Euler’s Elegant Identity *ei*π + 1 = 0

*a mathematical mystery tour*

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# 5.1. The Elegance of Euler’s Identity

In 1749 Euler set forth in his *Introduction in Analysis Infinitorum* an identity considered to be among the most elegant in all of mathematics:

.

Euler’s equations, succinctly and elegantly combines five of the most fundamental constants of mathematics.

}

0 The two *natural numbers* essential for arithmetic, the *identity elements* for *addition* and *multiplication*,

1

π Two ubiquitous *real numbers* π and *e,* the ratio of the circumference of a circle to its diameter

} }

and the base of an exponentiation function whose rate of chance is identical to itself, real

*e* numbers that are not only *irrational* but *transcendental*,

*i* The mysterious square root of –1, the basis of the *imaginary* or *complex numbers*.

Euler’s equation conceptually links numbers from three different number systems (the natural, real, and complex numbers) using three of the most fundamental arithmetic operations (addition, multiplication, and exponentiation.) Carl Friedrich Gauss (1777 - 1855), the “Prince of Mathematicians,” reportedly said that anyone to whom Euler’s identity was not immediately apparent would never become a first-class mathematician.

# 5.2. Demystifying Euler’s Equation

*“An ordinary genius is a fellow that you and I would be just as good as, if we were only many times better. There is no mystery as to how his mind works. Once we understand what he has done, we feel certain that we, too, could have done it. It is different with magicians... the working of their minds is for all intents and purpose incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark.”*

—Mark Kac (1914-1984) (quoted in Nahim, p. 9)

Euler’s identity, it must be admitted, appears to be the work of a magician. However, whether or not one aspires to become a first-class mathematician, it is not immediately apparent how Euler’s equation even makes sense.

*Exponentiation* is typically explained in terms of a repeated multiplication. For example, the number 2 raised to the exponent 3 is defined as the product of 2 multiplied by itself three times:

Euler’s identity uses transcendental numbers with exponentiation. The transcendental number π is defined as the ratio of the circumference to the diameter of a circle. Like the irrational number the √2, it is represented by an infinite non-periodic decimal; however, unlike, such *algebraic* irrational numbers, π is not the solution or root of any polynomial such as *x*2 – 2 = 0. The transcendental number *e* is defined as the base of an exponential function whose rate of change, or derivative, is equal to itself. In other words, *ex* is the unique function such that it is its own derivative.

Now the transcendental numbers *e* and π are numbers in the neighborhood of the natural numbers 2 and 3, respectively, so it makes sense to think of exponentiation a purely numerical functions. (To remember the digits of π remember the question: *“May I have a large container of coffee right now?”* and “*To express e remember to remember a sentence to remember this”.*) Thinking of exponentiation as a function of real numbers, we can compute:

2.7182818284…3.141592653… = 23.14069262….

However, Euler’s identity also includes am exponential factor of the imaginary number *i*, where *i* is defined to be the positive square root of – 1. Now *i* called *imaginary* number because no real number can be the root of a negative number. Real numbers are either positive or negative, and a positive number times a positive is positive and a negative number times a negative is also positive. Euler in his *Algebra* (1770) wrote:

All such expressions as , etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities, and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.

Obviously, there must be some meaningful mathematical way of thinking of exponentiation other than as a real-valued function.

The Nobel physicist Richard P. Feynman called Euler’s equation “our jewel” and “the most remarkable formula in mathematics.”[[2]](http://en.wikipedia.org/wiki/Euler%27s_formula#cite_note-2) Feynman recounts how he, as a young boy, was fascinated and in his notebooks figured out for himself, rather than taking it on authority, the mystery of why Euler’s identity was true. In Feynman’s spirit of discovery, let’s take a mathematical journey in which we figure out the meaning of Euler’s remarkable identity for ourselves—not being content with mere proofs but instead taking the time to demystify the elements of Euler’s beautiful equation.

The remarkable beauty of Euler’s derivation of his identity derives not only from its deep *connections* of five fundamental mathematical constants, but also how it illustrates the *cognitive* strategies mathematicians deploy they are creating or discovering mathematical truths. The last phrase “creating or discovering mathematical truths” poses an important philosophical question: do mathematicians discover mathematical truths or do they creatively construct them?

On this magical mystery tour, we will discover that mathematics is more than merely crunching numbers, that it is more than presenting formal proofs, and that mathematics involves such creative discoveries as:

* *constructing* new number systems to solve previously impossible equations;
* *discovering* in geometric representations higher-level *symmetries*;
* *transforming* anomalous *singularities* into *systematic closure properties*;
* *generalizing* arithmetic operations by *expanding* their mathematical meanings;
* *unifying* conceptual domains by making bold conjectural *identifications*.

In demystifying Euler’s equation, our goal is not to dispel its beauty. Indeed, by taking the time to understand how mathematicians create, we will finally come to appreciate—in a mathematically precise rather than a mystically vague way—the true beauty of Euler’s equation.

# 5.3. A Concise, but Conceptually Incomplete, Proof

*“It [Euler’s identity] is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."*

—Benjamin Peirce (1809-1880)

The Harvard mathematician Benjamin Peirce, father of the pragmatist philosopher C. S. Peirce (1839-1914), is reported to have made the above remark after proving Euler’s identity. This quotation reminds me of a joke told by Raymond Smullyan. A mathematics professor, teaching a class of mystified undergraduates, is writing a proof on the board and claims that the theorem is “trivial.” A brave student asks why the proof is trivial. The professor turns back to the board and is lost in thought for five minutes, and then announces, “aha, yes, it is trivial!” Too often in a typical mathematics class, the lectures consist of presenting proofs to silent, of silenced, students are left on their own to figure out what is going on mathematically by doing exercise.

Perhaps the students are silent because they are afraid to “look stupid” or perhaps they have learned that asking questions can be dangerous. When some student asks the professor to explain some step of the proof, perhaps the professor “explains” the proof by merely repeating the same words, maybe talking more slowly or perhaps by filling in a few details. But the students can follow the step but they still don’t understand the mathematical ideas. This pedagogical impasse is created by the assumption that teaching mathematics is communicating formal structure or proofs. However, often understanding the mathematical ideas requires going behind the formal structure of proofs. Teaching is not merely proving theorems but communicating the mathematical ideas so that the truth of the theorem can be not only followed, but understood or comprehended, in a deeper, more intuitive, way.

So we have reason to be skeptical about Peirce’s pronouncement:

*“It [Euler’s identity] is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."*

Here’s how Euler, in his *Introduction in Analysis Infinitorum*, derived his equation from the following trigonometric identity:

Here if we simply substitute π for the variable *x*, we have we have:

*,*

from which Euler’s identity immediately follows. This proof, while concise, is conceptually incomplete.

Beginning the proof with a puzzling *complex* trigonometric equation begs the conceptual question: what does exponentiation has to do with trigonometry? After all, both the *sine* and *cosine* functions are *periodic* and *bounded*, but the *exponential* function is *non-periodic* and *unbounded*. How than can an exponential function like *ex* which rapidly expands to positive infinity be defined in terms of periodic trigonometric functions whose values are confined to interval from 1 to 1?

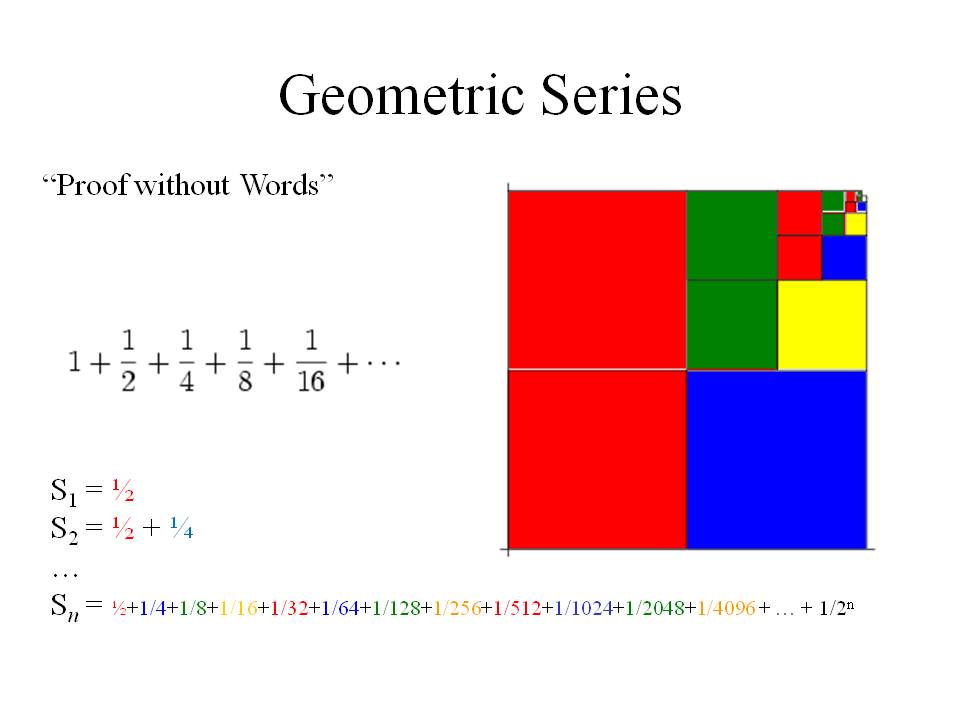
# 5.4. Expanding Exponentiation: From Powers of Numbers to Power Series

The answer to this question involves thinking of exponentiation not merely as a *power function* that produces a number but as a *power series.* We have already encountered infinite series when discussing Zeno’s paradoxes. For example, Zeno’s dichotomy paradox involved the *geometric* series

Figuring out the sum of such an infinite geometric series depended on the observation that multiplying the series by the common factor of ½ results in truncating the first term:

So by subtracting half the series from the entire series, we simply have:

Another way of seeing this result geometrically is the following “proof without words”:



The same idea for summing an infinite geometric series works, not only for *r* = ½, but for any *r* whose absolute value is < 1. In general, the infinite sum exists when |*r*| < 1,

Now the idea of a *power series* comes about by replacing *numbers* with a variable resulting in the sum of an infinite series of *power functions*. Generally, a power series is of the form

where *a0*, *a1*, *a2*, *a3*, *a4*, *a5*,…, *an*,… is some sequence of numbers. The reason for representing a function as a power series is that it is easy to compute the derivative of such a series term by term using the fact that the derivative of

is

.

Now remember that the defining characteristic of the exponential function is that the derivative of *ex* is just *ex*. How can we create a power series in which the derivative of the series, taken term by term, results in the very same series?

The derivative of 1 = 0

The derivative of *x* = 1

The derivative of =

The derivative of =

The derivative of =

So the taking the term-wise derivative is the infinite series itself! Now this is precisely the defining feature of the exponential function.

This power series can be used to estimate the value of *e*.

The following spreadsheet, computing the value of *e* for the first 12 terms, gives an approximation accurate to the first 9 decimal places.

|  |  |  |
| --- | --- | --- |
|  | **Power Series Approximation of *e*** | |
|  | 1 | 1 |
| 1 | 1 | 2 |
| 2 | 0.5 | 2.5 |
| 3 | 0.166666667 | 2.6666666667 |
| 4 | 0.041666667 | 2.7083333333 |
| 5 | 0.008333333 | 2.7166666667 |
| 6 | 0.001388889 | 2.7180555556 |
| 7 | 0.000198413 | 2.7182539683 |
| 8 | 2.48016E-05 | 2.7182787698 |
| 9 | 2.75573E-06 | 2.7182815256 |
| 10 | 2.75573E-07 | 2.7182818011 |
| 11 | 2.50521E-08 | 2.7182818262 |
| 12 | 2.08768E-09 | 2.7182818283 |

Euler’s power series gives a quick and accurate estimate of the value of *e* because the factorials in the denominator grow quickly diminishing the contribution of the successive terms, which quickly approach zero. However, the value of power series does not necessarily lie in computing approximate values.

Leibniz was proud of his discovery (1674) of the quadrature or Leibniz-Gregory series (Gregory discovered the series in 1671):

In an infinitely-approximating way, Leibniz’s series provides a “solution” to the classic geometrical problem of squaring the circle. But Leibniz’s series is practically useless in calculating the digits of π. Not only does it converge extremely slowly but it does not yield digits that are *stable*. For example, if you compute the first five terms of the series then the partial sum is 0.8349206, which is within 1/11 of the true value of π/4, but not a single digit of the partial sum is correct!

Similarly, the mathematical value of the exponential power series lies not so much as its usefulness in *calculating* the digits of *e*, but in its providing *conceptual* insight into how to generalize exponentiation to unify two formerly distinct mathematical domains. The catalyst for the intuitive leap is the curious way in which the power series for the trigonometric functions of sine and cosine appear to be “contained” in the exponential power series.

The terms of the *sine* power series are the *odd* powers and the terms of the *cosine* power series are the even powers. The questions is how to account or the alternation of + and – signs. And to repeat, the *sine* and *cosine* functions are *periodic* and *bounded*, but the *exponential* function is *non-periodic* and *unbounded*. How than can an exponential function like *ex* which rapidly expands to positive infinity be defined in terms of periodic trigonometric functions whose values are confined to interval from 1 to 1?

To solve this part of the puzzle we need to explore the undue prejudice directed against imaginary numbers.

# 5.5. A Tale of Two Negatives: From Negative to Imaginary Numbers

*Minus times minus is plus.*

*The reason for this we need not discuss.*

— W. H. Auden

Skepticism about the negative numbers persisted until the 18th century. Negative numbers are often explained in terms of debt. If Abe owes Beatrice $3 dollars and Calliope $7, but he has only $5, then Abe is in debt $5 or he has, so to speak, a negative $5. Now if Abe is in debt for $5 five times over, then Abe has, so to speak, a negative $25 dollars. This way of think raises a puzzle: suppose Abe is in debt $5 and negative five times. How does it make sense to say that Abe is now out of debt and in fact has a positive profit of $25?

Euler himself in his *Algebra* (1770), used the analogy of debt, but did not fall into the usually mistake of trying to explain that the product of two debts must be positive. Instead, Euler gave a negative “explanation” of why “minus times minus is plus” by considering logically exhaustive cases: *– a × – b* cannot be *– ab* since this what is given by *– a × b* and the change in sign of *b* should give a change in sign of the answer. Therefore, Euler argued, *– a × – b* must *ab.* Notice that Euler’s argument doesn’t give an *explanation* of why a negative times a negative is a positive, but instead gives a *systematic* argument for why the product of two negative can’t be assigned a negative.

Where do negative numbers come about conceptually in mathematics? They come from wanting to have solutions to equations such as

*x* + 1 = 0 .

A skeptical argument about *negative* numbers goes like this. The left-hand side of the equation is the *successor* of the number *x*. The right-hand side is zero. However, according to Peano’s postulates, zero is not the successor of any natural number.

The logicist answers this skeptical conservatism about negative numbers by *logical construction.* The logicist constructs negative numbers as a set of ordered pairs of natural numbers. First, one defines an equivalence relation between ordered pairs of natural numbers as follows:

(*a*, *b*) ≈ (*c*, *d*) ↔ *a* + *d* = *b* + *c*

The integers are then defined as the equivalence classes induced by this relation. One can define the operations of addition and multiplication for integers in terms of the corresponding operations for natural numbers.

(*a*, *b*) ⊕ (*c*, *d*) ↔ (*a* + *c, b + d*)

(*a*, *b*) ⊗ (*c*, *d*) ↔ (*ac + bd, ad + bc*)

The integers have the satisfying property, lacking in the natural numbers, of being closed under the operation of *subtraction*.

Negative numbers can be obtained from the positive numbers by multiplying by −1. Geometrically speaking, this is a rotation of 180 degrees along the number line for the natural numbers, which are now extended in the negative direction.

The puzzle about justifying the law that a “negative times a negative is a positive” by finding an interpretation for multiplying debts is based on an inadequate conception of a negative number. A more general way of thinking about negative numbers is that multiplying by – 1 indicates changing direction on the number line. To add – 5 and – 6 is to continue on the number line from the origin in the negative direction for a total of 11 units. To multiply – 5 by 6 to go 5 × 6 = 30 units in the negative direction. However, to multiply – 5 by – 6 is to multiply 5 by 6 changing directions twice and ending up on the positive direction. Skepticism towards negative numbers can now be dismissed as philosophical confusion of thinking of negative numbers too literally in terms of debt.

The construction of imaginary numbers from real numbers can be done in a way that is completely analogous to the construction of negative numbers from natural numbers.

*Imaginary i times i is minus one.*

*Rotation is how this can be done!*

Similar conceptual puzzles arise from thinking of the imaginary number as a quantity. Where does the imaginary number come about conceptually in mathematics? The number from wanting to have solutions to equations such as

*x2* + 1 = 0 .

The imaginary number *i* is defined to be . The question arises whether is *really* a number. Leibniz waxed mystical when it came to this question: “The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, the amphibian between being and not-being, which we call the imaginary root of negative unity.”[[1]](#footnote-1)

A skeptical argument comes from the axioms for real numbers. The skeptic about *imaginary* numbers can argue that this equation can have no real solutions: the square of any real number, positive or negative, is positive. Therefore, first term of (2) can never be equal to −1 and so there are no real solutions to the equation. Euler in his *Algebra* (1770) expressed his skepticism as follows:

All such expressions as , etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities, and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.

The logicist answers this skeptical conservatism about imaginary numbers by *logical construction.* The logicist constructs imaginary numbers as a set of ordered pairs of real numbers. One can define the operations of addition and multiplication for complex numbers in terms of the corresponding operations for natural numbers.

(*a*, *b*) ⊕ (*c*, *d*) ↔ (*a* + *c, b + d*)

(*a*, *b*) ⊗ (*c*, *d*) ↔ (*ac* − *bd, ad + bc*)

Note that these definitions for addition and multiplication of complex numbers are formally identical to the above definitions for addition of multiplication of integers except for the negative sign (here indicated in red).

By convention a complex number is expressed as *a* + *ib*, where *a* is the real component and *b* the imaginary component of the complex number. *Notice that the plus sign doesn’t really signify addition.* The plus sign is merely used to separate the real and imaginary components, and so it may be less confusing to simply use an ordered pair of real numbers to represent a complex number.

What does mean in terms of rotation? If multiplying by is a rotation of 180 degrees or π radians, then, geometrically speaking, multiplication by is represented by that rotation such that two successive rotations is equal to . In other words, a rotation by *i* can be represented by a rotation of 90 degrees or π/2 radians.

Complex numbers are conventionally plotted on the Argand plane. The real component of a complex number is plotted along the *x*-axis and the imaginary component is plotted along the *y*-axis. As we have seen, multiplying −1, is, by geometrically speaking, a rotation of 180 degrees or π radians. To geometrically represent *i*, we need a transformation which when applied to itself result in a multiplication by −1 or a rotation of π radians. By convention, a counterclockwise) rotation (CCW)of 90 degrees or π/2 radians is a natural way to represent this transformation. Repeated CCW rotations of 90 degrees of π/2 result in the cycle of values *i1* = *i*, *i2* =−1, *i3 =* −*i*, *i0* = *i4 =* 1. This cycle of values is the conceptual clue to why the imaginary number *i* in the exponent of Euler’s equation is related to rotation in the geometric representation of complex numbers.

We still need to find the conceptual connection between imaginary numbers a trigonometry. The values for trigonometric functions are proportional for all similar triangles, so it is convenient to represent these values within a circle of radius 1. Rather than using a pair of *Cartesian coordinates* to specify a point (*x*, *y*), it is more natural to use *polar coordinates* specified by a central angle θ and a radius *r* = When confining our attention to the unit circle, the trigonometric functions are defined in terms of θ. The Pythagorean theorem, for example, can be expressed by the trigonometric equation:

1 = cos2(θ) + sin2(θ) .

The values of cos(θ) are read on the *x*-axis and the values of sin(θ) are read on the *y*-axis for a right triangle whose right angle is formed perpendicular to the *x*-axis and whose hypotenuse is a radius from the origin to the unit circle.

*sin*(θ)

θ

−1 0 *cos*(θ) 1

One of the unsymmetrical properties of real numbers is that positive numbers have two square roots, whereas negative numbers have none. All real numbers have exactly one cube root, fifth root, and so on. One of the elegant properties of working in context of the complex numbers, as opposed to real numbers, is that every arithmetic equation or polynomial has a solution—in fact, within the complex numbers every *n*-degree polynomial has exactly *n* roots.

The formula that connects complex number with trigonometry is *De Moive’s formula*

.

De Moive’s formula follows from Euler’s trigonometric equation

for a complex exponent *z* = *x* + *iy*:

As real-valued functions, exponentiation and trigonometry are *incompatible*, but here they are *unified* within the complex numbers. Since exponential functions have the property that

we also have *De Moivre’s theorem*:

(*cos* θ + *i sin* θ)*n* = *cos*(*n*θ) + *i sin*(*n*θ) .

This theorem overcomes the anomaly within the real numbers that, for example, has two roots +1, whereas all *negative* numbers have no roots but all positive real numbers have exactly one cube root, exactly one fifth root, etc. for all the *odd* roots. This irregular behavior is, to the mathematical mind, inelegant. De Moivre’s theorem states, for example, the roots of are solutions to:

(*cos* θ + *i sin* θ)*2* = *cos*(2θ) + *i sin*(2θ) = .

Geometrically becomes the question: which angles, when doubled, give or an angle of π = 180 degrees? The two angles are π/2 = 90 degrees and 3π/2 = 2 = 270 degrees, which correspond to respectively to *i* and

In general, there will be exactly *n* roots for the *n*th root of because the unit circle can be equally divided into *n* evenly spaced parts. Here is a diagram for the eight roots of which divide the unit circle in two eight arcs of π/4.

*i*

1

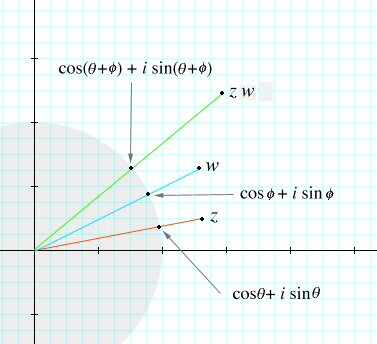
*–i*

It happens there is only one natural way to extend the real-valued functions to the complex numbers.

This formula tells us that complex numbers have two geometric representations—a point with Cartesian coordinates (*x, y*) and point with radius *ex* = *r* = and central angle θ which is the angle *x + iy* makes with the *x*-axis.

A convenient way to multiply vectors is to use *polar coordinates* instead of Cartesian coordinates, a method that will yield *geometric* insight into the mathematical connections with *chaos.* The polar coordinates of a point (*x*, *y*) in the Cartesian coordinate plane is given by (***r***, θ), where radial distance *r* = |*z*| = and θ is the central angle. We have the following trigonometric identities:

*z* = *x* + *iy* = (*r cos*θ + *r* *i sin*θ) = *r*(*cos*θ + *i sin*θ) =|*z*|(*cos*θ + *i sin*θ) .

******

Given polar coordinates is easy to *multiply* two vectors:

*z* = |*z*|(*cos*θ + *i sin*θ)

*w* = |*w*|(*cos*φ + *i sin*φ)

by simply *multiplying* their radial distances and *adding* their angles:

*z w* = *z w* [*cos*(θ + φ) + *i sin*(θ + φ)] .

**Fig**. **9.** Polar coordinates provide geometric insight into the connection

between trigonometric squaring, angle doubling and chaos.

We suggest terminology of *versimilitude* for the *length* of the truth vector *r* = |*z*| = and the *spin* for the clockwise or counter-clockwise *direction* of its central angle θ.

|  |  |
| --- | --- |
| Cartesian Coordinates | Polar Coordinates |
| *veridicality* | *magnitude* |
| *skew* | *spin* |

Whereas *skew* is a *linear* dimension along the *y*-axis, *spin* is an *angular rotation* in either the CW or CCW direction.[[2]](#footnote-2)

In summary, we have show how the equations

*x* + 1 = 0

and

*x2* + 1 = 0 .

give rise to the negative and imaginary numbers, respectively. Now we may return to the equation

and show it gives rise to the unification of trigonometry and complex exponentiation.

# 5.6. A Conceptual Unification: Understanding Euler’s Identity

Euler’s equation is creative, not as an identity of quantities, but as a bold conceptual conjecture that unifies previously separate mathematical domains—the domain of trigonometry and the domain of complex or imaginary numbers—by a meaningful expansion of exponentiation into the realm of imaginary numbers. Euler’s discovery is the result of creating a mathematical context which satisfies mathematical desiderata that every *n*-degree polynomial has exactly *n* roots. In fact the extension to the complex numbers provides a unification that is not apparent when restricted to the real numbers.

Recall that we noticed that the infinite trigonometric power series:

correspond to the odd and even summands in the power series for the exponential function. We can in fact show this to be precisely the case by introducing *imaginary* exponentiation:

Finally, for *x* = π, *cos*(π) = 1 and *sin*(π) = 0, so we have derived Euler’s elegant equation:

.

This beautiful equation connects the ideas of *imaginary numbers* *i* and *rotations* in terms of π to *imaginary*  *exponentiation* in a way that unifies *complex numbers, trigonometry,* and, as we shall see, *computation*.

# 5.7. From Truth Values to Truth Vectors.

Rather than thinking of truth as measured *linearly* (as in done in Tarski’s and Rescher’s schemas) as its absolute distance from 1 within the interval [0, 1], we can think of measuring truth in terms of *two* *dimensions*: its *degree* of *veridicality* or *accuracy* and its *skew* or *spin*. The natural mathematical representation of such an entity not a single *truth* *value* but a two-dimensional *truth* *vector* ***c*** = ***v*** + *i****s***.

First, we extend the range of values for ***v*** from the real-valued interval [1, 0] representing *degrees of truth* to the real-valued interval [1, −1] representing *degrees of accuracy* or inaccuracy. In ordinary language, to be *inaccurate* is more than merely *lacking accuracy* but to be positively misleading. Here the value ***v*** ranges from ***v*** = 1 for complete accuracy and ***v*** = −1 for complete inaccuracy.

Secondly, we include a scale for measuring skew ranging from *i* to –*i*. (One way of thinking about the *imaginary value* *i* = , is to use Zedah’s proposed reading as “*fairly (CCW) skewed.*”) From a mathematical point of view, we have *truth vectors* ***c*** = ***v*** + *i****s***, where the value ***v*** stands for the degree of *veridicality* or *accuracy* and the value ***s*** stands for the degree of *skew*. These vectors can be combined by the standard rules for vector addition and multiplication.

Thirdly, to make our graphs more intuitive, we shall sometimes plot the degree of veridicality along the *vertical* axis and the degree of skew along the *horizontal* axis. Other times we use the standard conventions.

*completely accurate*

1

*–i* *i*

*completely skewed to the left completely skewed to the right*

*–*1

*completely inaccurate*

The unit circle for two-dimensional truth vectors

Functions involving imaginary number often involve rotation and produce waves and so have proved, in terms of applications in physics, to be quite useful.

*Example 1.* *Vector addition.* Suppose we have three media outlets measured by two-dimensional vectors with a parameter ***v*** for veridicality or accuracy and another parameter ***s*** for skew. Suppose a random poll measures the accuracy and skew of the political views of MSNBC viewers and FOX viewers and finds them to be as follows:

* MSNBC viewers have beliefs that are 40% accurate and 50% skewed a *liberal* direction.
* FOX viewers have views that are 80% *inaccurate* and 80% skewed in the *conservative* direction.
* Polling reveals that the public’s political views are 50% accurate with a conservative skew of 20%.

Assume that truth vector for public opinion results from the vector *sum* of the truth vectors for these three media outlets. Calculate the truth vector (i.e., the *accuracy* and *skew*) of NPR.

*Solution*: NPR’s coverage is estimated to be 90% accurate with a liberal spin of 10%.

|  |  |  |
| --- | --- | --- |
|  | **Accuracy** | **Skew** |
| MSNBC | 0.40 | 0.50 |
| FOX | −0.80 | −0.80 |
| NPR | 0.90 | 0.10 |
| SUM | 0.50 | −0.20 |

*Example 2. A Fixed-Point Theorem for Truth Vectors*

Suppose that there are two media outlets—a blue station espousing a liberal point of view and a red station espousing a conservative point of view. Using a unit circle we can plot the dynamic states of truth vectors operating on, or influencing, the state of public opinion. We can plot the state of public opinion using a vert*i*cal axis for *veridicality* and a *horizontal* axis for political *spin*. The veridicality axis ranges from

*i* (completely *accurate* information) to –*i* (completely inac*c*urate misinformation). The horizontal axis ranges from –1 (complete liberal or *left*-wing politics) to 1 (complete conservative *right*-wing politics).

*i* = completely accurate

*–* 1 = completely liberal (left) 1 = completely conservative (right)

*–i* = completely inaccurate

The state of public opinion can be represented by a vector, for example, by the vector whose coordinates are say - 0.4 for accurate information and 0.3 for political spin in the conservative direction.

Given this, or any state of public opinion, and assuming the blue station aims at completely liberal broadcasting and the red station aims at completely conservative broadcasting, what additional broadcasting to the blue station and what additional broadcasting to the red station is required so that the balance of opinion as computed by the midpoint between the vectors of influence of the media outlets results in a state of public information which is approximately close to completely accurate information?

We can prove a fixed-point theorem that answers this question. First, we illustrate this fixed-point theorem with a classic puzzle due to George Gamow that demonstrates the power of complex numbers.

Imagining Gamow’s Treasure. There is hidden treasure on an island on which there are two palm trees. To find the hidden treasure, Captain Jack Sparrow must count his paces from the grave of Davey Jones to the palm tree on the left, turn 90 degrees clockwise, pace the same distance, and then plant a flag. Returning to the grave, Captain Jack must count his paces to the palm tree on the right, turn 90 degrees counterclockwise, pace the same distance, and then plant a second flag. The treasure is to be found midway between the two flags. The problem is that the location grave of Davey Jones is unknown. How can Captain Jack Sparrow still find the hidden treasure?

*Solution*: Assign coordinates to the palm trees. The palm tree on the left is assigned (–1,0) and the palm tree on the right is assigned (1,0). We will show that no matter where the grave is, the treasure is buried at (0, 1).

In the diagram below, the vector G from the Davey Jones’s Grave 🕱 to the origin is yellow. The vector from the Grave 🕱 to the palm tree at (-1, 0) is blue. The vector to from the Grave 🕱 to the palm tree at (1, 0) is red.

(1,0) *G*  (1,0)

🕱

The usefulness of imaginary numbers is this: the multiplication by *i* of a vector in the complex plane results in a 90 degree counter-clockwise (CCW) turn, and multiplication by – *i* results in a 90 degrees clockwise (CW) turn.

To plot where the first flag is planted, we first translate to blue vector to an origin at (–1, 0) and multiply the transformed vector by –*i*. Notice that the transplanted vector ends up at [(G – 1 ) – 1], and so the location of the first flag is –*i* [(G – 1 ) – 1]:

🏳

(1,0) (1,0)

🕱

Similarly, we translate the red vector to [(*G* + 1) + 1] with an origin at (1, 0) and then multiply by *i* to determine where the second flag is planted.

🏳

🏳

(-1,0) (1,0)

🕱

Now we can find the buried treasure by plotting the *midpoint* of the line between the two flags:

🏳

🏳*i*

(-1,0) (1,0)

🕱

Now the *midpoint* between the two flags can be computed as the *average* of the locations:

Notice that the vector G drops out of the equation. Therefore the position of Davey Jones’s grave is irrelevant: we have a fixed-point theorem. The treasure lies buried on the perpendicular bisector directly above the origin at a distance of one half the distance between the two palm trees.

Returning to our original problem, let’s define the *liberal aspirational vector* to be the vector from the current state of public opinion to (– 1, 0), which is pure liberalism, and define the *conservative aspirational vector* to be the vector from the current state of public opinion to (1, 0), which is pure conservatism.

*Fixed-Point Theorem*: Given any state of public opinion, if the liberal and conservative media outlets are approximately achieving their aspirational vectors, then the solution to moving the public towards accurate information is to supplement media influence by vectors equal to multiplying the liberal aspirational vector by – *i* and the conservative aspirational vector by *i*.

In the next chapter, we use this geometric interpretation of imaginary numbers as *rotation* unify logic with complex numbers, providing insight into mathematical properties of chaos.

1. Félix Klein, *Elementary Mathematics From an Advanced Standpoint: Arithmetic, Algebra, Analysis* (1924), 56. [↑](#footnote-ref-1)
2. *Verisimilitude* was a philosophical term, coined by Karl Popper [1976] to capture the notion of the *deductive power* of a scientific theory. How can one explain the paradox that *truth* is the aim of scientific inquiry although even the greatest scientific theories in the history of science are, strictly speaking, *false*? Popper wanted to acknowledge the intuitive belief that, for example, Newtonian mechanics is “closer to the truth” than Aristotelian physics even though both theories are *false*. Popper proposed distinguishing between the *truth* of a scientific hypothesis and its *content* in terms of the truth-content of its logical *consequences*. Popper never succeeded in giving a satisfactory formal account. Here we use the same term for a different account: our *two-dimensional* proposal of truth vectors is not, like Popper’s theory, committed to giving a *one-dimensional* measure of “closeness to the truth.” [↑](#footnote-ref-2)