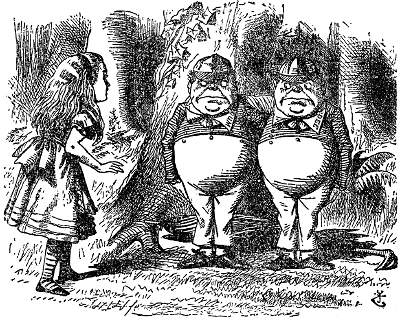
Chapter 7. ‘*Must*’ and ‘*Might*’

*The Modal Logic of Necessity and Possibility*.[[1]](#footnote-1)



*“I know what you’re thinking about,” said Tweedledum; “but it isn’t so, nohow.”*

*“Contrariwise,” continued Tweedledee, “if it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic.”*

Lewis Carroll (1832-1898), *Through the Looking Glass* [1871]

1. Modes of Truth and Modal Logics. 2

Exercises 7

2. Translation and Symbolization. 10

Exercises 10

3. Axioms for Modal Logics. 11

4. Modal Inference Rules. 15

Exercises 16

5. Natural Deduction for (T), (B), (S4), and (S5). 16

Exercises 28

6. Fallacies. 29

Exercises 35

7. Possible World Semantics. 38

Exercises 44

8. Theorems. 51

Exercises 57

9. Counterfactuals “Would” and “Could” 59

Exercises 63

10. Modal Provability Systems of Gödel and Löb. 64

11. Philosophical Remarks (due to Nathan Salmon) 70

References 78

Appendix A. List of Modal Theorems 79

# 1. Modes of Truth and Modal Logics.

Historically, notions like *necessity*, *possibility*, *impossibility,* and *contingency* were thought of as modes of truth or ways in which a proposition could be true or false. *Modal logic* began as the study of the logic of theses modes of truth.

Aristotle, in Chapter 9 of *De Interpretatione*, discusses modality in his famous example of the sea battle. Suppose the sea battle will be fought tomorrow. Then it was true yesterday that it would be fought tomorrow. So if all past truths are necessarily true, then it is necessarily true now that the battle will be fought tomorrow. A similar argument holds on the supposition that the sea battle will not be fought tomorrow. Aristotle proposed solving this problem of *logical fatalism* by denying that future contingent propositions have definite truth-values.

Using the ‘□’ for ‘it is necessary that’, the principle that all necessary truths are in fact true is captured by Axiom (**T**):

(**T**) □ P→ P ,

but adding its converse that all truths are necessary truths:

(**V**) P → □P

collapses the notions of truth and necessary truth.

Medieval philosophers, concerned with such theological issues as articulating the nature of the Trinity, appealed to such modal notions as essence and accident, contingency and necessity in their labyrinthine theological reflections.[[2]](#footnote-2) Akin to the problem is logical fatalism is the problem of *theological fatalism*: the problem of reconciling divine foreknowledge and human freedom. Saint Augustine (354 - 430) in his treatise *On the Free Choice of Will* considers an argument for *theological fatalism* proposed by Evodius. Evodius argued that “God foreknew that man would sin, that which God foreknew must necessarily come to pass.” We may set forth this argument for theological fatalism for a particular case as follows:

If God knew that Adam would sin, then, necessarily Adam sinned.

God knew that Adam would sin (because God is omniscient).

Therefore, Adam necessarily sinned.

St. Thomas Aquinas (1225 - 1274) in his *Summa Contra Gentiles* (part I, chapter 67) criticized this kind of argument as resting on an amphiboly. The critical first premise “if God knew Adam would sin, then, *necessarily*, Adam sinned” is ambiguous between

(1*a*) It is necessarily the case that if God knows that Adam will sin then Adam will sin.

(1*b*) If God knows that Adam will sin, then it is necessary that Adam will sin.

Aquinas called (1*a*) the *necessity of the consequence* contrasting it with (*1b*) the *necessity of the consequent*. Using the ‘□’ to abbreviate ‘it is necessary that’, the difference between these two can be made more perspicuous with symbols:

(1*a*) □(P → Q)

(1*b*) (P → □Q)

Solving the famous theological problem reconciling divine foreknowledge with human freedom may turn on exposing ambiguities of this sort.

Perhaps the most famous theological application of modal logic is Saint Anselm’s modal ontological argument. According to Saint Anselm (1033 - 1109), it follows from God’s nature that it is necessary that God exists if God exists at all. Moreover, this conditional itself, being a conceptual truth, is itself necessarily true. We then have the following argument:

Necessarily, if God exists, then God necessarily exists.

It is possible that God exists.

Therefore, God (actually) exists.

Using ‘’ for ‘it is possible that’, the above argument can be symbolized:

(2) □(G → □G) . G ∴ G

The question of whether Anselm’s argument is valid became a precise question when various systems of modal logic were proposed and developed in the 1960s.

Gottlob Frege (1848 - 1925), the inventor (or discoverer) of modern predicate-quantifier logic, relegated modality to autobiographical information about the speaker, and for many years logicians only investigated extensional logic.

One of the most puzzling validities, at least to beginning logical students is known as *Lewis’s Dilemma*:

P  ~P  Q ,

which states *“a contradiction implies anything”*. This implication follows from the inference rules of simplification, addition, and *modus tollendo ponens*, which are themselves not particularly puzzling.

1. *~~Show~~* P  ~P  Q 6, CD

2. P  ~P Assume (CD)

3. P 2, S

4. ~P 2, S

5. P  Q 3, ADD

6. Q 5, 4 MTP

The following theorems are known as the *paradoxes of material implication*:

T18 ~P  (P  Q) *law of denying the antecedent*

T2 Q  (P  Q) *law of affirming the consequent*

T58 (P  Q)  (Q  R) *conditional excluded middle*

T114 (~P  P) ↔P *Consequentia Mirabilis*

C. I. Lewis investigated modal logic in order to find a stricter form of the conditional which would not result in such paradoxes. Lewis defined *strict implication* P ⇒ Q (read “P *strictly implies* Q”) by combining modality with the truth-functional conditional:

P ⇒ Q :=(P  ~ Q) ,

or alternatively,

P ⇒ Q :=□(P  Q) .

The notion of strict implication was characterized by such axioms as:

The philosopher Leibniz (1646 - 1716) explicitly invoked that language of possible worlds to explain the difference between necessary and contingent truths. What is logically or necessarily true are those truths truth in *all* possible worlds, whereas a contingent truth is one that is true in *some* possible world.

Drawing upon this logical connection between universal and existential quantification and the modal notions of necessity and possibility, we obtain a modal version of the classical Aristotelian Square of Opposition and the duality of modal laws known as the laws of modal negation.

🞎~P

*It is impossible that P*

🞎P

*It is necessary that P*

Contraries

~P

*It is actually not the case that P*

P

*It is actually the case that P*

Subaltern

Subaltern

Contradictories

Subcontraries

◊P

*It is possible that P*

◊~P

*It is possible that not P*

Figure 1 An Aristotelian Diamond of Opposition

In the modern development of modal logic, logicians noticed that a host other phenomena—such as deontic notions of obligation and permissibility, epistemic notions of knowledge and belief, as well as temporal operators—share these logical relations and hence can be represented as modal logics.

In *deontic logic*, □ is read “it is morally obligatory that” and  is read “it is morally permissible that”. Kant’s maxim that “ought implies can”, that is, whatever is obligatory is permissible, is captured by modal axiom (**D**):

(**D**) □P → P .

In *epistemic logic*,  is read for some subject S “it is known that” and  is read “it is believed that”. Some modal axioms for epistemic logic that have been considered include:

(**K**) □(P → Q) → (□P → □Q) *logical omniscience*

(**T**) □P → P *veridicality*

(**4**) □P → □□P *positive introspection*

(**E**) ~□P → □~□P *negative introspection*

The axiom (**K**) expresses logical omniscience insofar as this axiom requires that the knowledge of an agent is closed under *modus ponens*; and hence such knowers know all the logical consequences of their knowledge.

Axiom (**T**) states the truism that whatever is known is true. Notice that this axiom would be too strong for deontic logic insofar as an action’s being obligatory doesn’t imply that the agent actually performs that action.

Axiom (**4**) expresses a high degree of *positive introspective knowledge*: if someone knows P, then she knows that she knows that P. Axiom (**E**)m on the o ther hand, expresses a high degree of *negative* introspective knowledge: if someone doesn’t know that P, then he knows he doesn’t know P. This axiom is contrary to the experience of Socrates: as the gadfly of Athens, Socrates found through his questioning that many of his fellow Athenians did not know what they were talking about but also didn’t know that they didn’t know. The gadfly of Athens believed his vocation wsa to sting his fellow Athenians into the awareness they were own ignorance, a service for which they did not always show adequate appreciation.

The *temporal logic* or Diodorian temporal logic was studied by the logician A. N. Prior (1914 – 1969). To model temporal language, we introduce a pair of modal operators for the future and a pair of modal operators for the past.

□ It *is always going* [i.e., in *all* futures] *to be the case that*

 It *will* [i.e., in *some* future] be the case that

■ It *has always been* [i.e., in *all* pasts] the case that

¿ It *was once* [i.e., in *some* past or *“once upon a time”*] the case that

The axioms for minimal tense logic include version of Axiom (**K**) for the two necessity operators:

(**K**■) ■(ϕ → ψ) → (■ϕ → ■ψ) *Whatever has always followed from what always has been,*

*always has been.*

(**K**□) □(ϕ → ψ) → (□ϕ → □ψ) *Whatever will always follow from what always will be,*

*always will be*

It also contains two axioms concerning the interaction of the past and future that have the form of the so-called Brouwersche axiom with alternating valences:

(B□¿) ϕ → □¿ϕ *What is, will always have been*

(B■) ϕ → ■ϕ *What is, has always been going to happen*

The Brouwersche (**B**) axiom was so-named by the logician Oskar Becker (*Zur Logik der Modalitäten* [1930] translated into English in 2022) after the charismatic Dutch mathematician L. E. J. Brouwer (1881 – 1966), who championed a philosophy of mathematics known as *intuitionism*. It happens that when the  can be paraphrased as ~~, the resulting axiom has the form of the acceptable form of double negation in intuitionistic logic:

(**B**) ϕ → ~~ϕ

According to intuitionism, mathematical objects do not exist as eternal Platonic objects but are constructions in intuition. Intuitionists read the propositional connectives as involving not merely truth, but proof, and so they rejected such classical forms of reasoning as *reductio ad absurdum* and theorems such as the *law of excluded middle.* Intuitionists reject the following mathematical proof.

Classical Theorem: There exists two irrational numbers *x* and *y* such that *xy* is rational.

*Proof*: Consider = = = 2, which is rational.

Either is rational or irrational.

If is rational, then *x* = *y* = are both irrational and *xy* is rational.

If is irrational, then *x* = and *y* = are both irrational and *xy* is rational.

Either way there exists *x* and *y* such *xy* is rational.

Intuitionists reject this classical argument by separation of cases because it does not actually construct numbers *x* and *y* such that *xy* is irrational. The idea behind intuitionistic logic is that the connectives are reinterpreted as involving a kind of provability.

Around the 1970s it was noticed that the famous incompleteness theorems of Gödel (1931) were propositional in character and that their logic could be captured in propositional modal logic. These modal provability logics added to Axiom (**K**) the following axiom known as the Gödel-Löb axiom or also the well-ordering axiom.

(**W**) □(□ϕ → ϕ) → □ϕ .

*Modal Provability Logics* proliferated from the 1950s -1970s, but the genesis of the idea goes back to a short note of Gödel’s [1933] in which he noted that intuitionistic truth defined in terms of proof since provability is a kind of necessity. The above axiom can be read as a kind of soundness theorem.

*if it is provable thatϕ being provable implies it is true, then ϕ is provable.*

Later we will show use a modal provability logic to exhibit the propositional logic of key parts of Gödel’s First and Second Incompleteness Theorems.

In contemporary logic, modal logic has grown beyond these philosophical origins and is at the interface of a number of disciplines including the studies information flow and dynamics, game-theory, and computability.

### Exercises

1. Symbolize the following modal arguments.
2. Eratosthenes must either be in Syene or Alexandria. Eratosthenes cannot be in Syene. Therefore, Eratosthenes must be in Alexandria.
3. Assume that justice can be defined as paying your debts and telling the truth. Then it is *morally obligatory* for Cephalus to comply to a madman’s request that Cephalus return a borrowed sword and that Cephalus to tell the truth about the whereabouts of a friend whom the madman wants to kill with the sword. However, if this act is *morally obligatory*, then it is *morally permissible*. However, it is *morally impermissible* (or *morally forbidden*) for Cephalus to comply. So it isn’t *morally obligatory* for Cephalus to comply. Justice, therefore, cannot be defined as paying your debts and telling the truth.
4. It is conceivable that I am having experiences qualitatively identical to those I am having now on the supposition that I am being deceived by an evil genius. If that is conceivable, then I do not have indubitable knowledge that the external world exists.

2. Johan van Bentham [2010, p. 12] was asked to symbolize the philosophical claim that “nothing is absolutely relative”. He came up with the following:

~□(ϕ ∧ ~ϕ) .

Use familiar equivalences from propositional logic and the modal negation laws to show that this symbolization is equivalent to

(*McKinsey’s Axiom*) □ϕ → □ϕ .

3. Match the following symbolizations with the best corresponding translation below.

|  |  |
| --- | --- |
| ¿■P | *It was always the case that it will sometime be the case that* P |
| □P → P | *It will sometime be the case that it was once the case that* P |
| ■P | *Whatever will always be, will be* |
| ¿P | *Once upon a time, it was always the case that* P |

**2. The Symbolic Language for Modal Propositional Logic.**

So far we have been using the new modal operators informally, but we will now formally characterize our symbolic language. The *symbolic language* with which we shall now deal is that of chapter II with ‘□’ and ‘’ added to its symbols as modal operators.

We enrich the class of symbolic sentences to include sentences that are the result of prefixing a symbolic sentence by a modal operator. Thus expression such as

□ ~ P

~ (P ∧ Q)

□(P → □P)

will be counted among the symbolic sentence of our language.

To be more explicit, the class of symbolic sentences is exhaustively characterized as follows:

1. *Sentence letters are symbolic sentences.*
2. *If ϕ and ψ are symbolic sentences, then so are*: ~ ϕ , (ϕ → ψ) , (ϕ ∧ ψ) , (ϕ ∨ ψ) , and (ϕ ↔ ψ) .
3. *If ϕ is a symbolic sentence, then so are:*

□ϕ

*and*

ϕ .

The sentences obtained by clause (3) are *modal sentences* and we shall call the occurrence of ϕ the *scope* of the modal operator.

Symbolic sentences can be parsed into *grammatical trees* to systematically display all the sub-sentences involved it its construction. Here, in addition to the kinds of nodes of Chapter II, non-branching nodes are also reached by clauses (3). For example, the symbolic sentence ‘(□(□P → Q) → ~~R)’ can be parsed into the following grammatical tree.

(□(□P → Q) → ~~R)

□(□P→ Q) ~~ R

(□P→ Q) ~ R

□P Q ~ R

P R

Figure 2 A Grammatical Tree for a Modal Proposition

Parsing a symbolic sentence into its grammatical tree provides a precise description of its grammatical structure. The above symbolic sentence is a conditional whose antecedent is the sentence formed by prefixing the necessity operator to a conditional whose antecedent comes from prefixing the necessity operator to ‘P’ and whose consequent is ‘Q’. The consequent of the main conditional is a negation of a sentence that is the result of prefixing the possibility operator to the negation of ‘R’.

We shall continue to use the informal conventions of chapters I and II for omitting outer parentheses, replacing pairs of matching parentheses with matching pairs of square brackets, and for resolving questions of scope among the various connectives and operators. Now, in addition, the modal operators ‘□’ and ‘’ and ‘~’ will, by convention, have the narrowest scope; ‘∧’, and ‘∨’ will have narrower scope than ‘→’ and ‘↔’, and, as before, repeated conjunctions or repeated disjunctions are grouped by association to the left. For example,

□(~P ∧ Q) → [(~R ∨ ~□S ∨ T) ∧ ~□P ↔ ◊Q ∧ ~(□R ∧ S ∧ □~T)]

becomes officially

(□(~P ∧ Q) → ((((~R ∨ ~□S) ∨ T) ∧ ~□P) ↔ (Q ∧ ~((□R ∧ S) ∧ □~T)))) .

# 2. Translation and Symbolization.

The English sentence

* 1. It is necessary that God exists

has a number of stylistic variants in English:

Necessarily, God exists ,

It is necessary for God to exist ,

God must exist ,

God has to exist .

Similarly, the sentence

* 1. It is possible that God exists

may also be expressed with a number of stylistic variants:

Possibly, God exists ,

It is possible for God to exist ,

God can exist ,

God might exist .

Using the modal operators with the sentential connectives of chapters I and II, we can develop stylistic variants of a number of English expressions. For instance, the expression

It is impossible that God exists

may be paraphrased as either

It is *necessary* that God *fails* to exist ,

or

It is *not possible* that God exists .

Modal operators, when they interact with other operators and connectives, can create *scope ambiguities* such that that between the necessity of the consequence and the necessity of the consequent, which we discussed in the previous section.

### Exercises

1. Symbolize the following sets of sentences using the given the scheme of abbreviation. If a sentence is ambiguous briefly explain the different possible readings.

P: Plato is rational. R: Theatetus is a cyclist.

Q: Plato is risible. S: Theatetus is bipedal.

1. It is possible that Plato is rational, but it is not necessary that Plato is rational.

It is impossible that Plato is not rational.

It is contingently true that Plato is rational.

It is neither contingently true nor contingently false that Plato is rational.

1. Necessarily, if Plato is risible, then Plato is rational.

If Plato is risible, then it is necessary that Plato is rational.

If it is possible that Plato is risible, then it is possible that Plato is rational.

If Plato is risible, then it is necessarily possible that Plato is rational.

It is impossible for Plato to be risible if he is not rational.

1. Theatetus could be a cyclist provided he is bipedal.

It is possible that Theatetus is both a cyclist and bipedal.

It is impossible that Theatetus is a cyclist if he is not bipedal.

If it is possible for Theatetus to be cyclist and necessarily if he is a cyclist then he is bipedal, then it is compatible that Theatetus is cyclist and bipedal.

1. Fill in the following chart by finding expressions using only the connective and operators at the top of the chart equivalent to the expressions in the first column.

* A sentence P is *contingent* (in symbols, ∇P) if it is neither necessary nor impossible.
* A sentence P *strictly implies* Q (in symbols, P ⇒ Q) if it is impossible for P to be true while Q is false.
* P and Q are *compatible* if ◊(P ∧ Q).
* P and Q are *incompatible* if ~ (P ∧ Q).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | { ~, □} | {~, } | {~, ⇒} | {~, ∇} |
| P is *necessarily true* |  |  |  |  |
| P is *impossible* |  |  |  |  |
| P is *possibly true* |  |  |  |  |
| P is *possibly false* |  |  |  |  |
| P is *contingently true* |  |  |  |  |
| P is *contingently false* |  |  |  |  |
| P is *contingent* |  |  |  |  |
| P *strictly implies* Q |  |  |  |  |
| P is *compatible* with Q |  |  |  |  |
| P is *incompatible* with Q |  |  |  |  |

1. Use the distinction between the necessity of the consequence and the necessity of the consequent to explain why someone might accept Axiom (**B)** ϕ → □ϕ as intuitively obvious while rejecting its logically equivalent form Axiom (**B**) □ϕ → ϕ. What are the two way in which one might interpret “if ϕ then necessarily it is possible that ϕ”?

# 3. Axioms for Modal Logics.

The logical relations in the previous section capture what is common to the many of the various interpretations for necessity. The distinctive notions of necessity discussed initially can be characterized by formulating distinctive modal axioms. For example, the axiom

□P → P

says that whatever is necessary is actually the case. This axiom characterizes what some philosophers have called ‘tautological necessity.’ This axiom would hold for temporal logics since whatever is, was, or will be the case, is the case. This axiom, known as the (**T**) axiom, is named after a system of modal logic studied by Gödel, Feys, and von Wright in which *tautologies* are necessary truths.

A noted above, the (**T**) axiom is too strong for deontic logics. George Washington, should he fail to live up to his reputation for veracity, might be morally obligated to tell the truth but fail to do so. It is a sad reality of moral life that obligations remain unfulfilled. However, whatever is morally obligatory is at least morally permissible. The (**D**) axiom (named for *deontic logic*) weakens the (**T**) axiom and states that what is necessary is possible.

□P → P .

Let’s consider another axiom, known as the (**4**) axiom (named after the strict implication system **S4** studied by C. I. Lewis). Jaakko Hintikka developed an epistemic interpretation using the (**4**) axiom:

□P → □□P .

This interpretation says that if an individual knows that P, then that individual knows that s/he knows that P. Axiom (**4)** has also been used to characterize systems of temporal logic in which *what is always going to be true is always always going to be true.*

Another axiom has been championed by philosophers such as Alvin Plantinga, who claims that the notion of “broadly logical necessity” is best captured by axiom (**5)** (named after C. I. Lewis’ fifth axiom for strict implication):

P → □P .

Whereas Axiom **4** states that necessary truths are necessary, Axiom **5** extends this necessity to the modality of possibility: if P is possible, then it is necessary that P is possible. Intuitively, axioms **4** and **5**, together, require that the modal status of a truth is not accidental, but the same in all possible world.

As noted above, the (**B**) axiom is syntactically similar to the form of double negation valid in intuitionistic logic, the law of double negation introduction:

P → ~~P .

By the logical equivalences noted earlier, this axiom can also be expressed by the more familiar form:

P → □P .

The (**B**) axiom also plays a role in modelling linear time as noted earlier. Ancient philosophers like Aristotle wrote about modal logic, and the medieval logicians such as Jean Buridan (c. 1301 – c. 1359/62)discussed modal notions with great subtlety. However, the modern treatment of modal logic began around 1912 when C. I. Lewis, after reading Russell and Whitehead’s *Principia* *Mathematica*, proposed various axioms (such as **4** and **5**) to find a connective more suitable than the material conditional to express our informal concept of entailment. In the 1930s Gödel discussed an interpretation of the modal operator as ‘it is provable in system M that,’ and in the 1950s *epistemic*, *doxastic*, *deontic* and *tense-logic* interpretations were intensively investigated.

Around the 1960s modal logic was being investigated by numerous logicians—including, among others, Rudolf Carnap, Jaakko Hintikka, Richard Montague, and von Wright. However, it was Saul Kripke’s remarkable paper *A Completeness Theorem in Modal Logic* (1961) published while he was still a high school student in Nebraska that presented possible world semantics in an elegant way that ignited decades of logical and philosophical research. Kripke’s possible world semantics gave formal expression to the Leibnizian idea that a proposition is *necessary* if it is true in *all* possible worlds, possible if it is true in *some* possible world, and *contingent* if it is neither necessary nor impossible.

The above axioms turned out to be valid in modal systems corresponding to some natural conditions on the relation of modal accessibility or relative possibility:

|  |  |  |  |
| --- | --- | --- | --- |
| Modal System | Characteristic Axiom | Named For | Accessibility Condition |
| System **D** | □P → P | **D**eontic Logic | R is *serial* |
| System **T** | □P → P | Modal systems modeling **T**autologies | R is *reflexive* |
| System **B** | P → □P | Named for **B**rouwer | R is *symmetric* |
| System **S4** | □P → □□P | Lewis’ Axiom **4** for Strict Implication | R is *transitive* |
| System **S5** | ◊P → □P | Lewis’ Axiom **5** for Strict Implication | R is *euclidean* |

Normal systems of modal logic are those in which the laws of logic, such as *modus ponens*, are assumed to hold. Axiom (**K**), which characterizes such normal systems, is named after Kripke:

□(P → Q) → (□P → □Q) .

In this chapter we will develop natural deduction systems for propositional modal logic and set forth the possible world semantics for them. In a subsequent chapter we’ll set forth these further axioms and the Provability Logic that enables us to capture the propositional modal logic structure of Gödel’s Incompleteness Theorems.

Exercises

1. Construct a diagram to chart these notions in logical space and give English examples.

(A) The realm of the *necessary*, i.e., propositions are necessarily true or necessarily false).

(B) The realm of the *contingent*, i.e., propositions which are neither necessary nor impossible, or alternatively, propositions that are either contingently true or contingently false).

(C) The realm of the *possible*, i.e., propositions are not impossible, or alternatively, necessarily true or contingent).

(D) The realm of the *true*, i.e., propositions that are necessarily true or contingently true.

(E) The realm of the *false*, i.e., propositions that are either impossible or contingently false.

2. Find succinct expressions using only ‘□’ and “~’ for each of the following notions:

|  |  |
| --- | --- |
| P is *contingently true* |  |
| P is *contingently false* |  |
| P is *contingent* |  |
| P and Q are *incompatible* |  |
| P and Q are *compatible* |  |
| P *strictly implies* Q |  |

3. Find succinct equivalent expressions using only ‘’ and ‘~’ and the relevant sentence letters.

|  |  |  |
| --- | --- | --- |
| D | □P → P |  |
| **T** | □P → P |  |
| **B** | P → □P |  |
| **4** | □P → □□P |  |
| **5** | ◊P → □P |  |

1. Discuss whether the following pairs of axioms are intuitively acceptable in the logics specified.

(A) P ∧ Q → (P ∧ Q); (P ∨ Q) → (P ∨ Q) in the modal logic of physical possibility.

(B) □ϕ → ϕ ; □(□ϕ → ϕ) in Deontic Logic.

(C) ϕ → □⧫ϕ ; ϕ → ■ϕ n Temporal Logic.

(D) □ϕ → □□ϕ ; □ϕ → ϕ in Epistemic Logic.

(E) ϕ → □ϕ ; ϕ → ϕ in the Logic of Conceivability.

5. Consider the following axioms for temporal logic:

(4) □P → □□P (*transitive*)

(4) P →P (*dense*)

(Hamblin) P → ◊□(P ∨P) (*discrete*)

Match the above with the following informal translations:

*“if P is the case, then it will be the case that it was always the case that either P is or P will be the case”*

*“if it will be the case that P, then it will, in between, be that it will be P”.*

*“if it will always be the case that P, then it will always be the case that it will always be the case that P”*

6. Express the following generalities as axioms in the language of propositional modal logic.

(A) Whatever follows from logically necessary propositions is logically necessary.

Whatever is logically implied by what is possible is possible.

(B) Whatever is necessary is possible.

Whatever is impossible might be false.

(C) Whatever is necessary is necessarily necessary.

Whatever is possibly possible is possible.

(D) Whatever is possible is necessarily possible.

Whatever is possibly necessary is necessary.

(E) Whatever is possible is necessarily possible.

Whatever is possibly necessary is necessary.

7. Interpret the modal operators in different ways to formulate modal axioms for the following.

(A) Whatever is physically necessary is actual.

Whatever isn’t actual isn’t physically necessary.

(B) Whatever is obligatory is permissible.

Whatever is impermissible is not obligatory.

(C) Whatever will always be the case will always be the case at all future times.

Whatever was possible with regard to a possible past is possible with regard to the past.

1. Whatever is known is true.

Whatever isn’t true couldn’t possibly be known.

(E) Whatever is conceivable is conceivably conceivable.

Whatever is conceivable conceptual necessary is conceptually necessary.

8. Translate the following theorems into English statements of modal principles.

1. ~□P → □(P → Q)

□Q → □(P → Q)

□(P → Q) ∨ □(Q → P)

1. □(P → Q) → (□P → □Q)

□(P → Q) → (P → Q)

□(P → Q) ↔ ~(P ∧ ~ Q)

1. ∇P ↔ ~ (□P ∨ ~P)

∇P ↔ ∇~P

□P ↔ (P ∧ ~∇P)

# 4. Modal Inference Rules.

To sentential logic of chapter II, we shall add new rules of inference to our original stock and state some derived rules that will be justified later. The new rules are intuitively analogous to rules from quantifier logic. This parallelism is not accidental since the semantics for sentential modal logic can be given in terms of quantification over possible worlds.

The modal sentences at opposite ends of the diagonals of the square are *contradictories*. These logical relations are captured in the rules of *modal negation* [MN].

~□ϕ ~ϕ ~ϕ □~ϕ

~ϕ ~□ϕ □~ϕ ~ϕ

The modal sentences at the top of the square are not contradictories, but *contraries*, a sentence cannot be both necessary and impossible, but it may be neither. A sentence that is neither necessary nor impossible is *contingent*. Contingency is the opposite of necessity. A sentence is *contingently true* if it happens to be true but could have been false, and a sentence is *contingently false* if it happens to be false but could have been true. The realm of the possible is everything other than the impossible, and so includes what is contingently true, contingently false, and necessarily true.

The analogy to quantification also implies corresponding laws of *modal distribution*:

□(P ∧ Q) ↔ (□P ∧ □Q)

(P ∨ Q) ↔ (P ∨ Q)

The rule of *necessity instantiation* [NI] is analogous to an application of the rule of universal instantiation.[[3]](#footnote-3)

∀α ϕ □ϕ

ϕ ϕ

Here the rule of NI allows us to drop an initial box. Intuitively, this rule says that whatever is necessary is also the case. The rule of *Possibility Generalization* [PG] is a derived rule analogous to existential generalization:

ϕ ϕ

∃αϕ ϕ

Here the rule of PG allows us to infer ‘ϕ’ from ϕ. Intuitively, this rule says that whatever is the case is also possibly the case. Note that PG follows from NI and the third form of MN.

### Exercises

1. Which of the following inferences come by an application of NI, PG, or MN? Briefly explain.

(A) □(P → Q) □(P → □Q) □P ∧ □Q □P → □Q □ P

P → Q P → Q P ∧ □Q □P → Q ◊◊ P

(B) □(P → Q) □(P → □Q) □P ∧ □Q\_\_ □◊P → □Q\_\_\_ □P\_\_

◊□(P → Q) □◊(P → □Q) □P ∧□Q □◊P → □Q □P

(C) ~ □(P ∨ Q) ~ (P → Q) □(P → ~◊~P) □ ~(P ↔ Q) ~ □□P

 ~ (P ∨ Q) □P → Q □(P → □P) ~ (P ↔Q) □~P

1. Verify that PG can be derived from NI and the third form of MN.
2. Using the notion that necessity is truth in *all* possible worlds and possibility is truth in *some* possible world, explain the analogies between the inference rules and forms of derivation in modal logic and the quantifier inference rules UI, EI, EG, and QN and the form of derivation UD.

# 5. Natural Deduction for (T), (B), (S4), and (S5).

In addition to the inference rules NI and MN, we add two form of strict derivation.

The strict form of derivation known as *necessity derivation* [ND] appears as follows:

*Show* □ϕ

χ1

.

.

.

χm

where the scope ϕ occurs unboxed among χ1 through χm.

In a *necessity derivation* for *Show* □ϕ, one derives ϕ using premises or antecedent lines that are themselves necessary. To ensure that this form of strict derivation only allows us to derive necessary truths, we must restrict the lines that may be imported into the strict derivation.

To understand these restrictions, we first supplement the notion of an *antecedent* line from propositional logic to obtain the notion of an *accessible* line for modal logic. An *antecedent* *line*, as it is defined for propositional logic, is a preceding line that is neither boxed nor contains uncancelled ‘*Show’*. An *accessible line* is defined as an antecedent line such that there is at most one intervening modal ‘*Show’* line, that is, a ‘*Show’* line of one of the forms

*Show* □ϕ

or

*Show* ϕ .

The basic restrictions on for a strict derivation are that none of the lines χ1 through χn are (*i*) premises or (*ii*) comes by an application of an inference rule to an inaccessible line, or (*iii*) comes by an application of an inference rule other than an admissible *strict importation rule* to an accessible line.

Next we formulate the *admissible* *strict importation rules* for the various systems of modal logic. These strict importation rules can be elegantly formulated as restrictions on the rule of repetition.

1. The characteristic strict importation rule for **T** is NI: when importing an accessible sentence of the form ‘□ϕ’ into a strict derivation, we must drop the initial ‘□’.
2. The characteristic strict importation rule for **B** is PG: when importing an accessible line into a strict derivation, we must prefix a ‘◊’.
3. The characteristic strict importation rule for **S4** is □R: when importing an antecedent line into a strict derivation, you may apply the rule of repetition R to an accessible line of the form ‘□ϕ’.
4. The characteristic strict importation rule for **S5** is R: when importing an antecedent line into a strict derivation, you may apply the rule of repetition R to an accessible line of the form ‘◊ϕ’.

Each of the modal systems **B**, **S4**, and **S5** has NI as a strict importation rule in addition to its characteristic strict importation rule. In addition, **S5** also has the strict importation rules for **B** and for **S4**, namely, PG and □R. (The annotation for these strict importation rules will be the line number of the sentence involved and either ‘NI**’**, ‘PG**’**, ‘□R**’**, or ‘R**’**.)

The second form of strict derivation is *possibility derivation* [Annotation: PD]. If we have a ‘*Show’* line of the form

*Show* ϕ ,

and there is some accessible line ◊ψ, then on the next line immediately after th*e ‘Show’* line one may enter

ψ ,

as an assumption for *possibility derivation* [Annotation: Assume(PD)]. The restrictions for PD are the same as those for the first form of strict derivation: none of the lines χ1 through χn are premises or comes by an application of an inference rule to an inaccessible line, or comes by an application of an inference rule other than an admissible strict importation rule to an accessible line.

Intuitively, *possibility derivation* allows us to Show ψ by deriving ψ using necessary truths from the assumption that ϕ, where ◊ϕ is an accessible line. The subsidiary derivation can be thought of as confined to a particular possible world in which ϕ is assumed to be true. The intuitive basis for this form of strict derivation is captured by T502:

□(P → Q) → (P → Q) ,

which states that whatever is logically entailed by something possible it itself possible.

The foregoing remarks on derivations constitute only an informal introduction. The following is an explicit set of directions for constructing a derivation.

1. *If ϕ is a symbolic sentence, then*

*Show* ϕ

*may occur as a line.*

1. *Any one of the premises may occur as a line.* [*Annotation:* ‘Premise’]
2. *If* ϕ *and* ψ *are symbolic sentences such that*

*Show* (ϕ → ψ)

*occurs as a line, then* ϕ *may occur as the next line.* [*Annotation:* ‘Assume (CD)’*.*]

1. *If ϕ is a symbolic sentence such that*

*Show* ϕ

*occurs as a line, then*

~ϕ

*may occur as the next line; if ϕ is a symbolic sentence such that*

*Show* ~ϕ

*occurs as a line, then*

ϕ

*may occur as the next line.* [*Annotation:* ‘Assume (ID)’*.*]

1. *If ϕ is a symbolic sentence such that*

*Show* ϕ

*occurs as a line, then*

ψ

*may occur as the next line provided that* ◊ψ *is an accessible line, that is, a preceding line that is neither boxed nor contains uncancelled ‘Show’ and is such that there is at most one intervening ‘Show’ line of the form*

*Show* □ϕ

*or*

*Show* ϕ *.*

[*Annotation: the line number of the modal sentence involved and ‘*Assume (PD)*’.*]

1. *A symbolic sentence may occur as a line if it follows by an inference rule* 
   * 1. *of sentential logic applied to antecedent lines, that is, preceding lines which neither are boxed nor contain uncancelled ‘Show’* [*here the annotation should refer to line numbers of the preceding lines involved and the inference rule employed*]; *or*
     2. *of modal logic applied to accessible lines that is, antecedent lines for which there is at most one intervening modal ‘Show’ line* [*here the annotation should refer to line numbers of the preceding lines involved and the inference rule employed*].
2. *When the following arrangement of lines has appeared:*

*Show* ϕ

χ*1*

.

.

.

χ*n*

*where none of the* χ1*through* χ*n contains uncancelled ‘Show’ and either*

1. *the conclusion to be shown* ϕ *occurs unboxed among* χ1 *through* χ*n,and neither* χ1 *nor* ϕ *was introduced by an assumption for Possibility Derivation,*
2. ϕ *is of the form*

(ψ*1* → ψ*2*)

*and the consequent* ψ*2 occurs unboxed among occurs unboxed among* χ*1 through* χ*n and* χ*1 but was not introduced by an assumption for Possibility Derivation,*

1. *for some sentence* χ*, both* χ *and its negation occur unboxed among* χ1 *through* χ*n and neither of the contradictory sentences was introduced by an assumption for Possibility Derivation,*
2. ϕ *is of the form*

□ψ

*or*

ψ

*and* ψ *occurs unboxed among* χ1 *through* χ*n  and none of* χ1 *through* χ*n are premises or comes by an application of an inference rule to an inaccessible line, or by an application of an inference rule other than an admissible strict importation rule to an accessible line antecedent to the displayed occurrence of*

*Show* ϕ *,*

*then one may simultaneously cancel the displayed occurrence of ‘Show’ and box all subsequent lines.* [*When we say that a symbolic sentence* ϕ *occurs among certain lines, we mean that one of those lines is either* ϕ *or* ϕ *preceded by ‘Show’. Furthermore, annotations for clause (7), parts (i), (ii), (iii),and (iv) are, respectively,* ‘DD’, ‘CD, ‘ID’*, and either* ‘ND’ or ‘PD’, *to be entered together with the relevant line numbers justifying the cancelled ‘Show’.*]

A derivation is said to be *complete* if each of its lines is either boxed or contains cancelled ‘*Show’*. A symbolic sentence ϕ is said to be *derivable* from given symbolic sentences in *modal system* M if, by using only clauses (1) - (6) and the strict importation rules for M, a complete derivation from those premises can be constructed in which

*~~Show~~* ϕ

occurs as an unboxed line.

Applications of the above clauses are familiar except for clauses (5), (6) part (ii), and (7).

We can illustrate clause 6, part (ii), by verifying the claim made above that *Possibility Generalization* [PG] can be derived from Necessity Instantiation and the third form of Modal Negation.

P ∴ P

1. *~~Show~~* P 3, 5 ID
2. ~P Assume (ID)
3. P Premise
4. □~P 2, MN
5. ~P 4, NI

An annotated derivation in modal system **T** will illustrate a strict derivation corresponding to the form of derivation in quantificational logic known as *universal derivation*. We will construct a derivation for the (**K**) axiom:

□(P → Q) → (□P → □Q) .

To use the modal premises properly one must be careful to employ the accompanying strict importation rules associated with the various modal systems.

1. *~~Show~~* □(P → Q) → (□P → □Q)
2. □(P → Q)
3. *~~Show~~* □P → □Q
4. □P
5. *~~Show~~* □Q

To complete the strict (necessity) derivation began in line 5, we use two applications of NI, the strict importation rule for **T**, and then apply MP.

1. *~~Show~~* □(P → Q) → (□P → □Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* □P → □Q 5, CD
4. □P Assume (CD)
5. *~~Show~~* □Q 8, ND
6. P → Q 2, NI
7. P 4, NI
8. Q 6, 7 MP

We may box and cancel line 4 by strict derivation because we have derived the *scope* of the modal symbolic sentence in line 5 and lines 6 - 8 have only come from accessible lines by the strict importation rule for **T**.

Intuitively, the first form of strict derivation *necessity derivation* amounts to showing ‘□Q’ by deriving ‘Q’ in an arbitrary possible world in the box surrounding lines 6 - 8. The restriction on strict derivation here is that we are only allowed to enter the scope ϕ of accessible lines of the form ‘□ϕ’, that is, the lines entered into the strict derivation in lines 5-7 are sentences stripped of an initial ‘□’. If we can derive ‘Q’ for this arbitrary possible world, we are justified in concluding that ‘Q’ holds in all possible worlds.

The second form of strict derivation is *possibility derivation* [Annotation: PD]. The initial ‘*Show’* line is

*Show* ϕ .

With possibility derivation, if ◊ψ occurs as an accessible line, then on the next line one may enter

ψ

as an assumption for *possibility derivation* [Annotation: Assume (PD)]. The restrictions are the same as those for the first form of strict derivation: none of the lines χ1 through χn are premises or comes by an application of an inference rule to an inaccessible line, or comes by an application of an inference rule other than an admissible strict importation rule to an accessible line. Intuitively, this form of strict derivation allows us to show a sentence is possibly true by deriving it using necessary truths from a sentence that is itself possible.

The intuitive basis for PD is modal theorem T502 □(P → Q) → (P → Q), which states that whatever is logically entailed by something possible it itself possible.

We may prove this theorem T502, which plays a prominent role in modal ontological arguments, to illustrate PD.

1. *~~Show~~* □(P → Q) → (P → Q)
2. □(P → Q) Assume (CD)
3. *~~Show~~* P → Q
4. ◊P Assume (CD)
5. *~~Show~~* ◊Q
6. P Assume (PD)

Line 6 comes from ‘◊P’ in the modally accessible line 4 by possibility instantiation, which is analogous to existential instantiation in quantifier logic in which we have the restriction that which we must instantiate to a variable that is completely new to the derivation. Here, in contrast to the previous example, the derivation of ‘Q’ is in a *particular*, rather than an *arbitrary*, possible world, namely, a world in which ‘P’ is assumed to be true. Therefore, unlike the necessity derivation in the previous example, since we begin with an assumption for PD for a *particular* possible world, we are only justified in concluding ‘◊Q’ by a strict *possibility* derivation.

1. *~~Show~~* □(P → Q) → (P → Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* P → Q 5, CD
4. P Assume (CD)
5. *~~Show~~* Q 8, PD
6. P 4, Assume (PD)
7. P → Q 2, NI
8. Q 7, 6 MP

Readers are encouraged to check their grasp of the ideas in this example by annotating and boxing and cancelling the derivation for

(P ∨ Q) ∴ ~P → Q .

1. *Show* ~P → Q

2. ~P

3. □~P

4. (P ∨ Q)

5. *Show* Q

6. (P  Q)

7. ~P

8. Q

Next we illustrate the new strict importation rule PG which supplements the strict importation rule NI in the modal system **B**. We shall construct a derivation for the Anselmian modal ontological argument discussed in the introductory section above. Letting ‘G’ abbreviate the sentence ‘God exists’, we may symbolize the Anselmian argument as follows:

□(G → □G) G ∴ □G

T570 P → □P Axiom (B)

1. *~~Show~~* P → □P 3, CD

1. P Assume (CD)
2. *~~Show~~* □P 4, ND
3. P 2, PG

T571 □P → P

T571, the dual form of (**B**), comes from T570 by contraposition and a generalized modal negation.

1. *~~Show~~* □G 11, DD
2. □(G → □G) Premise
3.  G Premise
4. *~~Show~~* □G 7, PD
5. G 3, Assume (PD)
6. G → □G 2, NI
7. □G 5, 6 MP
8. □G → G T571
9. G 4, 8, MP
10. G → □G 2, NI
11. □G 9, 10 MP

We suggest that the reader test her knowledge of the modal system **B** by annotating the following version of the above Anselmian argument:

□(G → ~~G) . G ∴ G

We shall construct a *B* derivation of the conclusion from its premises. We begin the derivation indirectly and then enter a critical modal ‘*Show’* line for strict derivation.

1. *~~Show~~* G 5, 12 ID
2. ~G Assume (ID)
3. □(G → ~~G) Premise
4. G Premise
5. *~~Show~~* ~~G 8, PD
6. G 4, Assume (PG)
7. G → ~~G 3, NI
8. ~◊~G 6, 7 MP
9. *~~Show~~* □~~~G 11, ND
10. ~G 2, PG
11. ~ ~~G 10, DN
12. ~~~G 9, MN

To illustrate the strict importation rule for the modal system **S4**, we construct a derivation for

∴ □(P → Q) → □(□P → □Q) .

To show the theorem we assume its antecedent and show its consequent, which is a modal sentence. To show the consequent, therefore, we commence a strict derivation. In order to maintain the accessibility of the initial assumption, which is a necessity, we apply the rule of □R obtain line 4 before entering the next ‘*Show’* line in line 5 for the scope of modal sentence in line 3.

* 1. *Show* □(P → Q) → □(□P → □Q)
  2. □(P → Q) Assume (CD)
  3. *Show* □(□P → □Q)
  4. □(P → Q) 2, □R
  5. *Show* □P → □Q

We then proceed by CD until we reach the consequent, which again, is a modal sentence.

* 1. *Show* □(P → Q) → □(□P → □Q)
  2. □(P → Q) Assume (CD)
  3. *Show* □(□P → □Q)
  4. □(P → Q) 2, R
  5. *Show* □P → □Q
  6. □P Assume (CD)
  7. *Show* □Q

We can now complete the derivation by applying NI to lines 4 and 6. Note that the ‘*Show’* line in 7 is the only intervening modal ‘*Show’* after line 2 since the ‘*Show’* line in line 5 is for conditional derivation.

1. *~~Show~~* □(P → Q) → □(□P → □Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* □(□P → □Q) 5, ND
4. □(P → Q) 2, □R
5. *~~Show~~* □P → □Q 8, CD
6. □P Assume (CD)
7. □(P → Q) 4, □R
8. *~~Show~~* □Q 11, ND
9. P → Q 7, NI
10. P 6, NI
11. Q 9, 10 MP

Technically, in applying □R we can pass over only one uncancelled modal ‘*Show’* at a time—that is, a ‘*Show’* of the form ‘*Show* □ϕ’ or ‘*Show* ϕ’. Notice, however, with repeated applications of □R, we can, in effect, pass over any number of modal ‘*Show’* lines for strict derivations.

This follows from the fact that the characteristic axiom for **S4** allows us to prefix any finite number of ‘□’s to a necessary sentence. In order to eliminate repetitious use of □R, we shall allow □R to be applied across *n*-nested strict derivations (annotation □R*n*). We may use this short-cut in our previous example,

1. *~~Show~~* □(P → Q) → □(□P → □Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* □(□P → □Q) 4, ND
4. *~~Show~~* □P → □Q 6, CD
5. □P Assume (CD)
6. *~~Show~~* □Q 11, ND
7. □(P → Q) 2, □R2
8. P 5, NI
9. P → Q 7, NI
10. Q 9, 8 MP

Intuitively, we may derive necessary truths only from premises which are themselves necessary. Syntactically, this condition is satisfied if we require that the lines used to show a necessary statement are themselves either necessary or follow from necessary statements by the laws of propositional logic and modal negation. This requirement is met by a restriction on importation: whenever we import a sentence into a strict derivation by crossing over an uncancelled ‘*Show*’ for a strict derivation, we must drop an initial ‘□’. When we have a series of nested strict derivations, we may apply the rule of □R across one strict derivation ‘*Show*’ line at a time, or we may generalize the rule of □R to allow for importation across multiple nested strict derivations as illustrated above.

Lastly, we wish to construct a derivation in **S5** that illustrates the strict importation rule R.

Alvin Plantinga’s modal version of an ontological argument is based on the intuition that God exists if and only if God is the greatest possible being. If God is the greatest possible being, Plantinga argues, then God has the greatest form of existence possible, namely, necessary existence. Using the resources of propositional modal logic, we shall formulate God’s having necessary existence as the proposition that ‘it is necessary that God exists’. Then Plantinga’s modal ontological argument can be stated as follows:

If it is possible that God exists, then it is possible that it is necessary that God exists.

It is possible that God exists. Therefore, God exists.

◊G → □G . G ∴ G .

We shall construct an indirect derivation.

1. *~~Show~~* G
2. ~G Assume (ID)
3. G → □G Premise
4. G Premise
5. □G 3, 4 MP

Notice that applying PG to the indirect assumption yields ~G. Since modalities in **S5** are necessarily true, we may import this modal statement into a strict derivation by the characteristic strict importation rule for **S5**, R. This allows us to construct a strict derivation for □~□G, which is, by modal negation, the contradictory of line 5.

1. *~~Show~~* G 5, 10 ID
2. ~G Assume (ID)
3. G → □G Premise
4. G Premise
5. □G 3, 4 MP
6. ~G 2, PG
7. *~~Show~~* □~□G 9, ND
8. ~G 6, R
9. ~□G 9, MN

10. ~□G 7, MN

Now in **S5** we also have all the previous strict importation rules including NI and PG, and in particular both □R and R. Technically, in applying □R and R, we can pass over only one uncancelled modal ‘*Show’* at a time—that is, a ‘*Show’* of the form ‘*Show* □ϕ’ or ‘*Show* ϕ’. Notice, however, in **S5** we can, in effect, prefix any finite number of ‘□’s to any modal sentence, and so, in effect, pass over any number of modal ‘*Show’* lines with any modal sentence. In order to eliminate repetitious uses of □R and R, we shall allow □R and R to be applied across *n*-nested strict derivations (annotations □R*n* and R*n*).

Plantinga’s modal ontological argument uses these strong modal assumptions of **S5** in which the modal status of any modal proposition is the same is all possible worlds. Hence, if it is possible that God exists, then it is necessarily possible that God exists (and, inversely, if it is possible that God does not exist, then it is impossible that God exists). Even if Plantinga’s argument is valid (and, indeed, *sound* if there is *at least one* sound arguments for the existence of God), it need not be modally persuasive: the way the argument goes—either theistically or atheistically—depends on the loaded possibility premise.

These derivations in the various modal systems can be tedious due to the various restrictions. We can make the modal natural deduction system less restricted by proving modal theorems. It is particularly useful to generalize MN for a string of modal operators, and to be able to employ the rule of Interchange of Equivalence using previously proven biconditional theorems. Modal theorems for the various modal systems are listed in a subsequent section. However, first, we shall explain why the various restrictions as set forth in the formal characterization of a modal derivation are required.

### Exercises

In constructing modal derivations, the following strategic hints may be useful.

*Hint 1. After entering the ‘Show’ line for the conclusion and taking any useful assumptions, enter the premises.*

*Hint 2. If your initial ‘Show’ line is a modal statement, you may need to reenter the ‘Show’ line to commence a strict derivation.*

*Hint 3. To show a sentence of the form* ‘□ϕ’*, use strict derivation in the form of a necessity derivation, that is derive* ϕ *from other necessary truths.*

*Hint4. To show a sentence of the form ‘*ϕ*’, use strict derivation in the form of possibility derivation, i.e., judiciously choose from the accessible lines of the form* ‘ψ’ *to get an assumption for possibility derivation that will enable you to derive* ϕ*.*

*Hint 5. If you have an iterated modal statement as an accessible line, you may need to get a contradiction for indirect derivation by modal negation.*

1. Construct derivations for theorems in the different modal systems.

* + 1. T553◊P ∨ ~P in **D**
    2. T550 P→ P in **T**
    3. T571 □P → P in **B**
    4. T561 P → P in **S4**
    5. T571 ◊□P → □P in **S5**

2. Construct **T**-derivations for the following.

(A) □(P → Q) . □(Q → R) . ~R ∴ □~P

(B) □(P ∨ Q) . ~P ∴ □Q

(C) □P ∨ ◊Q ∴ □(P ∨ Q)

(D) □(P ∨ Q) . □(P → R) . □(Q → S) ∴ □(R ∨ S)

(E) □P ↔ □Q ∴ ~~P → ~~Q

3. Construct **B**-derivations for the following.

(A) □P ∴ □P

(B) □(P → ~~P) ∴ P → □P

(C) □(P ∧ □Q) ∴ P ∧ Q

(D) □(P ∨ Q) ∴ □P ∨ Q

(E) □(P → Q) ∴ (P → □Q)

4. Construct **S4**-derivations for the following.

(A) □P ∴ □(Q → □P)

(B) P → □Q ∴ P → □(R → □Q)

(C) □(P → Q) ∴ □(R → □(P → Q))

(D) ∴ □P ↔ □□P

(E) ∴ P ↔ P

5. Construct **S5**-derivations for the following.

(A) □(P ↔ □P) . P ∴ □P

(B) □(P ↔ ◊□P) ∴ P → □P

(C) ◊~P . □(~P → ~P) ∴ ~P

(D) ∴ □P ↔ □P

(E) ∴ P ↔ □P

6. In this exercise, we construct modal derivations of the following Anselmian modal arguments. For the purpose of this exercise, we allow ourselves to use the letter ‘G’ for the proposition ‘God exists.’

(A) Construct a **T**-derivation for the following argument**:**

It is necessary that if God exists then it is necessary that God exists. It is possible that God does not exist. Therefore, God does not exist.

(B) Construct a **B**-derivation for the following argument:

*“No one...doubts that if it [that than which a greater cannot be thought] did exist, its nonexistence, either in actuality or in the understanding, would be impossible. For otherwise it would not be that than which a greater cannot be thought.”*

□(G → ~~G)

*“Therefore, if that than which a greater cannot be thought can even be thought, it cannot be nonexistent.”*

∴ G → G

(C) Construct an **S4**-derivation for the following argument taken from Anselm’s *Reply to Gaunilo* where he writes: *“But as to whatever can be conceived, but does not exist—if there were such a being, its non-existence, either in reality or in the understanding, would be possible.”*

This first premise is actually a schema since Anselm substitutes for the statement denying that God has existence, statements denying that God has various perfections:

□(G ∧ ~G → ~P)

If, however, God is a being than which none greater can be conceived, then God necessarily has all perfections. Hence, we have as a second premise:

□P .

From these two premises, derive that it is necessary that if God’s existence is possible then God exists:

□(G → G) .

(D) Taking the conclusion of exercise (C) as a premise, namely ‘□(G → G)’, construct an **S5**-derivation for the conclusion that it is necessary that God exists if it is possible that God exists:

G → □G .

(E) Construct an **S5**-derivation for the argument:

It is necessary that if God exists then it is necessary that God exists. It is possible that God exists. Therefore, it is necessary that God exists.

# 6. Fallacies.

In the directions for constructing a modal derivation, a number of restrictions appear whose significance is not perhaps immediately obvious. Recall that a *fallacy* is a procedure that permits the validation of a *false English argument*, that is, an argument whose premises are true sentences of English and whose conclusion is a false sentence of English (or at least are regarded so for the sake of argument). Here we show that the neglect of the various new restrictions on modal derivation would lead to fallacies.

Consider the following argument:

If Descartes is thinking, then Descartes exists. Descartes is thinking. Therefore, it is necessary that Descartes exists.

Descartes is a continent being and so the conclusion of the argument is false. Let us take it for granted that Descartes exists if he is thinking and even that Descartes is thinking. It does not follow from these premises that it is necessary that Descartes exists.

P → Q . P ∴ □Q

Where is the fallacy in the following attempted derivation?

1. *~~Show~~* □Q 4, ND

2. P → Q Premise

3. P Premise

4. Q 2, 3 MP

The fallacy occurs in line 1, where one attempts to box and cancel by strict derivation. One of the restrictions on strict derivation is that premises cannot be among the lines that are boxed. Intuitively, premises are true in the actual world but may not be true in every possible world, and therefore, cannot enter into strict derivations.

Consider the argument:

Roses could be green, but they aren’t. Therefore, is possible that the moon is made of green cheese.

P ∧ ~P ∴ Q

Here the premise does not assert an actual contradiction. Instead the premise asserts that it is contingently false that roses are green. The conclusion asserts something that is not only false but clearly irrelevant to the premises. Where is the fallacy in the following attempted derivation?

1. *~~Show~~* Q 6, PD

2. P ∧ ~P Premise

3. P 2, S

4. *~~Show~~* P → ~P 5, CD

5. ~P 2, S

6. *~~Show~~* Q 7, PD

7. *~~Show~~* Q 8, 9 ID

8. P 3, Assume (PD)

9. ~P 8, 4 MP

An assumption for possibility derivation can only occur immediately after a sentence of the form ‘*Show* ◊ϕ’. Therefore a fallacy occurs in line 8. Moreover, one cannot box and cancel by indirect derivation if the derivation begins with an assumption for possibility derivation. Consequently, a fallacy also occurs in line 7 when one attempts to box and cancel by indirect derivation.

The following argument is clearly invalid. In fact, assuming it is only contingently true that roses are red, it is a *false* English argument because its premise would be true while the conclusion is false.

Roses are red and violets are blue. Therefore, it is necessary that roses are red.

Where is the fallacy in the following attempted derivation?

P ∧ Q ∴ □P

1. *~~Show~~* □P 3, DD

2. P ∧ Q Premise

3. *~~Show~~* □P 4, ND

4. P 2, S

One of the restrictions on strict derivation is that inference rules cannot be applied to inaccessible lines. In this example, the premise in line 2 is not accessible from line 4. The rule of simplification cannot be applied to line 2 to obtain line 4 in the strict derivation. This line prohibits the boxing and cancelling in line 3. Intuitively, premises are true in the actual world but may not be true in every possible world.

The following English argument is not only invalid but false, assuming that the premise is a true future contingent proposition.

The sea battle will happen. Therefore, it is necessary that the sea battle will happen.

P ∴ □P

Where is the fallacy in the following attempted derivation?

* 1. *~~Show~~* □P 4, DD
  2. P Premise
  3. ◊P 2, PG
  4. *~~Show~~* □P 5, ND
  5. P 3, Assume (PD)

Here the assumption for possibility derivation in line 5 is incorrect. Such an assumption must occur immediately after a ‘*Show’* line of the form ‘*Show* ◊ϕ’, whereas in this case, the ‘*Show’* line is of the form ‘□ϕ’, a necessary statement. Furthermore, one may not box and cancel by strict derivation in line 4 even though we have obtained the scope of the sentence in the ‘*Show’* line 4, namely, P, in line 5. The reason is not that line 5 is a premise, or comes by an application of an inference rule to an inaccessible line, but because it was *obtained by a means other than an admissible strict importation rule to an accessible antecedent line.*

Consider the following two arguments. The first is intuitively valid:

It is possible that Schödinger’s cat is dead, or it is possible that Schrödinger’s cat is alive.

Therefore, it is possible that Schrödinger’s cat is dead or alive.

Conversely, the following argument is also intuitively valid:

It is possible that Schrödinger’s cat is dead or alive. Therefore, it is possible that Schödinger’s cat is dead, *o*r it is possible that Schrödinger’s cat is alive.

However, the argument obtained by *conjoining* the two separate possibility statements under a single possibility operator is clearly invalid:

It is possible that Schrödinger’s cat is dead. It is possible Schrödinger’s cat is alive.

Therefore, it is possible that Schrödinger’s cat is both dead *and* alive.

P . Q ∴ (P ∧ Q)

Where is the fallacy in the following attempted derivation?

1. *~~Show~~* (P ∧ Q) 4, DD

2.  Premise

3. Q Premise

4. *~~Show~~* P ∧ Q) 7, PD

5. P 2, Assume (PD)

6. Q 3, Assume (PD)

7. P ∧ Q 5, 6 ADJ

The restriction on entering as assumption for possibility derivation states that such an assumption must occur *immediately after* the uncancelled ‘*Show*’ line. Line 5 is legitimate since occurs immediately after line 4, which is of the form ‘*Show* ϕ’. However, line 6, the second application of entering as assumption for PD is fallacious since it does not occur immediately after the uncancelled ‘*Show’* line 4.

Consider next the following more intricate fallacious argument which is a variation on Ivan’s famous argument in *The Brother Karamazov*.

If God is dead, then everything is permitted. It is possible that God is dead.

Therefore, it is possible that everything is permitted.

P → Q . P ∴ Q

Where is the fallacy in the following attempted derivation?

1. *~~Show~~* Q 4, DD

2. P → Q Premise

3. P Premise

4. *~~Show~~* Q 6, PD

5. P 3, Assume (PD)

6. Q 2, 5 MP

Here the fallacy occurs in line 4 in the attempt to box and cancel by strict derivation. In a strict derivation, none of the boxed lines may come from an application of an inference rule to an inaccessible line, or by an application of an inference rule other than an admissible strict importation rule to an accessible line. Notice that line 6 comes from an application of an inference rules to the premise in line 2 and so violates the above restriction.

Sometimes when we say that P is possible, what we mean is not that P is *metaphysically* possible, but only that “for all we know” P is the case. This is an *epistemic* interpretation of possibility. In 1637 Fermat read Diophantus’ *Arithmetic*, a 3rd century treatise and noted in the margins “dividing a cube into two cubes, or in general an *n*th power of two *n*th powers, is impossible if *n* is larger than 2. I have found a remarkable proof of this fact, but the margin is too narrow to contain it.” This enigmatic remark led to the centuries of mathematicians trying to prove “Fermat’s Last Theorem”. The status of Fermat’s conjecture was not established until 1995 when Andrew Wiles, together with his graduate student Richard Taylor, published a proof of Fermat’s Last Theorem.

Now consider the following English argument (given before 1995):

It is possible (*for all we know*) that Fermat’s Last Theorem is a theorem (and hence, necessarily true). Therefore, if it is possible (*for all we know*) that Fermat was right in believing he had a proof of his theorem, then Fermat’s Last Theorem is a true.

□P ∴ Q → P

(Here it could be argued that the premise is not properly symbolized because it mixes metaphysical and epistemic modalities.) Where is the fallacy in tehe following attempted derivation?

1. *~~Show~~* Q → P 3, DD

2. □P Premise

3. *~~Show~~* Q → P 5, CD

4. □P 2, Assume (PD)

5. P 4, NI

One may enter an assumption for possibility derivation only after a ‘*Show’* line of the form ‘*Show* ◊ϕ’. However, the sentence in line 3 is conditional, not a modal sentence. One may enter an assumption for possibility derivation only after a ‘*Show’* line of the form ‘*Show* ◊ϕ’. Therefore, a fallacy occurs in line 4. It happens that if we interpret the modal operators as uniformly metaphysically and assume the modal system **S5**, then we can construct a derivation of the conclusion from the premise. The verification of this claim is left as an exercise for the reader.

Consider one final intuitively fallacious English argument. Let’s suppose that the universe is “fine-tuned”, namely, that it is physically impossible for this universe to exist and for the fundamental constants of physics to be other than they actually are. We could ask the further speculative question of whether this universe could have existed if it operated according to a different system of physical laws together with a different set of constants. Suppose the answer to this question is ‘yes’. Then the following argument is intuitively fallacious:

It is possible that it is possible that the laws of physics could have been different than they are.

Therefore, the law of physics could have been different than they are.

One way to symbolize this argument is as follows:

P ∴ P

Can you identify the fallacy in the following attempted derivation?

1. *~~Show~~* P 3, DD

2. P Premise

3. *~~Show~~*  4, DD

4. P 2, Assume (PD)

One cannot box and cancel the direct derivation in line 3 because line 4 was introduced by an assumption for possibility derivation. One of the new restrictions on a direct derivation for ‘*Show* ϕ’ in modal propositional logic is that ϕoccurs unboxed among subsequent lines χ1 through χ*n*and χ1 but was *not introduced by an assumption for Possibility Derivation*. Therefore, a fallacy occurs in the boxing and cancelling of line 3. It turns out, however, that this argument can be validated in **S4**. The verification of this claim is left as an exercise for the reader.

The above examples justify, in part, the new restrictions set forth in our official syntactical characterization of the natural deduction systems for modal propositional logic. In the next section, we shall see how these restrictions—in particular, the rules of strict importation—are related to the characteristic axioms of the various modal systems. The reason for these *syntactical* restrictions on derivations can be more clearly understood in light of the *semantics* for propositional modal logic, which we set forth in the next section.

### Exercises

1. Identify all the fallacies steps in the following attempted derivations for intuitively fallacious (i.e., “false”) English arguments. Accurately state the restriction or restrictions that are violated and explain why how the restriction has been violated in the particular case.

(A) Clearly the following argument involves a fallacious wishful thinking.

It is possible that I will win the lottery. Therefore, I will win the lottery.

P ∴ P

Where are the fallacies committed in the following three attempted derivations?

1. *~~Show~~* P 7,8 ID

2. ~ P Assume (ID)

3. *~~Show~~* □~P 4, ND

4. ~P 2, R

5. ~P 3, MN

6. P Premise

1. *~~Show~~* P 3, DD

2. P Premise

3. *~~Show~~* P 4, 5 ID

4. ~ P Assume (ID)

5. P Assume (PD)

1. *~~Show~~* P 5, DD

2. P Premise

3. *~~Show~~* ~P → P 4, CD

4. P 3, Assume (PD)

5. *~~Show~~* P 6, 7, ID

6. ~P Assume (ID)

7. P 6, 3, MP

(B) Evil exists and it’s necessary God exists. Therefore, necessarily, both evil exists and God exists.

P ∧ □Q ∴ □(P ∧ Q)

1. *~~Show~~* □(P ∧ Q) 7, 8 ID
2. P ∧ □Q Premise
3. P 2, S
4. □Q 2, S
5. *~~Show~~* □(P ∧ Q) 8, ND
6. P 3, R
7. Q 4, NI
8. P ∧ Q 6, 7 ADJ

(C) It is possible that Theatetus is sitting. Therefore, if it is possible that Theatetus is standing, then it is possible that Theatetus is both sitting and standing.

P ∴ Q → (P∧ Q)

1. *~~Show~~* Q → P ∧ Q) 4, CD
2. Q Assume (CD)
3. P Premise
4. *~~Sho~~w* (P ∧ Q) 7, PD
5. P 3, Assume (PD)
6. Q 2, Assume (PD)
7. P ∧ Q 5, 6 ADJ

2. Show that allowing the following violations of the restrictions would result in the validation of a fallacious (i.e., ‘false’) English argument.

(A) Allowing the result of applying inference rules to premises into strict derivations.

(B) Allowing the application of a strict importation law to an inaccessible line.

(C) Allowing boxing and canceling by CD when the derivation begins with an assumption for possibility derivation.

(D) Allowing a necessity derivation to commence with an assumption for possibility derivation.

(E) Allowing an assumption for possibility derivation to be entered into a strict derivation but not immediately after the relevant uncancelled ‘*Show’* line.

# 7. Possible World Semantics.

The Leibnizian idea of characterizing necessity and possibility in terms of truth in all or some possible worlds was given an elegant formalization by Saul Kripke (1959, 1963) when he was only a teenager. According to Leibniz, a sentence is *necessary* if it is true in *every* possible world, and a sentence is *possible* if it is true is *some* possible world. Kripke showed that by placing very natural conditions on a relation of *relatively possibility* or *accessibility* on a set of possible world, the various systems of modal logic could be validated.

Intuitively, a possible world tells us for each sentence letter whether it is true or false in that world. Stripping away inessentials, we can represent a possible world by a subset of sentence letters. A *modal structure* **M** is an ordered triple <*W*, R, α>, where *W* is a set of possible worlds, R is a relation on *W* x *W* known as the *accessibility relation* or the *relative possibility relation*, and α is a distinguished element of *W* known as the *actual world*.

We can exhaustively characterize the notion of the *truth of a sentence in a possible world ββ*

⊨ β ϕ .

(1) If ϕ is a sentence letter S, then

⊨β S ⇔ S ∈ β (i.e., S is a member of β) .

(2) If ϕ is a ~ψ, then

⊨β ~ψ ⇔ ⊭ β ψ (i.e., not ⊨β ψ) .

(3) If ϕ is (ψ ∧ χ), then

⊨β (ψ ∧ χ) ⇔ ⊨β ψ  ⊨β χ .

If ϕ is (ψ ∨ χ), then

⊨β (ψ ∨ χ) ⇔ (⊨β ψ **∨** ⊨ β χ ) (i.e., **∨** is the inclusive ‘*or’* in the metalanguage) .

If ϕ is (ψ → χ), then

⊨β (ψ → χ) ⇔ (⊨β ψ ⇒ ⊨β χ) (i.e., if ⊨β ψ then ⊨β χ , or either ⊭β ψ or ⊨β χ) .

If ϕ is (ψ ↔ χ), then

⊨β (ψ ↔ χ) ⇔ (⊨β ψ ⇔ ⊨β χ) (i.e., either both ⊨β ψ & ⊨β χ or both ⊭β ψ & ⊭β χ) .

Finally, the law clause gives the Leibnizian truth conditions for necessity and possibility:

1. If ϕ is □ψ, then

⊨β □ψ ⇔ ∀γ ∈ W(Rβγ ⇒ ⊨γ ψ) (i.e., ψ is true in *all* possible worlds γ possible relative to β) .

If ϕ is ψ, then

⊨β ψ ⇔ ∃γ ∈ W(Rβγ ⊨γ ψ) (i.e., ψ is true in *some* possible world γ possible relative to β) .

This completes the definition of truth in a model for modal propositional logic. Using this definition of truth, we can now defined what it means for a sentence to be *semantic valid.*

⊨ ϕ (i.e., ϕ is *semantically valid*) if and only if ∀α ∈ W, ⊨α ϕ .

Next we obtain different systems of modal logic when various conditions are placed on the accessibility or relative possibility relation R. We say that a relation R is a *series* if R is serial. R is a *reflexivity* if R is reflexive. R is a *similarity* if R is reflexive and symmetric. R is a *partial ordering* if R is reflexive and transitive. R is an *equivalence relation* if R is reflexive and euclidean. It turns out that the axioms of modal logic discussed above are validated when natural conditions are imposed on the accessibility or relative possibility relation R.

|  |  |  |  |
| --- | --- | --- | --- |
| **D** | □ϕ → ϕ | R is *serial* | ∀α∃β Rαβ |
| **T** | □ϕ → ϕ | R is *totally reflexive* | ∀α Rαα |
| **B** | ϕ → □ϕ | R is *symmetric* | ∀α∀β( Rαβ ⇒ Rβα) |
| **4** | □ϕ → □□ϕ | R is *transitive* | ∀α∀β∀γ( Rαβ & Rβγ ⇒ Rαγ) |
| **5** | ◊ϕ → □◊ϕ | R is *euclidean* | ∀α∀β∀γ (Rαβ & Rαγ ⇒ Rβγ) |

Figure 3 Properties of Accessibility

Systems of modal logic are *normal* when everything derivable from necessary truths are themselves necessary. This will be the case if the rule of *modus ponens* and axiom **K** (named after Kripke) are valid:

□(ϕ → ψ) → (□ϕ → □ψ) .

Axiom **K** expresses the intuition that *necessary truths imply only necessary truths*.

We can conveniently summarize the above systems of modal logic in a chart. The various modal systems can be characterized by the axioms that are valid in them. The smallest normal modal logic system **K** contains axiom **K**. The four most famous modal logics are system (T) named after the Gödel-Feyes-von Wright modal logic to model tautologies, system **S4** and **S5**, named after C. I. Lewis’s axioms for strict implication, and the unduly neglected Brouwersche system **B,** named by Becker after the intuitionist L. E. J. Brouwer due to the characteristic axiom’s similarity to intuitionistic double negation. All these systems contain **K** and **T**. The system (D) which is weaker system than (**T**), containing axioms K and **D**, isnamed for deontic logic.

Notice that relationships of containment among the modal systems follow from the logic of relations. Axiom **B** requires that R be symmetric and **4** requires that R be transitive. System **S5** with axioms **T** and **E** require R be an equivalence relation (i.e., reflexive, symmetric, and transitive); hence, **S5** could also be specified by requiring axioms **T**, **4**, and **B** to be valid. Therefore, **S5** contains **S4** and **B**, neither of which contains the other. Systems **S5**, **S4** and **B** all contain system **T**, which contains **D**.

A convenient way of describing these modal logics is by their *Lemmon code* listing the axioms valid in them. For example, **S5** = **KTE** = **KT4B** = **KD4B**. We can represent these containment relations in a diagram in which downward paths represent containment.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| System | Named For |  | Axiom | Code | Accessibility |
| **D** | **D**eontic Logic | D | □P → P | KD | R is a seriality |
| **T** | **T**autology lLgic | **T** | □P → P | **KT** | R is a reflexivity |
| **B** | **B**rouwersche System | **B** | P → □P | **KTB** | R is a similarity |
| **S4** | Lewis’s **4** for Strict Implication | **4** | □P → □□P | **KT4** | R is a partial ordering |
| **S5** | Lewis’s **5** for Strict Implication | **E** | P → □P | **KTE** | R is an equivalence relation |

Figure 4. Modal Axioms and Accessibility

**S5**

**B S4 .**

**T .**

**D .**

Figure 5 Logical Containment of the Modal Systems

|  |  |  |  |
| --- | --- | --- | --- |
| **T** | **KT** | Deontic **D** | **KD** |
| **B** | **KTB** |  |  |
| **S4** | **KT4** | Deontic **S4** | **KD4** |
| **S5** | **KTE = KT4B = KD4B** | Deontic **S5** | **KD4E** |

Figure 6 Lemmon Codes for Deontic Modal Systems

The logical relationships among the above systems of modal logic can be set forth in a diagram (called “Picasso’s Chair” by Krister Segerberg), which omits **KD5** and **K45**). As before, a modal logic is included in another if it is connected to it, directly or indirectly, by an upward path.

S5

**S4**

**dS5**

**T dS4**

**K5**

**K4**

**D**

**K**

Figure 7 Picasso's Chair



Figure 8 Picasso's Electric Chair (After Krister Segerberg)

One way to visualize how the conditions on the accessibility relation validate their respective axioms is use the definitions of ‘□’ and ‘’ in terms of possible worlds and use directed graphs from chapter IV to represent the accessibility relation R. Here the accessibility relation Rαβ (read “β is possible relative to α” or “β is accessible to α”) is represented by an arrow from a circle representing possible world α to a circle representing possible world β.

Rαβ

α β

Figure 9 Accessibility Represented by Directed Graphs

We can, using the directed graphs from the theory of relations, translate properties of accessibility relations into geometric properties of directed graphs. Symmetry, for example, requires that all accessibility arrows are double arrows. Reflexivity requires that every world be accessible to itself and so every world has a loop, a special case of a double arrow. Seriality requires that every world is a tail of an arrow. Transitivity requires that for every indirect path has a shortcut. Being euclidean and serial is tantamount to being an equivalence relation, that is, to being reflexive, symmetric and transitive. Expressing R in terms of love and worlds in terms of persons, we have the following intuitive translations:

|  |  |  |
| --- | --- | --- |
| R is *serial* | ∀α∃β Rαβ | *Everyone is a lover.* |
| R is *reflexive* | ∀α (∃βRαβ ∨∃βRβα → Rαα) | *Anyone who loves or is loved loves themselves.* |
| R is *totally* *reflexive* | ∀α Rαα | *Everyone loves themselves.* |
| R is *symmetric* | ∀α∀β(Rαβ ⇒ Rβα) | *All love is requited, returned, mutual.* |
| R is *transitive* | ∀α∀β∀γ(Rαβ & Rβγ ⇒ Rαγ) | *Everyone loves the beloveds of their beloveds.* |
| R is *euclidean* | ∀α∀β∀γ (Rαβ & Rαγ ⇒ Rβγ) | *Anyone’s beloveds love each other (including themselves).* |

Figure 10 Relational Properties of Directed Graphs

Using the definition of truth in a modal system set forth above, we can rigorously demonstrate that if R is transitive, then axiom **4** is valid. This demonstration is carried out in the meta-language. We use the use the symbols ‘∀’, ‘∃’, ‘&’, ‘**∨’,** ‘⇒’, ‘’⇔’, and ‘∈’ in the meta-language for ‘*all’*, ‘*some’*, ‘*and’*, ‘*or’* ‘*if… then’*, ‘*if and only if’* and ‘*is an element of’*, respectively. Once the truth clauses are unpacked, the logical demonstration is no more complicated than a derivation in the theory of relations.

We can show that imposing transitivity on R implies the validity of the (**4**) axiom as follows:

1. *~~Show~~* R is *transitive* ⇒ ⊨ (□ϕ → □□ϕ) 3, CD
2. ∀α∀β∀γ(Rαβ & Rβγ ⇒ Rαγ) Assume (CD), Def. *transitive*
3. *~~Show~~* ⊨ (□ϕ  □□ϕ) 24, DD
4. *~~Show~~* ∀α ∈ *W* (⊨α □ϕ ⇒ ⊨α □□ϕ) 5, UD
5. *~~Show~~* ⊨α □ϕ ⇒ ⊨α □□ϕ 7, CD
6. ⊨α □ϕ Assume (CD)
7. ∀β ∈*W* (Rαβ ⇒ ⊨β ϕ) 6, Truth-Def. □
8. *~~Show~~* ⊨α □□ϕ 23, DD
9. *~~Show~~* ∀β∈*W* (Rαβ ⇒ ⊨β □ϕ) 10, UD
10. *~~Show~~* Rαβ ⇒ ⊨β □ϕ 12, CD
11. Rαβ Assume (CD)
12. *~~Show~~* ⊨ β □ϕ 22, DD
13. *~~Show~~* ∀γ ∈ *W* (Rβγ ⇒ ⊨γ ϕ) 14, UD
14. *~~Show~~* Rβγ ⇒ ⊨γ ϕ 16, CD
15. Rβγ Assume (CD)
16. *~~Show~~* ⊨γ ϕ 22, DD
17. Rαβ & Rβγ 11, 15 ADJ
18. Rαβ & Rβγ ⇒ Rαγ 2, UI3
19. Rαγ 18, 17 MP
20. Rαγ ⇒ ⊨γ ϕ 7, UI
21. ⊨γ ϕ 20, 19 MP
22. ⊨β □ϕ 12, Truth-Def. □
23. ⊨α □□ϕ 8, Truth-Def. □
24. ⊨ (□ϕ  □□ϕ) 4, Def. ⊨

We can show that the above implication holds using semantic diagrams. We will show that if

□ϕ → □□ ϕ

is invalid, then the accessibility relation cannot be transitive. Assuming that the above axiom fails, then its antecedent ‘□ϕ’ is true but its consequent ‘□□ ϕ’ is false in α.

□ϕ

~□□ ϕ

So by a generalized form of modal negation, the latter is equivalent to

 ~ϕ .

So both ‘□ϕ’ and ‘~ϕ’ are true in some possible world α. By the definition of the truth for , we have that ◊~ϕ is true in some world β accessible to α. Applying the definition of truth for  again, we have that for some world γ accessible to β, ~ϕ is true in γ. By the definition of the truth for □, since we have that □ϕ is true in α, ϕ must be true in all worlds accessible from α including β. Can γ be accessible to α? No, because ~ϕ is true in γ, and since □ϕ is true in α and so ϕ must be true in all worlds accessible to α. Can R be transitive? There is an indirect path from α to β and from β to γ, therefore, if R were transitive, there would have to be a shortcut from α to γ. However, we have already seen that γ cannot be accessible to α. Therefore, if axiom **4** fails to be valid, then the accessibility relation R cannot be transitive. By contraposition, if the accessibility relation R is required to be transitive, then axiom **4** is valid.

#

□ϕ ϕ ~ϕ

~ □□ϕ ~ϕ

~ϕ

γ

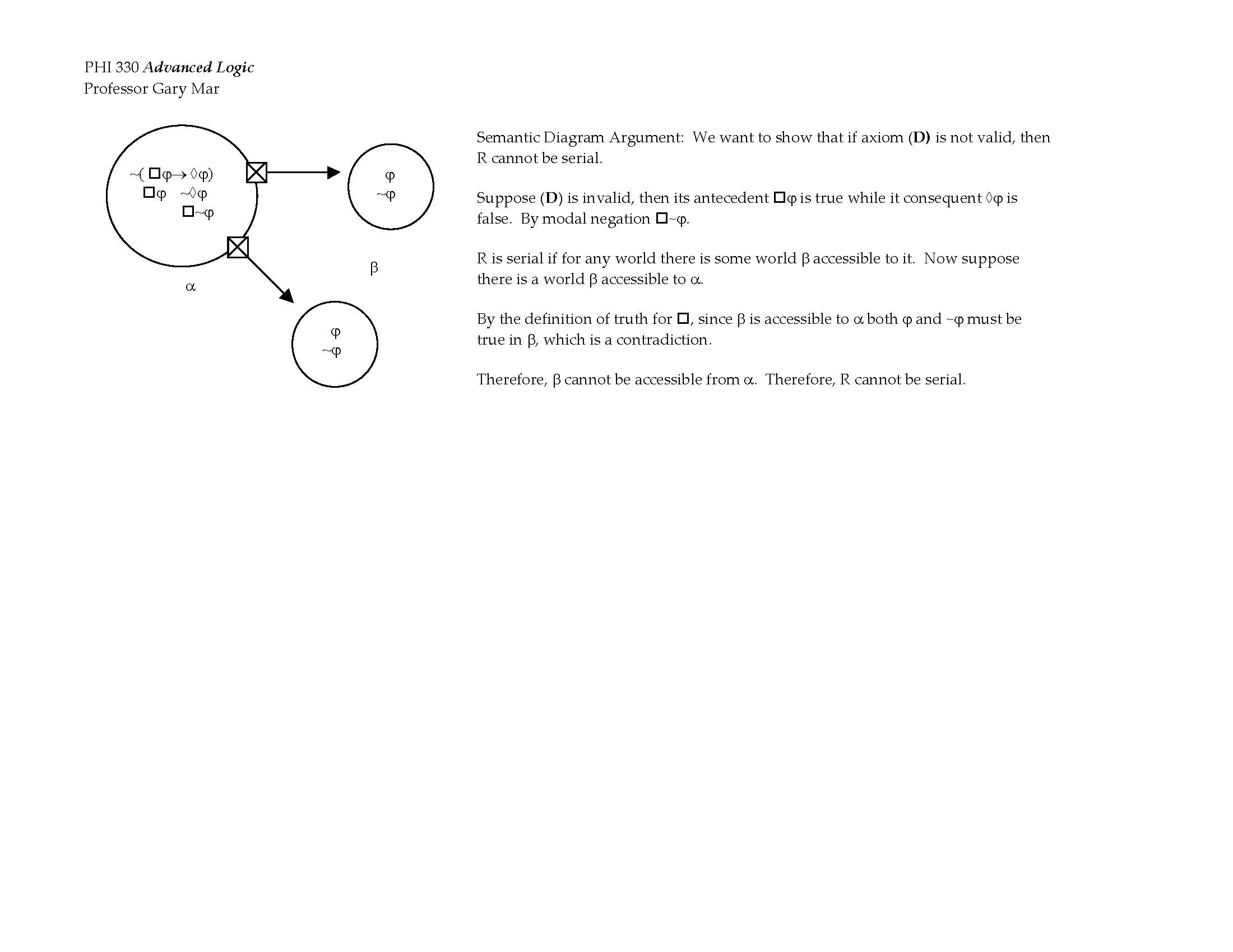
β

α

Figure 11 Transitivity Validates Axiom (4)

### Exercises

1. R is the relative possibility or accessibility relation in a modal system. Provide both an intuitive diagrammatic justification and a more rigorous derivation in the meta-language to establish the following.
   1. If R is a serial then axiom **D** is valid.



1. *~~Show~~* Ris *serial* ⇒ |= (□ϕ → ◊ϕ) 3, CD

2. \_\_\_\_\_\_\_\_\_\_\_\_\_ Assume (CD), Def. *serial*

3. *~~Show~~* |= (□ϕ → ◊ϕ) 18, DD

4. *~~Show~~* ∀α∈*W* |=α (□ϕ → ◊ϕ) 5, UD

5. *~~Show~~* |=α (□ϕ  ◊ϕ) 17, DD

6. *~~Show~~* |=α □ϕ ⇒ |=α◊ϕ 16, CD

7. |=α □ϕ Assume (CD)

8. ∀β∈W (Rαβ ⇒ \_\_\_\_\_ ) 7, Truth-Def. □

9. *~~Show~~* |=α◊ϕ 16, DD

10. *~~Show~~* ∃β∈W(Rαβ & \_\_\_\_) 15, DD

11. Rαγ 2, \_\_\_, \_\_\_

12. Rαγ ⇒ |=γ ϕ 8, \_\_\_

13. |=γ ϕ 11, 12 \_\_\_

14. Rαγ & |=γ ϕ 11, 13 \_\_\_

15. ∃β∈W(\_\_\_\_\_\_\_\_\_\_\_\_) 14, EG

16. \_\_\_\_\_\_ 10, Truth-Def. ◊

17. \_\_\_\_\_\_\_\_\_\_ 6, Truth-Def. 

18. |=□ϕ → ◊ϕ 4, Def. |= (*logical consequence*)

* 1. If R is reflexive then axiom **T** is valid. Annotate the following semantic derivation.

1. *~~Show~~* Ris *reflexive* ⇒ |= (□ϕ → ϕ) 3,CD

2. ∀αRαα Assume (CD), Def. *Reflexive*

3. *~~Show~~* |= (□ϕ → ϕ) 14, DD

4. *~~Show~~* ∀α∈ *W* |=α (□ϕ → ϕ) 5, UD

5. *~~Show~~* |=α (□ϕ  ϕ) 13, DD

6. *~~Show~~* |=α □ϕ ⇒ |=αϕ 9, CD

7. |=α □ϕ Assume (CD)

8. ∀β∈W (Rαβ ⇒ |=β ϕ ) 7, Def. Truth 🞎

9. *~~Show~~* |=α ϕ 12, DD

10. Rαα ⇒ |=αϕ 8, UI

11. Rαα 2, UI

12. |=α ϕ 10, 11 MP

13. |=α (□ϕ  ϕ) 6, Def. Truth 

14. |= (□ϕ  ϕ) 5, Def. |= (*logical consequence*)

Construct the semantic diagram to show that if reflexivity fails, then the (**T**) axiom cannot be valid.

* 1. If R is symmetric then axiom **B** is valid.

Provide the missing lines for the following semantic derivation.

1. *~~Show~~* R is *symmetric* ⇒|= (ϕ → □◊ϕ) 3, CD
2. Assume(CD), Def. *symmetric*
3. *~~Show~~* |= (ϕ → □◊ϕ) 21, DD
4. *~~Show~~*  5, UD
5. *~~Show~~* 20, DD
6. *~~Show~~* |=α ϕ ⇒ |=α □◊ϕ 8, CD
7. Assume (CD)
8. *~~Show~~* 19, DD
9. *~~Show~~* 10, UD
10. *~~Show~~* Rαβ ⇒ |=β ◊ϕ 12, CD
11. Assume(CD)
12. *~~Show~~* 18, DD
13. *~~Show~~* 17, DD
14. 2, UI2
15. 11, 14 MP
16. Rβα  |=α ϕ 15, 7 ADJ
17. ∃γ ∈*W* (Rβγ  |=γ ϕ) 16, EG
18. |=β ◊ϕ 13, Truth-Def. ◊
19. |=α □◊ϕ 9, Truth-Def. 🞎
20. |=α (ϕ → □◊ϕ)] 6, Truth-Def. 
21. |= ϕ → □◊ϕ 4, Def. |= (*logical consequence*)

Construct the semantic diagram to show that if (**B**) is invalid, then R cannot be symmetric.

* 1. Show (weak) reflexivity and euclideanness implies a partial equivalence relation.

*y*

*x*

*Given*:

∀*x*[∃*y*(R*xy* ∨ R*yx*) → R*xx*] (*weak* *reflexivity*)

∀*x*∀*y*∀*z*(R*xy* ∧ R*xz* → R*yz*) (*right* *euclidean*)

∀*x*∀*y*∀*z*(R*yx* ∧ R*zx* → R*yz*) (*left* *euclidean*)

*z*

*x*

*y*

*Lemma*: ∀*x*∀*y*(R*xy* → R*xx*) (*left reflexivity*)

R*xy*

R*xy* ∨ R*yx*

R*xy* ∨ R*yx* → R*xx* (*weak reflexivity*)

R*xx*

*z*

*x*

*y*

*~~Show~~* ∀*x*∀*y*(R*xy* → R*yx*) (*symmetry*)

R*xy*

R*xx* (*reflexivity*)

R*xy* ∧ R*xx* → R*yx* (*right* *euclidean*)

R*yx*

*z*

*x*

*y*

*~~Show~~* ∀*x*∀*y*∀*z*(R*xy* ∧ R*yz* → R*xz*) (*transitivity*)

R*xy* ∧ R*yz*

R*yx*

R*yx (symmetry)*

R*yx* ∧ R*yz* → R*xz* (*left euclidean*)

R*xz*

*z*

* 1. If R is euclidean then **E** is valid. Provide the missing lines and annotations.

1. *~~Show~~* R is *euclidean* ⇒ ⊨ (ϕ → □ϕ)
3. *~~Show~~*
5. *~~Show~~* ∀α∈W|⊨α (ϕ → □ϕ)

5. *~~Show~~* ⊨ αϕ → □ϕ

6. *~~Show~~* ⊨ αϕ ⇒ ⊨ α □ϕ

7.

8. ∃β∈W(Rαβ & \_\_\_\_\_)

9. *~~Show~~* ⊨α □ϕ

10. *~~Show~~* ∀β(Rαβ ⇒ \_\_\_\_\_)

11. *~~Show~~* Rαβ ⇒ ⊨β ϕ

12.

13. *~~Show~~* ⊨ β ϕ

14. *~~Show~~* ∃γ∈W(Rβγ & ⊨γ ϕ)

15. Rαγ′ & ⊨ γ′ ϕ

16. Rαβ & Rαγ′

17.

18.

19. Rβγ′ & ⊨ γ′ ϕ

20. ∃γ∈W (Rβγ & ⊨γ ϕ)

21. ⊨ β ϕ

22. ⊨ α □ϕ

23. ⊨ αϕ → □ϕ

 ⊨ϕ → □ϕ

Construct the Semantic Diagram

1. There are different axiomatizations for different philosophical conception of time. According to the *tensed theory of time* championed by the philosopher C. D. Broad, the past is real and the present moves up the tree of possible futures turning the unreal future into the real past. Another theory, due to A. N. Prior, known as *presentism*, holds that only the present is real and the past and the future are unreal. What do the following axioms express about the structure of time?

(A) □(ϕ → ψ) → (□ϕ → □ψ); ■(ϕ → ψ) → (■ϕ → ■ψ)

(B) □ϕ → □ □ϕ; ■ϕ → ■■ϕ

(C) □ □ϕ → □ϕ; ■ ■ϕ → ■ϕ [*Hint*: see exercise 4 below for **4C**.]

(D) □ϕ → ϕ; ■ϕ → ⧫ϕ [*Hint*: *seriality* toward the future implies there is no last moment of time].

(E) ϕ → □⧫ϕ; ϕ → ■ϕ [*Hint*: these are the *interaction* theorems.]

3. Prior’s axiomatization of temporal logic known as the *Diodorean System* consists of **KT4** and the following axiom **Dum** (named after the philosopher Michael Dummett, a champion of intuitionism). Give a reading of his axiom in temporal logic:

□(□(P→ □P) → P)→ (□P → P) .

4. Demonstrate that the following axioms require the corresponding conditions on R.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| DC | Converse **D** | ϕ→□ϕ | ∀α∀β∀γ (Rαβ & Rαγ ⇒ β = γ) | R is *functional* |
| **□T** | Necessitation **T** | □(□ϕ → ϕ) | ∀α∀β(Rαβ ⇒ Rββ) | R is *point reflexive* |
| **4C** | Converse **4** | □□ϕ → □ϕ | ∀α∀β[Rαβ ⇒ ∃γ (Rαγ & Rγβ)] | R is *dense* |
| **C** | Convergent | □ϕ→ϕ | ∀α∀β∀γ[R αβ & Rαγ ⇒ ∃η(Rαη & R γη)] | R is *convergent* |

Formulate each of the conditions on R in terms of a directed graph. Then demonstrate that if each of the axioms is assumed to be invalid, then the corresponding condition on R must fails to hold.

5. An elegant result due to Lemmon and Scott (1977) states a correspondence between conditions on R and axioms of the generalized form:

*h*□*i*ϕ → □*j**k*ϕ .

Here ◊*m* and □*n* represents *m* diamonds and *n* boxes in a row, respectively. The corresponding condition on R imposed by the generalized axiom is given by

R*h*αβ & R*j*αγ ⇒ ∃η(R*i*βη & R*k*γη) .

Here the *composition* R ° S of two relations R and S (in symbols, R ° S) can be defined as

R ° S αβ =df ∃γ(Rαγ & Sγβ) .

For example, if R is the relation *being a sister of* and S is the relation *being a parent of*, then the composition of the relations R ° S is the relation *being* *an aunt of*, that is

R ° S αβ ↔ ∃γ(Rαγ & Sγβ) .

If R is the relation *being a parent of*, then R ° R = R 2 is the relation *being a grandparent of*, and R ° R ° R = R3 is the relation *being a great-grandparent of*. R 0 is the *identity relation*, i.e., R0αβ if and only if α = β, and R 1 = R.

Verify that the values for *h, i, j, k* as indicated in the chart yield the standard axioms, and that the standard condition on R for each of the axioms is equivalent to the one specified by the Lemmon-Scott Generalized Correspondence theorem:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | *h* | *i* | *j* | *K* | Condition on R | |
| **B** | 0 | 0 | 1 | 1 | *Symmetric* | ∀α∀β(Rαβ ⇒ Rβα) |
| **4** | 0 | 1 | 2 | 0 | *Transitive* | ∀α∀β∀γ(Rαβ & Rβγ ⇒ Rαγ) |
| **5** | 1 | 0 | 1 | 1 | *Euclidean* | ∀α∀β∀γ (Rαβ & Rαγ ⇒ Rβγ) |
| **C** | 1 | 1 | 1 | 1 | *Convergent* | Rαβ & Rαγ ⇒ ∃η(Rβη & R γη) |

5. Another interpretation of the ‘□’ is provability. The *Gödel-Löb Provability System* is the system **K4W,** wherethe **(W)** axiom (here ‘W’ stands for “well-ordering”) is as follows:

(**GL**) □(□ϕ → ϕ) → □ϕ .

Give an informal argument to show that (**GL**) requires the accessibility relation to be transitive, irreflexive, and *finite* (i.e., indirect paths of R chains terminate in a *finite* number of steps.) It is interesting to note that being ‘finite’ cannot be expressed in a sentence of first-order logic. This follows from the compactness meta-theorem that states that any infinite set of sentences that is finitely satisfiable (i.e., any finite subset has a model) must also be satisfiable (i.e., there is a model that satisfies the entire infinite set).

# 8. Theorems.

Characteristic Axioms

T501 □(P → Q) → (□P → □Q) (**K**)

T538 □P → P (**D**)

T560 □P → P (**T**)

T570 P → □P (**B**)

T580 □P → □□P (**4**)

T590 ◊P → □◊P (**5**)

Laws of Modal Negation

T503 ~□P ↔ ~P

T504 ~P ↔ □~P

T505 □P ↔ ~~P

T506 P ↔ ~□~P

Laws of Modal Distribution

T501 □(P → Q) → (□P → □Q)

T502 □(P → Q) → (P → Q)

T507 (P ∨ Q) ↔ P ∨ Q

T508 □(P ∧ Q) ↔ □P ∧ □Q

T509 ◊(P ∧ Q) → P ∧ Q

T510 □P ∨ □Q → □(P ∨ Q)

T511 (P → Q) → (P → Q)

T512 (□P → □Q) → (P → Q)

T513 □(P ↔ Q) → (□P ↔ □Q)

T514 □(P ↔ Q) → (P ↔ Q)

T515 ~P → ~□P

T516 □(P ∨ Q) → P ∨ □Q

T517 ~P → ~P

T518 (P → □Q) → □(P → Q)

T519 (P → □Q) → (□P → □Q)

T520 (P → □Q) → (P → Q)

T521 (~P ∧ Q) → (~P ∧ Q)

The following theorems are modal counterparts to *consequentia mirabilis*:

T522 (~□P → P) ↔ P

T523 (~P → P) ↔ P

T524 (P → ~P) ↔ ~P

T525 (P → □~P) ↔ ~P

T526 (P → ~P) ↔ ~P

T527 (~P → □P) ↔ P

Modal analogues Smullyan’s Drinking Theorems” are **T** theorems despite having nested modalities:

T529 (P →P)

T530 (P → □P)

Corresponding to quantifier theorems T234-T247 are the modal theorems T534-T547.

T533 □(P → Q) → □(P ∧ R → Q)

T534 □[(P → Q) ∧ ( Q→ R) → (P → R)]

T535 □(P → Q) ∧ □(Q → R) → □(P → R)

T536 □(P ↔ Q) ∧ □(Q ↔ R) → □(P ↔ R)

T537 □(P → Q) ∧ □(P → R) → □(P → Q ∧ R)

T538 □P → P

T539 (□P ∧ Q) → P ∧ Q)

T540 □(P → Q) ∧ (P ∧ R) → (Q ∧ R)

T541 □(P → Q ∨ R) → □(P → Q)∨ (P ∧ R)

Laws of Strict Implication

T544 states the equivalence of two characterizations of strict implication. Theorems T533 and T535 above state laws of transitivity and strengthening hold for strict implication. As we shall see, these laws do not hold for *counterfactual conditionals.*

T542 ~□(P → Q) ↔ (P ∧ ~Q)

T543 ~◊(P ∧ Q) ↔ □(P → ~ Q)

T544 □(P → Q) ↔ ~(P ∧ ~Q)

T545 ~◊P ↔ □(P → Q) ∧ □(P → ~Q)

T546 ~P ∧ ~Q → □(P ↔ Q)

T547 (P → Q) ↔ (🞎P → Q)

Paradoxes of Strict Implication

T548 ~□(P → Q) → P

T549 ~□(P → Q) → ~□Q

T550 🞎Q → 🞎(P → Q)

T551 ~P → 🞎(P → Q)

T552 (P → Q) ∨ 🞎(Q → P)

Theorems for System **D**

T538 □P → P

T553 P ∨ ~P

T554 ~(□P ∧ □~P)

T555 □(P → ~Q) → (□P → ~□Q)

Theorems for System **T**

T560 □P → P

T561 P → P

T562 □(□P → P)

T563 □(P → P)

T564 □□P → □P

T565 P → P

T566 □P → P

T567 □P → □P

Theorems for the Brouwersche System **B**

T570 P → □P

T571 □P → P

T572 P → ~~P

Theorems T573 and T574 are valid in **KB**, that is, their proofs do not require NI as an inference rule, but only NI and ◊R as admissible strict importation rules.

T573 □(P → Q) → (P → □Q)

T574 □(P → □Q) → (P → Q)

T575 □P → □P

Theorems T576 and T577 are related to Anselmian arguments and can be validated in **B**.

T576 □(P → □P) ∧ P → P

T577 □(P → □P) ∧ P → □P

Theorems for **S4**

Theorems T580-T588 are theorems of the system **S4** (also known as **KT4**). T580 **i**s the characteristic theorem of the system and T581is its logical dual.

T580 □P → □□P

T581 P → P

Theorems T582-T584 are valid in **K4**, that is, they do not require NI as an inference rule but only NI and □R as admissible strict importation rules.

T582 □(P → Q) → □(□P → □Q)

T583 □(P ∨ Q) → □(P ∨ □Q)

T584 □(□(P → Q) → R) → □(□(P → Q) → □R)

Theorems T585 and T586 are the *modal reductions laws* for **S4** :one may collapse any string of iterated uniform operators to a single occurrence of that operator.

T585 □P ↔ □□P

T586 P ↔ P

Theorem **T587** shows that the **G** schema is valid in any normal **KB** system. **T588** is an alternative basis for **S4**.

T587 □P → □P

T588 □(P → Q) → □(□P → □Q)

Theorems for **S5**

Theorems T590-T595 are valid in **S5** (also known as **KT5**). T590 is the characteristic theorem of the system and T571 is its logical dual.

T590 □P → □P

T591 □P → □P

Theorems T592-T598 are valid in **K5**, that is, they do not require NI as an inference rule, but they may employ NI, □R, and ◊R as admissible strict importation rules.

T592 □P → □P

T593 □P → P

T594 ◊□P → □□P

T595 □P → □P

T596 P → □P

T597 □(□P ∨ Q) → □P ∨ □Q

T598 P ∧ Q → (P ∧ Q)

Theorems T599-T610 are valid in any normal **K5** system.

T599 □(□P ↔ □□P)

T600 □P ↔ P)

T601 □(□P ↔ □P)

T602 □(P ↔ 🞎P)

T603 □□P ↔ □□□P

T604 □□P ↔ □□P

T605 □P ↔ □□P

T606 □P ↔ □P

T607 P ↔ P

T608 P ↔ □P

T609 □P ↔ □P

T610 □P ↔ □□P

Theorems T611-T612 are the *modal reduction laws* for **S5**, which together with the modal reduction laws for **S4** and **B** imply that iterated modalities in **S5** collapse to the innermost operator.

T611 □P ↔ □P

T612 P ↔ □P

### Exercises

1. Verify the following claims.

* 1. **K** is derivable in **T**.
  2. PGisderivablein each of the modal systems **T**, **B**, **S4**, and **S5**.

(C) Show that in each of the modal systems if ϕ is a theorem of classical sentential logic, then □ϕ is derivable (this is known as the rule of necessitation **Nec**).

2. Prove by induction that Interchange of Equivalents (see pp. 362-363 of KMM) is derivable in each of the modal systems.

3. Construct **T**-derivations of the following theorems:

(A) T501 □(P → Q) → (□P → □Q)

(B) T502 □(P → Q) → (P → Q)

(C) T541 P → P

(D) T542 □(□P → P)

(E) T523 (P → Q) ∨ □(Q → P)

4. Construct **B**-derivations for the following theorems:

(A) T550 □P → P

(B) T551 P → □P

(C) T552 □(P → Q) → (P → □Q)

(D) T553 □(P → □Q) → (P → Q)

(E) T554 □P → □P

5. Construct **S4**-derivations for the following theorems:

(A) T560 □P → □□P

(B) T561 P → P

(C) T562 □(P → Q) → □(□P → □Q)

(D) T563 □(P ∨ Q) → □(□P ∨ □Q)

(E) T564 □(□(P → Q) → R) → □(□(P → Q) → □R)

6. Construct **S5**-derivations for the following theorems:

(A) T571 □P → □P

(B) T573 P ∧ Q → (P ∧ Q)

(C) T574 □P → □P

(D) T594 □P ↔ □□P

(E) T597 P ↔ □P

7. Prove that if a relation R is reflexive, then R is euclidean if and only if R is both symmetric and transitive, that is prove the following theorem:

∀*x*R*xx* ⇒ (∀*u*∀*t*∀*v*(R*ut* & R*uv* ⇒ R*tv*) ⇔ [∀*x*∀*y*(R*xy* ⇒ R*yx*) & ∀*x*∀*y*∀*z*(R*xy* & R*yz* ⇒ R*xz*)]) .

8. For each of the following pairs of statements, one is a theorem in **T**, but the other is not. If a statement is a theorem, construct a **T**-derivation of its conclusion from its premises. If statement is not a theorem, provide an invalidating semantic diagram.

(A) □(P → Q) → (P → Q) ; (P → Q) → □(P → Q)

(B) (P ∧ Q) → (P ∧ Q) ; (□P ∧ Q) → (P ∧ Q)

9. Construct a semantic diagram and a semantic derivation in our metalanguage to show that the axiom

□(□P → Q) ∨ □(□Q → P)

is valid in any modal structure in which the accessibility relation R is *weakly connected*, i.e.,

∀*x*∀*y*∀*z*(R*xy*  R*xz* ⇒ R*yz* **∨** R*zy*) .

10. Prove the following theorems about strict implication and strict equivalence in system (**T**) to which we add the following deductive rules:

*Strict Implication* [SI]: From P ⇒ Q to infer □(P → Q), and conversely.

*Strict Equivalence* [SE]: From P ⇔ Q to infer □(P ↔ Q), and conversely.

(A) ~(P ∧ ~Q) ↔ (P ⇒ Q)

(B) (P ⇒ Q) ∧ (P ⇒ ~Q) ⇒ 🞎Q

(C) (~P ∧ ~Q) ⇒ (P ⇔ Q)

(D) □Q ∴ P ⇒ Q

(E) ~◊P ∴ P ⇒ Q

11. A statement P is *contingent* if it is neither necessary nor impossible. We may define the *contingency operator* ∇ as follows:

∇P ↔ ~(□P  □~P) .

Translate the following axioms about contingency into English, and then, using the above definition, provide derivations of each of them in **S5** in which you may use both Interchange of Equivalences (IE) and biconditional derivation (BD).

(A) ∇P ↔ ∇~P

(B) ~∇(P ↔ Q) → (∇P → ∇Q)

(C) ∇(P → Q) → (~∇P → P)

(D) ~∇(P → ∇Q)

(E) □P ↔ (P ∧ ~∇P)

# 9. Counterfactuals “Would” and “Could”

Frege argued that if the conditional is truth-functional then it must have the values it is assigned in the traditional truth table.

|  |  |  |  |
| --- | --- | --- | --- |
|  | P | Q | P  Q |
| 1 | T | T | T |
| 2 | T | F | F |
| 3 | F | T | T |
| 4 | F | F | T |

Recall that an argument is *semantically valid* if and only if its logical form is such that it is impossible for its premises to be true while its conclusion is false. Conditional arguments occur with enough frequency to have acquired traditional names for its valid, and fallacious, inferences.

|  |  |  |
| --- | --- | --- |
| Valid Inferences | P Q  P  ∴ Q | P  Q  ~ Q  ∴ ~P |
|  | *Modus Ponens* | *Modus Tollens* |
| Common Fallacies | P  Q  Q  ∴ P | P  Q  ~ P  ∴ ~Q |
|  | *Fallacy of Affirming the Consequent* | *Fallacy of Denying the Antecedent* |

Notice that row 3 of the truth table for the conditional must be T if the fallacies of affirming the consequent or denying the antecedent are invalid. By the definition of invalidity, it must be possible for the following argument to have true premises and a false conclusion (steps 1-3 and 3’). However, given that a negation must have the opposite value of the sentence it negative, P must be **F** (step 4 and 4’).

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| P |  | Q |  | ~ | P |  | ∴ | Q |  |
| **F** | **T** | **F** |  | **T** | **F** |  |  | **F** |  |
| 4’ | 1 | 3’ |  | 2 | 4 |  |  | 3 |  |

Therefore, if the argument is to be invalidated, the conditional must be true when its antecedent is **T** and its consequent is **F**. The value of the conditional in row 3 must therefore be **T**.

Next we note that we must be able to distinguish a *conditional* (P  Q) from its *converse* (Q P). Therefore, the value of the second row of the truth table for the conditional must be *different* from its value in the third row. The value of the conditional in row 2 must therefore be **F**.

Beside the common fallacies noted above, there are some uncommon, or deviant, fallacies, which we shall call *modus tortoise* *ponens*, and *modus tortoise tollens*.

|  |  |  |
| --- | --- | --- |
| Uncommon Fallacies | P  Q  P  ∴ ~ Q | P  Q  ~ Q  ∴ P |
|  | *Modus Tortoise Ponens* | *Modus Tortoise Tollens* |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| P |  | Q |  |  | P |  | ∴ | ~ | Q |
| **T** | **T** | **T** |  |  | **T** |  |  | **F** | **T** |
| 2’ | 1 | 4’ |  |  | 2 |  |  | 3 | 4 |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| P |  | Q |  | ~ | Q |  | ∴ |  | P |
| **F** | **T** | **F** |  | **T** | **F** |  |  |  | **F** |
| 3’ | 1 | 2’’ |  |  | 2 |  |  |  | 3 |

To invalidate *modus tortoise ponens*, row 1 of the truth table for the conditional must be T, and to invalidate *modus tortoise tollens*, row 4 of the truth table for the conditional must also be T.

Given this standard truth-functional definition of the conditional, other validities also follow. Some of these have a paradoxical quality if the conditional is interpreted “implies” and so these are known as the *paradoxes of material implication*.

One of the most puzzling validities is known as *Lewis’s Dilemma*:

P  ~P  Q ,

which states *“a contradiction implies anything”*. This implication follows from the inference rules of simplification, addition, and *modus tollendo ponens*, which are themselves not particularly puzzling.

1. *~~Show~~* P  ~P  Q 6, CD

2. P  ~P Assume (CD)

3. P 2, S

4. ~P 2, S

5. P  Q 3, ADD

6. Q 5, 4 MTP

Therefore, the justification for the truth table for the conditional, the definition of semantic validity, and the derivation of other theorems using conditional derivation and standard rules of inference are all interconnected. To question one of these elements requires revisions of all the other concepts to form a coherent whole.

The following theorems (and tautologies) are known as the paradoxes of material implications because of their variance from the natural language uses of the conditional.

T18 ~P  (P  Q) False Antecedent—*“whenever ~P, P implies Q”*

T2 Q  (P  Q) True Consequent—*“whenever Q, P implies Q”*

T58 (P  Q)  (Q  R) Conditional Excluded Middle—*“Q is either an antecedent or consequent”*

T114 (~P  P) ↔P *Consequentia Mirabilis—“P is equivalent to ~P implies P”*

P  ~P  Q Lewis’s Dilemma—*“a contradiction implies anything”*

C. I. Lewis investigated modal logic in order to find a conditional—strict implication—which did not result in these paradoxes. Lewis defined strict implication P ⇒ Q (read “P *strictly implies* Q”) by combining modality with the truth-functional conditional:

P ⇒ Q :=(P  ~Q) ,

or alternatively,

P ⇒ Q :=□(P  Q) .

Lewis then added axioms such as axiom **4** and axiom **5** to characterize different implication relations.

Nevertheless, strict implication inherited some of the paradoxical qualities of the material implication.

T551 ~P → □(P → Q) *an impossible antecedent implies anything*

T550 □Q → □(P → Q) *a necessary consequent is implied by anything*

T552 (P → Q) ∨ □(Q → P) *every proposition is either the consequent of a possible conditional or*

*the antecedent of a necessary conditional*

The *counterfactual conditional* (or subjunctive condition) states what would be the case if its antecedent were true (although it is not true). In contrast, the material (or indicative) conditional indicates what is (in fact) the case if its antecedent is (in fact) true (which may or may not be true).

The difference between *indicative* and *subjective* conditionals can be illustrated:

(1) If Oswald *did* not shoot Kennedy, then someone else *did*. Indicative conditional

(2) If Oswald *had* not shot Kennedy, then someone else *would* *have*. Subjunctive conditional

In the first sentence, the antecedent may or may not be true, and the consequent may or may not be true, but what the speaker asserts (in the past tense *indicative* mood) is what would have had to have been the case if the antecedent were in fact true. It is a conditional statement about the actual past.

In the second sentence, the speaker takes it for granted that the antecedent (formulated in the past perfect subjunctive, or what is called the *pluperfect* subjunctive, mood) is contrary to fact and asserts what would have been the case if under that supposition.

Neither the truth-functional conditional nor the strict conditional represents the logic of the counterfactual conditional. The antecedents of counterfactuals are false, but that does not make them all true, as in the case of the material conditional. Neither are counterfactual conditionals strict conditionals.

*Strengthening*: P ⇒ Q ∴ P  R ⇒ Q

*Transitivity*: P ⇒ Q . Q ⇒ R ∴ P ⇒R

Counterexample to Strengthening the Antecedent:

1. If the Nazis had succeeded in constructing an atomic bomb before the U.S., Germany would have won WWII.

2. If the Nazis had succeeded in constructing an atomic bomb before the U.S. and had accidentally detonated it in Berlin, then Germany would have won WWII.

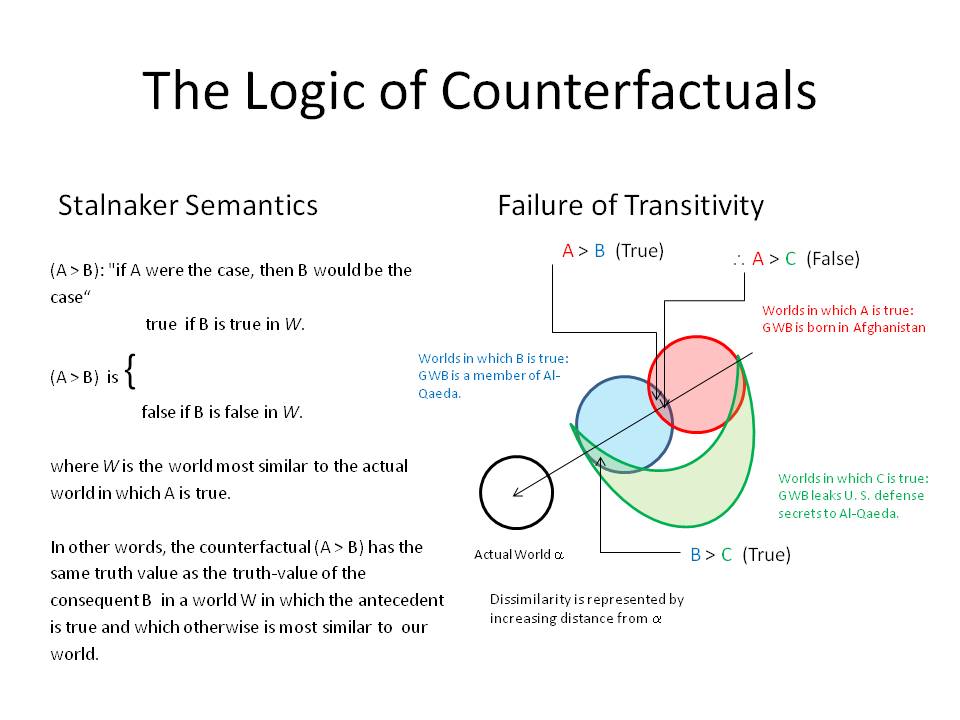
Counterexample to Transitivity:

1. If George W. Bush had been born in Afghanistan, then he would have been a member of Al-Qaeda.

2. If George W. Bush had been a member of Al-Qaeda, then he would have been leaking U. S. defense secrets to Al-Qaeda prior to 9-11.

3. So, if George W. Bush has been born in Afghanistan, then he would have been leaking U. S. defense secrets to Al-Qaeda prior to 9-11.

Why does transitivity fail for counterfactuals? The antecedents of the counterfactuals don’t relate to the same states of affairs. The first premise has to do with states of affairs (or possible worlds) in which--other things being as close as possible to the actual world—GWB was born in Afghanistan. The second premise, however, has to do with states of affairs (or possible worlds) in which--other things being as close as possible to the actual world—in which GWB as President of the U. S. was secretly an Al-Qaeda operative. The states of affairs that the first premise has to do with need not coincide with those of the second: if GWB had been born in Afghanistan, he wouldn't be President of the U. S. and so presumably wouldn't have access to U. S. defense secrets.



### Exercises

1. The counterfactual ‘*would’* and ‘*might’*.

P □Q If P had been the case, then Q *would* have been the case.

P ◊Q If P had been the case, then Q *might* have been the case.

Negation of a Stalnaker Counterfactual [NSCF]:

~(ϕ □ ψ) ∴ ϕ □ ~ψ

Stalnaker Counterfactual Excluded Middle: (ϕ □ ψ)  (ϕ □ ~ψ)

Lewis’s “*Might*” Counterfactual: (ϕ ◊ψ) := ~(ϕ □ ~ψ)

Notice that given the Stalnaker Counterfactual Excluded Middle, Lewis’s definition of “*might*” would collapse “*would*” and “*might*” counterfactuals.

How can Stalnaker define the “*might*” counterfactuals in terms of the “*would*” counterfactual?

P Q := (P □Q)

P □Q is true in a world α iff Q is true is *all* the worlds that make P true and are closest to Q.

P Q is true in a world α iff Q is true is *at least one* of the worlds that make P true and are closest to Q.

# 10. Modal Provability Systems of Gödel and Löb.

Soon after publishing his famous Incompleteness Theorems in 1931, Gödel wrote a short note on the embedding intuitionistic logic in modal logic entitled “An interpretation of intuitionistic propositional calculus [1933f].” What was Gödel’s central insight? Intuitionistic *truth* is characterized in terms of *proof*. Now since *provability* is a kind of *necessity*, modal logic be used in the investigation of the logic of provability.

Gödel’s idea is to interpret Heyting’s intuitionistic propositional calculus using classical propositional calculus supplemented with the modal notion “*p* is provable”, which he symbolized “B*p*” (for *beweisbar*). Gödel gave a particularly perspicuous axiomatization. Gödel’s three proposed axioms (in an equivalent form with now standard code letter) are as follows:

(**T**) B*p* → *p*

(**K**) B(*p* → *q*) → (B*p* → B*q*)

(**4**) B*p* → BB*p*

Gödel’s insight anticipated the discovery and development of modal provability systems in the 1970s.

|  |  |
| --- | --- |
| 1932 | Gödel in a short note “*Zum intuitionistischen Aussagenkalkül*” describes an interpretation of intuitionistic propositional logic in Lewis’s modal system S4. |
| 1942 | Haskell Curry formulates “Curry’s Paradox”, a negationless form of the semantic paradoxes. |
| 1950 | Leon Henkin poses the question, “Is the statement ‘I am provable’ is provable?” |
| 1955 | Arthur N. Prior recasts the Curry Paradox in a more perspicuous way. |
| 1963 | Richard Montague in “Syntactic treatments of modality, with corollaries on reflexion principles and finite *axiomatizability*” [*Acta Philosophica Fennica*, 16:153-167] notes the connection between provability as a modal operator and self-reference. |
| 1965 | Martin Löb answers Henkin’s question in the affirmative by proving “Löb’s theorem” |
| 1970’s | Development of abstract provability logics to model Gödel’s Incompleteness Theorems. |
| 1974 | Boolos and Jeffrey, *Computability and Logic* |
| 1979 | Boolos, *The Unprovability of Consistency: An Essay in Modal Logic* |
| 1984 | Barwise and Entchemendy point out the parallel between Curry’s paradox and Löb’s theorem |
| 1989 | Boolos and Jeffrey rehearse the Curry paradox without citation, referring to it as Löb’s theorem |
| 1993 | Boolos, *The Logic of Provability* |
| 2007 | Burgess updates Boolos and Jeffrey’s *Computability and Logic* (4th edition) |
| 2009 | Boolos, *The Unprobability of Consistency* (2nd edition) |

Gödel’s First Incompleteness Theorem (1931) is based on constructing a sentence in a formal system that intuitively says, “I am *not* provable.” In 1950 Leon Henkin asked if the sentence “I *am* provable” is provable. In 1954 Martin Löb answered this question in the affirmative. Then in 1963 Richard Montague stated the connections between provability, modality and self-reference. However, it was not until the 1970s that logicians fully formalized Gödel’s Incompleteness Theorems showing that the abstract logic of provability could be captured in propositional modal logic.

Modal logic is useful in clarifying our understanding of central results concerning provability in the foundations of mathematics (see, Boolos [1993]). Provabilty logics are systems where the propositional variables range over formulas of some mathematical system such as Peano Arithmetic. Gödel showed that arithmetic has strong expressive powers. Using Gödel numbering as a method for coding arithmetic sentences, he was able to demonstrate an effectively calculable correspondence between sentences of arithmetic and facts about which sentences are and are not provable in Peano Arithmetic.

In Provability Logics, ‘□P’ is interpreted as expressing that ‘P is provable’ in a system M such as Peano arithmetic. Using ‘⊥’ (read “*eet*”, the reverse of ‘*tee*”, alternatively ‘*tac’* in contrast to ‘*top’*) as a constant of provability logic denoting falsehood or a logical contradiction, the sentence

~ □⊥

says that a contradiction is not provable in system M and so says that M is consistent. We define Consis(M) to be the above sentence. The best-known Provability logic adds an axiom **GL** (for ‘Gödel-Löb)):

□(□ϕ → ϕ) → □ϕ .

Axiom **T** ‘□ϕ → ϕ’ says that Peano Arithmetic is *sound*, namely, that if ϕ were provable, then ϕ would be true. This claim is too strong if we wish to countenance the possibility that Peano Arithmetic might prove something that is false. Intuitively, **GL** is the more modest claim that if Peano Arithmetic proves a sentence that claims soundness for a given sentence ϕ, then ϕ is already provable in Peano Arithmetic. Peano Arithmetic does not insist that a proof of ϕ entails ϕ is true unless it already has a proof of ϕ to back up that claim.

We specify a symbolic language for Gödel-Löb Provability logic as follows. The symbols of this language will be

P, Q, R, …., Z sentence letter*s P through Z, without or without numerical subscripts;*

⊥ *falsehood (eet, tac*);

→ conditional sign,

□ provability operator

), ( pair of parentheses.

The sentences of this language are exhaustively characterized by the following clauses:

* + - 1. *The logical constant* ‘⊥’ *is a symbolic sentence*
      2. *Sentence letters are symbolic sentences.*
      3. *If* ϕ *and* ψ *are symbolic sentences, then so is*

(ϕ → ψ) .

* + - 1. *If* ϕ *is a symbolic sentence, then so is*

□ϕ .

Using von Bentham’s notation, we can express the above formal language concisely:

AT (for ‘atomic’) := P, Q, R, S, …, Z

ϕ : : = |AT|⊥|~ϕ| (ϕ → ψ) | □ϕ|

The intended interpretations for these expressions are

* 1. Sentence letters are sentences of Peano Arithmetic.

(2) ⊥ is some designated falsehood such as ‘0 = 1’.

(3) Expressions of the form (ϕ → ψ) are conditionals.

(4) ‘□ϕ’ is interpreted as saying that sentence ϕ is provable in Peano Arithmetic.

Note that {→, ⊥} is tautologically complete or ‘adequate’ to express all binary truth functions.

A *provability system* M of type **4** consists of the inference rule MP together with the axioms:

(**K**) □(ϕ →ψ) → (□ϕ → □ψ)

(**4**) □ϕ → □□ϕ .

The Gödel-Löb Provability Logic is a provability system type **4** together with the axiom

(**GL**) □(□ϕ → ϕ) → □ϕ .

Next we can state three equivalent formulations of the fixed-point properties of provability systems:

* A system M is *self*-*reflexive* if ∀ formula ϕ, there exists a sentence σ such that

M ⊢ σ ↔ (□σ → ϕ).

* A system M has the *Löb property* if ∀ formula ϕ,

M⊢ (□ϕ → ϕ) ⇒ M|− ϕ .

* A system M is *Gödelian* if ∀ formula ϕ,

M ⊢ □(□ϕ → ϕ) ⇒ M|− □ϕ .

Fixed-Point Metatheorem. If M is type **4**, then the following are equivalent:

* 1. M is *self-reflexive*;
  2. M has the *Löb property*; and
  3. M is *Gödelian*.

We can now state abstract forms of the Gödel Incompleteness Theorems within our modal provability systems. By the fixed-point lemma, there is a sentence G such that

⊢ G ↔ ~□G ,

This sentence, the *Gödel sentence*, intuitively says, “I am not provable.” Notice that if the Gödel sentence is provable in some system S then, if the above fixed-point biconditional is also provable, it would be provable that the Gödel sentence is *not* provable, rendering system S inconsistent.

Gödel’s First Incompleteness Theorem states that if Peano arithmetic is consistent, then it cannot prove the Gödel sentence. Now the consistency of Peano arithmetic can be expressed by a sentence stating that the arithmetical falsehood 0 = 1 is not provable:

Cons(PA): = ~□ 0 = 1 .

Using these ideas, we can express Gödel’s First Theorem in modal provability logic as follows:

Gödel’s First Incompleteness Theorem. Given that

⊢ G ↔ ~□G ,

if Peano arithmetic is consistent, then it cannot prove the Gödel sentence G, i.e.,

⊢ Cons(PA) → ~□ G .

*Proof*:

1. ⊢ G ↔ ~□G Fixed-point theorem

2. *~~Show~~* Cons(PA) → ~□G 12, DD

3. ⊢ G → ~□G 1, BC

4. ⊢ □G → □~□G (**K**) Lemma

5. ⊢ □G → □□G Axiom (**4**)

6. *~~Show~~* ⊢ □G → □ 0 = 1 10, DD

7. ⊢ ~□G → (□G → 0 = 1) T18

8. ⊢ □~□G → □(□G → 0 = 1) (**K**) Lemma

9. ⊢ □~□G → (□□G → □ 0 = 1) Axiom (**K**)

10. ⊢ □G → □ 0 = 1 4, 9, 5

11. ⊢ ~□ 0 = 1 → ~□G 6, contraposition

12. ⊢ Cons(PA) → ~□G 11, Def. Cons(PA)

Gödel’s Second Incompleteness Theorem exploits the fact that it is provable that if Cons (PA) is provable, then G is provable. Let’s take a closer look at the subsidiary derivation for what we might call *Gödel’s lemma*:

□G → □ 0 = 1 .

Notice that taking the contraposition of Gödel’s lemma, we have

~□ 0 = 1 → ~□G ,

whose consequent, by the fixed-point lemma, implies G. Thus, we have by the transitivity of the conditional:

~□ 0 = 1 → ~□G

↓

G

Now by the (K) lemma, we distribute a □ of the antecedent and consequent of the conditional theorem:

□~□ 0 = 1 → □G ,

whose consequent, by the Gödel Lemma again, implies □ 0 = 1:

□~□ 0 = 1 → □ 0 = 1 .

By contraposition once more, we have:

~□ 0 = 1 → ~□~□ 0 = 1 ,

which, by the definition of Cons(PA) as ~□ 0 = 1, becomes:

Gödel’s Second Incompleteness Theorem. If Peano Arithmetic is consistent, the it cannot prove its own consistency, i.e.,

Cons(PA) → ~□ Cons(PA) .

*Proof*:

1. ⊢ G ↔ ~□G Fixed-point theorem

2. *~~Show~~* ⊢ Cons(PA) → ~□ Cons(PA) 13, DD

3. ⊢ G → ~□G 1, BC

4. ⊢ □G → □~□G (**K**) Lemma

5. ⊢ □G → □□G Axiom (**4**)

6. *~~Show~~* ⊢ □G → □ 0 = 1 Gödel’s Lemma

*Same as before*

7. ⊢ ~□ 0 = 1 → ~□G 6, contraposition

8. ⊢ ~□G → G 1, BC

9. ⊢~□ 0 = 1 → G 7, 8 transitivity

10. ⊢ □~□ 0 = 1 → □G (**K**) Lemma

11. ⊢ □~□ 0 = 1 → □ 0 = 1 10, 7

12. ⊢ ~□ 0 = 1→ ~□~□ 0 = 1 11, contraposition

13. ├⊢ons(PA) → ~□ Cons(PA) 12, def. Cons(PA)

What exactly was Gödel’s Second Incompleteness Theorem and what effect did it has on Hilbert’s program? In addition to constructing the Gödel statement G for the formal system S, the argument establishing the implication “if S is consistent, then G is not provable in S” could be carried out with S itself. Moreover, the property of being a Gödel number of a proof in S is a computable one, and so ‘S is consistent’ is a Goldbach-like statement, a statement which if false, can be shown to be false by a computation. Thus, Gödel’s Second Incompleteness theorem follows: if S proves the statement Con(S) expressing ‘S is consistent’ in the language of S, then S proves G, and hence S is in fact inconsistent. Hilbert’s metamathematical program calling for consistency proofs for formal systems such as arithmetic in which all finitistic arguments can be formalized was effectively dashed by the Second Incompleteness Theorem.

Franzén carefully points out three common misconceptions about the Second Incompleteness Theorem. “First, it is often said that Gödel’s proof shows G to be true, or to be ‘in some sense’ true. But the proof does not show G to be true. What we learn from the proof is that G is true if and only if S is consistent. In this observation, there is no reason to use any such formulation as ‘in some sense true’…” Secondly, Gödel’s theorem does not rule out consistency proofs using methods not formalizable within Peano Arithmetic. Thirdly, “[a]nother aspect of the second incompleteness theorem that needs to be emphasized is that it does not show that S can only be proven consistent in a system that is stronger than S.” For example, Gentzen proved the consistency of Peano Arithmetic (PA) in 1936 by application of an arithmetically expressible instance of transfinite induction up to Cantor’s ordinal ε0 (the least fixed point under ordinal exponentiation to the base ω), while otherwise using arguments that can be formalized in a very weak subsystem of PA. So the consistency of PA is proved in a system that is overlaps in part with PA but is not an extension of it.

# 11. Philosophical Remarks (*due to Nathan Salmon; slighted edited, redundancies remain*).

Modal logic—the logic of what might have been and of what must be—raises difficult issues and questions that remain a matter of significant controversy. Some of these issues concern modal sentential logic, others the blending of the logic of quantification with that of modality.

The principal issue concerning modal sentential logic involves nested modality, wherein some modality is attributed to a state of affairs that is itself already modal. Consider the following argument:

It is mathematically necessary that 2 + 3 = 5 .

Therefore, that 2 + 3 = 5 could not have been merely accidental .

It is unclear whether the premise entails this nested modal conclusion, even if it is true, as opposed, say, to the unnested modal conclusion that 2 + 3 = 5is not accidental.

The sentential logic of un-nested modality is known as ‘**T**’. Let us say that the antecedent of a necessary conditional *strictly implies* the consequent. Then we may say that that **T** is motivated by four principles: (1) that necessity and possibility are duals (i.e., something is necessary just in case its denial is impossible and that something is impossible if its denial is necessary); (2) that whatever is necessary is true; (3) that the logical truths of classical and modal logic are necessary truths; and (4) that whatever is strictly implied by a necessary truth is itself a necessary truth (i.e., if the antecedent of a necessary conditional is itself necessary, then the consequent is also necessary). Any deductive apparatus for modal logic that respects these four principles is known as a *normal system*.

System ***T*** is the weakest normal system. It assigns no special logic to the nesting or iteration of modality. Whereas the sentential logic of un-nested modality is relatively uncontroversial, there exist numerous rival systems dealing with nested modality. The most important normal systems that impose a special logic on nested modality are **B** (named for the intuitionist, L. E. J. Brouwer) and **S4** and **S5** (named after C. I. Lewis’ systems for strict implication). Each of these three systems builds on **T** through the addition of a specific principle governing an iterated modality. Recall that system **B** adds the principle that whatever is the case is such that its possibility is necessary:

P → □P ,

(or equivalently, whatever is possibly necessary is the case). System ***S4*** adds the principle that whatever is necessary is such that its necessity is itself necessary:

□P → □□P ,

(or equivalently, whatever is possibly possible is possible). System ***S5*** adds the principle that whatever is possible is such that its possibility is necessary:

P → □P ,

(or equivalently, whatever is possibly necessary is necessary). System **S5** is the strongest of the four systems under discussion. Recall that the ***S5*** principle, together with the basic sentential modal logic **T**, yields both the **B** and **S4** principles. Whereas **S5** thus includes **T**, **B**, and **S4**, neither of **B** and **S4** includes the other.

The study of modal logic underwent a quantum leap with the advent of a semantic analysis due to Saul Kripke [1963], among others, based on Leibniz’ notion of a *possible world*. A possible world, on one conception, is an entire world history that might have obtained. Socrates might have been a Spartan if and only if there is a possible world in (or, according to) which Socrates is a Spartan. In possible-world semantics, the classical extensional semantic attributes, like truth and denotation, are intensionalized by relativization to a possible world.

A sentence is deemed true with respect to a possible world just in case the proposition expressed is true in that world. Necessity of a proposition is identified with its truth in all possible worlds, possibility with truth in some possible worlds. More accurately, to say that a proposition P is *necessary* in a world *w*1 is to say that Pis true in every world *w*2 that is a possible world in *w*1—i.e., P is true in every world *w*2 such that in *w*1,*w*2 is a genuine possibility (or, possible relative to) *w*1. To say that a proposition Pis *possible* in a world *w*1 is to say that Pis true in at least one world *w*2 that is possible in (or, relative to) *w*1. With these identifications, the sentential logic of modality is easily derived from the classical logic of ‘*all’* and ‘*some’*, in much the same way that the logic of ‘*all’* and ‘*some’* may be based on the logic of ‘*and’* and ‘*or’*. The four basic principles of **T** emerge as analogues, respectively, of Quantifier Negation, Universal Instantiation, Universal Derivation, and T201.

This conception of possible world semantics extends straightforwardly to the iteration of modality. To say that Pis necessarily necessary in a world *w*1 is to say that for every world *w*2 that is possible in *w*1, for every world *w*3 that is possible in *w*2, Pis true in *w*3, i.e., P is true in every world that is possible in any world that is possible in *w*1. [What is possible in that world is possible in any world.] To say that *p* is necessarily possible in *w*1 is to say that for every world *w*2 possible in *w*1, there is least one world *w*3 that is possible in *w*2 and in which *p* is true. In the terminology of Kripke, a world *w*2 is said to be *accessible to* a world *w*1 if *w*2 is possible in *w*1. The binary relation of accessibility is reflexive in any normal sentential modal logical system. Kripke demonstrated a close connection between each of the four main sentential modal logical systems and accessibility. If accessibility is symmetric, then the **B** principle is verified. If accessibility is transitive, the **S4** principle is verified. And if it is an equivalence relation⎯reflexive, symmetric, and transitive⎯the **S5** principle is verified.

There are many notions of necessity. Whereas it is logically necessary that either nine is odd or it is not, it is *mathematically* necessary that nine is indeed odd. It is *metaphysically* necessary, and arguably also logically necessary, that if *x* and *y* are one and the same object, then *x* and *y* are exactly alike in every respect. (As mentioned in chapter V, this Leibnizian principle of the *indiscernibility of identicals* is reflected in the inference rule of Leibniz’ Law.) Whereas it is *physically* necessary that the force acting on a physical object is equal to the product of the mass of the object with its acceleration, this is neither *logically* nor *mathematically* nor *metaphysically* necessary. Not all notions of necessity need share the same logic.

Physical necessity may be seen as a restricted notion of necessity: A proposition Pis *physically necessary* in a world *w* just in case the laws of the physics of *w* strictly imply it, in the sense that Pis true in every world that is accessible to *w* and in which the physical laws of *w* are true. Though any actual physical law is thus *physically* necessary, that same law may be *metaphysically* contingent; there may be a world that is accessible to the actual world and in which the law fails. Had there been an additional physical law over and above those that actually obtain (perhaps because of the physical possibility of some universal generalization that does not in fact obtain, or of the evolution of some species that does not in fact exist), the actual physical laws together with the additional law would all be physically necessary, and the actual state of affairs thus physically impossible. This would rule out B, hence also **S5**, as the logic of physical modality.

It is often maintained that metaphysical necessity is unrestricted necessity. A proposition is *metaphysically* necessary in a world *w*, it is observed, just in case it is true in every possible world accessible to *w*, without exception. This would yield the result that the sentential logic of metaphysical modality is **S5**. The argument for the **S5** principle that *whatever is metaphysically possible is metaphysically necessarily so* is straightforward: Assume that *p* is metaphysically possible. Then there is at least one possible world *w* (which may or may not bear any special relation to the actual world) in which Pis true. In that case, for every possible world *w′*, without exception, there is at least one world (which may or may not bear any special relation to *w*′) in which Pis true, *viz*., *w*. Though many philosophers have been persuaded by this argument, it is in fact fallacious.

Whether a given world is metaphysically possible or not might be a contingent fact. The same world that is metaphysically possible in (i.e., accessible to) one world might be metaphysically impossible in another. Even metaphysical modality thus involves one crucial restriction: P is metaphysically necessary in a world *w* just in case *p* is truein every world *that is metaphysically possible in* *w*; and P­­­­­­ is metaphysically possible in *w* just in case *p* is true in at least one world *that is metaphysically possible in* *w*1. Suppose *p* is metaphysically possible in some world *w*1 (e.g., the actual world),so that there is a world *w*2 that is metaphysically possible in *w*1 and in which Pis true. It does not follow that for every world *w*3 metaphysically possible in *w*1, there is a world that is metaphysically possible in *w*3 and in which *p* is true. What follows instead is that for every world *w*3 possible in *w*1, there is a world—*viz*., *w*2—that is possible *in a world in which w3 is possible* and in which *p* is true. But we have so far no reason to suppose that *w*2 is possible in *w*3, and hence no reason to suppose that there is a world that is possible in every world possible in *w*1 and in which *p* is true.

To presuppose that because *w*2 and *w*3 are each possible in *w*1, *w*2 is likewise possible in *w*3 is tantamount to the assumption that accessibility is *euclidean*:

∀*x*∀*y*∀*z*(R*xz* & R*yz* ⇒ R*xy*) .

A binary relation is both reflexive and euclidean if and only if it is an equivalence relation. But the argument provides no reason to suppose that accessibility is euclidean.

On the contrary, there is a compelling reason to suppose that metaphysical accessibility is *not* transitive (let alone an equivalence relation)—or at least, that logic does not preclude the prospect that accessibility is intransitive. Some metaphysicians (including the present writer) maintain that a typical material artifact, such as a bicycle or a ship, might have originated from at least slightly different matter (molecules) from its actual original matter, but could not have originated from altogether different matter. This suggests that for each material artifact, there is a threshold or limit such that the artifact might have originated from matter different from its actual original matter up to that limit, but could not have originated from matter that differs any more than that.

Consider the *Paradox of the Ship of Theseus* goes back to a Greek legend reported by Plutarch:

The ship wherein Theseus and the youth of Athens returned [from Crete] had thirty oars, and was preserved by the Athenians down even to the time of Demetrius Phalereus, for they took away the old planks as they decayed, putting in new and stronger timber in their place, insomuch that this ship became a standing example among the philosophers, for the logical question of things that grow; one side holding that the ship remained the same, and the other contending that it was not the same.

There is also an additional perplexity: if the replaced parts were stored in a warehouse and later used to reconstruct the ship, which—if either—would be the original ship of Theseus?

Suppose, without any loss of generality, that in a world *w* (perhaps the actual world), a given ship *S* which originated from particular matter *m* might have originated instead from the same amount of matter but with as many as 10% different molecules, and could not have originated from the same amount of matter as *m* while differing from *m* by more than 10%. Then there is a world *w*′ that is metaphysically possible in (i.e., accessible to) *w* and in which *S* originated from particular matter *m*′ that is 90% of *m* together with 10% different matter. Now in *w*′, *S* might have originated from matter that differs from *m′* by as much as 10%. Hence, there is a world *w″* that is possible in *w*′ and in which ship *S* originated from particular matter *m*″ that is more than 90% the same as *m*′, but less than 90% of *m* with more than 10% different matter from *m*. But in *w*, *S* could not have originated from *w*″. Hence, though *w*″ is possible in *w*′, which is itself possible in *w*, *w″* is not possible in *w*. In *w*, *w*″ is an impossible (albeit possibly possible) world. It is only contingently necessary in *w* that ship *S* does not originate from matter *m*″. This yields a counter-instance to the **S4** principle that whatever is necessary is necessarily so.

On this metaphysics of materiality, metaphysical accessibility is intransitive. Opponents may argue that any such metaphysic is not only incorrect but necessarily so. Nevertheless, it cannot be plausibly argued that such a metaphysic is logically inconsistent. On the contrary, whether it is necessarily correct or necessarily incorrect, the theory in question is obviously coherent. Hence the logic of metaphysical modality is not as strong as **S5**, or even **S4**. No equally plausible counter-instance to **B** (as a deductive system governing metaphysical modality) is presently known. But even if the **B** principle is correct, here also it seems that its truth is not required by logic. As far as logic is concerned, the actual world might have been a metaphysically impossible world instead of a metaphysically possible one (even if this prospect is metaphysically impossible).

The logic of combining of modality with quantifiers is known as *quantified modal logic*. Quantifiers and Leibniz’ Law call for special attention in modal logic. According to the logician Willard van Orman Quine, identity is governed by Leibniz’s Law stated as a *principle of substitutivity*: the terms of a true identity statement are everywhere intersubstitutive, preserving truth (or *salva veritate)*. As soon at the principle is stated, however, one is confronted with counterexamples. For example, although there are (let us suppose) exactly nine planets in the solar system, this is surely a matter of contingent fact—there might have been eight planets instead of nine. Though it is mathematically necessary that nine is odd, it is no necessary truth that the number of planets is odd. A puzzle, which we will call *Quine’s conundrum*, points to a needed restriction on Leibniz’ Law formulated as a principle of substitutivity. Relative to the scheme of abbreviation,

A : 9

F : *a* is odd

G : *a* numbers the planets,

The sentence

(0) □FA

is true, whereas

(1) □Fι*y*G*y*

is false, even though

A = ι*y*G*y*

is true. Where a symbolic term ζ stands within a formula jζ within the scope of an occurrence of a modal operator, and either ζ or η is a definite description, Leibniz’ Law may be weakened as follows:

□ζ = η

jζ

jη

where jη is like jζ except for having free occurrences of η where jζ has free occurrences of ζ. The stronger form of Leibniz’s Law given in Chapter V remains operative when neither of the symbolic terms ζ nor η is a definite description, and also when jη is like jζ except for having free occurrences of η where jζ has free occurrences of ζ not within the scope of an occurrence of a modal operator. A more general form of Leibniz’s Law is employed in this chapter, specifically:

□□…□ ξ = η

jζ

jη

where ‘□□…□’ represents a string of occurrences of ‘□’ whose length is unrestricted if neither ζ nor η is a definite description, and is otherwise at least as great as the largest number of occurrences in φζ of symbolic formulas of the form

□ψ

or of the form

ψ

where ψ is a symbolic formula, such that there is a free occurrence of ζ that stands within each (i.e., the largest number of modal-operator occurrences having the same free occurrence of ζ in their scope).

A choice needs to be made concerning the range of quantification with respect to a given world. *Possibilist quantification* allows the quantifiers to range over all possible individuals with respect to all worlds. So-called *actualist* (or internal) quantification, which is the alternative employed in this chapter, allows ~~lets~~ each quantifier range with respect to a world *w* only over the individuals that exist in *w*. Universal derivation and the rules of existential instantiation, universal instantiation, and existential generalization are modified accordingly. This choice also excludes as theorems the *Barcan formula*

∀*x*□F*x* → □∀*x*F*x ,*

and its converse, as well as the *Martin*

*formula*,

∃*x*□F*x* → □∃*x*F*x* ,

and its converse. Instead there is a *weakened version of the converse Barcan formula*,

□∀*x*F*x →* ∀*x* □(∃*y* *x* = *y* → F*x*) ,

as well as a *weakened version of the Buridan formula*,

∃*x*□(∃*y x* = *y* ∧ F*x*) → □∃*x*F*x .*

Each of these formulas involves quantification across a modal operator into a modal context. Quine argued that such constructions are problematic at best, perhaps even incoherent. Consider the number, which is both nine and the number of planets, apart from any particular manner of specification. Is it a necessary truth about this number that it is odd? Abstracting from any particular manner of specification (or what Frege [1892] called a *mode of presentation*) of the number, then the grounds for saying that it is a necessary feature of the number that it is odd—*viz*., that it is necessary that *nine* is odd—seem perfectly counterbalanced by equally good grounds for saying that the oddness is only a contingent feature: it is not necessary that *the number of planets* is odd. Yet Leibniz’ Law prohibits one from consistently saying that oddness is a necessary feature of nine and, at the same time, only an accidental feature of the number of planets. For the number of planets *is* nine; hence, any property of the number of planets (e.g., being contingently odd) is equally a property of nine, and vice versa.

More formally, there seems to be no coherent answer to the question of whether the number that is both nine and the number of planets satisfies the open formula

(2) □F*x*

relative to the scheme of abbreviation displayed above. A symbolic formula of the form

j ,

where φ is a symbolic formula, is true with respect to a world *w*, and under an assignment *s* of values to variables, just in case the proposition expressed by j under assignment *s* is a proposition that is a necessary truth in *w*—i.e., just in case j is true, under *s*, with respect to every world accessible to *w*. In particular, a symbolic formula of the form

□Fα ,

where α is a symbolic term that denotes the number nine, is true on the scheme of abbreviation displayed above, and under an assignment *s* of values to variables, if and only if the proposition expressed by

Fα ,

on the same scheme of abbreviation and under assignment *s*, is a necessary truth. But which proposition is expressed depends on the manner in which the term α specifies nine. If α specifies nine as *the successor of eight* (or as *the predecessor of ten*, *the square of three*, etc.), a necessary truth is expressed. If α specifies nine as *the number of planets* (or as *Bill’s favorite odd number*, etc.), a contingent truth is expressed. When the variable ‘*x*’ occurring in (2) is assigned the number of planets as its value, it denotes nine but does not specify nine in any particular manner. We are thus at a loss concerning whether the formula ‘F*x*’ expresses a necessary truth under that assignment. The result of substituting ‘A’ for ‘*x*’ in (2) is true, whereas the result of substituting ‘ι*y*G*y*’ is false, even though each of the two substituends—‘A’ and ‘ι*y*G*y*’—denotes nine. This situation appears to render the construction ‘∃*x*□F*x*’ without a coherent interpretation.

As Arthur Smullyan [1948] noted against Quine, Russell’s [1905] theory of descriptions, when applied to quantified modal logic, yields a straightforward solution to this conundrum. According to Russell’s theory, a definite description is not a term, and symbolic sentence (1) is subject to an ambiguity of scope. One reading of the sentence (the Russellian “*secondary occurrence*” reading) is given by:

□∃*x*[∀*y*(G*y* ↔ *x* = *y*) ∧ F*x*] ,

i.e., necessarily, something that uniquely numbers the planets is odd. This is false, since there might have been an even number of planets. The other reading of (1) (the “*primary occurrence*” reading) is given by:

∃*x*[∀*y*(G*y* ↔ *x* = *y*) ∧ □F*x*] ,

i.e., something that uniquely numbers the planets is necessarily odd. This is evidently true, since the number in question (nine, i.e., the number of planets) is such that it could not have been an even number instead of odd.

The matter of interpreting (2) is straightforward on a Russellian theory. (Notice, however, that Russell’s theory of descriptions plays no role in the interpretation.) The proposition expressed by ‘F*x*’ relative to the scheme of abbreviation displayed above, and under the assignment of the number of planets as value for the variable ‘*x*’, is the proposition about the number in question that it is odd. The latter is a *singular proposition*, one that is about a particular individual that occurs as a propositional component, representing itself in the proposition. The singular proposition about nine that it is odd is true in any possible world in which nine is an odd number (whether or not it numbers the planets in that world). Formula (2), when evaluated under the assignment in question, functions in a manner exactly analogous to (0).

Semantic theories—like those of Frege [1892], Church [1946], and Quine [1950]—that reject singular propositions may not avail themselves of this theoretically elegant and satisfying solution to Quine’s conundrum. Such theories fail to provide a proposition to serve as the content of an open formula like ‘F*x*’, leaving (2) in need of interpretation. The insistence that denotation is invariably determined by an accompanying manner of specification leads to a famous problem in modal metaphysics: the so-called problem of *cross-world identification*.

The open formula ‘F*x*’ is true relative to the scheme of abbreviation displayed above, and under the assignment of nine as value for the variable ‘*x*’, if and only if for every possible world *w* (i.e., for every world *w* accessible to the actual world), the individual denoted by the variable ‘*x*’ with respect to *w*, and under the same assignment of nine as value, is odd in *w*. The problem is that, though the variable ‘*x*’ is assigned nine (*qua* the number of planets) as its value with respect to the actual world, it is not assigned any particular manner of specification of nine (e.g., as *the number of planets*, as *the successor of eight*, or in some alternative manner). This fails to provide any fact of the matter concerning which item from *w*, if any, is the value of ‘*x*’ with respect to *w*.

There is a question concerning which object from *w* is to be identified with the actual number of planets? This alleged problem does not even arise once it is recognized that the variable ‘*x*’ is what Russell called a *logically proper name* of its value. That is, the proposition expressed by the open formula ‘F*x*’ relative to the scheme of abbreviation displayed above, and under the assignment of nine as value for ‘*x*’, is simply the singular proposition about nine that it is odd. One very significant benefit of countenancing singular propositions, in fact, is that the “problem” of cross-world identification is thereby exposed as a pseudo-problem.

Section 12. Historical Remarks.

*Modal logic* is the study of the logical structure of these modes of truth. Ancient philosophers like Aristotle tried to study their logic, and the medieval logicians such as Jean Buridan discussed modal notions with great subtlety.

4th century B. C. Diodorus Cronus poses Paradox of Future Contingents, discussed by Aristotle

1077-78 Anselm’s modal ontological argument *Proslogiom*, Book III (1033-1109)

14th Century Jean Buridan (1300- 1358)

1663–90 Leibniz (1646 - 1716) interprets necessity in terms of possible worlds

1912 C. I. Lewis’s formulates axioms for systems of strict implication

1933 Gödel system **T,** also studied by Feys (1937) and von Wright (1951)

1950’s Deontic and epistemic logics explored

1952 Leon Henkin poses question about sentences that assert their own provability

1955 Martin Löb answers Henkin’s question

1957 A. N. Prior develops temporal logics

1. Saul Kripke, “A Completeness Theorem for Modal Logic”

1970’s Provability Logics, de Jongh, Kripke, Sambin, Segerberg, Smorynski, Solovay

Figure 12. An Incomplete Chronology for Modal Logic

However, the modern treatment of modal logic began around 1912 when C. I. Lewis, after reading Russell and Whitehead’s *Principia* *Mathematica*, proposed various axioms (such as **4** and **5**) to find a connective more suitable than the material conditional to express our informal concept of entailment. In the 1930s Gödel discussed an interpretation of the modal operator as ‘it is provable in system M that,’ and in the 1950s *epistemic*, *doxastic*, *deontic* and *tense-logic* interpretations were intensively investigated.

Around the 1960s modal logic was being investigated by numerous logicians—including, among others, Rudolf Carnap, Jackko Hintikka, Richard Montague, and von Wright. However, it was Saul Kripke’s remarkable paper *A Completeness Theorem in Modal Logic* (1961) published while he was a high school student in Nebraska that presented possible world semantics in an elegant way that ignited decades of logical and philosophical research. Kripke’s *possible world semantics* gave formal expression to the Leibnizian idea that a proposition is *necessary* if it is true in *all* possible worlds, possible if it is true in *some* possible world, and *contingent* if it is neither necessary nor impossible.

Around the 1970s it was noticed that the famous incompleteness theorems of Gödel (1931) were propositional in character and that their logic could be captured in propositional modal logics known as Provability Logics based on Löb’s axiom and Gödel’s well-ordering axiom.

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# Appendix. Modal Theorems

Characteristic Axioms

T501 □(P → Q) → (□P → □Q) **K**

T530 □P → P **D**

T540 □P → P **T**

T550 P → □P **B**

T560 □P → □□P **4**

T570 P → □P **5**

Modal Derivation Principles

T501 □(P → Q) → (□P → □Q)

T502 □(P → Q) → (P → Q)

Laws of Modal Negation

T503 ~□P ↔ ~P

T504 ~P ↔ □~P

T505 □P ↔ ~~P

T506 P ↔ ~□~P

Laws of Modal Distribution

T507 (P ∨ Q) ↔ P ∨ Q

T508 □(P ∧ Q) ↔ □P ∧ □Q

T509 ◊(P ∧ Q) → ◊P ∧ ◊Q

T510 □P ∨ □Q → □ (P ∨ Q)

T511 (P → Q) → (P → Q)

T511.1 (□P → Q) ↔ ~P ∨ Q

T512 (□P → □Q) → P → Q)

T513 □(P ↔ Q) → (□P ↔ □Q)

T514 □(P ↔ Q) → (P ↔ Q)

T515 (□P ∧ Q) → (P ∧ Q)

T516 □(P ∨ Q) → P ∨ □Q

T517 ◊(P → Q) ↔ (□P → Q)

T518 (◊P → □Q) → □(P → Q)

T519 (P → □Q) → (□P → □Q)

T520 (P → □Q) → (P → Q)

T521 (~P ∧ Q) → (~P ∧ Q)

Modal Laws of Strict Implication.

T522 □(P → Q) ↔ ~(P ∧ ~Q)

T523 ~□(P → Q) → P

T524 ~□(P → Q) → ~□Q

Modal Laws Not Applicable to Counterfactual Conditionals

T525 □(P → Q) ∧ □(Q → R) → □(P → R)

T526 □(P → Q) → □(P ∧ R → Q)

Paradoxical Laws of Strict Implication

T527 □Q → □(P → Q)

T528 ~◊P → □(P → Q)

T529 ◊(P → Q) ∨ □(Q → P)

System **D**

T530 □P → P

T531 P ∨ P `

T532 ~(□P ∧ □~P)

T533 □(P → ~Q) → (□P → ~□Q)

T534 □[(P → Q) ∧ (Q→ R) → (P → R)]

T535 □(P → Q) ∧ □(P → R) → □(P → Q ∧ R)

T536 □(P ↔ Q) ∧ □(Q ↔ R) → □(P ↔ R)

T537 (□P ∧ ◊Q) → ◊(P ∧ Q)

T537.1 (□P ∨ Q) → (□P ∨ ◊Q)

T537.2 □P∨ □Q → □(P ∨ □Q)

T538 □(P → Q) ∧ (P ∧ R) → (Q ∧ R)

T539 □(P → Q ∨ R) → □(P → Q) ∨ (P ∧ R)

System**T**

T540 □P → P

T541 P → P

Modal Theorems of **KT**

T542 □(□P → P)

T543 □(P → P)

T544 □□P → □P

T545 P → P

T546 □P → P

T547 □P → □P

Miscellaneous Theorems of **T**

T548 □(□(P → Q ∧ ~Q) → □~P)

T549 □(□(P → Q) ∧ □(P → ~Q) → □Q)

Brouwersche System**B**

T550 P → □P

T551 ◊□P → P

T552 P → ~~P

Modal Theorems of **KB**

T553 □(P → Q) → (P → □Q)

T554 □(P → □Q) → (P → Q)

T555 □P → □P

T556 □(P → □P) ∧ P → P

T557 □(P → □P) ∧ P → □P

Miscellaneous Theorems of **B**

T558 □(P ∧ □Q) → (P ∨ Q)

T559 □(P ∧ Q) → (□P ∨ Q)

System**S4**

T560 □P → □□P

T561 P → P

T562 ◊□P → □P

Modal theorems of **K4**

T563 □(P → Q) → □(□P → □Q)

T564 □(P ∨ Q) → □(P ∨ □Q)

Modal Reduction Laws for **S4**

T565 □P ↔ □□P

T566 P ↔ P

Miscellaneous Theorems of **S4**

T567 □(P → Q) → □(Q → □P)

T567.1 □(P → Q) → □(R → □(P → Q))

T568 □(□□(P → Q) → R) → □(□(P→Q) → □R)

T569 □(□(P → Q) → □(□(Q→R) → □(P → R)))

System **S5**

T570 ◊P → □◊P

T571 ◊□P → □P

Modal Theorems or **K5**

T572 ◊□P → □◊P

T573 ◊□P → ◊P

T574 ◊□P → □□P

T575 □P → □P

T576 P → □P

T577 □(□P ∨ Q) → □P ∨ □Q

T578 □(P ∨ □Q) → □P ∨ □Q

T579 (P ∧ □Q) → (P ∧ □Q)

T580 P ∧ Q → ( ∧ Q)

T581 P ∧ Q → (P ∧ Q)

T582 □(□P ↔ □□P)

T583 □(P ↔ P)

T584 □(□P ↔ □P)

T585 □(P ↔ □P)

T586 □□P ↔ □□□P

T587 □□P ↔ □□P

T588 □P ↔ □□P

T589 □P ↔ □P

T590 P ↔ P

T591 P ↔ □P

T592 □P ↔ □P

T593 □P ↔ □□P

Modal Reduction Laws for **S5**.

T565 □P ↔ □□P

T566 P ↔ P

T594 □P ↔ □P

T595 P ↔ □P

Miscellaneous Theorems for **S5**

T596

T597

T598

T599

List of Figures

Figure 1 An Aristotelian Diamond of Opposition 5

Figure 2 A Grammatical Tree for a Modal Proposition 9

Figure 3 Properties of Accessibility 39

Figure 4. Modal Axioms and Accessibility 40

Figure 5 Logical Containment of the Modal Systems 41

Figure 6 Lemmon Codes for Deontic Modal Systems 41

**Figure 7 Picasso's Chair** 41

Figure 8 Picasso's Electric Chair 42

Figure 9 Accessibility Represented by Directed Graphs 42

Figure 10 Relational Properties of Directed Graphs 42

Figure 11 Transitivity Validates Axiom (4) 44

Figure 12. A Chronology for Modal Logic 77

Index

*accessibility relation*, 37

*accessible line*, 17

*actual world*, 37

Aquinas, St. Thomas, 2

Augustine, Saint, 2

Axiom **5**, 11

Axiom **D**, 11

Axiom **Dum**, 48

Axiom **K**, 13, 38

Axiom **T**, 11

Axiom **W**, 49

Broad, C. D., 48

C. D. Broad, 48

*composition*, 49

Consis(M), 63

*contingency operator*, 57

*contingently false*, 15

*contingently true*, 15

*Diodorean System*, 48

Dummett, Michael, 48

*equivalence relation*, 38

Evodius, 2

*fallacy*, 30

*false English argument*, 30

Gödel, Kurt, 12, 74

*Gödel-Löb Provability System*, 49

grammatical tree, 8

Henkin, Leon, 62

Interchange of Equivalents

modal, 56

Kripke, Saul, 12, 74

Lemmon and Scott

Generalized Correspondence Theorem, 48

*Lemmon code*, 39

Lewis, C. I., 12, 74

*modal reductions laws* for **S4**, 54

*normal*, 38

*partial ordering*, 38

possible world semantics, 12, 75

*Principia* *Mathematica*, 12, 74

Prior, A. N., 48

provability system M of type **4**, 64

Provabilty logics, 62

*reflexivity*, 38

*relative possibility relation*, 37

*scope ambiguities*. *See* scope ambiguities, modal

*scope* of the modal operator, 8

*series*, 38

*similarity*, 38

soundness, Peano Arithmetic, 63

*strict derivation*, 17

Strict Equivalence, 57

Strict Implication, 57

*strict importation*, 17

*tensed theory of time*, 48

*truth of a sentence in a possible world β*, 37

1. This exposition of the standard systems of propositional modal logic in the style of Kalish, Montague and Mar was written with helpful comments from Nathan Salmon, who originally published the elegant idea of incorporating the axioms for the various system of propositional modal logic as restrictions on the rule of repetition within a strict derivation. These restrictions for **T**, **B**, **S4** and **S5** correspond to Kripke’s accessibility conditions of relative possibility in his axiomatic formulation of propositional modal logic. [↑](#footnote-ref-1)
2. The theological, if not the political, roots Great Schism of 1054, which can be traced back to a disagreement about the modalities of the Persons of the Trinity. The Nicean Creed (325) use the term *homoousios* (from the Greek *homo* = ‘same‘ and *ousios* = ‘essence’ or ‘substance’), in contrast to *homoiousios* (from the Greek *homoi* = ‘similar’) making the solitary *i* the jot and tittle of Nicean Creedal Orthodoxy. The Greek Church preferred the latter term since the former had been used by the Syrian Bishop of Antioch to espouse *modal monarchism*, the heresy thatthe Heavenly Father, Resurrected Son and Holy Spirit are not three distinct Persons, but are rather different *modes* or *aspects* of one monadic God perceived by believers as distinct persons. The Latin Church adopted the former siding with Athanasius against the heretic Arius, who denied that Jesus was co-equal and co-eternal with the Father. The Second Council of Nicea (381), among other changes, inserted the word *filioque* (from the Latin *filio* = the son, and *que* = “and”) into the Nicean-Constantinopolitan Creed and the Latin mass. In Latin theology, the three Persons of the Trinity are logically distinguished by the formal relations of “proceeding from” citing such proof texts as Phil. 1:9, Titus 3:6, Acts 2:33. Orthodox theology, citing the words of Jesus in proof texts such John 15:26, regarded the insertion of *filioque* into the Nicean-Constantipolitan creed, is as Semi-Sabellianism. The great church historian Jaroslav Pelikan (*The Melody of Theology*, Cambridge, Mass: Harvard University Press, 1988, p. 90) opined: “If there is a special circle of the inferno described by Dante reserved for historians of theology, the principal homework assigned to that subdivision of Hell for at least the first several eons of eternity may well be a thorough study of all the treatises… devoted to the inquiry: Does the Holy Spirit proceed from the Father only, as Eastern Christendom contends, or from both the Father and the Son as the Latin Church teaches?” In 1989 Pope John Paul II and Patriarch Demetrius knelt together in Rome and recited the Nicene Creed without the *filioque*. [↑](#footnote-ref-2)
3. van Bentham [2010], p. 51, notes “this often-cited analogy is not quite right, if you think about it.” [↑](#footnote-ref-3)