A Mathematical Meditation on Euler’s Elegant Equation

*ei*π + 1 = 0

[5.1. The Elegance of Euler’s Identity 1](#_Toc324197070)

[5.2. Demystifying Euler’s Equation 2](#_Toc324197071)

[5.3. A Concise, but Conceptually Incomplete, Proof 5](#_Toc324197072)

[5.4. Making Connections/Creating Mathematics 6](#_Toc324197073)

[Connection 1. From Negativity to Complexity 6](#_Toc324197074)

[Connection 2. From in Cartesian Coordinates to in Polar Coordinates 8](#_Toc324197075)

[Connection 3. From the Derivative to Infinite Series 10](#_Toc324197076)

[Connection 4. The Derivative to the Natural Exponential Function 12](#_Toc324197077)

[Connection 5. From Infinite Series to Power Series 14](#_Toc324197078)

[Connection 6. Interlacing the Power Series for Sin(*x*), Cos(*x*) and *ex* 18](#_Toc324197079)

[Connection 7. Deriving Euler’s Equation from Interlacing Power Series 19](#_Toc324197080)

[5.5. From Euler’s Equation to the Riemann Zeta Function 22](#_Toc324197081)

[5.7. From the Riemann Zeta Function to Places Yet Unknown 24](#_Toc324197082)

# 5.1. The Elegance of Euler’s Identity

In 1749 Euler published in his *Introductio in Analysis Infinitorum* an equation considered by many to be the most beautiful in all of mathematics:

.

Euler’s equation is succinct in its elegance and depth. Euler’s identity uses the three basic arithmetic operations of addition, multiplication, and exponentiation to conceptually link three number systems connecting five of the most fundamental constants of mathematics.

}

0 The two *natural numbers* essential for arithmetic,

1

π The two ubiquitous *transcendental numbers* π and *e,*

} }

*e*

*i* The square root of –1, the basis of the *imaginary or complex numbers*.

Carl Friedrich Gauss, the “Prince of Mathematicians,” reported said that anyone to whom Euler’s identity was not immediately apparent would never become a first-class mathematician. Whether or not one wishes to become a first-class mathematician, our goal is to demystify Euler’s equation and to explain the mathematical ideas in Euler’s beautiful equation in a way accessible to those with a minimal mathematical education. The remarkable beauty of Euler’s derivation of his equation derives not only from its deep *connections* of five fundamental mathematical constants, but also in its masterful illustration of the *cognitive* strategies mathematicians deploy they are creating new mathematical discoveries. This last phrase “creating… mathematical discoveries” may seem paradoxical: do mathematicians discover mathematical truths or creatively construct them?

As we shall see in this case study, the answer is ‘yes’. Mathematical thinking includes:

* creating new number systems by constructing solutions to equations;
* studying geometric representations as new mathematical objects;
* turning anomalous singularities into higher-level symmetries;
* generalizing equations by expanding the domain of mathematical objects;
* unifying conceptual domains by making bold conjectural identifications.

# 5.2. Demystifying Euler’s Equation

*“An ordinary genius is a fellow that you and I would be just as good as, if we were only many times better. There is no mystery as to how his mind works. Once we understand what he has done, we feel certain that we, too, could have done it. It is different with magicians... the working of their minds is for all intents and purpose incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark.”*

—Mark Kac (1914-1984) (quoted in Nahim, p. 9)

In this quotation, Mark Kac distinguishes ordinary geniuses from mathematical magicians. Euler’s equation, it must be admitted, appears to be the work of a magician. Let’s enumerate some of the mysteries of Euler’s equation, which can be expressed:

.

First, there is the mystery of *negative numbers* whose existence was considered dubious by mathematicians even up to the 18th century. One intuitive way to motivate negative numbers is to use the analogy with debt. This analogy makes sense with addition of credits and debits, but breaks down when one tries to continue the analogy and work out what the product of two debts might be.[[1]](#footnote-1) Euler himself in his *Algebra* (1770), used the analogy of debt, but did not fall into the usually mistake of trying to explain that the product of two debts must be positive. Instead, Euler gave a negative “explanation” of why “minus times minus is plus” by considering logically exhaustive cases: *– a × – b* cannot be *– ab* since this what is given by *– a × b* and the change in sign of *b* should give a change in sign of the answer. Therefore, Euler argued, *– a × – b* must *ab.* Notice that Euler’s argument doesn’t give an *explanation* of why a negative times a negative is a positive, but instead gives a *systematic* argument for why the product of two negative can’t be assigned a negative. But negativity isn’t all that’s mystifying about Euler’s formula.

Secondly, there is the mystery of the *exponential function*:

.

As our math teacher taught us exponentiation, the meaning of 23 is that you take the *base* 2 and multiply itself the number of times indicated the *exponent* 3. In other words,

23 = 2 × 2 × 2 = 8 .

In Euler’s formula instead of integers like 2 and 3, we have the *transcendental* *real* numbers *e* and *π* and the *imaginary* number *i*. The real numbers π and *e* are not only *irrational* numbers—that is, they are *not* expressible as the *ratio* of integers (e.g., is irrational and so has a non-repeating decimal representation), but they are also *non-algebraic*—that is, they are not the solutions or roots of any polynomial with integer coefficients. The transcendental number π is the ratio of the circumference to the diameter of a circle and is non-repeating decimal which begins with 3.1415926… (to remember the digits of π you can count the letters of the words in the sentence *“may I have a large container of coffee”.* The transcendental number *e* is defined as the base of an exponential function such that its rate of change, or derivative, is equal to itself (to remember the digits of e you can count the letters in the words in the sentence “*to express e remember to remember a sentence to remember this”*). We we round these transcendental numbers doesn to the near integer, then is similar to the number .

Thirdly, there is the mystery of the *imaginary component* in the exponent of the exponential function.

.

If negative numbers and the fact that –1 –1 = +1 can be mystifying, then the real possibility of the imaginary number is even more mystifying. Since the square of any real number—negative or positive—must be positive, no real number can be the square root of a negative number. Euler himself in his *Algebra* (1770) wrote:

*“All such expressions as , etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities, and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.”*

Fourthly, there is the question of the nature of the identity in Euler’s equation:

Identities, equations, equivalences are used in various ways in mathematics. In the equation

*x* + 7 = 12 ,

the identity sign is used to equate two ways of expressing the name number and the variable *x* stands for an specific unknown number. In the following equation

(*x* + *y*) × (*x* – *y*) = *x*2 – *y*2

the identity is used to express a law known as the *difference of two squares* and the variables are used to stand generality for any integers.

Another mathematical use of the identity sign in mathematics is for stating *equivalence relations*. When the integers are logically constructed from the natural numbers, an equivalence relation = is defined on ordered pairs of natural numbers using the previously defined realtion of addition + on natural numbers:

(*a* – *b*) = (*c* – *d*)

if and only if

*a* + *d* = *b* + *c* .

The former relation is proven to be an *equivalence relation* by showing that the relation is reflexive, symmetric and transitive.

Still another use of the identity sign is to define the operations of addition + and multiplication × on the equivalence classes of ordered pairs of natural numbers and the previous relations of addition + and multiplication × on natural numbers.

Finally, identities can be used to make *bold conjectures*—a proposed identification which organizes ideas from previously separate cognitive domains into a new paradigm. Perhaps the most famous identity of this sort is Einstein’s equation proposing the mass-energy equivalence in his famous 1905 paper, *“Does the inertia of a body depend on its energy-content?”*

.

In this equivalence, the total internal energy *E* of a body at rest is equal to the product of its rest mass *m* and a suitable conversion factor, here expressed in terms of the square of the speed of light, to transform units of mass to units of energy. Einstein was not the first to propose the mass-energy equivalence, but he rightly is given the credit for proposing that this identity in the contexts of a new paradigm created by the theory of relativity.

Stringing these mysteries together, we have in Euler’s equation the identification of a transcendental number raised to an exponent which is the product of a transcendental number and an imaginary number with the negative number . It is easy to understand why the Nobel physicist Richard P. Feynman called Euler’s formula “one of the most remarkable formulas in mathematics.” Feynman was known for figuring things out for himself rather than taking things on authority and his youthful notebooks have the workings out of Euler’s equation. In Feynman’s spirit, let’s take a mathematical journey in which we figure out the meaning of Euler’s remarkable identity for ourselves—not being content with mere proofs but instead taking the time to demystify the elements of Euler’s beautiful equation.

In demystifying Euler’s equation, our goal is not to not dispel its beauty, but perhaps by taking the time to explain how mathematicians create, we will finally come to appreciate—in a mathematically precise rather than a mystically vague way—the beauty of Euler’s equation.

# 5.3. A Concise, but Conceptually Incomplete, Proof

*“It [Euler’s identity] is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."*

—Benjamin Peirce (1809-1880)

The 19th century Harvard mathematician Benjamin Peirce--father of logician and pragmatist philosopher C. S. Peirce (1839-1914)—is reported to have made the above remark after proving Euler’s identity. The quotation underscores a fact overlooked on a pedagogy of mathematics that is based on reproducing proofs: proofs by themselves need not yield any deep conceptual understanding of the mathematics underlying the proof.

In his above mentioned *Introductio in Analysis Infinitorum* Euler derived his equation from the following trigonometric equation:

Here we simply substitute π for the variable *x*. Since *cos*(πand *sin*(πwe have

*ei*π = 

from which we can immediately deduce Euler’s equation: *e*π** + 1 = 0 .

The proof given here, while concise, is not conceptually illuminating. Beginning the proof with a puzzling complex trigonometric equation as the first premise begs the conceptual question of what exponentiation has to do with trigonometry. How can an exponential function like *ex* which rapidly expands to positive infinity be defined in terms of periodic trigonometric functions whose values are confined to interval from 1 to 1?

# 5.4. Making Connections/Creating Mathematics

## Connection 1. From Negativity to Complexity

How does negativity, in particular −1, come about conceptually in mathematics? The negative integers are added to (“logically constructed from” ) the natural numbers when we want to have solutions to equations such as

*x* + 1 = 0 .

No natural number can be a solution to this equation. (Proof: the left hand side of the equation is the successor of *x*, and according to Peano’s third postulate, zero is not equal to the successor of any natural number.) By embedding the natural numbers in the richer context of integers with positive and negative numbers, there is a solution to every arithmetic equation involving subtraction, the inverse operation of addition.

When multiplication by 1 is represented geometrically in terms of the number line, the negative sign not only creates new *numbers*, but also adds the concept of *direction* on the number line. A minus sign indicates a change of direction—a *rotation* of 180 degrees—on the number line. Understood in these terms, it is perfectly understandable that a change of direction followed by another change of direction cancel each other out:

.

1 l

The puzzle about justifying the law that a “negative times a negative is a positive” by finding an interpretation for multiplying debts is based on an inadequate conception of a negative number. A better way to think about multiplying by – 1 is as a sign that indicates changing directions on the number line. To add – 5 and – 6 is to continue on the number line from the origin in the negative direction for a total of 11 units, and to multiply – 5 by 6 to go 5 × 6 = 30 units in the negative direction. However, to multiply – 5 by – 6 is to multiply 5 by 6 changing directions twice and ending up on the positive direction.

The construction of imaginary numbers from real numbers can be done in a way that is completely analogous to the construction of negative numbers from natural numbers.

Negative Numbers: A Tale of Two Minuses

*Minus times minus is plus.*

*The reason for this we need not discuss.*

— W. H. Auden

There was skepticism about the negative numbers up until the 18th century. Negative numbers were introduced to have solutions to equations such as

No natural number can be a solution to this equation. According to one of Peano’s postulates, 0 is not the successor of any natural number. We can expand the natural numbers by adding the negative integers. Negative numbers can, for example, be introduced to make it easy to calculate debts. If you owe 5 dollars to Abe and 10 dollars to Beatrice, but you only have 2 dollars, then you owe a total of 13 dollars—you have, so to speak, dollars. If you owe five dollars five times over, then your total debt is .

However, does it make sense to ask how much you own if you owe 5 dollars a *negative* 5 times over?

Negative numbers can be modeled by a pair of natural numbers that obey the following laws for addition and multiplication:

.

Even the great mathematician Euler commented (incorrectly) upon the perplexing law, which we now take for granted:

.

Prove the above law that a minus times a minus is plus follows from the *distributive law*

together with the following facts:

; ;

Extending the natural number line to the left, one generates the negative integers by multiply the natural numbers by . Geometrically speaking, multiplication by is a rotation of 180 degrees. The initially puzzling fact that corresponds to the geometric fact that two 180 degree rotations brings you back to the beginning. The integers are constructed as equivalence classes of ordered pairs of natural numbers and the negative integers are geometrically generated by multiplication by .Complex Numbers: An Imaginary Journey

*Imaginary i times i is minus one.*

*Let me tell you why it can be done!*

There has also been widespread skepticism about imaginary numbers even up to the present day. Imaginary numbers were introduced to have solutions to equations such as

No real number r can be a solution to this equation. A real number is either positive or negative; but a positive times a positive is a positive, and a negative times a negative is also a positive and so . Negative numbers can be seen as involving a rotation of 180 degrees. Complex numbers can be introduced to make it easier to calculate rotations without tedious trigonometry.

What does mean in terms of rotation?

Multiplying by is a rotation of 180 degrees. Geometrically speaking, multiplication by is a counter-clockwise rotation of 90 degrees or π/2 radians. Two successive rotations is equal to .

Complex numbers can be modeled by a pair of real numbers that obey the following laws for addition and multiplication:

.

Euler wrote “All such expressions , , etc. are consequently impossible or imaginary numbers….” (*Algebra*, 1770). We have the following cycle of facts:

; ; ; .

If *c* = *a* + i*b*, then the *modulus* of *c* is By the Pythagorean theorem. this is the distance from the origin to the point (*a, b*). The *argument* of a complex number *c* = *a* + i*b* is the *angle* formed by the positive real *x*-axis and the line from (0, 0) to (*a, b*).

Every complex number can be represented by a modulus and an argument using polar coordinates. Multiplication of complex numbers in this form

*c* = *r*[cos(θ) + *i*sin(θ)]

*d* = *s*[cos(ρ) + *i*sin(ρ)]

amounts to this: (1) the *modulus* of the product is the *product* of the moduli; and (2) the *argument* of the product is the *sum* of the *arguments, i.e,*

.

Multiplication by a unit length is equivalent to pure rotation.

## Connection 2. From in Cartesian Coordinates to in Polar Coordinates

*“All such expressions as , etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities, and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.”*

* Euler, *Algebra* (1770)

No real number r can be a solution to this equation. A real number is either positive or negative; but a positive times a positive is a positive, and a negative times a negative is also a positive and so . Negative numbers can be seen as involving a rotation of 180 degrees. Complex numbers can be introduced to make it easier to calculate rotations without tedious trigonometry.

What does mean in terms of rotation?

Multiplying by is a rotation of 180 degrees. Geometrically speaking, multiplication by is a counter-clockwise rotation of 90 degrees or π/2 radians. Two successive rotations is equal to .

Complex numbers can be modeled by a pair of real numbers that obey the following laws for addition and multiplication:

.

Adding complex numbers to the real numbers, leads to the elegant result that every polynomial of degree *n* will have *n* solutions or roots. For example, the above polynomial of degree 2 has 2 solutions, namely, *i* and *–i*.

By convention a complex number is expressed as *a* + *ib*, where *a* is the real component and *b* the imaginary component of the complex number. *Notice that the plus sign doesn’t really signify addition.* The plus sign is merely used to separate the real and imaginary components, and so we can less confusingly used an ordered pair of real numbers to represent a complex number.

Given the ordered pair (*a, b*) it is natural to represent the complex number on the Cartesian coordinate plane. The *x*-axis is the real component and the *y*-axis is the imaginary component of the complex number.

As we have seen a rotation of 180 degrees or π radians is equivalent to multiplying by −1. To geometrically represent *i*, we need a transformation which when applied to itself result in a multiplication by −1 or a rotation of π radians. A rotation of 90 degrees or π/2 radians is a natural way to represent this transformation. The rotations of π result in −1 and + 1 along the horizontal *x*-axis, and rotations of π/2 result in the cycle of values *i*, −1, −*i*, 1 with *i* and –*i* along the vertical *y*-axis at the values of θ equal to π/2 and 3π/2, respectively. This is the third clue linking the representation of the imaginary number *i* in terms of rotation in the geometric representation of complex numbers.

*i*

*–*1 l

*–i*

To summarize our steps so far, we have a conceptual connection between −1 as multiplication and rotation of 180 degrees or π radians, and the trigonometric functions cos(θ) and sin(θ) within the unit circle. To geometrically represent the imaginary unit *i* such that *i* x *i* = −1, we need a rotation which when applied twice is equal to a rotation of 180 degrees or π radians. This requirement is satisfied by a rotation of 90 degrees or π/2 radians. This rotation results in the cycle of repeating values *i0* = 1, *i1* = *i*, *i2=* −1, *i3=* −*i*, *i4=* 1, *i5* = *i*, *i6=* −1, *i7=* −*i*, *i8=* 1, … The next conceptual link to be forged is with the transcendental number *e* and the exponential function with base *e*.

To obtain the *Cartesian coordinate plane* to represent ordered pairs (*x, y*) we place another copy of the number line for the integers (the *y*-axis), perpendicular or at 90 degrees to the original number line (the *x*-axis). In this way the ordered pairs in the equivalence classes constructing the integers can be modeled as ordered pairs on the lines parallel to the line *x* = *y.*

The same Cartesian coordinate plane can be used to represent rational numbers. When *x* and *y* take on integer values with *y* ≠ 0, the line determined by the origin and the point (*x, y*) passes through all the ordered pairs equal to the rational number *x/y.* All such lines whose rays in the first quadrant form an angle with the *x*-axis less than 45 degrees represent all the rational larger than 1 and the lines whose rays in the first quadrant lie between 45 degrees and 90 degrees represent the rational numbers less than 1.

Next we need to find the conceptual link is between and the imaginary number

The problem with thinking of the imaginary unit as a quantity is that it can’t be any real number. As we have seen it, like the number 1, is a rotation.

Cartesian coordinate plane for the real numbers when *x* and *y* can take on any real numbered values such as the values for the trigonometric functions sine and cosine. However, since the values of these trigonometric functions are proportional for all similar triangles, it is convenient to represent these values within a circle of radius 1.

Therefore, rather than using a pair of *Cartesian coordinates* to specify a point (*x*, *y*), it is more natural to use *polar coordinates* specified by a central angle θ and a radius *r* = When confining our attention to the unit circle, the trigonometric functions are defined in terms of θ. The Pythagorean theorem, for example, can be expressed by the trigonometric equation:

1 = cos2(θ) + sin2(θ) .

The values of cos(θ) are read on the *x*-axis and the values of sin(θ) are read on the *y*-axis for a right triangle whose right angle is formed perpendicular to the *x*-axis and whose hypotenuse is a radius from the origin to the unit circle.

sin(θ)

θ

−1 0 cos(θ) 1

Using polar coordinates, a rotation of 180 degrees can be expressed as a rotation of π radians, and rotation of 2π brings us back to the starting point. Notice that at the value of θ = π, *cos*(θ) = −1 and *sin*(θ) = 0. This is the second clue linking −1, or a rotation of π radians, to the trigonometric functions *cos*(θ) and *sin*(θ) for the value θ = π.

## Connection 3. From the Derivative to Infinite Series

Consider Zeno’s arrow paradox. If you look at the arrow in flight at an instant—or indivisible moment of time—it is indistinguishable from the arrow at rest. However, the flight of the arrow is just the infinite collection of moments of time. How can an infinite collection of motionless arrows become the arrow in motion? One way to begin to unravel Zeno’s paradox is to develop the concept of *instantaneous velocity.* In order to distinguish the arrow in motion from the arrow at rest, we need to look at what is happening in the interval of time surrounding that instant of time. The notion of a non-zero *instantaneous velocity* is what is needed to solve Zeno’s paradox of the arrow. This concept is captured by the mathematical concept of the derivative.

The derivative is one of the two fundamental ideas of calculus. The derivative is a mathematical model of change. Given a real-valued function *f*(*x*), the derivative of *f* is the function *f’*(*x*) which is the rate of change of *f*.

Conceptually, the derivative of a function is the rate of change of that function at a point in time. The *derivative of a function f*(*x*)is the limit of the function in the interval of *dx* around *x* as *dx* goes to zero:

Geometrically, the derivative of a function *f*(*x*) is the slope of the tangent line to its graph. In graphical terms, if we have the graph of

*y* = *f*(*x*) ,

then *f*’(*x*) is the slope of the line which is tangent to the graph of *f*(*x*) at the point *x* is the derivative.[[2]](#footnote-2) We can write this using Leibniz’s notation:

= *f’*(*x*) .

In our example, the derivative of *f*(*x*) = *x2* is computed as follows:

So the derivative, or rate of change, of the function *f*(*x*) = *x2* is the function *f*(*x*) = 2*x.*

Here’s a chart of some common derivatives:

|  |  |  |
| --- | --- | --- |
| ***Function*** | ***Derivative*** | ***Comment*** |
| *c*  (*any constant number*) | 0 | A constant, by definition, doesn’t change. |
| *xn* | *nxn-1* | The iterated derivative of x*n* is related to *n*! |
| *ex* | *ex* | The base *e* of the natural exponential function is chosen to be such that the rate of change of the function is equal to itself. |
| ln(*x*) | 1/*x* | The natural logarithm is the inverse of the exponential function. |
| *sin*(*x*) | *cos*(*x*) | The derivative of *sin*(*x*) is *cos*(*x*) |
| *cos*(*x*) | –*sin*(*x*) | The derivative of *cos*(*x*) is – *sin*(x) |

## Connection 4. The Derivative to the Natural Exponential Function

How can *e* both a transcendental number calculated to be approximately 2.7182818284… and the base for an exponential function *ex* whose rate of change is equal to itself?

Recall the fundamental idea of exponentiation and its relationship to multiplication and addition. We can compute the product of two numbers with the same base, say 23 and 27, by simply adding their exponents 23 x 27 = 23+7 . A standard table of logarithms with a base of 10 expresses numbers as exponents, and so multiplying two numbers is equivalent to adding their exponents and then using the table of logarithms backwards to compute their product. This is how a slide rule works: it has two logarithmic scales to compute products by adding logarithms. Just as 10 is not the most mathematically natural base for the number system, logarithms based on powers of 10 is not the most natural base for logarithms. In general, an *exponential function* with base *a* is a function of the form *y* = *ax*, where *a* > 1.

The defining property of the exponential function *ex* is that its rate of change is equal to itself. You may remember memorizing in calculus classes that the derivative of *ex* is *ex*. When the value of *x* = 1, the value of the function *ex* is simply the value of *e*. According to a mathematical characterization: *ex* is an exponential function whose *derivative* is equal to itself.

Many processes in the natural and financial worlds have this property. Euler’s antediluvian example in his *Introduction to the Analysis of the Infinite* (1784, published 1988, with an English translation by John Blanton, Springer-Verlag, New York, p. 92) is population growth:

“Since after the Flood all men descended from a population of six, if we suppose that the population after two hundred years was 1,000,000, we would like to find the annual rate of growth.”

A more worldly example, also studied by Euler, is the growth rate of *compounded interest*. In 1683 Jacob Bernoulli (1654-1705) arrived at the number *e* by modeling compounded interest. If a principal amount of earns interest at an annual rate of *x* compounded monthly, then the interest earned each month is *x*/12 times the current value, so each month the total value is multiplied by (1 + *x*/12) and the value at the end of the year is (1 + *x*/12)12. If interest is compounded daily, we have (1 + *x*/365)365. If the time intervals per year increase, the limit of this function is the exponential function. This is Euler’s formula for *ex* as continuous interest:

It is possible to derive Euler’s definition of the exponential function using the concept of the *derivative*.[[3]](#endnote-1)

Geometrically, the derivative of a function *f*(*x*) is the slope of the tangent line to its graph computed by taking the limit as the interval around the *x* value is smaller and smaller:

Substituting for the function *f(x)* an exponential function with base *b*, we obtain

Exponential was introduced to make multiplication easier. The basic idea is that multiplication of numbers expressed as powers with the same base is equivalent to addition of exponents. Using the idea that exponentiation turns multiplication into addition, we have the crucial step:

Now factoring out the *bx*, we obtain:

Now to solve for *e*, the base of the exponential function whose rate of change, *derivative*, is equal to itself, we set the limit equal to *b*.

The value when *x* = 1 of the function *bx*, by convention, is the constant base *b*, which can be brought out the limit expression.

Cancelling out common factor of *b* from both side and multiplying both sides of the equation by *dx,* we have

Dropping the limit notation temporarily, we can solve for *bdx*:

*b**dx* = 1 + *dx*

Now *dx* going to zero is equivalent to 1/*x* going to zero as *x* goes to infinity. Expressed in this form, we can element exponent by raising both sides to the power *n.*

But the exponential function whose base *b* is such that the function grows proportional to itself is just the definition of *ex* and so we obtain Euler’s famous formula noted above:

## Connection 5. From Infinite Series to Power Series

Zeno’s paradoxes involved an infinite series such that each term is the series was a number. For example, the dichotomy involved the *geometric* series

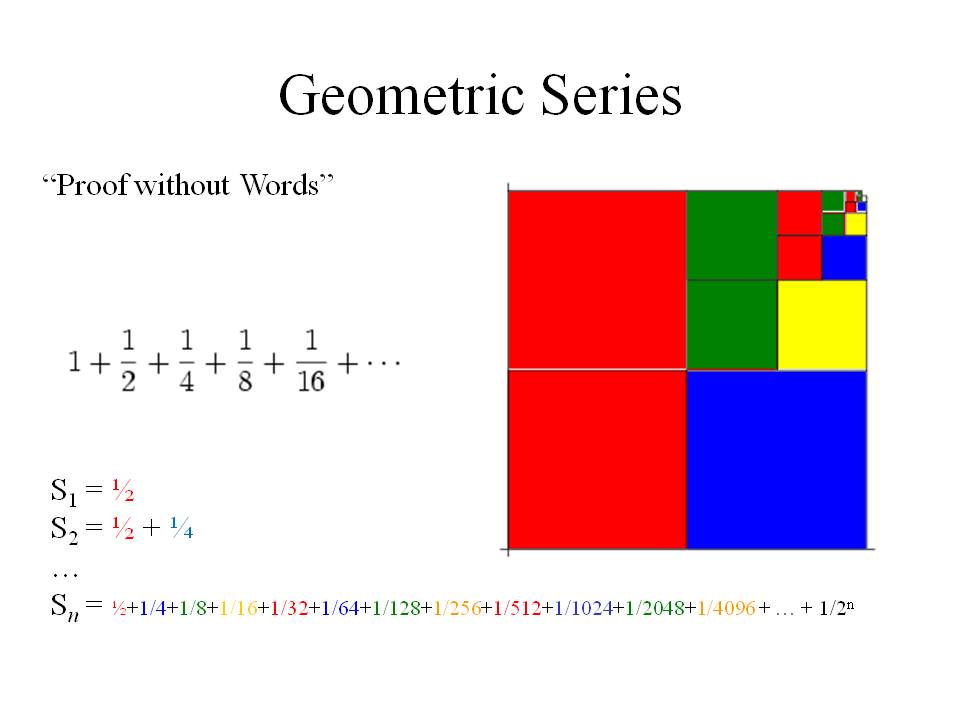
The infinite sum is defined as the limit of the finite sums Sn of the first *n* terms. So consider

Multiplying both sides by a factor of 1/2 we obtain a series:

Subtracting the latter from the former, we have:

and so

So the limit as *n* goes to infinity is:



The same proof for summing a geometric series works, not only for *r* = ½, but for any *r* whose absolute value is < 1. The infinite sum, this particular case, turned out to be 2. In general, the infinite sum exists when |*r*| < 1,

What happens at a = 1 and r = 1? The *harmonic series* is the infinite sum of the reciprocal of all the natural numbers:

1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 +…. + 1/*n* + ….

The proof of that the harmonic series diverges is to compare the series in blocks of powers of 2 to the clearly divergent series 1 + ½ + ½ + ½ + ….:

1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16) +….

1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + (1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16) + …

Each term in the harmonic series is greater than or equal to the corresponding term in the new series which diverges to infinity.

Now the idea of a *power series* comes about by replacing numbers with a variable resulting in the sum of an infinite series of functions. Generally, a power series is of the form

where *a0*, *a1*, *a2*, *a3*, *a4*, *a5*,…, *an*,… is some sequence of numbers. The reason for representing a function as a power series is that it is easy to compute the derivative of such a series term by term using the fact that the derivative of

is

.

Euler also discovered a power series expansion for *ex*. In his fast and loose manner, he took the term-wise derivative of each term this expansion:

The derivative of 1 = 0

The derivative of *x* = 1

The derivative of =

The derivative of =

The derivative of =

Euler noticed that the result of taking the term-wise derivative is the infinite series itself! Now this is precisely the defining feature of the exponential function, so

This infinite series has a dual nature as either an object or an operation, a number or an infinite sum or operation. To compute the value of the number *e*, we can set the exponent *x* = 1, and compute the first 12 terms.

As you can see from the following spreadsheet, computing the value of *e* for the first 12 sums gives an approximation accurate to the first 9 decimal places.

|  |  |  |
| --- | --- | --- |
|  | **Power Series Approximation of *e*** | |
|  | 1 | 1 |
| 1 | 1 | 2 |
| 2 | 0.5 | 2.5 |
| 3 | 0.166666667 | 2.6666666667 |
| 4 | 0.041666667 | 2.7083333333 |
| 5 | 0.008333333 | 2.7166666667 |
| 6 | 0.001388889 | 2.7180555556 |
| 7 | 0.000198413 | 2.7182539683 |
| 8 | 2.48016E-05 | 2.7182787698 |
| 9 | 2.75573E-06 | 2.7182815256 |
| 10 | 2.75573E-07 | 2.7182818011 |
| 11 | 2.50521E-08 | 2.7182818262 |
| 12 | 2.08768E-09 | 2.7182818283 |

Euler’s formula gives a quick and accurate estimate of the value of *e* because the factorials in the denominator grows quickly diminishing the contribution of the successive terms which quickly approach zero.[[4]](#footnote-3)

## Connection 6. Interlacing the Power Series for Sin(*x*), Cos(*x*) and *ex*

Let’s discover the power series for

If we set *z* = 0, then all the terms except the first disappear. There sin(0) = *a0*  but since sin(0) = 0, *a0* = 0. Recall that the derivative of is

We may compute the derivative of the power series term by term:

If we set *z* = 0, then all the terms except the first disappear. There cos(0) = 1 = *a1.*

Next take the derivative of the power series again:

If we set *z* = 0, then all the terms except the first disappear. There so *a2* = 0.

Taking the derivative of the power series again:

If we set *z* = 0, then all the terms except the first disappear and

.

The general pattern is that *an* = 0 whenever *n* is even and when *n* is odd, = with the sign alternating. Therefore,

*Exercise*: In a similar manner discover the power series expansion for cos(*x*).

## Connection 7. Deriving Euler’s Equation from Interlacing Power Series

Once Euler had computed the power series for *ex*, cos(*x*) and sin(*x*), it is hard not to notice an intriguing relationship—the power series expansions for the trigonometric functions cosine and sine appears—except for the feature of the alternating signs—to be interwoven in the power series expansion for exponentiation. The power series terms for the cosine are the odd terms, and the power series terms for the sine are the even terms, of the power series expansion for *ex*.

How can the exponential function, which rapidly grows to infinity, related to the cyclical sine and cosine functions of trigonometry? Formally, the sine function is composed of terms with the *odd* powers of the exponent and the cosine is composed of the terms with the *even* powers of the exponent—except for the alternating signs.

It was Euler’s great genius to see that a deep unification could be achieved if one substitutes a complex number *ix* for the exponent *x* in the power series for *ex*. This substitution makes sense because the laws of addition, multiplication, and exponentiation are naturally extended to the complex numbers. Recall the cycle of values:

*i*

*–*1 l

*–i*

Given the cycle of repeating values *i0* = 1, *i1* = *i*, *i2=* −1, *i3=* −1, *i4=* 1, *i5* = *i*, *i6=* −1, *i7=* −*i*, *i8=* 1, we obtain alternating terms with two positives and then two negatives:

Rearranging the terms, it turns out that the *odd* terms form the alternating infinite series for the cos(x) and the *even* terms form the alternating infinite series for the sin(x):

Substituting the trigonometric functions for their infinite expansions, we obtain the equation:

The last step is to evaluate the function for the value x = π:

But since cos(π) = —1 and sin(π) = 0, we have

and so we have derived Euler’s identity!

It remains to understand what this identity teaches us about the mathematics of exponentiation, trigonometry, and complex numbers and the deeper issue of unification.

* There is only one natural way to extend the real-valued exponential function to the complex numbers.
* The exponential function “contains” the sine and cosine functions “implicitly” but the unification comes within the context of complex numbers.
* The most significant benefit of working in context of the complex numbers, as opposed to real numbers, is that every arithmetic equation (i.e., polynomial) has a solution—in fact, it systematizes the theory of polynomials giving the elegant result that an *n*-degree polynomial has *n* roots.

Given , we may derive *De Moive’s formula*

This explains why the real-valued polynomials have such irregular results concerning roots—positive numbers have two square roots, whereas negative numbers have none; all real numbers have exactly one cube root, fifth root, and so on. The above formula tells us when we are looking for the square roots of we have that

which, in turn, implies that

The geometric question becomes what rotations when doubled result in 180 degrees or π? The roots are = π/2 and θ= 3π/2, giving the roots *i* and . In other words, every nonzero complex number has *n* distinct *n*th roots, which divides the circle evenly. For example, the 8th root of 1 divides the unit circle into eighths equal rotations of 45 degrees or π/4.

*i*

1

*–i*

There is only one natural way to extent the real-valued functions to the complex numbers.

This formula tells us that the Polar coordinates for is given by a radius equal to *ea* and central angle θ = *b.*

Euler’s equation is not fundamentally creative because it is an identity of quantities, but because it is a bold conceptual conjecture that unifies previously separate mathematical domains—the domain of trigonometry and the domain of complex or imaginary numbers by means of interlacing the power series expansions for *sin*(x) and *cos*(x) and the exponential function *ex* expanded to include imaginary components to the exponents. Euler’s discovery is the result of creating a mathematical context which satisfies mathematical desiderata due to the following facts:

* There is only one natural way to extend the real-valued exponential function to the complex numbers.
* The exponential function “contains” the sine and cosine functions “implicitly” but the unification comes by expanding exponents to complex numbers.
* The most significant benefit of working in context of the complex numbers, as opposed to real numbers, is that every arithmetic equation (i.e., polynomial) has a solution—in fact, it systematizes the theory of polynomials giving the elegant result that an *n*-degree polynomial has *n* roots.

# 5.5. From Euler’s Equation to the Riemann Zeta Function

*To see the world in a grain of sand*

*And a heaven in a wild flower,*

*Hold infinity in the palm of your hand*

*And eternity in an hour….*

—William Blake (1757-1827), “Auguries of Innocence”

As we have seen much of the development of mathematics can be seen as the development of number systems. These number systems emerge out of everyday experiences with counting, accounting, dividing, measuring, and rotating. Patterns in the world are a catalyst for Platonistic mathematical reflection—why is the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 21, 34, 55, 89, 144, ….

so “fascinatingly prevalent” (H. S. M. Coxeter, *Introduction to Geometry*) in nature? Lilies have 3 petals, buttercups 5, some delphiniums 8, marigolds 13, asters 21, and daises have 34, 55, or even 89 petals. Why do the pyramids of Egypt, the chambered nautilus, the star in the pentagon, the segments of our fingers, the limit of the ratios of successive Fibonacci numbers all exhibit the golden ratio φ? The beauty of mathematics and its intricate interconnections suggest something deeper than mere empirical generalization.

Platonism, moreover, doesn’t give us a way to improve mathematical abilities or a coherent pedagogy of mathematics—if memory is all there is to learning, then remembering is the key to becoming a better mathematician. If Platonic picture is the right one, then some of us have the math meme and others don’t. Don’t bother trying to learn mathematics, if don’t have the meme you’ll just scream. Platonism, however, rightly points out that the patterns of mathematics are only imperfectly reflected in the empirical world and that the finite poverty of empirical stimulus is not sufficient to account for the infinity of mathematical knowledge. Platonism becomes mathematically grounded in the interaction between formalism and logicism.

These abstract properties are first formalized as laws which are then hierarchically arranged as axioms from which theorems can be proved using deductive logic. This is the seductive model of mathematics we derive from Euclid’s *Elements*. Mathematics as a system of first truths and theorems. Mathematics yields certain knowledge and knowledge that is *a priori* or justified independently of sense experience.

Formalization is not sufficient to capture the mathematical experience, nor is formalization in itself sufficient to guarantee logical consistency. Frege, for example, was able to formalize a great deal of arithmetic but he had to logically construct the set of natural numbers to ensure that there was such a consistent mathematical object. It is not enough to intuit that the Peano Postulates are true of the natural numbers in Plato’s heaven. The Peano Postulates in Frege’s foundational system were proven as theorems about a logical construction. The logical construction was supposed to exhibit a set theoretical object whose existence established the consistency of the postulates and whose construction shed light on the inherent properties of the natural numbers.

In order to guarantee that the formal laws of the various numbers systems do not lead to contradiction, more complex numbers systems are logically constructed as equivalences classes of simpler number systems, and eventually, the natural numbers are constructed as set theoretical objects for which the Peano’s Postulates are no longer accepted as mere *postulates* but instead are derived as *theorems* within a foundational system such as ZF set theory.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | ***Empirical Use*** | ***Formalism*** | ***Logicism & Constructivism*** | ***Anomaly*** |
| Natural Numbers | N | Counting | Peano’s Postulates | Frege’s construction of **N** from the inductive sets | No solutions:  x + 1 = 0 |
| Integers | Z | Accounting | Ring Axioms | Equivalence Classes of Pairs of Natural Numbers | No solutions: |
| Rational Numbers | Q | Dividing | Field Axioms | Equivalence Classes of Pairs of Integers | No solutions: |
| Real Numbers | R | Measuring lengths | Field Axioms with Completeness | Dedekind Cuts  *Completeness*: every set bounded above has a least upper bound | No solutions:  + 1 = 0 |
| Complex Numbers | C | Rotations and Headings | Fields with *n* roots for *n*th degree polynomials | Equivalence Classes of Pairs of Real Numbers | All non-trivial solutions:  *z* = ½ + *bi* |

This interaction suggests a more dynamic philosophy of number—in which the standard *competing* schools of mathematics are, in fact, *complementary*.

Platonism

Intuitionism

Formalism

Empiricism

Rules of Thumb

Patterns

Predictions

Logicism

Math = Logic + Set Theory

Axioms as Theorems

Axioms

Theorems

Meta-Mathematics

This development is a dynamic interaction between diverse philosophies of mathematics—empiricism and Platonism, formalism and intuitionism, logicism and mathematical creationism, in which, in the words of Cantor, “the essence of *mathematics* is *freedom*.” This interaction suggests another approach to the philosophy of mathematics. This approach does not merely choose among the competing philosophies of mathematics to explain mathematics, but instead looks to mathematical experience to enrich our philosophizing about mathematics.

# 5.6. From the Riemann Zeta Function to Places Yet Unknown

*“I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?”* David Hilbert.

* The complex numbers, unlike the real numbers, has a geometry forming a two-dimensional plane—thus allowing for the combination of geometry and the differential calculus, the mathematical study of the patterns of continuous motion and change.
* A complex function associates with each point in the complex plane a “height” which is a complex number and the “landscape” you get in this way can provide insight into mathematical problems—including the pattern of the primes.

The natural numbers are points in the complex plane—all the integers lying on the positive half of the x-axis. By studying the properties of the complex plane, mathematicians can sometimes deduce facts about the natural numbers. The *Riemann Hypothesis* is just this sort of problem in analytic number theory.

How can we find a function that provides information about the primes? The story begins with Euler who in 1740 introduced the zeta function.

When s < 1, the sum is *infinite* and so divergent, but when s > 1, the sum is *finite* and so convergent. Intuitively, when s > 1, the denominator grows fast enough to that even if you are adding infinitely many terms, the terms diminish rapidly enough to result in a finite sum.

What does the zeta function have to do with the primes?

The Euler Product Formula (1740)

can be derived from the Riemann Zeta function using the idea of the *Sieve of Eratosthenes*.

Given the zeta function’s infinite sum, we first “factor out” all the denominators with factors of the first prime 2. We do this by multiplying the infinite series by a factor of

Next we subtract this multiplied series from the original:

Notice that this subtraction removes all the denominators that have a factor of 2:

Next we multiply both sides using the next prime raised to the power *s* as the denominator:

which allows us to “factor out” the terms whose denominators have a factor of the prime 3 by subtracting from the previous sum:

Notice that now all the denominators having a factor of 3 or 2 (or both) have been removed from the right-hand sum.

Repeating this procedure for each of the primes in turn, we can see the right side is being systematically sieved:

Dividing both sides by the all products on the left to get ζ(*s*) by itself we obtain:

We have derived *Euler’s Product Formula*:

The Euler product converges when the real part of s is > 1.

As a real-valued function, the zeta function is a 1-dimensional object and thus has no geometric structure to help you uncover the pattern of the primes.

To do that you need to move up two dimensions, which is the key step that Riemann made when he replaced the real number s with complex numbers. This was in a remarkable paper of 1859.

The *Riemann zeta function* ζ(s) is a function of a complex variable *s* which is defined by a power series:

In his one and only paper on number theory, *“On the Number of Prime Numbers Less than a Given Quantity”* (1859), Riemann wrote in eight pages one of the most influential paper ever written. Riemann noticed that:

1. The zeta function could be extended to the complex plane.

2. Euler’s product formula was only valid when the real part is greater than 1.

3. There is another formula for the zeta function which is valid everywhere except for s =1.

When working with real-valued functions you can chop their parts and combine them to get smooth functions.

The *analytic continuation theorem* (ACT) says that the situation is completely different with complex functions. If we know the values of an analytic function on a patch of plane, then there is exactly one way to extend it to the whole complex plane.

Riemann wanted to prove Gauss’ conjecture that for large numbers *n*, the density of the primes below *n* is approximately 1/ln(*n*), a result known as the *Prime Number Theorem.* This theorem was proven by Hadamard and de la Vasllee Poussin in 1896.

Riemann in this paper – 8 pages in length—made a bold conjecture about the zeros of the zeta function. He began by observing that zeta(s) = 0 whenever s was one the even negative numbers were zeros -2, -4, -6, etc….. Call these the *trivial zeros*. Then he noticed that the zeta function must have infinitely many zeros and calculated the first 9 zeros, noticing they were all on the line *z* = ½ + *bi*, for some real number *b*. In geometric terms, all nonreal zeros of the zeta function lie on a straight line in the complex plane that runs vertically through the point ½ on the x-axis—known as the *critical line*.

Riemann showed that if all the complex zeros of the zeta function have real part equal to ½, then the degree to which the density function differs from the curve 1/ln(*n*) varies in a systematically random proportion…. Even though you can’t predict when the next prime will occur the overall pattern of primes is extremely regular (analogy: coin flipping).

With the single exception of z = 1, we can calculate this function.

The zeta function converges when the real part of *s* is greater than 1. Euler’s product formula is the special case when *s* = 1, and makes the connection between the harmonic series and the infinity of the primes.

Now ζ(1) is the harmonic series

and so we have a concise way of expressing the harmonic series as the product of all primes of the form 1/(1− 1/*p*):

This identity gives us another proof of the infinity of the primes.

The infinite sum of the harmonic series, we have already proven, diverges to infinity and so the left hand side is infinite. Now assume that the number of primes is finite. Then the right hand side of the equation would be finite. Contradiction. Therefore, the number of primes must be infinite. This completes an alternative proof of Euclid’s theorem about the infinity of the premise.

The next step in this development involves what we might call the “mathematics of philosophy”—the meta-mathematical approach elaborated by Hilbert, but masterfully and creatively deployed by Kurt Gödel. This approach to mathematics not only yielded profound mathematical results but yielded mathematical theorems which had profound consequences for philosophy and for formalist approaches to mathematics. This intellectual symphony by Gödel was a decisive turn from the philosophy of mathematics (the use philosophy to explain and clarify mathematics) to the use of mathematical methods to obtain philosophical results (the “mathematics of philosophy”).

*Conjecture*: The hitherto unprovability of the Riemann Hypothesis is related to Chaitin’s complexity interpretation of Gödel Incompleteness: since prime numbers can be used to code up information of arbitrary complexity, if you were able to predict the primes with Reimann-like constraints, then you would be able to decode information of arbitrarily high complexity, which is a contradiction.

1. It seems like voodoo economics to say that a debt of $5 multiplied a negative 5 times is equal to a positive $25. It’s like saying that some persons are so *negative* that when they finally *leave* it seems like someone has arrived! [↑](#footnote-ref-1)
2. The Harvard mathematician and satirical song writer Tom Lehrer put the definition of the derivative to music to a tune “There’ll Be Some Changes Made.”

   You take a function of *x* and you call it *y*,

   Take any *x0* that you care to try.

   You make a little change and call it δ*x*,

   The corresponding change in *y* is what you find nex’,

   And then you take the quotient, and now carefully

   Send δ*x* to zero and I think you’ll see

   That what the limit gives us, if our work all checks,

   Is what we call Is just [↑](#footnote-ref-2)
3. [↑](#endnote-ref-1)
4. Leibniz was proud of his discovery (1674) of the quadrature or Leibniz-Gregory series (Gregory discovered the series in 1671):

   +

   In an infinitely-approximating way, the series provided a “solution” to the classic geometrical problem of squaring the circle. But Leibniz’s series is practically useless in calculating the digits of π--not only does it converges extremely slowly but it does not yield digits that are *stable*. For example, if you compute the first five terms of the series then the partial sum is 0.8349206, which is within 1/11 of the true value of , but not a single digit of the partial sum is correct! [↑](#footnote-ref-3)