\exp : T_e T^k \longrightarrow T^k.$$

Hint: use the basis from (a) for $T_e T^k$.

(c) Consider an $n \times m$ matrix with integer coefficients $A = \{a_{i,j} \in \mathbb{Z}\}_{1 \leq i \leq n, 1 \leq j \leq m}$. Show that the map

$$F_A : (T^m, \cdot) \longrightarrow (T^n, \cdot),$$

$$F_A(z_1, \ldots, z_m) = (z_1^{a_{1,1}} \ldots z_m^{a_{1,m}}, \ldots, z_1^{a_{n,1}} \ldots z_m^{a_{n,m}})$$

is a Lie group homomorphism.

(d) Show that any Lie group homomorphism $F : T^m \to T^n$ is of the form described in (c). *Hint: use Proposition 11.1.14.*

Exercise 11.4. Consider the Lie group $GL(n)$ of invertible $n \times n$-matrices. Denote the space of all $n \times n$-matrices by $\mathfrak{gl}(n)$. Since $GL(n)$ is an open subset of $\mathfrak{gl}(n)$, we identify:

$$T_g GL(n) = \mathfrak{gl}(n), \quad \text{for all} \; g \in GL(n).$$

(a) Prove that, with the above identification, the differential of left translation is simply given by matrix multiplication:

$$d_h \lambda_g : T_h GL(n) = \mathfrak{gl}(n) \longrightarrow T_{gh} GL(n) = \mathfrak{gl}(n),$$

$$d_h \lambda_g(v) = g \cdot v.$$

(b) Prove that the left invariant extension of $v \in T_I GL(n) = \mathfrak{gl}(n)$ is the vector field
$$v^L \in \mathfrak{X}_L(GL(n)), \quad v_g^L = g \cdot v \in T_g GL(n) = \mathfrak{gl}(n).$$

(c) Prove that the flow of the left invariant extension $v^L \in \mathfrak{X}_L(GL(n))$ of $v \in T_I GL(n) = \mathfrak{gl}(n)$ is given by
$$\phi_{v^L}^t(g) = g \cdot e^{tv},$$
where $e^w = \sum_{n \geq 0} \frac{w^n}{n!}$ is the matrix exponential.

(d) Using Proposition 10.4.1 and (c) prove that the Lie bracket on $\mathfrak{gl}(n)$ is the commutator of matrices:
$$[v, w] = v \cdot w - w \cdot v, \quad v, w \in \mathfrak{gl}(n).$$

Exercise 11.5. The **quaternions** are the real 4-dimensional associative algebra $\mathbb{H}$ with basis $1, i, j, k$, where the multiplication is determined by the rules:
$$i^2 = j^2 = k^2 = -1, \ ij = k, \ jk = i, \ ki = j, \ ji = -k, \ kj = -i, \ ik = -j.$$

One can also represent $\mathbb{H}$ as the space of all 2×2-matrices with complex entries of the form:
$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \quad z, w \in \mathbb{C}.$$

The correspondence between the two descriptions is given by
$$i \longleftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ j \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ k \longleftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(a) Prove that every element in $\mathbb{H}^* = \mathbb{H} \backslash \{0\}$ is invertible, and conclude the $\mathbb{H}^*$ is a 4-dimensional Lie group.

(b) Prove that the map following map is a Lie group homomorphism:
$$F : (\mathbb{H}^*, \cdot) \longrightarrow (\mathbb{R}_{>0}, \cdot), \quad F(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2.$$

(c) Prove that $G := F^{-1}(1)$ is a Lie subgroup of $\mathbb{H}^*$ which as a manifold is diffeomorphic to S^3.

(d) Identifying $T_1 \mathbb{H}^* = \mathbb{H}$, prove that the exponential map of $\mathbb{H}^*$ is given by the usual exponential series:
$$\exp : \mathbb{H} \longrightarrow \mathbb{H}^*, \quad A \mapsto \sum_{n \geq 0} \frac{A^n}{n!}.$$

(e) Prove that the Lie algebra of G is given by
$$\mathfrak{g} = T_1 G = \{bi + cj + dk \ : \ b, c, d \in \mathbb{R}\}.$$

(f) Consider the set of "square roots of the unity" in $\mathbb{H}$:
$$\Sigma := \{X \in \mathbb{H} \ : \ X^2 = -1\}.$$
Prove that $\Sigma \subset \mathfrak{g}$, and that Σ is diffeomorphic to S^2.

(g) Prove that every element in $A \in \mathbb{H}$ can be decomposed as
$$A = \lambda + \mu X, \quad \lambda, \mu \in \mathbb{R}, \quad X \in \Sigma.$$

(h) In the decomposition $A = \lambda + \mu X$ from (g), prove that $A \in G$ iff $\lambda^2 + \mu^2 = 1$.

(i) In the decomposition $A = \lambda + \mu X$ from (g), prove that
$$\exp(A) = e^\lambda (\cos(\mu) + \sin(\mu)X).$$

(j) On $\mathfrak{g}$ we consider the metric such that Σ is the unit sphere around the origin, and denote by $B_r \subset \mathfrak{g}$ the open ball of radius $r > 0$. Using the previous items, prove the following about the exponential map $\exp : \mathfrak{g} \to G$:

    (i) $\exp$ is a diffeomorphism between B_π and $G \backslash \{-1\}$;

    (ii) $\exp$ sends the spheres of radii $2k\pi$ to $1 \in G$, and the spheres of radii $(2k+1)\pi$ to $-1 \in G$;

    (iii) $\exp$ restricts to a diffeomorphism between $B_{(k+1)\pi} \backslash \overline{B}_{k\pi}$ and $G \backslash \{1, -1\}$.

(k) Show that every flow line of a left invariant vector field on G is a circle.

LECTURE 12

12.1. The exterior algebra

There are (at least) two important algebras associated to any vector space V. First, the algebra $S(V^*)$ of polynomial functions on V, which is also called the *symmetric algebra* of V^*. One can construct $S(V^*)$ as the algebra generated by the elements of V^* subject to the relations that these generators commute (in fact this gives a presentation of $S(V^*)$ in the realm of unital associative algebras). The second algebra, which will be constructed in this section, is called the *exterior algebra* of V^* and is denoted by $\bigwedge V^*$. The algebra $\bigwedge V^*$ is the algebra generated by elements of V^* where these generators are subject to the condition that they *anti-commute*, i.e. $\alpha\beta = -\beta\alpha$.

Definition 12.1.1. *An **alternating k-form** on a vector space V is a map*

$$\omega : \underbrace{V \times \ldots \times V}_{k \; \times \; V} \longrightarrow \mathbb{R}$$

which is multilinear

$$\omega(v_1, \ldots, v_{i-1}, av + bw, v_{i+1}, \ldots, v_k) =$$
$$= a\omega(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k) + b\omega(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_k),$$

and satisfies

$$\omega(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -\omega(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k).$$

*The space of alternating k-forms on V is denoted by $\bigwedge^k V^*$ and is called the k-**th exterior power** of V^*. We have that $\bigwedge^1 V^* = V^*$, and we define $\bigwedge^0 V^* := \mathbb{R}$.*

Fixing $v \in V$ to be the first argument of $\omega \in \bigwedge^k V^*$, we obtain an alternating $k - 1$ form, denoted by $\iota_v\omega \in \bigwedge^{k-1} V^*$,

$$(\iota_v\omega)(v_1, \ldots, v_{k-1}) := \omega(v, v_1, \ldots, v_{k-1}).$$

*The operation $\omega \mapsto \iota_v\omega$ is called the **interior product by** v, or the **contraction** by v. For $k = 1$ and $\eta \in V^*$, we have that $\iota_v\eta = \eta(v)$. For $k = 0$ and $a \in \mathbb{R} = \bigwedge^0 V^*$, we set $\iota_v a := 0$.*

Remark 12.1.2. The symmetric group $\mathfrak{S}_k$ consists of permutations of the set $\{1, \ldots, k\}$. Recall that $\mathfrak{S}_k$ is generated by the transpositions $i \leftrightarrow (i+1)$. If $\sigma \in \mathfrak{S}_k$ can be written as a product of p transpositions, then the parity of p is independent of this decomposition, and the sign of σ is defined as $\text{sign}(\sigma) = (-1)^p$. This implies that an alternating k-from ω satisfies

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \text{sign}(\sigma)\omega(v_1 \ldots, v_k),$$

for all $\sigma \in \mathfrak{S}_k$. In particular, ω returns 0 whenever two entries of coincide:

$$\omega(\ldots, v, \ldots, v, \ldots) = 0.$$

Exercise 12.2 gives a more general version of this last observation.

The **tensor product** of $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^l V^*$ is the multilinear map:

$$\alpha \otimes \beta : \underbrace{V \times \ldots \times V}_{(k+l) \ \times \ V} \longrightarrow \mathbb{R},$$

$$(\alpha \otimes \beta)(v_1, \ldots, v_k, w_1, \ldots, w_l) := \alpha(v_1, \ldots, v_k)\beta(w_1, \ldots, w_l).$$

Note that $\alpha \otimes \beta$ is not alternating: if we permute the v_i's among themselves, or the w_j's among themselves, the sign does change correspondingly, however, if the v_i's and the w_j's get mixed up, one obtains a completely different outcome. For example, if $k = l = 1$, then $\alpha(v)\beta(w)$ and $\alpha(w)\beta(v)$ can be quite different. Nevertheless, note that their difference is alternating:

$$(\alpha \otimes \beta - \beta \otimes \alpha)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

This operation can be extended to all degrees $k, l \geq 1$, by considering the sum (with signs) of all possible ways the v_i's and the w_j's can mix. To make this precise we introduce the following notion:

Definition 12.1.3. *A (k, l)-**shuffle** is a permutation $\sigma \in \mathfrak{S}_{k+l}$ which is increasing on the first k integers and on the last l integers:*

$$\sigma(1) < \sigma(2) < \ldots < \sigma(k), \quad \sigma(k+1) < \sigma(k+2) < \ldots < \sigma(k+l).$$

We denote the set of all (k, l)-shuffles by $S(k, l)$.

A (k, l)-shuffle for $k + l = 52$.

Definition 12.1.4. *The **wedge product** (or **exterior product**) is the following operation on alternating forms:*

$$\wedge : \bigwedge^k V^* \times \bigwedge^l V^* \longrightarrow \bigwedge^{k+l} V^*,$$

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S(k,l)} \text{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

For $a \in \mathbb{R} = \bigwedge^0 V^*$ and $\alpha \in \bigwedge^k V^*$ note that:

$$a \wedge \alpha = a\alpha = \alpha \wedge a.$$

It is not obvious from the definition that $\alpha \wedge \beta$ is alternating; this will be shown this in the proof of the following:

Theorem 12.1.5. *The wedge product has the following properties:*

*(1) the interior product with any $v \in V$ is a **graded derivation**:*

$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_v \beta);$$

*(2) the wedge product is **graded commutative**:*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha,$$

*(3) the wedge product is **associative**:*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma),$$

for all $\alpha \in \bigwedge^k V^*$, $\beta \in \bigwedge^l V^*$ and $\gamma \in \bigwedge^m V^*$.
Moreover, if $n = \dim(V^*)$ and $\{e^1, \ldots, e^n\}$ is a basis of V^*, then the set

$$\left\{ e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k} \; : \; 1 \le i_1 < i_2 < \ldots < i_k \le n \right\}$$

forms a basis of $\bigwedge^k V^*$. Hence

$$\dim\left(\bigwedge^k V^*\right) = \begin{cases} \binom{n}{k} = \frac{n!}{k!(n-k)!}, & 0 \le k \le n \\ 0, & n < k. \end{cases}$$

Proof. First we prove (1). Note that the set of (k,l)-shuffles admits the disjoint decomposition $S(k,l) = S'(k,l) \sqcup S''(k,l)$, where

$$S'(k,l) = \{\sigma \in S(k,l) \; : \; \sigma(1) = 1\} \quad S''(k,l) = \{\sigma \in S(k,l) \; : \; \sigma(k+1) = 1\}.$$

Therefore,

$$(\iota_{v_1}(\alpha \wedge \beta))(v_2, \ldots, v_{k+l}) = (\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) =$$

$$= \sum_{\sigma \in S(k,l)} \text{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) =$$

$$= \sum_{\sigma' \in S'(k,l)} \text{sign}(\sigma')(\iota_{v_1}\alpha)(v_{\sigma'(2)}, \ldots, v_{\sigma'(k)})\beta(v_{\sigma'(k+1)}, \ldots, v_{\sigma'(k+l)})+$$

$$+ \sum_{\sigma'' \in S''(k,l)} \text{sign}(\sigma'')\alpha(v_{\sigma''(1)}, \ldots, v_{\sigma''(k)})(\iota_{v_1}\beta)(v_{\sigma''(k+2)}, \ldots, v_{\sigma''(k+l)}).$$

Next, note that there are one-to-one correspondences

$$\theta \in S(k-1,l) \leftrightarrow \sigma' \in S'(k,l) \qquad \eta \in S(k,l-1) \leftrightarrow \sigma'' \in S''(k,l)$$

$$\theta(i) = \sigma'(i+1) - 1 \qquad \eta(i) = \begin{cases} \sigma''(i) - 1, & 1 \le i \le k \\ \sigma''(i+1) - 1, & k+1 \le i \le k+l-1 \end{cases}$$

and moreover, that $\text{sign}(\sigma') = \text{sign}(\theta)$ and $\text{sign}(\sigma'') = (-1)^k \text{sign}(\eta)$. Using this fact, the above formula becomes:

$$(\iota_{v_1}(\alpha \wedge \beta))(v_2, \ldots, v_{k+l}) =$$

$$= \sum_{\theta \in S(k-1,l)} \text{sign}(\theta)(\iota_{v_1}\alpha)(v_{\theta(1)+1}, \ldots, v_{\theta(k)+1})\beta(v_{\theta(k+1)+1}, \ldots, v_{\theta(k+l)+1}) +$$

$$(-1)^k \sum_{\eta \in S(k,l-1)} \text{sign}(\eta)\alpha(v_{\eta(1)+1}, \ldots, v_{\eta(k)+1})(\iota_{v_1}\beta)(v_{\eta(k+1)+1}, \ldots, v_{\eta(k+l-1)+1}) =$$

$$= (\iota_{v_1}\alpha \wedge \beta)(v_2, \ldots, v_{k+l}) + (-1)^k(\alpha \wedge \iota_{v_1}\beta)(v_2, \ldots, v_{k+l}).$$

This proves (1).

Next, we prove by induction on $n := k + l \geq 0$ that, for all $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^l V^*$, $\alpha \wedge \beta$ is alternating. Clearly, this holds for $n = 0$. Assuming the induction hypothesis for $n-1$, it follows that $(\iota_{v_1}\alpha) \wedge \beta + (-1)^k\alpha \wedge (\iota_{v_1}\beta) \in \bigwedge^{n-1} V^*$. Hence, if we switch v_i and v_{i+1} in $(\alpha \wedge \beta)(v_1, \ldots, v_{k+l})$, for $1 < i < n-1$, the result changes the sign. For v_1 and v_2, we need to check that

$$\iota_{v_1}\iota_{v_2}(\alpha \wedge \beta) = -\iota_{v_2}\iota_{v_1}(\alpha \wedge \beta).$$

For this, we apply the derivation rule twice, and use that α and β are alternating:

$$\iota_{v_2}\iota_{v_1}(\alpha \wedge \beta) = \iota_{v_2}\left((\iota_{v_1}\alpha) \wedge \beta + (-1)^k\alpha \wedge (\iota_{v_1}\beta)\right) =$$

$$= (\iota_{v_2}\iota_{v_1}\alpha) \wedge \beta + (-1)^{k-1}(\iota_{v_1}\alpha) \wedge (\iota_{v_2}\beta) +$$

$$+ (-1)^k(\iota_{v_2}\alpha) \wedge (\iota_{v_1}\beta) + \alpha \wedge (\iota_{v_2}\iota_{v_1}\beta) =$$

$$= -(\iota_{v_1}\iota_{v_2}\alpha) \wedge \beta - (-1)^k(\iota_{v_1}\alpha) \wedge (\iota_{v_2}\beta) +$$

$$- (-1)^{k-1}(\iota_{v_2}\alpha) \wedge (\iota_{v_1}\beta) - \alpha \wedge (\iota_{v_1}\iota_{v_2}\beta) =$$

$$= -\iota_{v_1}\iota_{v_2}(\alpha \wedge \beta),$$

where in the last step we have performed first two steps backwards, with v_1 and v_2 interchanged. This finishes the inductive argument, and proves that the wedge product of alternating forms is indeed alternating.

Properties (2) and (3) are proven similarly, by induction on $n = k + l$ and $n = k + l + m$, respectively, and by using the derivation rule. We will prove only (2), and leave it to the reader to check (3). Clearly, (2) holds for $n = k + l = 0$. Assume that it holds for $n - 1$. Let $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^l V^*$. For $v \in V$, we have

$$\iota_v(\alpha \wedge \beta) = (\iota_v\alpha) \wedge \beta + (-1)^k\alpha \wedge (\iota_v\beta) =$$

$$= (-1)^{(k-1)l}\beta \wedge (\iota_v\alpha) + (-1)^{k+k(l-1)}(\iota_v\beta) \wedge \alpha =$$

$$= (-1)^{kl}\left((-1)^l\beta \wedge (\iota_v\alpha) + (\iota_v\beta) \wedge \alpha\right) =$$

$$= (-1)^{kl}\iota_v(\beta \wedge \alpha),$$

where we have used the induction hypothesis in the second line. Since this holds for all v, (2) follows.

For the last part, let $e_1, \ldots, e_n \in V$ be the dual basis, i.e. these elements are defined by $e^i(e_j) = \delta_j^i$, where $\delta_j^i = 1$ if $i = j$ and $\delta_j^i = 0$ if $i \neq j$. By repeatedly applying the derivation rule, one can easily show that, for $1 \leq i_1 < \ldots < i_k \leq n$ and $1 \leq j_1 < \ldots < j_k \leq n$, the following holds:

$$\left(e^{i_1} \wedge \ldots \wedge e^{i_k}\right)(e_{j_1}, \ldots, e_{j_k}) = \delta_{j_1}^{i_1} \ldots \delta_{j_k}^{i_k},$$

in other words the result is zero unless $i_p = j_p$, for all $1 \le p \le k$, and then the result is 1.

Consider now a linear combination:

$$\omega = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k} \in \bigwedge^k V^*.$$

Then for $1 \le j_1 < \ldots < j_k \le n$, we have that $\omega(e_{j_1}, \ldots, e_{j_k}) = a_{j_1 \ldots j_k}$. Thus if $\omega = 0$, then all coefficients $a_{j_1 \ldots j_k} = 0$. This shows that the k-forms $e^{i_1} \wedge \ldots \wedge e^{i_k}$ are linearly independent.

To show that these forms span $\bigwedge^k V^*$, let $\omega \in \bigwedge^k V^*$. Consider the k-form:

$$\eta := \omega - \sum_{i_1 < \ldots < i_k} \omega(e_{i_1}, \ldots, e_{i_k}) e^{i_1} \wedge \ldots \wedge e^{i_k} \in \bigwedge^k V^*.$$

Then $\eta(e_{j_1}, \ldots, e_{j_k}) = 0$ for all $1 \le j_1 < \ldots < j_k \le n$. Since η is alternating, this implies that, $\eta(e_{l_1}, \ldots, e_{l_k}) = 0$, for all $l_1, \ldots, l_k \in \{1, \ldots, n\}$ (see Remark 12.1.2). Finally, consider k vectors $v_1, \ldots, v_k \in V$. By writing these in the basis $v_i = \sum_{l=1}^n v_i^l e_l$, and using that η is multilinear, we obtain:

$$\eta(v_1, \ldots, v_k) = \sum_{l_1=1}^n \ldots \sum_{l_k=1}^n v_1^{l_1} \ldots v_k^{l_k} \eta(e_{l_1}, \ldots, e_{l_k}) = 0.$$

Thus $\eta = 0$, and therefore:

$$\omega = \sum_{i_1 < \ldots < i_k} \omega(e_{i_1}, \ldots, e_{i_k}) e^{i_1} \wedge \ldots \wedge e^{i_k}.$$

We conclude that the set $\{e^{i_1} \wedge \ldots \wedge e^{i_k} : 1 \le i_1 < \ldots < i_k \le n\}$ forms a basis of $\bigwedge^k V^*$. Note that sequences of the form $1 \le i_1 < \ldots < i_k \le n$ are in one-to-one correspondence with subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ with exactly k elements; this implies that $\dim(\bigwedge^k V^*) = \binom{n}{k}$. $\square$

As a consequence of the graded commutativity, we obtain:

Corollary 12.1.6. *If k is odd and $\omega \in \bigwedge^k V^*$ then $\omega^2 = \omega \wedge \omega = 0$.*

Putting the exterior powers of V^* together, one obtains the exterior algebra:

Definition 12.1.7. *Let V be a vector space. The **exterior algebra** of V^*, denoted by $(\bigwedge V^*, +, \wedge)$, is the direct sum of the exterior products of V^**

$$\bigwedge V^* := \bigoplus_{k \ge 0} \bigwedge^k V^*,$$

endowed with the bilinear extension of the wedge product

$$\wedge : \bigwedge V^* \times \bigwedge V^* \longrightarrow \bigwedge V^*.$$

The exterior algebra is an associative unital algebra of $\dim(\bigwedge V^*) = 2^{\dim(V)}$.

Remark 12.1.8. The unit of $\bigwedge V^*$ is just $1 \in \mathbb{R} = \bigwedge^0 V^*$. The "bilinear extension of the wedge product" is meant in the obvious way: if $\alpha, \beta \in \bigwedge V^*$ decompose as

$$\alpha = \alpha_0 + \alpha_1 + \ldots + \alpha_k, \quad \alpha_i \in \bigwedge^i V^*,$$

$$\beta = \beta_0 + \beta_1 + \ldots + \beta_l, \quad \beta_j \in \bigwedge^j V^*,$$

then their wedge product is given by:

$$\alpha \wedge \beta = \gamma_0 + \gamma_1 + \ldots + \gamma_{k+l}, \quad \gamma_s := \sum_{i+j=s} \alpha_i \wedge \beta_j \in \bigwedge^s V^*.$$

Definition 12.1.9. *Let V and W be vector space, and let $A : V \to W$ be a linear map. The **pullback map** induced by A is the algebra homomorphism*

$$A^* : \left(\bigwedge W^*, +, \wedge \right) \longrightarrow \left(\bigwedge V^*, +, \wedge \right),$$

defined by

$$A^* : \bigwedge^k W^* \longrightarrow \bigwedge^k V^*, \quad (A^*\omega)(v_1, \ldots, v_k) := \omega(Av_1, \ldots, Av_k),$$

for $k \geq 1$, and for $a \in \bigwedge^0 W^* = \mathbb{R}$, $A^*a = a \in \bigwedge^0 V^* = \mathbb{R}$.

It can be easily checked that A^* is indeed an algebra homomorphism:

$$A^*(\alpha \wedge \beta) = (A^*\alpha) \wedge (A^*\beta).$$

The **top exterior power** of V^*, i.e. the 1-dimensional vector space $\bigwedge^{\dim(V)} V^*$, is also called the **determinant** of V^*, because of the following:

Proposition 12.1.10. *For any linear map $A : V \to V$ and any $\omega \in \bigwedge^{\dim(V)} V^*$:*

$$A^*(\omega) = \det(A)\omega;$$

explicitly, for $v_1, \ldots, v_n \in V$, where $n = \dim(V)$,

$$\omega(Av_1, \ldots, Av_n) = \det(A)\omega(v_1, \ldots, v_n).$$

Remark 12.1.11. Note that (k, l)-shuffles are in one-to-one correspondence with pairs of subsets (A, B) of the set $\{1, \ldots, k + l\}$, such that:

$$\{1, \ldots, k + l\} = A \cup B, \quad |A| = k, \quad |B| = l.$$

The element $\sigma \in S(k, l)$ corresponds to the pair

$$A = \{\sigma(1), \ldots, \sigma(k)\}, \quad B = \{\sigma(k+1), \ldots, \sigma(k+l)\}.$$

Conversely, given such a pair (A, B), one first orders the elements of the two sets:

$$A = \{a_1 < a_2 < \ldots < a_k\}, \quad B = \{b_1 < b_2 < \ldots < b_l\},$$

and then the corresponding σ is given by:

$$\sigma(i) := a_i, \ 1 \leq i \leq k; \qquad \sigma(k+j) := b_j, \ 1 \leq j \leq l.$$

In particular, we have that

$$|S(k, l)| = \binom{k+l}{k} = \frac{(k+l)!}{k! \ l!}.$$

12.2. The cotangent bundle

Definition 12.2.1. *Let M be a manifold. The **cotangent space** at $p \in M$, denoted by $T_p^* M$, is defined as the linear dual of the tangent space $T_p M$; i.e. $T_p^* M$ consists of all linear maps $\alpha : T_p M \to \mathbb{R}$. Elements of $T_p^* M$ are called **cotangent vectors**. All the cotangent spaces together form the **cotangent bundle** of M:*

$$T^* M := \bigsqcup_{p \in M} T_p^* M.$$

Example 12.2.2. Let $f \in C^\infty(M)$ be a smooth function. The differential of f at $p \in M$ is a cotangent vector $\mathrm{d}_p f \in T_p^* M$ at p

$$\mathrm{d}_p f : T_p M \longrightarrow \mathbb{R}, \qquad \mathrm{d}_p f(v) := v(f),$$

where, as usually, we identify $T_{f(p)} \mathbb{R} = \mathbb{R}$.

Let $(U, \varphi = (x^1, \ldots, x^m))$ be a coordinates chart on M[1]. At $p \in U$, we have:

$$\mathrm{d}_p x^i \Big(\frac{\partial}{\partial x^j} \Big|_p \Big) = \frac{\partial x^i}{\partial x^j}(p) = \delta_j^i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

This shows that $\mathrm{d}_p x^1, \ldots, \mathrm{d}_p x^m$ is a basis of the cotangent space $T_p^* M$ dual to the basis $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^m}\big|_p$ of $T_p M$.

Theorem 12.2.3. *The cotangent bundle $\pi : T^* M \to M$ is a vector bundle over M of rank equal to $m = \dim(M)$. The vector bundle structure is uniquely determined by the condition that, for every chart $(U, \varphi = (x^1, \ldots, x^m))$ on M, the differentials of the coordinate functions:*

$$\mathrm{d}x^1, \ldots, \mathrm{d}x^m : U \longrightarrow T^* M|_U, \quad p \mapsto \mathrm{d}_p x^i \in T_p^* M,$$

form a local frame on $T^* M$.

The above result can be proven exactly like Theorem 8.1.2, therefore we omit the proof. However, it is instructive to write down the transition functions from one frame to another. Given a second set of coordinates $(V, \psi = (y^1, \ldots, y^m))$, for $p \in U \cap V$, we have that:

$$\mathrm{d}x^i|_p = \sum_{j=1}^m \frac{\partial x^i}{\partial y^j}(p) \mathrm{d}y^j|_p,$$

which can be easily seen by applying to both sides the basis vectors $\frac{\partial}{\partial y^k}\Big|_p$.

12.3. The exterior powers of the cotangent bundle

Next, we consider the exterior powers of the cotangent spaces.

Definition 12.3.1. *Let M be a manifold, and let $p \in M$. The k-**th exterior power** of the cotangent space at p is denote by $\bigwedge^k T_p^* M$. The collection of all these vector spaces is the k-**th exterior power of the cotangent bundle**:*

$$\bigwedge^k T^* M := \bigsqcup_{p \in M} \bigwedge^k T_p^* M.$$

---

[1]From this lecture on, we use simplified notation: instead of x_φ^i we denote the coordinates by x^i.

Consider a chart $(U, \varphi = (x^1, \ldots, x^m))$ on M. By Theorem 12.1.5, for each $p \in U$, a basis of $\bigwedge^k T_p^* M$ is given by the elements:

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}|_p, \quad 1 \le i_1 < \ldots < i_k \le m.$$

Generalizing the case $k = 1$ of the cotangent bundle, when choosing a second chart $(V, \psi = (y^1, \ldots, y^m))$, on $U \cap V$ the frame changes as follows:

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} = \sum_{l_1 \ldots l_k} \frac{\partial x^{i_1}}{\partial y^{l_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{l_k}} \mathrm{d}y^{l_1} \wedge \ldots \wedge \mathrm{d}y^{l_k},$$

where the sum runs over all $1 \le l_1, \ldots, l_k \le m$. After ordering the indices $l_1, \ldots, l_k$ increasingly, and dropping the terms where some indexes are repeated (are therefore are zero), we can rewrite:

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} = \sum_{j_1 < \ldots < j_k} c_{j_1, \ldots, j_k}^{i_1, \ldots, i_k} \mathrm{d}y^{j_1} \wedge \ldots \wedge \mathrm{d}y^{j_k},$$

where the coefficients are given explicitly by

$$c_{j_1, \ldots, j_k}^{i_1, \ldots, i_k} = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma) \frac{\partial x^{i_1}}{\partial y^{j_{\sigma(1)}}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_{\sigma(k)}}} = \frac{\partial(x^{i_1} \ldots x^{i_k})}{\partial(y^{j_1} \ldots y^{j_k})},$$

where the notation on the right is the standard notation for the minor consisting of the rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ in the Jacobian matrix $(\frac{\partial x^i}{\partial y^j})_{i,j=1}^m$ of the change of coordinates map $\psi \circ \varphi^{-1}$, i.e.

$$\frac{\partial(x^{i_1} \ldots x^{i_k})}{\partial(y^{j_1} \ldots y^{j_k})} = \det\left(\left(\frac{\partial x^{i_a}}{\partial y^{j_b}}\right)_{a,b=1}^k\right).$$

The precise formula is not that important; what matters is that these coefficients are smooth, because this allows us to apply the same argument as in the proof of Theorem 8.1.2, and conclude:

Theorem 12.3.2. *The k-th exterior power of the cotangent bundle*

$$\pi : \bigwedge^k T^* M \longrightarrow M$$

is a vector bundle over M of rank equal to $\binom{m}{k}$, where $m = \dim(M)$. A chart $(U, \varphi = (x^1, \ldots, x^m))$ on M induces a local frame on $\bigwedge^k T^* M$ over U

$$\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}, \quad 1 \le i_1 < \ldots < i_k \le m;$$

moreover, these frames define the vector bundle structure uniquely.

12.4. Differential forms

Definition 12.4.1. *A **differential form of degree** k (or a k-**form**) on M is a section $\bigwedge^k T^* M$. The space of differential k-forms is denoted by*

$$\Omega^k(M) := \Gamma\left(\bigwedge^k T^* M\right).$$

For $k = 0$, $\Omega^0(M) = C^\infty(M)$. The direct sum of these vector spaces is denoted by

$$\Omega(M) := \bigoplus_{k \ge 0} \Omega^k(M).$$

A **differential form** on M is an element

$$\omega \in \Omega(M), \quad \omega = \omega_0 + \ldots + \omega_m, \quad \omega_i \in \Omega^i(M),$$

where $m = \dim(M)$.

The pointwise wedge product induces the structure of an **associative unital algebra** on the space of differential forms:

$$(\Omega(M), +, \wedge), \quad (\alpha \wedge \beta)_p := \alpha_p \wedge \beta_p,$$

which is **graded commutative**:

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha, \quad \alpha \in \Omega^k(M), \ \beta \in \Omega^l(M).$$

The **interior product** with a vector field $X \in \mathfrak{X}(M)$ is the linear operator:

$$\iota_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \quad (\iota_X \omega)_p := \iota_{X_p}(\omega_p),$$

which is a **graded derivation** for the wedge-product:

$$\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_X \beta),$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega(M)$.

Given a local chart (U, φ) with coordinates $\varphi = (x^1, \ldots, x^m)$, a differential k-form $\omega \in \Omega^k(M)$ can be written as

$$\omega|_U = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

with smooth coefficients $\omega_{i_1 \ldots i_k} \in C^\infty(U)$. The notation where the indexes are not ordered is also useful:

$$\omega|_U = \sum_{i_1 \ldots i_k} \frac{1}{k!} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where the coefficients are defined to be fully skew-symmetric:

$$\omega_{i_{\sigma(1)} \ldots i_{\sigma(k)}} = \mathrm{sign}(\sigma) \omega_{i_1 \ldots i_k}, \quad \text{for all } \sigma \in \mathfrak{S}_k.$$

In this notation the interior by a vector fields $X|_U = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}$ is given by

$$(\iota_X \omega)|_U = \sum_{i_1 \ldots i_k} \frac{1}{(k-1)!} X^{i_1} \omega_{i_1 \ldots i_k} dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

12.5. The pullback of differential forms

Differential forms can be pulled back via any smooth maps:

Definition 12.5.1. *The **pullback** along a smooth map $f : M \to N$ is defined by:*

$$f^* : \Omega^k(N) \longrightarrow \Omega^k(M), \quad (f^* \omega)_p := (d_p f)^* \omega_{f(p)},$$

that is, for $v_1, \ldots, v_k \in T_p M$:

$$(f^* \omega)_p(v_1, \ldots, v_k) = \omega_{f(p)}(d_p f(v_1), \ldots, d_p f(v_k)).$$

The pullback satisfies the following:

Proposition 12.5.2. *The pullback along a smooth map $f : M \to N$ is an algebra homomorphism:*

$$f^* : (\Omega(N), +, \wedge) \longrightarrow (\Omega(M), +, \wedge),$$

i.e. for all $\alpha, \beta \in \Omega(N)$ and all $a, b \in \mathbb{R}$ it satisfies:

$$f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta),$$
$$f^*(a\alpha + b\beta) = af^*(\alpha) + bf^*(\beta).$$

Moreover, if $g : N \to P$ is also a smooth map, then $f^* \circ g^* = (g \circ f)^*$.

12.6. Exercises

Exercise 12.1. Prove the following formulas for the wedge product:

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \text{sign}(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

Exercise 12.2. Let $v_1, \ldots, v_p$ be p linearly dependent vectors in V. Show that for any alternating form $\omega \in \bigwedge^k V^*$, we have that

$$\iota_{v_1} \ldots \iota_{v_p}\omega = 0.$$

Exercise 12.3. Let $\alpha_1, \ldots, \alpha_k \in V^*$, and $v_1, \ldots, v_k \in V$. Prove that:

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(v_1, \ldots, v_k) = \det \left(\alpha_i(v_j) \right)_{i,j=1}^k.$$

Exercise 12.4. Prove Proposition 12.1.10.

Exercise 12.5. Denote the product of k-copies of V by $V^{\times k}$. For a $k \times k$-matrix $R = \{r_i^j\}_{i,j=1}^k$, consider the following map:

$$R^V : V^{\times k} \longrightarrow V^{\times k}, \quad R^V(v_1, \ldots, v_k) := \left(\sum_{j=1}^k r_1^j v_j, \ldots, \sum_{j=1}^k r_k^j v_j \right).$$

Prove that a map $\omega : V^{\times k} \to \mathbb{R}$ is an alternating k-form if and only if, for every $k \times k$-matrix R:

$$\omega \circ R^V = \det(R) \cdot \omega.$$

Exercise 12.6. The **rank** of a 2-form $\omega \in \bigwedge^2 V^*$ is the largest even integer $2r$ such that $\omega^r \neq 0$. Prove that ω has rank $2r$ if and only if V^* has a basis $e^1, \ldots, e^r, f^1, \ldots, f^r, g^1, \ldots, g^c$ such that

$$\omega = e^1 \wedge f^1 + \ldots + e^r \wedge f^r.$$

Exercise 12.7. Let V be a vector space of dimension n. Let $v \in V$ and $\xi \in V^*$ such that $\xi(v) = 1$. Consider the operators: interior product by v

$$\iota_v : \bigwedge V^* \longrightarrow \bigwedge V^*, \quad \omega \mapsto \iota_v\omega,$$

and exterior products by ξ

$$e_\xi : \bigwedge V^* \longrightarrow \bigwedge V^*, \quad \omega \mapsto \xi \wedge \omega.$$

(a) Prove that these operators satisfy:

$$e_\xi \circ \iota_v + \iota_v \circ e_\xi = \text{Id}_{\bigwedge V^*}.$$

(b) For $\omega \in \bigwedge V^*$, prove $\iota_v\omega = 0$ if and only if there $\alpha \in \bigwedge V^*$ such that $\omega = \iota_v\alpha$.
(c) For $\omega \in \bigwedge V^*$, prove $\xi \wedge \omega = 0$ if and only if there $\beta \in \bigwedge V^*$ such that $\omega = \xi \wedge \beta$.

Exercise 12.8. Let V be a vector space of dimension n. An element $\alpha \in \bigwedge V^*$ is called invertible if there exists $\beta \in \bigwedge V^*$ such that $\alpha \wedge \beta = 1$. Prove that an element

$$\alpha = \alpha_0 + \alpha_1 + \ldots + \alpha_n \in \bigwedge V^*, \quad \alpha_i \in \overset{i}{\bigwedge} V^*$$

is invertible iff $\alpha_0 \neq 0$ (*Hint: think about the geometric series* $\frac{1}{1-q} = 1 + q + q^2 + \ldots$).

Exercise 12.9. Consider the differential forms, the vector field and the map below:

$$\alpha = \cos(y)\mathrm{d}x + \mathrm{d}z - e^{-w^2}\mathrm{d}w \in \Omega^1(\mathbb{R}^4),$$
$$\beta = x^2\mathrm{d}x \wedge \mathrm{d}y + w^7\mathrm{d}w \wedge \mathrm{d}z \in \Omega^2(\mathbb{R}^4),$$
$$\gamma = w^2\mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}y \wedge (\mathrm{d}w + y\mathrm{d}w \wedge \mathrm{d}x) \in \Omega(\mathbb{R}^4),$$
$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + y^3\frac{\partial}{\partial w},$$
$$f : \mathbb{R}^2 \to \mathbb{R}^4, \quad f(u, v) = (u^2, e^{uv}, u - v^4, \sin(v)).$$

(a) Calculate the nine exterior products:

$$\begin{array}{ccc} \alpha \wedge \alpha & \alpha \wedge \beta & \alpha \wedge \gamma \\ \beta \wedge \alpha & \beta \wedge \beta & \beta \wedge \gamma \\ \gamma \wedge \alpha & \gamma \wedge \beta & \gamma \wedge \gamma \end{array} .$$

(b) Calculate the three interior products: $\iota_X\alpha$, $\iota_X\beta$, $\iota_X\gamma$.
(c) Calculate the pullback along f of the three forms: $f^*(\alpha)$, $f^*(\beta)$, $f^*(\gamma)$.
(d) Construct, or prove that it is impossible to do so, two one-forms $\eta, \theta \in \Omega^1(\mathbb{R}^4)$ such that $\beta = \eta \wedge \theta$.

LECTURE 13

13.1. The exterior derivative

Recall that, with our usual identification $T_t\mathbb{R} \cong \mathbb{R}$, the differential at $p \in M$ of a smooth map $a \in C^\infty(M)$ can be viewed as a covector

$$\mathrm{d}_p a \in T_p^* M, \qquad \mathrm{d}_p a(v) = v(a), \qquad v \in T_p M.$$

Moreover, the assignment $p \mapsto \mathrm{d}_p a$ defines a one-form $\mathrm{d}a \in \Omega^1(M)$. Globally, we obtain a linear maps from functions to one-forms:

$$\mathrm{d} : C^\infty(M) \longrightarrow \Omega^1(M), \quad a \mapsto \mathrm{d}a.$$

In local coordinates $(U, \varphi = (x^1, \ldots, x^m))$ the differential is given by

$$\mathrm{d}a = \sum_{i=1}^m \frac{\partial a}{\partial x^i} \mathrm{d}x^i.$$

The exterior derivative is an operator which extends the differential of functions to forms of higher degree, and is uniquely determined by the properties below.

Theorem 13.1.1. *For any manifold M there is a unique linear map*

$$\mathrm{d} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \quad k \geq 0$$

satisfying the following properties:

(1) on zero-forms $\Omega^0(M) = C^\infty(M)$ it is the differential of functions;
(2) it squares to zero: $\mathrm{d}^2\alpha = 0$, for all $\alpha \in \Omega(M)$;
(3) it is a graded derivation for the wedge product, i.e.

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta,$$

for all $\alpha \in \Omega^k(M)$ and all $\beta \in \Omega(M)$.
Moreover, the following holds:
(4) if $f : M \to N$ is a smooth map, then $\mathrm{d}f^*(\alpha) = f^*(\mathrm{d}\alpha)$ for all $\alpha \in \Omega(N)$.

Definition 13.1.2. *The operator $\mathrm{d} : \Omega(M) \to \Omega(M)$ from Theorem 13.1.1 is called the **exterior derivative**.*

Remark 13.1.3. It is important to note that (2), (3) and (4) are essentially equivalent to the following properties of smooth functions (this will be seen in the proof):

   (2) partial derivatives commute;

(3) the Leibniz rule;

(4) the chain rule.

We first prove Theorem 13.1.1 for open subsets of the Euclidean space.

Lemma 13.1.4. *Theorem 13.1.1 holds for open subsets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$.*

Proof. Let us deduce a formula for d by using properties (1)-(3); this argument will imply uniqueness. Let $\alpha \in \Omega^k(M)$. To calculate $\mathrm{d}\alpha$ we decompose α in the standard basis:

$$(*) \qquad \alpha = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}, \quad a_{i_1 \ldots i_k} \in C^\infty(M).$$

Using linearity of d and the derivation rule (3), we obtain

$$\mathrm{d}\alpha = \sum_{i_1 < \ldots < i_k} \mathrm{d}a_{i_1 \ldots i_k} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} + a_{i_1 \ldots i_k} \wedge \mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}\right),$$

where by (1) the first term is determined since $\mathrm{d}a_{i_1 \ldots i_k}$ is the usual differential of the smooth map $a_{i_1 \ldots i_k}$. For the second term, by applying (3), we obtain:

$$\mathrm{d}\left(\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}\right) = \mathrm{d}\left(\mathrm{d}x^{i_1}\right) \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k} - \mathrm{d}x^{i_1} \wedge \mathrm{d}\left(\mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k}\right).$$

Since d extends the usual derivation (1), and since $\mathrm{d}^2 = 0$ (2), the first term is zero. By repeatedly applying this argument, the entire expression is zero.

We conclude that there is a unique linear operator $\mathrm{d} : \Omega(M) \to \Omega(M)$ satisfying the conditions (1)-(3), and that on a form $\alpha \in \Omega^k(M)$ as in (*) it acts by:

$$(**) \qquad \mathrm{d}\alpha = \sum_{i_1 < \ldots < i_k} \mathrm{d}a_{i_1 \ldots i_k} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} =$$

$$= \sum_{i_1 < \ldots < i_k} \sum_i \frac{\partial a_{i_1 \ldots i_k}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

Next, we check that this linear operator satisfies indeed the properties from Theorem 13.1.1. Property (1) clearly holds.

We check (2)-(4) first on smooth functions. For $a \in C^\infty(M)$, we have that:

$$\mathrm{d}^2 a = \mathrm{d}\left(\sum_{i=1}^n \frac{\partial a}{\partial x^i} \mathrm{d}x^i\right) = \sum_{i=1}^n \mathrm{d}\left(\frac{\partial a}{\partial x^i}\right) \wedge \mathrm{d}x^i = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 a}{\partial x^i \partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^i =$$

$$= \sum_{i=1}^n \frac{\partial^2 a}{(\partial x^i)^2} \mathrm{d}x^i \wedge \mathrm{d}x^i + \sum_{i<j} \frac{\partial^2 a}{\partial x^i \partial x^j} \left(\mathrm{d}x^j \wedge \mathrm{d}x^i + \mathrm{d}x^i \wedge \mathrm{d}x^j\right) = 0,$$

where we have used that the partial derivatives commute, and the following algebraic properties of the exterior algebra:

$$\mathrm{d}x^i \wedge \mathrm{d}x^i = 0 \quad \text{and} \quad \mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i.$$

The derivation rule (3), for $a, b \in C^\infty(M)$, is immediate:

$$\mathrm{d}(ab)(v) = v(ab) = v(a)b + av(b) = (\mathrm{d}a \cdot b + a \cdot \mathrm{d}b)(v), \quad \forall\, v \in TM.$$

Similarly, for $f : N \to M$ and $a \in C^\infty(M)$, (4) holds by the chain rule:

$$f^*(\mathrm{d}a)(v) = \mathrm{d}a(\mathrm{d}f(v)) = \mathrm{d}(a \circ f)(v) = \mathrm{d}(f^*a)(v), \quad \forall\, v \in TN.$$

Next, we show that (3) holds in all degrees. Since the wedge product is bilinear, and d is linear, it suffices to show (3) for elements of the form $\alpha = a\mathrm{d}x^I \in \Omega^k(M)$

and $\beta = b\mathrm{d}x^J \in \Omega^l(M)$, where $a, b \in C^\infty(M)$ and, for $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_l\}$, we have denoted:

$$\mathrm{d}x^I := \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}, \quad \mathrm{d}x^J := \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_l},$$

Note that $\mathrm{d}(\mathrm{d}x^I \wedge \mathrm{d}x^J) = 0$. Using (3) for functions, we obtain (3) in all degrees:

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}(ab\,\mathrm{d}x^I \wedge \mathrm{d}x^J) = \mathrm{d}(ab) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J = ((\mathrm{d}a)b + a(\mathrm{d}b)) \wedge \mathrm{d}x^I \wedge \mathrm{d}x^J =$$
$$= \mathrm{d}a \wedge \mathrm{d}x^I \wedge b\,\mathrm{d}x^J + (-1)^k a\,\mathrm{d}x^I \wedge \mathrm{d}b \wedge \mathrm{d}x^J = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta.$$

Again, by linearity of d, it suffices to prove (2) for $\alpha = a\mathrm{d}x^I$. Using (3) and (2) for functions, we obtain (2) in general:

$$\mathrm{d}^2(\alpha) = \mathrm{d}\left(\mathrm{d}a \wedge \mathrm{d}x^I\right) = (\mathrm{d}^2 a) \wedge \mathrm{d}x^I - \mathrm{d}a \wedge \mathrm{d}(\mathrm{d}x^I) = 0.$$

By linearity of d and of the pullback map f^*, it suffices to check (4) for $\alpha = a\mathrm{d}x^I$. By (4) for functions, we have that

$$f^*(\mathrm{d}x^I) = \mathrm{d}f^*(x^{i_1}) \wedge \ldots \wedge \mathrm{d}f^*(x^{i_k}),$$

and therefore, applying (2) and (3) repeatedly,

$$\mathrm{d}f^*(\mathrm{d}x^I) = \sum_{p=1}^k (-1)^{p+1} \mathrm{d}f^*(x^{i_1}) \wedge \ldots \wedge \mathrm{d}^2 f^*(x^{i_p}) \wedge \ldots \wedge \mathrm{d}f^*(x^{i_k}) = 0.$$

Using this, and again (4) for functions, we obtain that (4) holds in general:

$$\mathrm{d}f^*(\alpha) = \mathrm{d}\left(f^*(a)f^*(\mathrm{d}x^I)\right) = \mathrm{d}(f^*a) \wedge f^*(\mathrm{d}x^I) + f^*(a)\mathrm{d}f^*(\mathrm{d}x^I) =$$
$$= f^*(\mathrm{d}a) \wedge f^*(\mathrm{d}x^I) = f^*(\mathrm{d}a \wedge \mathrm{d}x^I) = f^*(\mathrm{d}\alpha).$$

$$\square$$

Next, we prove existence of the operator d for any manifold:

Lemma 13.1.5. *For any manifold M there exists an operator $\mathrm{d} : \Omega(M) \to \Omega(M)$ with the properties (1)-(4) from Theorem 13.1.1.*

Proof. For $\alpha \in \Omega(M)$, let $\mathrm{d}\alpha$ be such that, for any chart (U, φ) on M,

$$(\mathrm{d}\alpha)|_U = \left(\varphi^* \circ \mathrm{d}^{\mathbb{R}} \circ (\varphi^{-1})^*\right)(\alpha|_U),$$

where on the right-hand-side we have denoted by $\mathrm{d}^{\mathbb{R}}$ the operator constructed in Lemma 13.1.4 for forms on $\varphi(U) \subset \mathbb{R}^m$. For this definition we need to check that, given a second chart (V, ψ) we obtain the same value for $\mathrm{d}\alpha$ on $U \cap V$. Consider the change of coordinates diffeomorphism between open sets in $\mathbb{R}^m$:

$$f : \psi(U \cap V) \xrightarrow{\sim} \varphi(U \cap V), \quad f = \varphi \circ \psi^{-1}.$$

By Lemma 13.1.4, we have that the operator $\mathrm{d}^{\mathbb{R}}$ satisfies property (4); therefore:

$$\mathrm{d}^{\mathbb{R}} \circ f^*\left((\varphi^{-1})^*(\alpha|_{U \cap V})\right) = f^* \circ \mathrm{d}^{\mathbb{R}}\left((\varphi^{-1})^*(\alpha|_{U \cap V})\right).$$

Writing $f^* = (\psi^{-1})^* \circ \varphi^*$, and composing both sides with ψ^* on the left, we obtain:

$$\left(\psi^* \circ \mathrm{d}^{\mathbb{R}} \circ (\psi^{-1})^*\right)(\alpha|_{U \cap V}) = \left(\varphi^* \circ \mathrm{d}^{\mathbb{R}} \circ (\varphi^{-1})^*\right)(\alpha|_{U \cap V}).$$

In other words, the two definition of $\mathrm{d}\alpha$ coincide on $U \cap V$. We conclude that d is well-defined. Clearly, it suffices to check properties (1)-(4) locally; and these follow from the respective properties of $\mathrm{d}^{\mathbb{R}}$ which were proven in Lemma 13.1.4. This concludes the proof. $\square$

Next, we show that any operator satisfying (3) is local.

Lemma 13.1.6. *Any linear operator $D : \Omega(M) \to \Omega(M)$ satisfying (3) is local, i.e. $D\alpha|_U$ depends only on $\alpha|_U$ for any $\alpha \in \Omega(M)$ and any open set $U \subset M$.*

Proof. We need to show that, if $\alpha_1, \alpha_2 \in \Omega(M)$ satisfy $\alpha_1|_U = \alpha_2|_U$, then also $D(\alpha_1)|_U = D(\alpha_2)|_U$. Since D is linear, by taking the difference of these forms, it suffices to show that $\alpha|_U = 0$ implies that $D\alpha|_U = 0$. Let $p \in U$, and let $\chi \in C^\infty(M)$ be a smooth function such that $\chi(p) = 1$ and $\mathrm{supp}(\chi) \subset U$ (this exists by Lemma 4.1.2). Then, $\chi\alpha = 0$. Applying D, we obtain:

$$0 = D(\chi\alpha) = D\chi \wedge \alpha + \chi D\alpha.$$

Evaluating at p, and using that $\alpha_p = 0$ and $\chi(p) = 1$, we obtain that $(D\alpha)_p = 0$. □

The following concludes the proof of Theorem 13.1.1

Lemma 13.1.7. *The operator $\mathrm{d} : \Omega(M) \to \Omega(M)$ constructed in Lemma 13.1.5 is the unique linear map satisfying (1)-(3).*

Proof. Assume $D : \Omega(M) \to \Omega(M)$ is a second linear map satisfying (1)-(3). As in the proof of Lemma 13.1.4, (1)-(3) imply that $D = \mathrm{d}$ on forms of the form $\alpha = a^0 \mathrm{d}a^1 \wedge \ldots \wedge \mathrm{d}a^k$, where $a^i \in C^\infty(M)$. Consider now a general $\alpha \in \Omega(M)$. Let $p \in M$ and choose a chart $(U, (x^1, \ldots, x^m))$ around p, and write

$$\alpha|_U = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

By Lemma 4.1.2, there are function $\widetilde{x}^i, \widetilde{a}_{i_1 \ldots i_k} \in C^\infty(M)$ and an open set $p \in V \subset U$ such that $\widetilde{x}^i|_V = x^i|_V$ and $\widetilde{a}_{i_1 \ldots i_k}|_V = a_{i_1 \ldots i_k}|_V$. Consider the form:

$$\widetilde{\alpha} := \sum_{i_1 < \ldots < i_k} \widetilde{a}_{i_1 \ldots i_k} \mathrm{d}\widetilde{x}^{i_1} \wedge \ldots \wedge \mathrm{d}\widetilde{x}^{i_k}.$$

By the above, $\mathrm{d}\widetilde{\alpha} = D\widetilde{\alpha}$. Since $\alpha|_V = \widetilde{\alpha}|_V$, Lemma 13.1.6 gives:

$$(D\alpha)|_V = (D\widetilde{\alpha})|_V = (\mathrm{d}\widetilde{\alpha})|_V = (\mathrm{d}\alpha)|_V.$$

Thus, $D\alpha$ and $\mathrm{d}\alpha$ are equal in a neighborhood of p. We conclude that $D = \mathrm{d}$. □

13.2. De Rham cohomology

Differential forms can be used to extract topological information about manifolds through the so-called **de Rham cohomology**.

Definition 13.2.1.
- *A k-form $\omega \in \Omega^k(M)$ satisfying $\mathrm{d}\omega = 0$ is called **closed**.*
- *A k-form $\omega \in \Omega^k(M)$ for which there exists $\alpha \in \Omega^{k-1}(M)$ such that $\mathrm{d}\alpha = \omega$ is called **exact**.*

Since $\mathrm{d}^2 = 0$, every exact form is closed.

Definition 13.2.2.
- *The k-th **de Rham cohomology group** of M is the quotient of the space of closed forms modulo the space of exact forms:*

$$H^k_{dR}(M) := \frac{closed\ k\text{-}forms}{exact\ k\text{-}forms} = \frac{\ker(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

- *The **de Rham cohomology** of M is the direct sum of all these spaces:*

$$H_{dR}(M) := \bigoplus_{k \geq 0} H_{dR}^k(M) = \frac{closed\ forms}{exact\ forms} = \frac{\ker \mathrm{d}}{\mathrm{im\ d}}.$$

- *The **cohomology class** of a closed k-form ω is the equivalence class:*

$$[\omega] = \omega + \mathrm{d}\Omega^{k-1}(M) \in H_{dR}^k(M).$$

- *The k-th **Betti number** of M is the dimension of the k-th de Rham cohomology group of M (provided this vector space is finite dimensional):*

$$b_k(M) := \dim\big(H_{dR}^k(M)\big).$$

Remark 13.2.3. The general algebraic framework in which cohomology groups are considered is that of a **chain complex** $(C^\bullet, \delta)$, i.e. a sequence of vector spaces $\{C^k\}_{k \in \mathbb{Z}}$ together with linear maps $\delta^k : C^k \to C^{k+1}$ satisfying $\delta^{k+1} \circ \delta^k = 0$. The **cohomology** of the chain complex are the quotient spaces:

$$H^k(C^\bullet, \delta) := \frac{\ker(\delta^k)}{\mathrm{im}(\delta^{k-1})}.$$

Remark 13.2.4. We have that $H_{dR}^0(M)$ is the space of function $a \in C^\infty(M)$ satisfying $\mathrm{d}a = 0$. In a chart, this condition is equivalent to all partial derivatives of a being zero. Therefore, $\mathrm{d}a = 0$ is equivalent to a being a locally constant function. Such functions are uniquely determined by their values on the connected components of M; thus, $H_{dR}^0(M) = \mathrm{Maps}(\pi_0(M), \mathbb{R})$ where $\pi_0(M)$ denotes the set of connected components of M. So, if finite, $b_0(M)$ counts connected components of M.

Remark 13.2.5. Let M be a connected manifold of dimension m. The top degree cohomology has the following interpretation: if M is compact and orientable (to be defined in the following lecture), then $H_{dR}^m(M) = \mathbb{R}$, otherwise, $H_{dR}^m(M) = 0$.

Example 13.2.6. Since S^1 is connected, we have that $H_{dR}^0(S^1) \cong \mathbb{R}$. Let $0 \leq \theta \leq 2\pi$ be the "angle coordinate" on S^1. Every one-forms on S^1 can be written as

$$f(\theta)\mathrm{d}\theta \in \Omega^1(S^1), \quad f \in C^\infty(S^1).$$

The notation $\mathrm{d}\theta$ is a bit misleading, because $\mathrm{d}\theta$ is not an exact 1-form (θ is not a smooth function on S^1); in fact, let us prove that $[\mathrm{d}\theta]$ is a basis of $H_{dR}^1(S^1)$, i.e.

$$H_{dR}^1(S^1) = \mathbb{R}[\mathrm{d}\theta] \neq 0.$$

Clearly, every one-form is closed. The exact one-forms have the form

$$f(\theta)\mathrm{d}\theta = \mathrm{d}(g(\theta)) = \frac{\partial g}{\partial \theta}(\theta)\mathrm{d}\theta,$$

for some $g(\theta) \in C^\infty(S^1)$. Note that an exact one-form integrates to 0 when viewed as a smooth function on $[0, 2\pi]$

$$\int_0^{2\pi} f(\tau)\mathrm{d}\tau = g(2\pi) - g(0) = 0.$$

Conversely, given $f \in C^\infty(S^1)$, we regard f as a 2π-periodic function on $\mathbb{R}$. Then

$$g(x) := \int_0^x f(\tau)\mathrm{d}\tau,$$

defines a smooth function $g : \mathbb{R} \to \mathbb{R}$, which satisfies

$$g(x + 2\pi) = g(x) + \int_0^{2\pi} f(\tau)\mathrm{d}\tau.$$

Thus, $g \in C^\infty(S^1)$ iff $\int_0^{2\pi} f(\tau)\mathrm{d}\tau = 0$; and so, $f(\theta)\mathrm{d}\theta$ is exact iff $\int_0^{2\pi} f(\tau)\mathrm{d}\tau = 0$. In particular, $[\mathrm{d}\theta] \neq 0$ in $H^1(S^1)$. One the other hand, for $f(\theta)\mathrm{d}\theta \in \Omega^1(S^1)$, we have that $(f(\theta) - \frac{c}{2\pi})\mathrm{d}\theta$ has integral 0 for $c = \int_0^{2\pi} f(\tau)\mathrm{d}\tau$, and so $[f(\theta)\mathrm{d}\theta] = \frac{c}{2\pi}[\mathrm{d}\theta]$. We conclude that:

$$H^1(S^1) = \mathbb{R}[\mathrm{d}\theta] \neq 0.$$

Remark 13.2.7. Assume that M is connected, and let $p \in M$. It can be proven that $H^1_{dR}(M)$ is isomorphic to the vector space of group homomorphisms from the fundamental group $(\pi_1(M, p), *)$ to $(\mathbb{R}, +)$:

$$H^1_{dR}(M) \cong \mathrm{Hom}(\pi_1(M, p); \mathbb{R}).$$

Even if the reader is not familiar with the notion of *fundamental group*, it is important to note that this notion is purely topological; it uses only the topology of M and not its differentiable structure. In fact, this is true in all degrees; namely, even though the groups $H_{dR}(M)$ are constructed using smooth forms, they encode information only about the topology of M. We state the precise result below:

Theorem ♣ 13.2.8 (De Rham). *We have that* $H^k_{dR}(M) \cong \mathrm{Hom}(H_k(M, \mathbb{Z}); \mathbb{R})$[1].

Next, we give some important properties of de Rham cohomology.

Proposition 13.2.9. *The wedge product on differential forms* $\Omega(M)$ *induces a product on de Rham cohomology* $H_{dR}(M)$*:*

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta],$$

which turns $(H_{dR}(M), +, \wedge)$ *into an associative algebra.*

Proof. If α and β are closed forms, using that d is a derivation for the wedge product (Theorem 13.1.1 (3)), we have that $\alpha \wedge \beta$ is closed; hence the class $[\alpha \wedge \beta]$ exists. Next, we show that $[\alpha \wedge \beta]$ depends only on $[\alpha], [\beta] \in H_{dR}(M)$, and not on the particular representatives $\alpha, \beta \in \Omega(M)$. Any other representatives are of the form $\alpha' = \alpha + \mathrm{d}\eta$ and $\beta' = \beta + \mathrm{d}\theta$. Again by the derivation rule, we have that:

$$\alpha' \wedge \beta' = \alpha \wedge \beta' + \mathrm{d}(\eta \wedge \beta') = \alpha \wedge \beta + \mathrm{d}((-1)^k \alpha \wedge \theta + \eta \wedge \beta'),$$

where k is the degree of α. Thus $[\alpha' \wedge \beta'] = [\alpha \wedge \beta]$. This proves that the product is well-defined. Clearly, associativity and the other algebraic properties (unital, graded commutativity) are inherited from the wedge product on $\Omega(M)$. □

Proposition 13.2.10. *The pullback of differential forms along any smooth map* $f : M \to N$ *induces an algebra homomorphism in de Rham cohomology*

$$f^* : H_{dR}(N) \longrightarrow H_{dR}(M), \quad f^*[\omega] := [f^*\omega].$$

Moreover, if $g : N \to P$ *is a second smooth map, then*

$$f^* \circ g^* = (g \circ f)^* : H_{dR}(P) \longrightarrow H_{dR}(M).$$

---

[1]The group $H_k(M, \mathbb{Z})$ is called the k-th singular homology of M; it will not be introduced in this course, but it is important to note that it is defined purely in topological terms

Proof. By item (4) of Theorem 13.1.1, we have that $f^* : \Omega(N) \to \Omega(M)$ sends closed forms to closed forms, and exact forms to exact forms; thus it induces indeed a map $f^* : H_{dR}(N) \to H_{dR}(M)$. The fact that f^* is an algebra homomorphism on differential forms, implies that it is an algebra homomorphism in cohomology, and similarly, the equality $f^* \circ g^* = (g \circ f)^*$ holds already at the level of forms. $\quad\square$

An immediate consequence of the Proposition, is that de Rham cohomology is an invariant of a manifold:

Corollary 13.2.11. *If M and N are diffeomorphic manifolds, then $H_{dR}(M)$ and $H_{dR}(N)$ are isomorphic algebras.*

Theorem 13.2.12 (Homotopy invariance of de Rham cohomology)**.** *Let $f_t : M \to N$ be a family of smooth maps depending smoothly on $t \in [0,1]$, then*

$$f_0^* = f_1^* : H_{dR}(N) \longrightarrow H_{dR}(M).$$

Proof. We can assume that f_t is defined for all $t \in \mathbb{R}$ (first, extend the family to a small open interval containing $[0,1]$, and then use a diffeomorphism of this interval to $\mathbb{R}$ which fixes $[0,1]$). The family can be regarded as a smooth map

$$f : \mathbb{R} \times M \longrightarrow N.$$

Differential forms on $\mathbb{R} \times M$ can be uniquely decomposed as

$$\mathrm{d}t \wedge \alpha_t + \beta_t \in \Omega(\mathbb{R} \times M),$$

where α_t, β_t are smooth families of differential forms on M. We denote the exterior derivative of forms on $\mathbb{R} \times M$ by $\mathrm{d}_{\mathrm{tot}}$, and the exterior derivatives on M and N as usually by d. In the above decomposition, $\mathrm{d}_{\mathrm{tot}}$ becomes:

$$\mathrm{d}_{\mathrm{tot}}\left(\mathrm{d}t \wedge \alpha_t + \beta_t\right) = -\mathrm{d}t \wedge \mathrm{d}\alpha_t + \mathrm{d}t \wedge \frac{\mathrm{d}}{\mathrm{d}t}\beta_t + \mathrm{d}\beta_t.$$

Note that the pullback of $\omega \in \Omega^k(N)$ by f decomposes as follows:

$$f^*(\omega) = \mathrm{d}t \wedge H_t(\omega) + f_t^*(\omega),$$

where $H_t(\omega) \in \Omega^{k-1}(M)$ is a smooth family of $k-1$-forms on M. Next, we calculate both sides of the equality $\mathrm{d}_{\mathrm{tot}} f^*(\omega) = f^*(\mathrm{d}\omega)$ (Theorem 13.1.1 (4)):

$$\mathrm{d}_{\mathrm{tot}}\left(\mathrm{d}t \wedge H_t(\omega) + f_t^*(\omega)\right) = \mathrm{d}t \wedge \left(-\mathrm{d}H_t(\omega) + \frac{\mathrm{d}}{\mathrm{d}t}f_t^*(\omega)\right) + \mathrm{d}f_t^*(\omega)$$

$$f^*(\mathrm{d}\omega) = \mathrm{d}t \wedge H_t(\mathrm{d}\omega) + f_t^*(\mathrm{d}\omega).$$

Using that also f_t^* commutes with the exterior derivative, and the uniqueness of the decomposition of differential forms on $\mathbb{R} \times M$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t^*(\omega) = H_t(\mathrm{d}\omega) + \mathrm{d}H_t(\omega).$$

Integrating from 0 to 1:

(*) $$f_1^*(\omega) - f_0^*(\omega) = h \circ \mathrm{d}(\omega) + \mathrm{d} \circ h(\omega),$$

where h denotes the following linear operator:

$$h : \Omega^k(N) \longrightarrow \Omega^{k-1}(M), \quad h(\eta) := \int_0^1 H_t(\eta)\mathrm{d}t.$$

This operator is called a *homotopy operator*, and relation (*) is all what is needed to conclude the proof. Namely, if $[\omega] \in H_{dR}^k(M)$, then $\mathrm{d}\omega = 0$, and so

$$f_1^*([\omega]) = [f_1^*(\omega)] = [f_0^*(\omega) + \mathrm{d} \circ h(\omega)] = f_0^*([\omega]).$$

Thus f_1 and f_0 induce the same map in cohomology. $\qquad\square$

Remark 13.2.13. The operator $H_t : \Omega^k(N) \to \Omega^{k-1}(M)$ can be given explicitly:

$$H_t(\omega) = f_t^*(\iota_{\frac{\mathrm{d}}{\mathrm{d}t} f_t} \omega).$$

Therefore, the homotopy operators are given by

$$h(\omega) = \int_0^1 f_t^*(\iota_{\frac{\mathrm{d}}{\mathrm{d}t} f_t} \omega) \mathrm{d}t.$$

Homotopy invariance can be used to calculate the cohomology of $\mathbb{R}^n$.

Corollary 13.2.14 (The Poincaré Lemma). *We have that*

$$H_{dR}^k(\mathbb{R}^n) = \begin{cases} 0, & k \neq 0; \\ \mathbb{R}, & k = 0. \end{cases}$$

Proof. Consider the family of maps $f_t : \mathbb{R}^n \to \mathbb{R}^n$, $f_t(x) := tx$. Then $f_1 = \mathrm{id}$, and so $f_1^*[\omega] = [\omega]$ for all $[\omega] \in H_{dR}^k(\mathbb{R}^n)$. On the other hand, $f_0(x) = 0$ for all $x \in \mathbb{R}^n$, and so $f_0^*[\omega] = 0$ for all $[\omega] \in H_{dR}^k(\mathbb{R}^n)$ and $k > 0$, and $f_0^*(c) = c$, for $c \in \mathbb{R} = H^0(\mathbb{R}^n)$. Applying Theorem 13.2.12, we obtain the result. $\qquad\square$

More generally, we have that:

Corollary 13.2.15. *Let $\pi : E \to M$ be a vector bundle. Then $\pi^* : H_{dR}(M) \to H_{dR}(E)$ is a linear isomorphism.*

Example 13.2.16. We show that

$$H_{dR}^1(S^n) = 0, \quad n \geq 2.$$

Consider a closed one-form $\alpha \in \Omega^1(S^n)$. We will prove that α is exact. Decompose $S^n = U_N \cup U_S$, where U_N is the complement of the north pole, and U_S is the complement of the south pole. Since $U_N \cong \mathbb{R}^n$ we have that $[\alpha|_{U_N}] = 0 \in H_{dR}^1(U_N)$. Therefore, there exists $f_N \in C^\infty(U_N)$ such that $\mathrm{d}f_N = \alpha|_{U_N}$. Similarly, we find $f_S \in C^\infty(U_S)$ such that $\mathrm{d}f_S = \alpha|_{U_S}$. On the sphere without the two poles, we have that

$$\mathrm{d}(f_N|_{U_N \cap U_S} - f_S|_{U_N \cap U_S}) = \alpha|_{U_N \cap U_S} - \alpha|_{U_N \cap U_S} = 0.$$

This implies that $f_N|_{U_N \cap U_S} - f_S|_{U_N \cap U_S}$ is a constant function $c \in \mathbb{R}$ (this is because $U_N \cap U_S$ is connected; here we are using that $n \geq 2$). Therefore, the following defines a smooth function on S^n:

$$f : S^n \longrightarrow \mathbb{R}, \quad f|_{U_N} := f_N - c, \text{ and } f|_{U_S} = f_S,$$

which clearly satisfies $\mathrm{d}f = \alpha$.

The general structure of the cohomology of spheres is given below (the proof is not difficult; see e.g. [**3**]):

Theorem ♣ 13.2.17. *The cohomology of spheres is given by:*

$$H_{dR}^k(S^n) = \begin{cases} 0, & 0 < k < n; \\ \mathbb{R}, & k = 0, \ n. \end{cases}$$

13.3. Cartan calculus

In this section we introduce the Lie derivative of a differential form, and we prove several relations between this operator, the exterior derivative and the interior product. This entire algebraic machinery, including the definition of the exterior derivative, was developed by Élie Cartan, and therefore it is commonly called *Cartan calculus*.

Definition 13.3.1. *The **Lie derivative** of differential forms along a vector field $X \in \mathfrak{X}(M)$ is the operator*

$$\mathscr{L}_X : \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\mathscr{L}_X \alpha := \frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^t)^*(\alpha)\big|_{t=0} = \lim_{t \to 0} \frac{1}{t}((\phi_X^t)^*(\alpha) - \alpha).$$

Theorem 13.3.2. *The following relations hold:*
(1) $\mathscr{L}_X(\alpha \wedge \beta) = \mathscr{L}_X \alpha \wedge \beta + \alpha \wedge \mathscr{L}_X \beta$
(2) $\mathscr{L}_X \circ \mathrm{d} = \mathrm{d} \circ \mathscr{L}_X$
(3) $\mathscr{L}_X \circ \iota_Y - \iota_Y \circ \mathscr{L}_X = \iota_{[X,Y]}$
(4) $\mathscr{L}_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}$ *(Cartan's magic formula)*
(5) $\mathscr{L}_X \circ \mathscr{L}_Y - \mathscr{L}_Y \circ \mathscr{L}_X = \mathscr{L}_{[X,Y]}$
for all $X, Y \in \mathfrak{X}(M)$ and all $\alpha, \beta \in \Omega(M)$.

Proof. Since the pullback by the flow of X preserves the wedge product, we obtain:

$$(\phi_X^t)^*(\alpha \wedge \beta) = (\phi_X^t)^*(\alpha) \wedge (\phi_X^t)^*(\beta).$$

Taking the derivative at $t = 0$, and using that $\phi_X^0 = \mathrm{id}$, we obtain (1).

For (2), we use that the pullback commutes with the exterior derivative:

$$(\phi_X^t)^*(\mathrm{d}\alpha) = \mathrm{d}(\phi_X^t)^*(\alpha),$$

and take the derivative at $t = 0$.

Formula (3) is a version of (1) for the interior product instead of the exterior product. Namely, directly from the definitions of the pullback, it follows that

$$(\phi_X^t)^*(\iota_Y \alpha) = \iota_{(\phi_X^t)^*(Y)}(\phi_X^t)^*(\alpha).$$

Taking the derivative at $t = 0$, we obtain that:

$$\mathscr{L}_X(\iota_Y \alpha) = \iota_{\mathscr{L}_X Y} \alpha + \iota_Y(\mathscr{L}_X \alpha).$$

Using that $\mathscr{L}_X Y = [X, Y]$ (see Proposition 10.3.3), we obtain (3).

Denote the operator in (4) by $D_X := \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}$. By using the derivation rule for d and ι_X, one easily checks that D_X satisfies the same derivation rule as $\mathscr{L}_X$ (this follows also from the comments in the previous subsection)

$$D_X(\alpha \wedge \beta) = D_X \alpha \wedge \beta + \alpha \wedge D_X \beta.$$

Using that $\mathrm{d}^2 = 0$, we obtain that D_X commutes with d:

$$D_X \circ \mathrm{d} = (\mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}) \circ \mathrm{d} = \mathrm{d} \circ \iota_X \circ \mathrm{d} = \mathrm{d} \circ (\mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}) = \mathrm{d} \circ D_X.$$

Next, note that D_X and $\mathscr{L}_X$ coincide on functions $a \in C^\infty(M)$:

$$D_X(a) = (\mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d})(a) = \iota_X \mathrm{d}a = \mathscr{L}_X(a).$$

Finally, since both operators $\mathscr{L}_X$ and D_X are local and linear, it suffices that show that they coincide on forms of the form $a^0 \mathrm{d}a^1 \wedge \ldots \wedge \mathrm{d}a^k$, with $a^i \in C^\infty(M)$. This

follows by applying the derivation rule, that both commute with d, and that both act the same on functions.

Formula (5) follows by using (2)-(4):

$$\begin{aligned}
\mathscr{L}_{[X,Y]} &= \mathrm{d} \circ \iota_{[X,Y]} + \iota_{[X,Y]} \circ \mathrm{d} = \\
&= \mathrm{d} \circ (\mathscr{L}_X \circ \iota_Y - \iota_Y \circ \mathscr{L}_X) + (\mathscr{L}_X \circ \iota_Y - \iota_Y \circ \mathscr{L}_X) \circ \mathrm{d} = \\
&= \mathscr{L}_X \circ \mathrm{d} \circ \iota_Y - \mathrm{d} \circ \iota_Y \circ \mathscr{L}_X + \mathscr{L}_X \circ \iota_Y \circ \mathrm{d} - \iota_Y \circ \mathrm{d} \circ \mathscr{L}_X = \\
&= \mathscr{L}_X \circ (\mathrm{d} \circ \iota_Y + \iota_Y \circ \mathrm{d}) - (\mathrm{d} \circ \iota_Y + \iota_Y \circ \mathrm{d}) \circ \mathscr{L}_X = \\
&= \mathscr{L}_X \circ \mathscr{L}_Y - \mathscr{L}_Y \circ \mathscr{L}_X.
\end{aligned}$$

$\square$

A differential k-form $\omega \in \Omega^k(M)$ can be viewed also as a $C^\infty(M)$-multilinear k-form on the $C^\infty(M)$-module $\mathfrak{X}(M)$ with values in $C^\infty(M)$:

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k \ \times \ \mathfrak{X}(M)} \longrightarrow C^\infty(M)$$

$$\omega(X_1, \ldots, X_k)(p) := \omega_p((X_1)_p, \ldots, (X_k)_p),$$

for $X_1, \ldots, X_k \in \mathfrak{X}(M)$. This allows for a coordinate-free definition of the exterior derivative:

Theorem 13.3.3. *The exterior derivative satisfies:*

$$\begin{aligned}
\mathrm{d}\omega(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i \mathscr{L}_{X_i}(\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k)) \\
&+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),
\end{aligned}$$

for all $\omega \in \Omega^k(M)$ and all $X_0, \ldots, X_k \in \mathfrak{X}(M)$, where the notation $\widehat{X}_i$ means that the vector field X_i is being skipped.

The proof is left as an exercise. In low degrees the formula gives:

$$\begin{aligned}
\mathrm{d}a(X_0) &= \mathscr{L}_{X_0}(a), \\
\mathrm{d}\theta(X_0, X_1) &= \mathscr{L}_{X_0}(\theta(X_1)) - \mathscr{L}_{X_1}(\theta(X_0)) - \theta([X_0, X_1]), \\
\mathrm{d}\omega(X_0, X_1, X_2) &= \mathscr{L}_{X_0}(\omega(X_1, X_2)) - \mathscr{L}_{X_1}(\omega(X_0, X_2)) + \mathscr{L}_{X_2}\omega(X_0, X_1) \\
&\quad - \omega([X_0, X_1], X_2) + \omega([X_0, X_2], X_1) - \omega([X_1, X_2], X_0),
\end{aligned}$$

for $a \in C^\infty(M)$, $\theta \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$.

13.4. The algebra of graded derivations

We explain the algebraic framework in which the relations of Theorem 13.3.2 fit best. A **graded algebra** is an algebra $(A, +, \cdot)$ which decomposes as a direct sum into vector space:

$$A = \bigoplus_{k \in \mathbb{Z}} A^k,$$

and satisfies

$$a \cdot b \in A^{k+l}, \quad \text{for all} \ \ a \in A^k, \ b \in A^l.$$

For this discussion, it is not important whether A is unital, associative, or (graded) commutative. A **derivation of degree** $p \in \mathbb{Z}$ of A is a linear map $D : A \to A$, satisfying $D(A^k) \subset A^{k+p}$, and the derivation rule:

$$D(a \cdot b) = D(a) \cdot b + (-1)^{kp} a \cdot D(b), \quad \text{for all} \ \ a \in A^k, \ b \in A.$$

Let us denote by $\mathrm{Der}^p(A)$ the space of all derivations of degree p of A. The **graded commutator** of $D_1 \in \mathrm{Der}^p(A)$ and $D_2 \in \mathrm{Der}^q(A)$ is the map

$$[D_1, D_2]_c := D_1 \circ D_2 - (-1)^{pq} D_2 \circ D_1 : A \longrightarrow A.$$

It can be easily checked that $[D_1, D_2]_c$ is a derivation of degree $p + q$. Consider

$$\mathrm{Der}(A) := \bigoplus_{p \in \mathbb{Z}} \mathrm{Der}^p(A).$$

This vector space endowed with the bilinear extension of the commutator is the **graded algebra of derivations** of A:

$$(\mathrm{Der}(A), +, [\cdot, \cdot]_c).$$

The commutator satisfies the following relations:

$$[D_1, D_2]_c = -(-1)^{pq}[D_2, D_1]_c,$$
$$[D_1, [D_2, D_3]_c]_c = [[D_1, D_2]_c, D_3]_c + (-1)^{pq}[D_2, [D_1, D_3]_c]_c,$$

for all $D_1 \in \mathrm{Der}^p(A)$, $D_2 \in \mathrm{Der}^q(A)$ and $D_3 \in \mathrm{Der}(A)$. These relations are called **graded commutativity** and the **graded Jacobi identity**, respectively. Note that the second relation is equivalent to the operator $[D_1, \cdot]_c$ being a derivation of degree p of $\mathrm{Der}(A)$, i.e. $[D_1, \cdot]_c \in \mathrm{Der}^p(\mathrm{Der}(A))$. A graded algebra which satisfies these two axioms is called a **graded Lie algebra**.

In our case, $A = \Omega(M)$, with multiplication the usual wedge product. In the language of graded algebras, the derivation rule for the interior product and the exterior derivative become:

$$\iota_X \in \mathrm{Der}^{-1}(\Omega(M)) \quad \text{and} \quad \mathrm{d} \in \mathrm{Der}^1(\Omega(M)).$$

Note that we have the relations $d^2 = 0$ and $\iota_X \circ \iota_Y = -\iota_Y \circ \iota_X$ can be expressed using the graded commutator:

$$[d, d]_c = 2d^2 = 0 \quad \text{and} \quad [\iota_X, \iota_Y]_c = \iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0.$$

Moreover, Theorem 13.3.2 can be restated as follows:

(1) $\mathscr{L}_X \in \mathrm{Der}^0(\Omega(M))$
(2) $[\mathscr{L}_X, \mathrm{d}]_c = 0$
(3) $[\mathscr{L}_X, \iota_Y]_c = \iota_{[X,Y]}$
(4) $\mathscr{L}_X = [\iota_X, \mathrm{d}]_c$
(5) $[\mathscr{L}_X, \mathscr{L}_Y]_c = \mathscr{L}_{[X,Y]}$

13.5. Exercises

The exterior derivative of a function $a \in C^\infty(M)$ can be calculated as follows:

$$\mathrm{d}a(v) = \frac{\mathrm{d}}{\mathrm{d}t} a(\gamma(t))\big|_{t=0},$$

where $t \mapsto \gamma(t)$ is a curve on M, with $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0) = v$. The following exercise extends this result to higher degrees:

Exercise 13.1. Let $v_1, \ldots, v_k \in T_pM$ be k-vectors. A small **parallelotope** with directions $v_1, \ldots, v_k$ is a smooth map $(t_1, \ldots, t_k) \mapsto \gamma(t_1, \ldots, t_k)$ defined on an open neighborhood of 0 in $\mathbb{R}^k$ such that

$$\gamma(0, \ldots, 0) = p, \quad \frac{\partial \gamma}{\partial t_i}(0, \ldots, 0) = v_i.$$

Prove that for any $k-1$-form $\omega \in \Omega^{k-1}(M)$, the exterior derivative is given by:

$$d\omega(v_1, \ldots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \frac{\partial}{\partial t_i} \omega\Big(\frac{\partial \gamma}{\partial t_1}, \ldots, \widehat{\frac{\partial \gamma}{\partial t_i}}, \ldots, \frac{\partial \gamma}{\partial t_k}\Big)\Big|_{(t_1, \ldots, t_k) = (0, \ldots, 0)}.$$

Hint: Calculate $d\gamma^*(\omega)$, and use item (4) from Theorem 13.1.1.

Exercise 13.2. Let $U \subset \mathbb{R}^3$ be an open set. Recall the standard operators from Calculus: the **gradient**, the **curl** and the **divergence**:

$$\text{grad} = \nabla : C^\infty(U) \longrightarrow \mathfrak{X}(U),$$
$$\text{curl} = \nabla \times : \mathfrak{X}(U) \longrightarrow \mathfrak{X}(U),$$
$$\text{div} = \nabla \cdot : \mathfrak{X}(U) \longrightarrow C^\infty(U),$$

which for $f \in C^\infty(U)$ and $F = F_x \frac{\partial}{\partial x} + F_y \frac{\partial}{\partial y} + F_z \frac{\partial}{\partial z} \in \mathfrak{X}(U)$ are given by:

$$\nabla f = \frac{\partial f}{\partial x} \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial}{\partial z},$$
$$\nabla \times F = \Big(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\Big) \frac{\partial}{\partial x} + \Big(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\Big) \frac{\partial}{\partial y} + \Big(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\Big) \frac{\partial}{\partial z},$$
$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

Denote by $\langle \cdot, \cdot \rangle$ the standard dot-product on $\mathbb{R}^3 = T_pU$.

(a) Show that the following give $C^\infty(U)$-linear isomorphisms:

$$A : \mathfrak{X}(U) \xrightarrow{\sim} \Omega^1(U), \quad A(F) := \langle F, \cdot \rangle$$
$$B : \mathfrak{X}(U) \xrightarrow{\sim} \Omega^2(U), \quad B(F) := \iota_F(dx \wedge dy \wedge dz)$$
$$C : C^\infty(U) \xrightarrow{\sim} \Omega^3(U), \quad C(f) := f dx \wedge dy \wedge dz.$$

(b) Prove that the following diagram commutes:

$$\begin{array}{ccccccc} C^\infty(U) & \xrightarrow{\nabla} & \mathfrak{X}(U) & \xrightarrow{\nabla \times} & \mathfrak{X}(U) & \xrightarrow{\nabla \cdot} & C^\infty(U) \\ \downarrow{\text{id}} & & \downarrow{A} & & \downarrow{B} & & \downarrow{C} \\ C^\infty(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \end{array}.$$

(c) Deduce the classical relations:

$$\text{curl}(\text{grad}(f)) = 0 \quad \text{and} \quad \text{div}(\text{curl}(F)) = 0,$$

and show that under the map A *conservative vector fields* correspond to exact 1-forms, *irrotational vector fields* correspond to closed 1-forms; and under the map B *divergence free vector fields* correspond to closed 2-forms. (*Hint: open a calculus book.*)

Exercise 13.3. (a) For $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \mathfrak{X}(M)$, prove that:

$$\mathscr{L}_{X_0}(\omega(X_1, \ldots, X_k)) = \mathscr{L}_{X_0}(\omega)(X_1, \ldots, X_k) + \sum_{i=1}^{k} \omega(X_1, \ldots, [X_0, X_i], \ldots, X_k).$$

Hint: either directly from the definitions, or use item (3) from Theorem 13.3.2.
(b) Prove Theorem 13.3.3.
    Hint: argue by induction; use (a) and item (4) from Theorem 13.3.2.

Exercise 13.4. Let $U \subset \mathbb{C}$ be an open set, and consider a smooth function

$$f : U \to \mathbb{C}, \quad f(z) = u(z) + iv(z), \quad u(z), v(z) \in \mathbb{R}.$$

Consider the one-forms $\theta_1, \theta_2 \in \Omega^1(U)$:

$$\theta_1(x, y) := v(x + iy)\mathrm{d}x + u(x + iy)\mathrm{d}y, \quad \theta_2(x, y) := u(x + iy)\mathrm{d}x - v(x + iy)\mathrm{d}y.$$

Prove that f is a holomorphic map if and only if both θ_1 and θ_2 are closed.

Exercise 13.5. Prove Corollary 13.2.15.

Exercise 13.6. (a) For $1 \le k \le n$, given a closed k-form on $\mathbb{R}^n$:

$$\omega = \sum_{i_1 < \ldots < i_k} \omega_{i_1, \ldots, i_k}(x^1, \ldots, x^n)\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \in \Omega^k(\mathbb{R}^n)$$

give an explicit formula for a $k - 1$-form $\alpha \in \Omega^{k-1}(\mathbb{R}^n)$ such that $\mathrm{d}\alpha = \omega$. (*Hint: extract a formula from the proofs of Theorem 13.2.12 and of Corollary 13.2.14, and then check directly that it works.*)
(b) Prove Corollary 13.2.14 directly, by using (a).

Exercise 13.7. Calculate the de Rham cohomology of the Möbius band.

LECTURE 14

In this lecture we construct the density bundle of a smooth manifold. The density bundle and smooth densities give a precise geometric meaning to the notion of "volume element" $\mathrm{d}x^1 \ldots \mathrm{d}x^n$ which appears when writing integrals

$$\int f(x^1, \ldots, x^n) \mathrm{d}x^1 \ldots \mathrm{d}x^n.$$

We will show that smooth densities, i.e. smooth section of the density bundle, can be canonically integrated. The discussion below follows [**10**].

14.1. Volume elements

Let us review some elementary facts about volumes of n-dimensional *parallelotope*, i.e. the n-dimensional generalization of the notions of: line segment, parallelogram, parallelepiped. Let $P \subset \mathbb{R}^n$ be an n-dimensional parallelotope which has a corner in the origin of $\mathbb{R}^n$, and is spanned by n linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^n$:

$$P = \{t^1 v_1 + \ldots + t^n v_n \ : \ 0 \le t^i \le 1\}.$$

Note that, up to a permutation, P determines the basis $(v_1, \ldots, v_n)$. Writing $v_j = (a_j^1, \ldots, a_j^n)$, we parameterize P by the standard n-cube $[0,1]^n$

$$A : [0,1]^n \xrightarrow{\sim} P, \quad (t^1, \ldots, t^n) \mapsto (x^1, \ldots, x^n), \quad x^i = \sum_{j=1}^n a_j^i t^j.$$

The volume of P is simply the absolute value of the determinant of A:

$$\mathrm{Vol}(P) = \int_P \mathrm{d}x^1 \ldots \mathrm{d}x^n = \int_{[0,1]^n} |\det(A)| \mathrm{d}t^1 \ldots \mathrm{d}t^n = |\det(A)|,$$

where we used the change of coordinate formula for multiple integrals. More generally, the same argument applies to the following situation: if Q is any other n-dimensional parallelotope spanned by $w_1, \ldots, w_n$, and $B : \mathbb{R}^n \to \mathbb{R}^n$ is linear isomorphism satisfying $B(v_i) = w_i$, for $1 \le i \le n$ (i.e. B is the transition matrix between the two bases) then we have that:

$$\mathrm{Vol}(Q) = |\det(B)| \mathrm{Vol}(P).$$

The following is the abstract notion of a volume of parallelotopes:

Definition 14.1.1. *Let V be an n-dimensional vector space. A **volume element** on V is a map*

$$\mu : \underbrace{V \times \ldots \times V}_{n \,\times\, V} \longrightarrow \mathbb{R}$$

such that, for all linear maps $A : V \to V$ and all $v_1, \ldots, v_n \in V$:

$$\mu(Av_1, \ldots, Av_n) = |\det(A)|\mu(v_1, \ldots, v_n).$$

We denote the space of volume elements on V by $\mathrm{Ve}(V)$.

Up to a constant, there is a unique non-zero volume element:

Lemma 14.1.2. *Let $e_1, \ldots, e_n$ be a basis of the vector space V. Then there is a unique volume element $\mu_0 \in \mathrm{Ve}(V)$ such that $\mu_0(e_1, \ldots, e_n) = 1$. Moreover, for any other $\mu \in \mathrm{Ve}(V)$ there is a unique $c \in \mathbb{R}$ such that $\mu = c\mu_0$; in other words the volume elements on V form a 1-dimensional vector space with basis μ_0.*

Proof. Let $v_1, \ldots, v_n \in V$, and define $\mu_0(v_1, \ldots, v_n) = |\det(A)|$, where $A : V \to V$ is the linear map determined by $Ae_i = v_i$, for $1 \le i \le n$. Note that μ_0 is indeed a volume element: for any linear map $B : V \to V$, we have that

$$\mu_0(Bv_1, \ldots, Bv_n) = \mu_0(BAe_1, \ldots, BAe_n) = |\det(BA)| =$$
$$= |\det(B)||\det(A)| = |\det(B)|\mu_0(v_1, \ldots, v_n).$$

Consider an arbitrary $\mu \in \mathrm{Ve}(V)$, and denote $c := \mu(e_1, \ldots, e_n)$. Let $v_1, \ldots, v_n \in V$, and let $A : V \to V$ be such that $Ae_i = v_i$, for $1 \le i \le n$. Then, we have that

$$\mu(v_1, \ldots, v_n) = |\det(A)|\mu(e_1, \ldots, e_n) = c\mu_0(v_1, \ldots, v_n).$$

Hence $\mu = c\mu_0$. This shows that $\mathrm{Ve}(V)$ is indeed 1-dimensional, and also that μ_0 is the unique volume element such that $\mu_0(e_1, \ldots, e_n) = 1$. $\qquad\square$

14.2. The density bundle

Let M be a manifold. For $p \in M$, denote the space of volume elements on T_pM by

$$D_p(M) := \mathrm{Ve}(T_pM).$$

The **density bundle** of M is the union of all these vector spaces:

$$D(M) := \bigsqcup_{p \in M} D_p(M).$$

Let $(U, \varphi = (x^1, \ldots, x^m))$ be a chart around a point $p \in M$. Lemma 14.1.2 shows that there exists a unique volume element

$$|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|_p \in D_p(M)$$

such that

$$|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|_p\Big(\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p\Big) = 1,$$

and moreover, that $|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|_p$ forms a basis of $D_p(M)$.

Consider a second chart $(V, \psi = (y^1, \ldots, y^m))$, with $p \in U \cap V$. Then,

$$\frac{\partial}{\partial y^i}\Big|_p = \sum_{j=1}^m \frac{\partial x^j}{\partial y^i}(p)\frac{\partial}{\partial x^j}\Big|_p,$$

and so, by Lemma 14.1.2, this implies that:

$$|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|_p\Big(\frac{\partial}{\partial y^1}\Big|_p, \ldots, \frac{\partial}{\partial y^m}\Big|_p\Big) = \left|\frac{\partial(x^1, \ldots, x^m)}{\partial(y^1, \ldots, y^m)}(p)\right|,$$

where we have denoted by $\frac{\partial(x^1,\ldots,x^m)}{\partial(y^1,\ldots,y^m)}$ the determinant of the Jacobian matrix of $\varphi \circ \psi^{-1}$, i.e.

$$\frac{\partial(x^1, \ldots, x^m)}{\partial(y^1, \ldots, y^m)}(p) = \det(\mathrm{d}_p\varphi \circ (\mathrm{d}_p\psi)^{-1}) = \det\Big(\frac{\partial x^j}{\partial y^i}(p)\Big).$$

Thus we have obtained the following change of coordinate formula:

$$|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|_p = \left|\frac{\partial(x^1, \ldots, x^m)}{\partial(y^1, \ldots, y^m)}(p)\right| \cdot |\mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^m|_p.$$

Using this, the result below can be proven exactly like Theorem 8.1.2:

Theorem 14.2.1. *The density bundle $\pi : D(M) \to M$ is a smooth vector bundle of rank 1. The vector bundle structure is uniquely determined by the condition that, for every chart $(U, \varphi = (x^1, \ldots, x^m))$ on M, $|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m| : U \to D(M)$ is a smooth section.*

14.3. Densities

Definition 14.3.1. *The space of sections of the density bundle will be denoted by*

$$\mathcal{D}(M) := \Gamma(D(M)).$$

*Sections of the density bundle are called **smooth densities** (or 1-densities, or just densities) on M. A density $\mu \in \mathcal{D}(M)$ is said to be **positive**, if $\mu_p(v_1, \ldots, v_m) > 0$ for every $p \in M$ and every basis $v_1, \ldots, v_m$ of T_pM.*

Given a chart $(U, \varphi = (x^1, \ldots, x^m))$ on M, the standard volume element defines a positive density over U

$$|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m| \in \Gamma(D(M)|_U) = \mathcal{D}(U).$$

Any other density $\mu \in \mathcal{D}(U)$ over U is of the form

$$\mu_p = f(p)|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|_p,$$

for a unique smooth map $f \in C^\infty(U)$.

Next, we show that $D(M)$ is a trivializable vector bundle (however, the trivialization is not canonical).

Proposition 14.3.2. *On any manifold M there exists a positive density $\mu \in \mathcal{D}(M)$. In particular, the density bundle $D(M)$ is trivializable.*

Proof. Consider an atlas on M, $\{(U_i, \varphi_i)\}_{i \in I}$, and let $\{\rho_i\}_{i \in I}$ be a subordinate partition of unity. Let $\mu_i = |\mathrm{d}x^1_{\varphi_i} \wedge \ldots \wedge \mathrm{d}x^m_{\varphi_i}| \in \mathcal{D}(U_i)$ be the local density corresponding to the chart φ_i. It is easy to check that $\mu := \sum_{i \in I} \rho_i \mu_i$ is a positive density on M. $\square$

14.4. Integration of densities

We define the pullback of densities under diffeomorphisms

Definition 14.4.1. *The **pullback** of a smooth density $\mu \in \mathcal{D}(N)$ along a local diffeomorphism $\varphi : M \to N$ is the smooth density $\varphi^*(\mu) \in \mathcal{D}(M)$ defined by:*

$$\varphi^*(\mu)_p(v_1, \ldots, v_m) := \mu_{\varphi(p)}(\mathrm{d}_p\varphi(v_1), \ldots, \mathrm{d}_p\varphi(v_m)).$$

Let us write the pullback of densities in local coordinates. Let $U \subset \mathbb{R}^m$ be an open set, and let $\mu \in \mathcal{D}(U)$ be a density on U. We can write:

$$\mu = f(x^1, \ldots, x^m)|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|, \quad f \in C^\infty(U).$$

Consider a diffeomorphism $\varphi : V \xrightarrow{\sim} U$, where $V \subset \mathbb{R}^m$ is another open set. Then,

$$\varphi^*(\mu) = f(\varphi(y^1, \ldots, y^m)) \left|\det\left(\frac{\partial \varphi^i}{\partial y^j}\right)\right| \cdot |\mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^m|.$$

Note that this is precisely the change of variables formula in multiple integrals! More precisely, let us recall the following:

Theorem 14.4.2 (Substitution rule for multiple integrals). *Let $U, V \subset \mathbb{R}^m$ be open sets, and let $\varphi : V \to U$ be a diffeomorphism. If $f \in C^\infty(U)$ is a compactly supported function on U, then we have that*

$$\int_U f(x^1, \ldots, x^m)\mathrm{d}x^1 \ldots \mathrm{d}x^m = \int_V f(\varphi(y^1, \ldots, y^m)) \left|\det\left(\frac{\partial \varphi^i}{\partial y^j}\right)\right| \mathrm{d}y^1 \ldots \mathrm{d}y^m.$$

Let $\mathcal{D}_c(M)$ denote the space of compactly supported densities on M. Compactly supported densities can be integrated canonically:

Theorem 14.4.3. *For each manifold M, there is a linear map*

$$\int_M : \mathcal{D}_c(M) \longrightarrow \mathbb{R},$$

which is uniquely determined by the following conditions:

(1) *It extends the usual Riemann/Lebesgue integration of compactly supported functions on $\mathbb{R}^m$: if $\mu \in \mathcal{D}_c(\mathbb{R}^m)$ is a compactly supported density, then*

$$\int_{\mathbb{R}^m} \mu := \int_{\mathbb{R}^m} f(x^1, \ldots, x^m)\mathrm{d}x^1 \ldots \mathrm{d}x^m,$$

where $\mu = f(x^1, \ldots, x^m)|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|$.[1]

(2) *It is local in the sense that: if $\mu \in \mathcal{D}_c(M)$ has support inside the open U, then*

$$\int_U \mu = \int_M \mu.$$

(3) *It is invariant under diffeomorphisms: if $\varphi : M \xrightarrow{\sim} N$ is diffeomorphism, and $\mu \in \mathcal{D}_c(N)$, then*

$$\int_N \mu = \int_M \varphi^*(\mu).$$

---

[1] This property looks quite tautological. However, note that the right hand side of the equation is the integral of the smooth compactly supported function f on U (the Riemann/Lebesgue integral), and $\mathrm{d}x^1 \ldots \mathrm{d}x^m$ plays a formal notational role (the "volume element" which one often forgets to write), whereas $\mu = f(x^1, \ldots, x^m)|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|$ is an "actual volume element", as defined in Definition 14.1.1.

We will make the following identification: if $U \subset M$ is an open set, we will identify $\mathcal{D}_c(U)$ with elements in $\mathcal{D}_c(M)$ with support in U. This is used implicitly in writing condition (2), and will be used throughout the proof.

Proof. We will construct the integration map in several steps:

Step 1. Let $U \subset \mathbb{R}^m$ be an open set, and let $\mu = f|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m| \in \mathcal{D}_c(U)$ be a compactly supported density on U. Conditions (1) and (2) force us to define $\int_U \mu = \int_{\mathbb{R}^m} f\mathrm{d}x^1 \ldots \mathrm{d}x^m$.

Step 2. Consider two open sets $U, V \subset \mathbb{R}^m$, and a diffeomorphism $\theta : V \xrightarrow{\sim} U$. Let $\mu \in \mathcal{D}_c(U)$ be a compactly supported density on U. Then $\theta^*(\mu)$ is a compactly supported density on V, and Theorem 14.4.2 can be restated as:

$$\int_U \mu = \int_V \theta^*(\mu).$$

This shows that (3) holds for open subsets of $\mathbb{R}^m$.

Step 3. Let $\mu \in \mathcal{D}_c(M)$, and assume that there is a coordinate chart (U, φ) on M such that $\mathrm{supp}(\mu) \subset U$. Then we define:

$$\int_M \mu = \int_{\varphi(U)} (\varphi^{-1})^*(\mu).$$

Note that this definition is forced by conditions (1)-(3). We prove that this definition does not depend on the chart. Consider a second chart (V, ψ) such that $\mathrm{supp}(\mu) \subset V$. Then, by applying Step 2 to the change of coordinates diffeomorphism $\theta = \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$, we obtain

$$\int_{\varphi(U \cap V)} (\varphi^{-1})^*(\mu) = \int_{\psi(U \cap V)} \theta^*(\varphi^{-1})^*(\mu) = \int_{\psi(U \cap V)} (\psi^{-1})^*(\mu).$$

Step 4. Consider a cover of M by coordinate charts $\{(U_i, \varphi_i)\}_{i \in I}$, and let $\{\rho_i\}_{i \in I}$ be a partition of unity subordinate to it. For $\mu \in \mathcal{D}_c(M)$, define

$$\int_M \mu := \sum_{i \in I} \int_M \rho_i \mu.$$

Since $\rho_i \mu$ is supported in the domain U_i of the chart φ_i, it follows that $\int_M \rho_i \mu$ is well-defined; since the support of μ is compact, and the family $\{\rho_i\}_{i \in I}$ is locally finite, only for a finite number of $i \in I$ we have that $\rho_i \mu \neq 0$, therefore the sum above is finite. Thus, $\int_M \mu$ is well-defined. Moreover, the definition is forced by linearity of the integral and by uniqueness of the integral on the domains of charts (see Step 3).

Step 5. Next, we prove that the definition is independent on the atlas and on the partition of unity. Consider a second atlas $\{(V_j, \psi_j)\}_{j \in J}$, with subordinate partition of unity $\{\sigma_j\}_{j \in J}$. For each $i \in I$, we have $\rho_i \mu = \sum_{j \in J} \rho_i \sigma_j \mu$ is a finite sum of densities supported in U_i. Since $\int_M : \mathcal{D}_c(U_i) \to \mathbb{R}$ is linear,

$$\int_M \rho_i \mu = \sum_{j \in J} \int_M \rho_i \sigma_j \mu.$$

This implies that:

$$\sum_{i \in I} \int_M \rho_i \mu = \sum_{i \in I} \sum_{j \in J} \int_M \rho_i \sigma_j \mu.$$

Since all sums involved are finite, we can change the summation order on the right hand side. Therefore, by applying the same argument with the two atlases interchanged, we obtain:

$$\sum_{i \in I} \int_M \rho_i \mu = \sum_{j \in J} \int_M \sigma_j \mu.$$

This shows that the integral is independent on the choice of the atlas.

We have proven existence of the integral, and we have proven that the conditions (1)-(3) imply uniqueness of the integral. We leave it to the reader to check that conditions (1)-(3) are indeed satisfied for the integral we have constructed. □

14.5. Orientations

Definition 14.5.1. *Let V be an n-dimensional vector space V, and let $\mathcal{B}(V)$ denote the space of all bases of V. An **orientation** on V is a map*

$$o : \mathcal{B}(V) \longrightarrow \{-1, 1\},$$

such that, for all linear isomorphisms $A : V \xrightarrow{\sim} A$, it satisfies:

$$o(Av_1, \ldots, Av_n) = \operatorname{sign}(\det(A))o(v_1, \ldots, v_n).$$

*We denote by $\operatorname{Or}(V)$ the set of orientations on V. Given an orientation o on V a basis $v_1, \ldots, v_n$ of V is said to be a **positive** basis for o (resp. **negative** basis), if $o(v_1, \ldots, v_n) = 1$ (resp. $o(v_1, \ldots, v_n) = -1$).*

Clearly, every vector space has precisely two orientations. This is true even for a 0-dimensional vector space, which has a unique basis, namely the empty set $\emptyset$. The two orientations are $o(\emptyset) = 1$ and $o(\emptyset) = -1$; we simply write $o = 1$ or $o = -1$.

The **standard orientation** on $\mathbb{R}^n$ is the orientation for which the standard basis $e_1, \ldots, e_n$ is positive. In general, a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ is positive with respect to the standard orientation iff the matrix $A = (v_i^j)_{1 \le i, j \le n}$ has positive determinant, where we have denoted $v_i = (v_i^1, \ldots, v_i^n)$.

Let M be a manifold. Putting together the orientations of all tangent spaces $T_p M$ one obtains the **orientation bundle** of M:

$$O(M) := \bigsqcup_{p \in M} \operatorname{Or}(T_p M).$$

Now $O(M)$ comes with a canonical projection $\pi : O(M) \to M$. For $p \in M$, $\pi^{-1}(p)$ is composed of the two orientations of $T_p M$. The orientation bundle is a smooth manifold and $\pi : O(M) \to M$ is a local diffeomorphism. To a chart (U, φ) on M we associate two charts on $O(M)$:

$$(U^+, \varphi^+) \quad \text{and} \quad (U^-, \varphi^-).$$

Let $\varphi = (x^1, \ldots, x^m)$. For each $p \in U$ consider the orientation on $T_p M$ denoted by

$$o_p^\varphi \in O_p(M) = \operatorname{Or}(T_p M)$$

which is determined by the condition that

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p$$

forms a positive basis. Then $o^\varphi : U \to O(M)|_U$ is a section of π, i.e. $\pi \circ o^\varphi = \mathrm{id}_U$. Let $-o^\varphi : U \to O(M)|_U$ be the opposite orientation. The charts are given by:

$$U^+ := o^\varphi(U), \qquad\qquad \varphi^+ := \varphi \circ \pi|_{U^+},$$
$$U^- := (-o^\varphi)(U) = -U^+, \qquad\qquad \varphi^- := \varphi \circ \pi|_{U^-}.$$

We leave it to the reader to check that $O(M)$, endowed with these family of charts, is a smooth manifold of the same dimension as M.

Example 14.5.2. The orientation bundle of the Möbius band is a cylinder; you can see this here: [**19**].

Definition 14.5.3. *An **orientation** on a manifold M is a smooth choice of orientations on all the tangent spaces of M. In other words, an orientation of M is a smooth section of the orientation bundle, i.e. a smooth map $o : M \to O(M)$ such that $\pi \circ o = \mathrm{id}_M$.*
   *A manifold M is said to be **orientable** if it admits an orientation.*
   *An **oriented manifold** is a manifold M with a fixed orientation o.*
   *If (M_1, o_1) and (M_2, o_2) are two oriented manifolds, then a local diffeomorphism $\varphi : M_1 \to M_2$ is said to be **orientation preserving**, if for every $p \in M_1$ and every positive basis $v_1, \dots, v_m$ of $T_p M_1$, we have that $\mathrm{d}_p\varphi(v_1), \dots, \mathrm{d}_p\varphi(v_m)$ is a positive basis of $T_{\varphi(p)} M_2$.*

Proposition 14.5.4. *A connected orientable manifold has two orientations.*

Proof. Since M is orientable, there exists an orientation $o : M \to O(M)$ on M. Now, also $-o : M \to O(M)$ is an orientation on M. Thus there are at least two orientations. In general, given any other orientation, $o_1 : M \to O(M)$, we can look at the quotient map $o/o_1 : M \to \{1, -1\}$, which returns 1 if the orientations agree, and -1 if they disagree. This map is smooth; and therefore it is locally constant. Since M is connected, o/o_1 is constant. Therefore, either $o_1 = o$ or $o_1 = -o$. $\qquad\square$

14.6. Exercises

Exercise 14.1. (a) Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional vector space together with an inner-product. Prove that the following expression defines a positive volume element on V:

$$\mu(v_1, \dots, v_n) := \sqrt{\det\left(\langle v_i, v_j \rangle_{i,j}\right)}.$$

(b) Let $M \subset \mathbb{R}^n$ be an embedded submanifold. For each $p \in M$, we regard $T_p M$ as a linear subspace of $T_p \mathbb{R}^n = \mathbb{R}^n$. The restriction of the standard dot-product on $\mathbb{R}^n$ gives an inner-product on $T_p M$. Show that the following defines a smooth density on M:

$$\mu_p^M(v_1, \dots, v_m) := \sqrt{\det\left(\langle v_i, v_j \rangle_{i,j}\right)},$$

where $v_1, \dots, v_m$ is a basis of $T_p M$.

(c) With the notation from (b), prove that $\mu^{\mathbb{R}^n} = |\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n|$.

(d) For $n = 3$, show that the construction from (b) recovers the usual notions of length of a curve and area of a surface, more precisely:
   - Let $\gamma : (a, b) \hookrightarrow \mathbb{R}^3$ be a smooth embedding, and let $\mathcal{C}$ be the image of γ. Prove that $\mu^{\mathcal{C}} = \mathrm{d}s$, where, at $p = \gamma(t)$, $\mathrm{d}s_p$ denotes the "arc-length

element", $\left|\frac{d\gamma}{dt}(t)\right| dt$. Conclude that

$$\int_{\mathcal{C}} \mu^{\mathcal{C}} = \int_a^b \left|\frac{d\gamma}{dt}(t)\right| dt,$$

assuming that both sides are bounded (note that the right hand side is the length of $\mathcal{C}$).

- Let $U \subset \mathbb{R}^2$ be an open set, and let $\sigma : U \hookrightarrow \mathbb{R}^3$ be a smooth embedding parameterizing the curve $\mathcal{S} := \sigma(U)$. Prove that $\mu_p^{\mathcal{S}} = dS_p$, where dS_p is the "area form":

$$dS_p := \left|\frac{\partial\sigma}{\partial u}(u,v) \times \frac{\partial\sigma}{\partial v}(u,v)\right| dudv, \quad \text{at } p = \sigma(u,v).$$

Conclude that the following holds:

$$\int_{\mathcal{S}} \mu^{\mathcal{S}} = \int_U \left|\frac{\partial\sigma}{\partial u}(u,v) \times \frac{\partial\sigma}{\partial v}(u,v)\right| dudv,$$

assuming that both sides are bounded (note that the right hand side is the area of $\mathcal{S}$).

Exercise 14.2. Let G be a Lie group with Lie algebra $\mathfrak{g} := T_e G$. A smooth density $\mu \in \mathcal{D}(G)$ is called **left invariant**, if $\lambda_g^*(\mu) = \mu$ for all $g \in G$ (recall that λ_g denotes left translation by g). Let $\mathcal{D}_L(G)$ denote the space of left invariant densities.

(a) Prove that evaluation at the identity

$$\mathcal{D}_L(G) \longrightarrow \text{Ve}(\mathfrak{g}), \quad \mu \mapsto \mu|_p$$

is a linear isomorphism.

(b) If G is compact, prove that there exists a unique left invariant density $\mu_G \in \mathcal{D}_L(G)$ such that $\int_G \mu_G = 1$ (*The density μ_G is called the **Haar measure** of G*).

(c) For $g \in G$ and $\mu \in \mathcal{D}_L(G)$, prove that $\rho_g^*(\mu) \in \mathcal{D}_L(G)$ (recall that ρ_g denotes right translation by g). Moreover, show that $\rho_g^*(\mu) = u(g)\mu$, where $u(g) \in \mathbb{R}\backslash\{0\}$ does not depend on $\mu \neq 0$. Prove that the map $u : G \to \mathbb{R}\backslash\{0\}$, $g \mapsto u(g)$ is a Lie group homomorphism.

(d) If G is compact, prove that $u(g) = 1$, for all $g \in G$.

(e) Consider the Lie group $G = \text{Aff}(1)$ consisting of affine transformation of the real line $\mathbb{R}$, i.e. the elements of $\text{Aff}(1)$ are maps $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax+b$, with $a \neq 0$, and the group structure is the composition of maps. Calculate the map $u : \text{Aff}(1) \to \mathbb{R}\backslash\{0\}$.

Exercise 14.3. Prove that a manifold M is orientable if and only if it admits an atlas $\{(U_a, \varphi_a)\}_{a \in I}$ for which all transition maps $\varphi_a \circ (\varphi_b)^{-1} : \varphi^b(U_a \cap U_b) \to \varphi_a(U_a \cap U_b)$, for $a, b \in I$, have positive Jacobian determinant:

$$\det\left(\frac{\partial x_a^i}{\partial x_b^j}\right) > 0,$$

where $\varphi^a = (x_a^1, \ldots, x_a^m)$.

Exercise 14.4. Let M be a connected manifold. Prove that M is orientable iff $O(M)$ is disconnected.

Exercise 14.5. Let (M, o) be a connected oriented manifold, and let $\varphi : M \to N$ be a local diffeomorphism. Assume that φ is not injective, and consider $p, q \in M$, with $p \neq q$, such that $\varphi(p) = \varphi(q) =: z$. Consider the orientation $o_z \in \mathrm{Or}(T_z N)$ for which $\mathrm{d}_p \varphi : (T_p M, o_p) \to (T_z N, o_z)$ is orientation preserving (i.e. it sends positive bases to positive bases). If $\mathrm{d}_q \varphi : (T_q M, o_q) \to (T_z N, o_z)$ is not orientation preserving, prove that N is not orientable.

Exercise 14.6. (a) Show that the n-sphere S^n is orientable.
(b) Let $\tau : S^n \to S^n$ be the antipodal map, $\tau(x) = -x$. Prove that τ is orientation preserving iff n is odd.
(c) Show that $\mathbb{P}^n(\mathbb{R})$ is orientable iff n is odd.
(d) If n is even, prove that $O(\mathbb{P}^n(\mathbb{R}))$ is diffeomorphic to S^n.

LECTURE 15

15.1. Manifolds with boundary

The simplest example of a manifold with boundary is the **upper half-space**

$$\mathbb{H}^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \ : \ x^n \geq 0\}.$$

The **boundary** of $\mathbb{H}^n$ is the $n-1$-dimensional space:

$$\partial \mathbb{H}^n := \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{H}^n,$$

and its complement is called the **interior** of $\mathbb{H}^n$:

$$\text{int}(\mathbb{H}^n) := \mathbb{R}^{n-1} \times (0, \infty) \subset \mathbb{H}^n.$$

Clearly, these coincide with the topological notions of boundary and interior as subsets in $\mathbb{R}^n$, but soon we will introduce the more general class of *manifolds with boundary*, which also are made of an "interior" and a "boundary" but which, intrinsically, are not subsets of a larger space.

The **boundary** and **interior** of an open set $U \subset \mathbb{H}^n$ are defined as follows:

$$\partial U := U \cap \partial \mathbb{H}^n \quad \text{int}(U) := U \cap \text{int}(\mathbb{H}^n).$$

Note that already in this case, these sets do not coincide with the topological notions of boundary and interior.

A map $f : U \to \mathbb{R}^m$ defined on an open subset $U \subset \mathbb{H}^n$ is called **smooth**, if there exists a smooth map $\widetilde{f} : \widetilde{U} \to \mathbb{R}^m$, where $\widetilde{U}$ is an open set in $\mathbb{R}^n$ containing U such that $\widetilde{f}|_U = f$. If U is already open as a subset of $\mathbb{R}^n$, this is equivalent to the usual smoothness condition on f. The **differential** of f at $p \in U$ is defined as

$$\mathrm{d}_p f := \mathrm{d}_p \widetilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

where $\widetilde{f}$ is an extension of f. The differential does not depend on the chosen extension of f: at $p \in \text{int}(U)$ this is clear; at $p \in \partial U$, this is because $\mathrm{d}\widetilde{f}$ is continuous and $\text{int}(U)$ is dense in U, and therefore:

$$\mathrm{d}_p \widetilde{f} = \lim_{\substack{q \to p \\ q \in \text{int}(U)}} \mathrm{d}_q \widetilde{f} = \lim_{\substack{q \to p \\ q \in \text{int}(U)}} \mathrm{d}_q f,$$

and clearly the right hand side is independent of $\widetilde{f}$.

Clearly the composition of smooth maps between open subsets of upper half-spaces is again smooth (its extension is the composition of the extensions), and the usual chain rule holds as well (it follows by the chain rule for the extensions).

The notion of a **diffeomorphism** generalizes accordingly: if $U, V \subset \mathbb{H}^n$ are open subsets, then $f : U \to V$ is a diffeomorphism if and only if f is smooth, bijective, and $f^{-1} : V \to U$ is also smooth.

The boundary and the interior are preserved under diffeomorphisms:

Lemma 15.1.1. *Let $U, V \subset \mathbb{H}^n$ be open sets, and let $f : U \to V$ be a diffeomorphism. Then*

$$f(\partial U) = \partial V \qquad f(\mathrm{int}(U)) = \mathrm{int}(V).$$

Proof. Note that $f|_{\mathrm{int}(U)} : \mathrm{int}(U) \to \mathbb{R}^n$ is an injective local diffeomorphism; hence $f(\mathrm{int}(U))$ is an open set (as a subset of $\mathbb{R}^n$). This implies that $f(\mathrm{int}(U)) \subset \mathrm{int}(V)$. The same argument for f^{-1} implies that $f^{-1}(\mathrm{int}(V)) \subset \mathrm{int}(U)$. Since f is a bijection, we conclude that $f(\mathrm{int}(U)) = \mathrm{int}(V)$. But then f restricts to a bijection between the complements of these sets; thus, $f(\partial U) = \partial V$. $\square$

Remark 15.1.2. In fact any homeomorphism $f : U \to V$, where $U, V \subset \mathbb{H}^n$ are open sets, satisfies the conclusions of the Lemma: $f(\partial U) = \partial V$ and $f(\mathrm{int}(U)) = \mathrm{int}(V)$. To show this, one needs to find a topological property which differentiates points on the boundary from interior points. For example, a point $p \in \partial U$ has a basis of neighborhoods which consists of contractible open sets O such that $O \backslash \{p\}$ is also contractible. This is not true for a point $p \in \mathrm{int}(U)$; actually, if O is any open neighborhood of $p \in \mathrm{int}(U)$ then $O \backslash \{p\}$ then is not contractible (a small $n-1$-sphere around p in O in a nontrivial element of $H_{n-1}(O; \mathbb{Z})$, because its image under the map $H_{n-1}(O; \mathbb{Z}) \to H_{n-1}(\mathbb{R}^n \backslash \{p\}; \mathbb{Z}) \cong \mathbb{Z}$ is the generator of this group).

In the usual definition of a smooth manifold, if one considers charts with values in $\mathbb{H}^m$, one obtains the concept of a manifolds with boundary. Let us give a brief outline of this construction.

First, we consider atlases with values in $\mathbb{H}^m$:

Definition 15.1.3. *Let M be a topological space. A C^∞-**atlas with values in** $\mathbb{H}^m$ on M is a family $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, where*

- *$\{U_\alpha\}_{\alpha \in I}$ is an open cover of M;*
- *$\varphi_\alpha : U_\alpha \to \mathbb{H}^m$ are maps such that $\varphi_\alpha(U_\alpha)$ are open sets in $\mathbb{H}^m$, and $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ are homeomorphisms,*

such that, for all $\alpha, \beta \in I$, the transition map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a diffeomorphism between open sets in $\mathbb{H}^m$.

*Two C^∞-atlases $\mathcal{A}$ and $\mathcal{B}$ with values in $\mathbb{H}^m$ on M are said to be **equivalent**, if $\mathcal{A} \cup \mathcal{B}$ is again such a C^∞-atlas with values in $\mathbb{H}^m$.*

Definition 15.1.4. *A **smooth m-dimensional manifold with boundary** is a Hausdorff, second countable topological space M endowed with an equivalence class of C^∞-atlases with values in $\mathbb{H}^m$.*

Note that a usual m-dimensional manifold can be regarded also as a manifold with boundary, by choosing an atlas with values in $\mathrm{int}(\mathbb{H}^m) \cong \mathbb{R}^m$.

Definition 15.1.5. *Let M be a manifold with boundary. A **boundary point** of M is a point $p \in M$ such that there exists a chart $\varphi : U \to \mathbb{H}^m$ around p with $\varphi(p) \in \partial\mathbb{H}^m$. By Lemma 15.1.1, the transition functions preserve $\partial\mathbb{H}^m$; therefore, if p is a boundary point, then every chart around p sends p to a point on $\partial\mathbb{H}^m$. The set of boundary points of M is called the **boundary** of M is denote by ∂M.*

*An **interior point** of M is a point $p \in M$ which is not a boundary point, i.e. for every chart $\varphi : U \to \mathbb{H}^m$ we have that $\varphi(p) \in \text{int}(\mathbb{H}^m)$ (which is equivalent to the existence of at least one such chart). The set of interior points of M is called the **interior** of M and is denoted by $\text{int}(M)$.*

Thus, any manifold with boundary M has a partition into two subsets $M = \partial M \sqcup \text{int}(M)$. Note that $\text{int}(M)$ is open in M and so ∂M is closed in M. These subsets are manifolds (without boundary):

Proposition 15.1.6. *Let M be an m-dimensional manifold with boundary. Then $\text{int}(M)$ and ∂M are smooth manifolds without boundary of dimensions m and $m-1$, respectively. Their differentiable structures are such that, if $\varphi : U \to \mathbb{H}^m$ is a chart on M, then*

$$\varphi|_{\text{int}(U)} : \text{int}(U) \longrightarrow \text{int}(\mathbb{H}^m) \quad \text{and} \quad \varphi|_{\partial U} : \partial U \longrightarrow \mathbb{R}^{m-1}$$

are smooth charts on $\text{int}(M)$ and ∂M, respectively, where we have denoted $\text{int}(U) := U \cap \text{int}(M)$ and $\partial U := U \cap \partial M$, and we have used the identification

$$\partial\mathbb{H}^m \cong \mathbb{R}^{m-1}, \quad (x^1, \ldots, x^{m-1}, 0) \mapsto (x^1, \ldots, x^{m-1}).$$

Example 15.1.7. The intervals $[0,1]$, $(0,1]$, $[0,1)$ and $(0,1)$ are manifolds with boundary; all have interior $(0,1)$, and their respective boundaries are:

$$\partial[0,1] = \{0,1\}, \quad \partial(0,1] = \{1\}, \quad \partial[0,1) = \{0\}, \quad \partial(0,1) = \emptyset.$$

Example 15.1.8. The closed ball in $\mathbb{R}^n$

$$\overline{B^n} := \{(x^1, \ldots, x^n) : (x^1)^2 + \ldots + (x^n)^2 \leq 1\}$$

is a manifold with boundary the $n-1$-dimensional sphere and interior the open ball in $\mathbb{R}^n$:

$$\partial\overline{B^n} = S^{n-1}, \quad \text{int}(\overline{B^n}) = B^n.$$

This can be checked by applying Exercise 15.1 to the map

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad f(x^1, \ldots, x^n) = 1 - ((x^1)^2 + \ldots + (x^n)^2).$$

All the construction we have performed for manifold without boundary generalize to manifolds with boundary. We briefly enumerate what one needs to keep track of. Let M and N be manifolds with boundary. Then

- The notion of a **smooth map** generalizes accordingly: $f : M \to N$ is smooth iff it is continuous and

$$\varphi \circ f \circ \psi^{-1} : \psi(U \cap f^{-1}(V)) \longrightarrow \mathbb{H}^n$$

  is smooth for every pair of charts chart $\psi : U \to \mathbb{H}^m$ and $\varphi : V \to \mathbb{H}^n$ on M and N, respectively.
- The tangent space T_pM at a point p is defined as for usual manifolds. Even at $p \in \partial M$, one has that $\dim(T_pM) = m$. The tangent bundle $\pi : TM \to M$ is a $2m$-dimensional manifold with boundary $\partial TM = \pi^{-1}(\partial M)$. Note that $\partial TM \neq T\partial M$: $\dim(\partial TM) = 2m-1$, but $\dim(T\partial M) = 2m-2$. The differential of smooth maps generalizes accordingly.

- A **vector bundle** $\pi : E \to M$ is defined exactly as in Definition 8.2.1, just that E is also a manifold with boundary $\partial E = \pi^{-1}(\partial M)$.
- The density bundle $D(M)$, the orientation bundle $O(M)$, the cotangent bundle T^*M and its exterior powers $\bigwedge^k T^*M$ are constructed exactly as in the case of manifold without boundary (note that all these were obtained by applying a linear algebra construction to T_pM for all $p \in M$).
- The exterior derivative d $: \Omega(M) \to \Omega(M)$ is defined in the same way, and has the usual properties. One can consider the associated de Rham cohomology $H_{dR}(M)$ (one can show that $H_{dR}(M) \cong H_{dR}(\mathrm{int}(M))$).
- There is a canonical integration of compactly supported densities defined as for usual manifolds $\int_M : \mathcal{D}_c(M) \to \mathbb{R}$.

Vectors along the boundary, not tangent to the boundary, are of two types:

Definition 15.1.9. *Let $p \in \partial M$. A vector $v \in T_pM$ such that $v \notin T_p\partial M$ is said to **point inwards** if there exists a smooth curve*

$$\gamma : [0, \epsilon) \longrightarrow M \qquad such \ that \qquad \gamma(0) = p \ \ and \ \ \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\big|_{t=0} = v,$$

*and v is said to **point outwards**, if $-v$ points inwards.*

In other words, the hyperplane $T_p(\partial M)$ divides T_pM into two components: vectors pointing inwards and vectors pointing outwards. Note that, for $p \in \partial M$, and $v \in T_pM$ precisely one of the following situations holds: v points inwards, v points outwards, or $v \in T_p(\partial M)$. In a chart, $\varphi : U \to \mathbb{H}^m$ around p, with $\mathrm{d}_p\varphi(v) = (v_1, \ldots, v_m) \in T_{\varphi(p)}\mathbb{R}^m \cong \mathbb{R}^m$ this can be checked as follows:

- v points inwards iff $v_m > 0$ iff $\mathrm{d}_p\varphi(v) \in \mathrm{int}(\mathbb{H}^m)$;
- v points outwards iff $v_m < 0$ iff $\mathrm{d}_p\varphi(v) \notin \mathbb{H}^m$;
- v is tangent to ∂M iff $v_m = 0$ iff $\mathrm{d}_p\varphi(v) \in \partial\mathbb{H}^m$.

In other words, the one-dimensional vector space $T_pM/T_p\partial M$, which consists of "directions transverse to ∂M", comes with a canonical orientation. This implies that an orientable manifold with boundary has an orientable boundary, as the following construction shows:

Definition 15.1.10. *Let (M, o) be an oriented manifold with boundary. The **boundary orientation** ∂o is the orientation on the manifold ∂M defined on the basis $v_1, \ldots, v_{m-1}$ of $T_p\partial M$ by:*

$$\partial o(v_1, \ldots, v_{m-1}) = o(v, v_1, \ldots, v_{m-1}),$$

where $v \in T_pM$ is any outwards pointing vector, and $m = \dim(M)$.

Example 15.1.11. Consider the manifold with boundary $[0, 1]$. Consider the orientation o on $[0, 1]$, such that $\frac{\partial}{\partial t} \in T_t[0, 1]$ is a positive basis for all $t \in [0, 1]$. Outwards pointing vectors at the boundary $\partial[0, 1] = \{0\} \cup \{1\}$ are given by:

$$-\frac{\partial}{\partial t}\big|_{t=0} \in T_0[0, 1] \quad \text{and} \quad \frac{\partial}{\partial t}\big|_{t=1} \in T_1[0, 1].$$

Recall that an orientation of a 0-dimensional vector space is the same as the choice of a sign (see the comments after Definition 14.5.1). The boundary orientation ∂o consists of the following two signs:

$$(\partial o)_0 = o\Big(-\frac{\partial}{\partial t}\big|_{t=0}\Big) = -1 \quad \text{and} \quad (\partial o)_1 = o\Big(\frac{\partial}{\partial t}\big|_{t=1}\Big) = +1.$$

The following result describes a manifold in a neighborhood of its boundary.

Theorem 15.1.12. *(Collar Neighborhood Theorem) Let M be a manifold with boundary. There exists a neighborhood $U \subset M$ of ∂M and a diffeomorphism $\varphi : \partial M \times [0,1) \xrightarrow{\sim} U$, such that $\varphi(x,0) = x$ for all $x \in M$.*

For the proof of the above, we will use:

Lemma 15.1.13. *Let M be a manifold with boundary. There exists a vector field $X \in \mathfrak{X}(M)$ such that X_p points inwards, for all $p \in \partial M$.*

Proof. Let $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ be a smooth atlas on M with valued $\mathbb{H}^m$. In each chart, consider the inwards pointing vector field $X_i := (\varphi_i)^*(\frac{\partial}{\partial x^m}) \in \mathfrak{X}(U_i)$. Let $\{\rho_i\}_{i \in I}$ be a partition of unity subordinated to the cover $\{U_i\}_{i \in I}$. Define $X := \sum_{i \in I} \rho_i X_i \in \mathfrak{X}(M)$. Since each X_i is pointing inwards, it follows that also X points inwards (note that pointing inwards is a convex condition). $\qquad\square$

We prove Theorem 15.1.12 only in the case of compact boundary:

Proof of Theorem 15.1.12 for compact ∂M. Let $X \in \mathfrak{X}(M)$ be a vector field pointing inwards. Because X points inwards, there is a neighborhood $U \subset [0,\infty) \times M$ of $\{0\} \times M$ on which the flow is defined:

$$\phi_X : U \longrightarrow M, \quad (t,p) \mapsto \phi_X^t(p).$$

Consider the restriction of the flow to the boundary of M:

$$\psi : U \cap \big([0,\infty) \times \partial M\big) \longrightarrow M, \quad \psi(t,p) := \phi_X^t(p).$$

For any $p \in \partial M$, we have that $\psi(0,p) = p$, hence $\mathrm{d}_{(0,p)}\psi(0,v) = v$ for any $v \in T_p\partial M$. On the other hand, since the curve $t \mapsto \psi(t,p)$ is a flow line of X,

$$\mathrm{d}_{(0,p)}\psi\big(\frac{\partial}{\partial t}, 0\big) = X_p.$$

Since $T_pM = T_p\partial M \oplus \mathbb{R}X_p$, we conclude that $\mathrm{d}_{(0,p)}\psi$ is a linear isomorphism. The Inverse Function Theorem 6.1.1, implies that ψ is a diffeomorphism in a neighborhood of p. On the other hand, since ∂M is compact and $\psi(0,p) = p$ for all $p \in \partial M$, Exercise 6.8 implies that ψ is a diffeomorphism between a neighborhood $V \subset U$ of $\{0\} \times \partial M$ and a neighborhood $O \subset M$ of ∂M. On the other hand, since ∂M is compact, there exists $\epsilon > 0$ such that $[0,\epsilon) \times \partial M \subset V$. Finally, define $\varphi : [0,1) \times \partial M \to M$, $\varphi(t,p) = \psi(\epsilon t, p)$. $\qquad\square$

15.2. Integration of differential forms on oriented submanifolds

The top power of V^* is closely related to volume elements on V. Proposition 12.1.10 implies the following:

Lemma 15.2.1. *Let V be a vector space of dimension n. An orientation $o \in \mathrm{Or}(V)$ induces a linear isomorphism:*

$$\bigwedge^n V^* \xrightarrow{\sim} \mathrm{Ve}(V), \quad \omega \mapsto o\omega,$$

$$(o\omega)(v_1, \ldots, v_n) := o(v_1, \ldots, v_n)\omega(v_1, \ldots, v_n),$$

for all $(v_1, \ldots, v_n) \in \mathcal{B}(V)$.

Consider an oriented manifold (M, o) of dimension m. The above lemma implies that the orientation o gives a vector bundle isomorphism

$$\bigwedge^m T^*M \xrightarrow{\sim} D(M), \quad \omega \mapsto o\omega.$$

Definition 15.2.2. *Let (M, o) be an oriented manifold of dimension m. The **integral of a compactly supported top-form** $\omega \in \Omega_c^m(M)$ on M is defined as the integral of the compactly supported density $o\omega \in \mathcal{D}_c(M)$:*

$$\int_{(M,o)} \omega := \int_M o\omega.$$

Remark 15.2.3. If the orientation changes, the sign of the integral changes:

$$\int_{(M,-o)} \omega = -\int_{(M,o)} \omega.$$

In notation, if the orientation o on M is fixed, it is common to omit the orientation under the orientation under the integral symbol; we also adopt this convention, and write

$$\int_M \omega \quad \text{instead of} \quad \int_{(M,o)} \omega.$$

Integration of forms is invariant under orientation preserving diffeomorphisms:

Proposition 15.2.4. *Let M and N be connected oriented manifolds, and let $\varphi : M \xrightarrow{\sim} N$ be a diffeomorphism. Then, for all compactly supported top-forms $\omega \in \Omega_c^m(N)$, we have that*

$$\int_M \varphi^*(\omega) = \epsilon \int_N \omega,$$

where $\epsilon = 1$ if φ is orientation preserving, and $\epsilon = -1$ if φ is orientation reversing.

Proof. Let o_M and o_N be the respective orientations. We have that following equality of densities: $\varphi^*(o_N\omega) = \epsilon o_M \varphi^*(\omega)$. Using that integration of densities is invariant under diffeomorphisms (see Theorem 14.4.3), we obtain the result:

$$\int_{(M,o_M)} \varphi^*(\omega) = \int_M o_M \varphi^*(\omega) = \epsilon \int_M \varphi^*(o_N\omega) = \epsilon \int_N o_N\omega = \epsilon \int_{(N,o_N)} \omega.$$

$\square$

Forms of lower degree can be integrated on oriented submanifolds:

Definition 15.2.5. *Let M be an m-dimensional manifold. Consider an embedded submanifold $N \subset M$ of dimension n, which is oriented, and consider an n-form $\eta \in \Omega^n(M)$. If $\operatorname{supp}(\eta) \cap N$ is compact, we define the **integral** of η over N by:*

$$\int_N \eta := \int_{(N,o)} i^*(\eta),$$

where o is the orientation on N, and $i : N \to M$ denotes the inclusion map.

Note that $\operatorname{supp}(i^*(\eta)) = \operatorname{supp}(\eta) \cap N$; thus the compactness assumption is equivalent to $i^*(\eta)$ being compactly supported.

15.3. Stokes' Theorem

Stokes' Theorem is the higher-dimensional version of the Fundamental Theorem of Calculus:

Theorem 15.3.1 (Stokes' Theorem). *Let (M, o) be an oriented manifold with boundary, and consider on ∂M the induced orientation ∂o. Then, for any compactly supported $m-1$-form $\omega \in \Omega_c^{m-1}(M)$, where $m = \dim(M)$, we have that:*

$$\int_{(M,o)} \mathrm{d}\omega = \int_{(\partial M, \partial o)} \omega.$$

Proof. We first prove the result for $\mathbb{H}^m$ endowed with the standard orientation o_{st}. Consider $\omega \in \Omega_c^{m-1}(\mathbb{H}^m)$. Then ω can be written uniquely as the sum:

$$\omega = \sum_{i=1}^m f_i(x)\mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^i} \wedge \ldots \wedge \mathrm{d}x^m,$$

where the coefficients $f_i \in C_c^\infty(\mathbb{H}^m)$ are compactly supported function on $\mathbb{H}^m$, and $\widehat{\mathrm{d}x^i}$ denotes that $\mathrm{d}x^i$ is missing in the product. The differential of ω is given by:

$$\mathrm{d}\omega = \left(\sum_{i=1}^m (-1)^{i-1} \frac{\partial f_i}{\partial x^i}(x) \right) \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m.$$

Note that, under the standard orientation on $\mathbb{H}^m$, we have that $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$ corresponds to the density $|\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m|$, and therefore:

$$\int_{\mathbb{H}^m} \mathrm{d}\omega = \sum_{i=1}^m (-1)^{i-1} \int_{\mathbb{H}^m} \frac{\partial f_i}{\partial x^i}(x)\mathrm{d}x^1 \ldots \mathrm{d}x^m.$$

Choosing a large enough number $N > 0$ such that

$$\mathrm{supp}(\omega) \subset [-N, N]^{m-1} \times [0, N].$$

Consider a term in the sum above with $1 \leq i < m$. By interchanging the order of integration, we first calculate the integral with respect to $\mathrm{d}x^i$:

$$\int_{-N}^{N} \frac{\partial f_i}{\partial x^i}(x)\mathrm{d}x^i = f_i(x^1, \ldots, x^{i-1}, N, x^{i+1}, \ldots, x^m)$$
$$- f_i(x^1, \ldots, x^{i-1}, -N, x^{i+1}, \ldots, x^m) = 0,$$

where we have used the Fundamental Theorem of Calculus and that $\mathrm{supp}(f_i) \subset [-N, N]^{m-1} \times [0, N]$. The term with $i = m$ gives:

$$\int_{0}^{N} \frac{\partial f_m}{\partial x^m}(x)\mathrm{d}x^m = f_m(x^1, \ldots, x^{m-1}, N) - f_m(x^1, \ldots, x^{m-1}, 0) =$$
$$= -f_m(x^1, \ldots, x^{m-1}, 0).$$

We conclude that

(*) $$\int_{\mathbb{H}^m} \mathrm{d}\omega = (-1)^m \int_{\mathbb{R}^{m-1}} f_m(x^1, \ldots, x^{m-1}, 0)\mathrm{d}x^1 \ldots \mathrm{d}x^{m-1}.$$

On the other hand, if $i : \partial \mathbb{H}^m \to \mathbb{H}^m$ denotes the inclusion, we have that:

$$i^*(\omega) = f_m(x^1, \ldots, x^{m-1}, 0)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{m-1}.$$

Note that the vector $-\frac{\partial}{\partial x^m}$ is pointing outwards, therefore, the boundary orientation on $\partial \mathbb{H}^m$ is such that

$$(\partial o_{\text{st}})\Big(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m-1}}\Big) = o_{\text{st}}\Big(-\frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m-1}}\Big) = (-1)^m.$$

We obtain that

$$(\partial o_{\text{st}})\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{m-1} = (-1)^m |\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{m-1}|,$$

and therefore

$$\int_{\partial \mathbb{H}^m} \omega = \int_{\partial \mathbb{H}^m} (\partial o_{\text{st}})\, i^*(\omega) = (-1)^m \int_{\mathbb{R}^{m-1}} f_m(x^1, \ldots, x^{m-1}, 0)\mathrm{d}x^1 \ldots \mathrm{d}x^{m-1}.$$

This and $(*)$ prove Stokes' Theorem for $\mathbb{H}^m$ with the standard orientation.

Consider an oriented manifold (M, o) with boundary. Let $\omega \in \Omega_c^{m-1}(M)$. Assume first that the support of ω lies inside the domain of a coordinate chart $\varphi : U \to \mathbb{H}^m$, such that U and $\partial U := U \cap \partial M$ are connected. By Proposition 15.2.4 and Stokes' Theorem for $\mathbb{H}^m$, we obtain that

$$\int_M \mathrm{d}\omega = \int_U \mathrm{d}\omega = \epsilon \int_{\varphi(U)} (\varphi^{-1})^*(\mathrm{d}\omega) = \epsilon \int_{\varphi(U)} \mathrm{d}(\varphi^{-1})^*(\omega) = \epsilon \int_{\mathbb{H}^m} \mathrm{d}(\varphi^{-1})^*(\omega) =$$

$$= \epsilon \int_{\partial \mathbb{H}^m} (\varphi^{-1})^*(\omega) = \epsilon \int_{\partial \varphi(U)} (\varphi^{-1})^*(\omega),$$

where $\epsilon = 1$ if $\varphi : U \to \varphi(U)$ is orientation preserving, and otherwise $\epsilon = -1$. If $\partial \varphi(U) = \emptyset$, then the last term is zero, but also $U \cap \partial M = \emptyset$, and therefore also $\int_{\partial M} \omega = 0$. On the other hand, if $\partial \varphi(U) \neq \emptyset$, then note that $\varphi : U \to \varphi(U)$ is orientation preserving if and only if $\varphi|_{\partial U} : \partial U \to \partial \varphi(U)$ is orientation preserving; therefore, applying Proposition 15.2.4 to $\varphi|_{\partial U}$, we obtain that

$$\epsilon \int_{\partial \varphi(U)} (\varphi^{-1})^*(\omega) = \int_{\partial U} \omega = \int_{\partial M} \omega.$$

In both cases, we obtain that $\int_M \mathrm{d}\omega = \int_{\partial M} \omega$.

In general, using partitions of unity, we can decompose any $\omega \in \Omega_c^{m-1}(M)$ as a finite sum $\omega = \omega_1 + \ldots + \omega_l$, with $\omega_i \in \Omega_c^{m-1}(M)$ having support inside the domain of a coordinate chart with the properties from above. Applying Stokes' Theorem to each ω_i, and summing up the resulting equalities, we obtain that Stokes' Theorem holds for ω. $\qquad\square$

Example 15.3.2. Consider the closed interval $[a, b]$ with the standard orientation. Stokes' Theorem in this case gives

$$\int_a^b \mathrm{d}F = \int_{\{a,b\}} F, \quad \text{for all} \ \ F \in \Omega^0([a,b]) = C^\infty([a,b]).$$

The left hand side is just $\int_a^b F'(t)\mathrm{d}t$, while the right hand side gives $o_a F(a) + o_b F(b) = F(b) - F(a)$, where $o_a, o_b \in \{-1, 1\}$ are the induced orientations on the boundary (see example 15.1.11). Thus, in this case, Stokes' Theorem is equivalent to the Fundamental Theorem of Calculus:

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a), \quad \text{where} \ \ f(t) = F'(t).$$

For manifolds without boundary, Stokes' Theorem gives:

Corollary 15.3.3. *Let M be a compact oriented manifold without boundary of dimension m. Then, for any $m - 1$ form $\omega \in \Omega^{m-1}(M)$, we have that $\int_M \mathrm{d}\omega = 0$. Equivalently, if $\mu \in \Omega^m(M)$ satisfies $\int_M \mu \neq 0$, then its cohomology class $[\mu] \in H^m_{dR}(M)$ is non-trivial $[\mu] \neq 0$.*

We have that:

Corollary 15.3.4. *Let M be a compact oriented manifold without boundary of dimension m. Then $H^m_{dR}(M) \neq 0$.*

Proof. Let $p \in M$, and consider an oriented chart $(U, \varphi = (x^1, \ldots, x^m))$, around p. Let $\chi : M \to [0, 1]$ be smooth function with compact support in U, and such that $\chi(p) > 0$. Then $\mu = \chi \cdot \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$ satisfies $\int_M \mu > 0$, thus, by the previous Corollary, $[\mu] \neq 0$ in $H^m_{dR}(M)$. $\qquad\square$

15.4. Exercises

Exercise 15.1. Let M be a manifold without boundary, and let $f : M \to \mathbb{R}$ be a smooth function. If $0 \in \mathbb{R}$ is a regular value of f, prove that $N := f^{-1}([0, \infty))$ can be given the structure of a manifold with boundary $\partial N = f^{-1}(0)$ such that the inclusion $N \to M$ is smooth. *Hint: use the local submersion Theorem 6.2.3.*

Exercise 15.2. Show that Stokes' Theorem implies the following classical results from Calculus:

(a) *Green's Theorem.* Let $\Omega \subset \mathbb{R}^2$ be an open set bounded by the simple closed smooth curve $\mathcal{C}$. For every smooth vector field $\mathbf{F} = (f_1, f_2)$ on $\mathbb{R}^2$, we have that:
$$\oint_{\mathcal{C}} f_1 \mathrm{d}x + f_2 \mathrm{d}y = \iint_{\Omega} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathrm{d}x \mathrm{d}y.$$

(b) *Divergence Theorem.* Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary. Then for every smooth vector field $\mathbf{F}$ on $\mathbb{R}^3$, we have that:
$$\iiint_D \mathrm{div}\mathbf{F} \; \mathrm{d}v = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \; \mathrm{d}A.$$

(c) *Classical Stokes' Theorem.* Let $S \subset \mathbb{R}^3$ be a smooth bounded surface with smooth boundary. Then for every smooth vector field $\mathbf{F} = (f_1, f_2, f_3)$ on $\mathbb{R}^3$, we have that:
$$\oint_{\partial S} f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z = \iint_S \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n} \; \mathrm{d}A.$$

Use Exercise 13.2 and a Calculus book (or Wikipedia) to make sense of all terms.

Exercise 15.3. Let $U \subset \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be a holomorphic function. Consider a closed disk $D \subset U$ with center a and boundary $\mathcal{C}$. Prove Cauchy's integral formula:
$$f(a) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - a} \mathrm{d}z.$$
Hint: Use Exercise 13.4 and Stokes' Theorem.

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EXERCISES AND ASSIGNMENTS

Problems for exercise classes

- Exercise class 11/9: 1.1, 1.2, 1.6, 1.7.
- Exercise class 18/9: 2.1, 2.2, 2.3.
- Exercise class 25/9: 3.2, 3.9.
- Exercise class 2/10: 4.1, 4.3.
- Exercise class 9/10: 5.1, 5.3.
- Exercise class 16/10: 6.3, 6.6.
- Exercise class 23/10: 6.5, 6.11.
- Exercise class 13/11: 8.1, 8.2, 8.3.
- Exercise class 20/11: 9.3, 9.7, 9.8.
- Exercise class 27/11: 10.2,10.3,10.4.
- Exercise class 4/12: 11.3.
- Exercise class 11/12: 12.3,12.9.
- Exercise class 18/12: 13.4, 13.5, 13.6.
- Exercise class 8/1: 14.1, 14.6.
- Exercise class 15/1: 15.2.

Homework assignments

- Homework for 15/9: 1.2, 1.7.
- Homework for 22/9: 2.3.
- Homework for 29/9: 3.9.
- Homework for 6/10: 4.3.
- Homework for 13/10: 5.1.
- Homework for 20/10: 6.3, 6.6.
- Homework for 27/10: 6.5, 6.11.
- Homework for 17/11: 8.1, 8.2, 8.3.
- Homework for 24/11: 9.3, 9.7, 9.8.
- Homework for 1/12: 10.2,10.3,10.4.
- Homework for 8/12: 11.3.
- Homework for 15/12:12.3,12.9.
- Homework for 22/12: 13.4, 13.5, 13.6.
- Homework for 12/1: 14.1, 14.6.
- Homework for 19/1: 15.2.