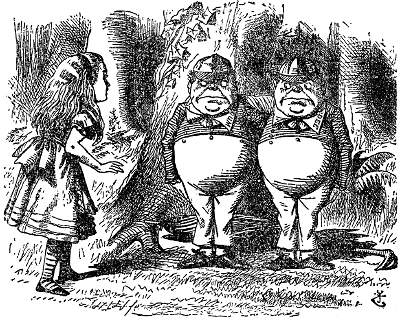
Chapter 7. “*Possibly*” and “*Necessarily*”

*“I know what you’re thinking about,” said Tweedledum; “but it isn’t so, nohow.”*

*“Contrariwise,” continued Tweedledee, “if it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic.”*

Lewis Carroll (alias Charles Dodgson) (1832-1898)

*Through the Looking Glass, and What Alice Found There* [1871]



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# 1. Modes of Truth and Modal Logic.

Historically, *necessity*, *possibility*, *impossibility,* and *contingency* were conceived of as *modes* of truth—ways in which a proposition could be true or false. From the very beginning, modal logic was a part of classical logic. Aristotle’s paradox of the Sea Battle tomorrow raised questions about the truth-values of future contingent propositions. Medieval philosophers—such as Peter Abelard (1079 –1142), John Buridan (1301 – c. 1359/62), and Louis de is de Molina (1535 – 1600)—debated *theologoumenon* or *possible* theological views).

During the Enlightenment, modality achieved its own category—*apodictic judgements* in Kant’s Table of Judgments.

In 1879 in his *Begrifftschrift*, “the founding document of modern logic”[[1]](#footnote-1), Gottlob Frege (1848 – 1925) dismissed modality as having “only grammatical significance”:

The apodictic judgment differs from the assertory in that it suggests the existence of universal judgments from which the proposition may be inferred, while in the case of the assertory one such a suggestion is lacking. By saying that a proposition is necessary I give a hint about the grounds for my judgment. *But, since this does not affect the conceptual content of the judgment, the form of the apodictic judgment has no significance for us*. (italics in text, van Heijenoort [1967], 13)

Frege’s generally negative assessment of the relevance of modality for logic was taken up by the American logician Willard van Orman Quine (1908 - 2000). Quine famously disparaged modal logic as being “conceived in sin” (by which he meant the confusion of use and mention). In a 1938 letter, Quine chastised his mentor and teacher Rudolf Carnap (1891 – 1970):

I proceed to inveigh against your recent intensional propensities... I fear your principle of tolerance may finally lead you even to tolerate Hitler.

To which Carnap gave the spirited reply:

Your sermon against my sin of intensionality has made a great impression on me. But I may say as an apology, I do not indulge in this vice generally and thoroughly.... Although we usually do not like to apply intensional languages, nevertheless we cannot help analyzing them. What would you think of an entomologist who refused to investigate fleas and lice because he dislikes them?[[2]](#footnote-2)

Quine’s conservative dogma counseled logicians to translate issues in philosophical logic into the Procrustean bed of “canonical” first-order logic. Quine’s prejudice against modal logic, and intensional logics more generally, cast a long shadow over the modern development of modal logic until it burst alethically developed from the work of Saul Kripke and othersin the 1960s and 1970s.

In his “whirlwind history” of modal logic, Johan van Bentham notes that the emphasis on extensional logic, as opposed to intensional logics like modal logic, was, in the short run, “immensely beneficial”:

“especially in the analysis of the foundations of mathematics, whose Golden Age was the 1930s with classical results on provability, completeness, computability, and definability by Hilbert, Post, Gödel, Tarski, Turing, and many others.” (p. 1)

In retrospect, the history of logic was neither as linear and progressive nor as periodic and confined as depicted by the victors who lived to tell the story.

After the publication of his famous Incompleteness Theorems in 1931, Gödel in his 1933 studies of intuitionistic logic used modal operator to axiomatize the intuitionistic notion of provability. These “Gödel logics” foreshadowed the emergence of modal provability logics that would flourish in the 1970s, providing elegant proofs of Gödel’s Second Incompleteness Theorem or the Unprovability of Consistency.

Following Frege, Russell and Whitehead championed extensional logic and “material implication” through the publication of *Principia Mathematica* [1910–1913]. However, C.I. Lewis (1883 – 1964) began in 1912 riticizing *Principia’s* use of “material implication” as leading to paradoxical theorems:

T7 ~P → (P → Q) Law of Denying the Antecedent

T2 Q → (P → Q) Law of Affirming the Consequent

T58 (P → Q) ∨ (Q → R) Law of Conditional Excluded Middle

Lewis championed “strict implication”

P **⥽** Q := ~(P ∧ ~Q)

characterizing it in terms of the axioms for strict implication (**S4**) and (**S5**), which would become a standard part of modal logic. Lukasiewicz in his contributions to temporal modal logic in the 1920s and 1930s endorsed the view taken by Aristotle in his Sea Battle Paradox that future contingent propositions do not definitive truth-values.

Working in relative isolation in New Zealand, insulated from Quine’s “dogmas” against modal logic, Arthur N. Prior began to develop temporal modal logics. Prior’s John Locke Lectures were published as Time and Modality [1957] caught the attention of the young Saul Kripke who wrote a letter in 1958, raising questions about branching time. Throughout the 1950s, modal and intentional logics were intensely studied by Rudolf Carnap, Stig Kanger, Jaakko Hintikka, Richard Montague, Robert Stalnaker, David Lewis. Saul Kripke, while still a high school student, produced beautiful semantics for standard modal logical systems in his “Semantical Considerations on Modal Logic” [1963].

Kripke proved completeness theorems for standard systems of modal logic with characteristic axioms—such as (**D**) in Deontic Logic requiring the accessibility relation to be *serial*, the logic of tautologies (**T**) with an accessibility relation that is *reflexive*, the (B) axiom requiring the accessibility relation be *symmetric*, axiom (**4**) requiring the accessibility relation be *transitive* and axiom (**5**) requiring that accessibility be an *equivalence* relation.

From the 1970 - 1990 the Philosophy of Language was dominated with the “New Theory of Reference” and “Possible World Semantics”. Medieval distinctions such as *the de re / de dicto* distinction were used by Saul Kripke to challenge Quine’s arguments against the coherence of “quantifying into” propositional attitudes and other intensional contexts. The ideas of the medieval Jesuit Louis de Molina about divine “middle knowledge” were revived by philosopher of religion Alvin Plantinga as the “counterfactuals of freedom” in his masterful discussion of the problem of evil.

After this early philosophical phrase, modal logic flourished. According to van Bentham, the philosophical phrase was “consolidated into a beautiful mathematical theory by … Block, Fine, Gabbay, Goldblatt, Segerberg and Thomason, in the 1970s “crossed over to linguistics and formal semantics with “Montague Grammars”, in the 1980s, modal logics found their way into computer science in the study of program verification, and into “economics in the study of knowledge of players in games Aumann, 1976)”. Van Bentham underscores the transition from viewing modal logic as *extensions* of classical logic in the 1960s to the view that they are *fragments* of classic logics with the virtue of lower computational complexity. These applications have broken down the distinction between pure and applied logic lowering comutational complexity in applications as diverse as image processing and branching programming.

# 2. Syntax for Modal Propositional Logic.

Propositional modal logic come from propositional logic by adding two new operators:

◇ diamond (read “it is possible that”)

□ box (read “it is necessary that”)

To be more explicit, the class of symbolic sentences is exhaustively characterized:

1. *Sentence letters are symbolic sentences.*
2. *If* ϕ *and* ψ *are symbolic sentences, then so are*: ~ ϕ , (ϕ → ψ) , (ϕ ∧ ψ) , (ϕ ∨ ψ) , (ϕ ↔ *ψ) .*
3. *If* ϕ *is a symbolic sentence, then so are*:

□ϕ

*and*

◇ϕ .

Sentences obtained by (3) are *modal sentences*. The occurrence of ϕ in the modal sentences □ϕ or ◇ϕ is called the *scope* of the corresponding modal operator.

A more succinct way of specifying our grammar is this notation:

| P, Q, …, Z | ~ϕ | (ϕ ∧ ψ) | (ϕ ∨ ψ) | (ϕ → ψ) |(ϕ ↔ ψ )| □ϕ | ◇ϕ |

Symbolic sentences can be parsed into *grammatical trees* to systematically display all the sub-sentences involved it its construction. Here, in addition to the kinds of nodes of Chapter II, non-branching nodes are also reached by clause (3).

(□(□P → Q) → ~◇~R)

□(□P→ Q) ~◇~ R

(□P→ Q) ◇~ R

□P Q ~ R

P R

We shall continue to use the informal conventions of KM2 for omitting outer parentheses, replacing pairs of matching parentheses with matching pairs of square brackets, for resolving questions of scope among the various connectives and operators:

1. The negation and modal operators ‘□’ and ‘◇’ and ‘~’ by convention have the *narrowest* scope.
2. The connectives ‘∧’, and ‘∨’ have *narrower* scope than ‘→’ and ‘↔’.
3. Repeated conjunctions or disjunctions are grouped by association to the left.

Thus, for example, the official sentence

□(~P → Q) ↔ ((◇(P ∧ Q) ∨ (P ∧ □Q)) ∨ (~□P ∧ Q))

becomes informally

□(~P → Q) ↔ ◇(P ∧ Q) ∨ (P ∧ □Q) ∨ (~□P ∧ Q) .

# 2. Translation and Symbolization.

The English sentence

* 1. It is necessary that God exists

has a number of stylistic variants in English:

*Necessarily*, God exists ,

*It is necessary for* God to exist ,

God *must* exist ,

God *has to* exist .

Similarly, the sentence

* 1. It is possible that God exists

may also be expressed with a number of stylistic variants:

*Possibly*, God exists ,

*It is possible for* God to exist ,

God *can* exist ,

God *might* exist .

Using the modal operators with the sentential connectives of chapters I and II, we can develop stylistic variants of a number of English expressions. For instance, the expression

It is impossible that God exists

may be paraphrased as either

It is *necessary* that God *fails* to exist ,

or

It is *not possible* that God exists .

Modal operators, when they interact with other operators and connectives, can create *scope ambiguities* such that that between the necessity of the consequence and the necessity of the consequent, which we discussed in the previous section.

### Exercises

1. Symbolize the following sets of sentences using the given the scheme of abbreviation. If a sentence is ambiguous briefly explain the different possible readings.

P: Plato is rational. R: Theatetus is a cyclist.

Q: Plato is risible. S: Theatetus is bipedal.

1. It is possible that Plato is rational.

It is impossible that Plato is not rational.

It is contingently true that Plato is rational.

It is neither contingently true nor contingently false that Plato is rational.

1. Necessarily, if Plato is risible, then Plato is rational.

If Plato is risible, then it is necessary that Plato is rational.

If it is possible that Plato is risible, then it is possible that Plato is rational.

If Plato is risible, then it is necessarily possible that Plato is rational.

It is impossible for Plato to be risible if he is not rational.

1. Theatetus could be a cyclist provided he is bipedal.

It is possible that Theatetus is both a cyclist and bipedal.

It is impossible that Theatetus is a cyclist if he is not bipedal.

If it is possible for Theatetus to be cyclist and necessarily if he is a cyclist then he is bipedal, then it is compatible that Theatetus is cyclist and bipedal.

2. Fill in the following chart by finding expressions using only the connective and operators at the top of the chart equivalent to the expressions in the first column.

* A sentence P is *contingent* (in symbols, ∇P) if it is neither necessary nor impossible.
* A sentence P *strictly implies* Q (in symbols, P ⇒ Q) if it is impossible for P to be true while Q is false.
* P and Q are *compatible* if (P ∧ Q).
* P and Q are *incompatible* if ~ (P ∧ Q).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | { ~, □} | {~, } | {~, ⇒} | {~, ∇} |
| P is *necessarily true* |  |  |  |  |
| P is *impossible* |  |  |  |  |
| P is *possibly true* |  |  |  |  |
| P is *possibly false* |  |  |  |  |
| P is *contingently true* |  |  |  |  |
| P is *contingently false* |  |  |  |  |
| P is *contingent* |  |  |  |  |
| P *strictly implies* Q |  |  |  |  |
| P is *compatible* with Q |  |  |  |  |
| P is *incompatible* with Q |  |  |  |  |

1. Use the distinction between the necessity of the consequence and the necessity of the consequent to explain why someone might accept Axiom (**B)** ϕ → □ϕ as intuitively obvious while rejecting its logically equivalent form Axiom (**B**) □ϕ → ϕ. What are the two ways in which one might interpret “if ϕ then necessarily it is possible that ϕ”?

Some philosophers have argued that propositions like “5 + 7 = 12” or “Socrates was snub-nosed or Socrates was not snub-nosed”—mathematical or logical truths—were *necessarily true*, or in Leibniz’s vivid terminology “true in all possible worlds”. Some medieval philosophers such as Saint Anselm argued that “God exists” if true is *necessarily true*; while some atheists have argued that the very idea of theism is incoherent and so “God exists” is *necessarily f*alse or *impossible*. Empirical truths like “the cat is on the mat” or “Vienna is the capital of Austria” are only contingently true or contingently false, because they depend on how the world happens to be rather than being necessarily true or how all possible world must turn out or to be “true in all possible worlds.”

Using the ‘□’ for ‘it is necessary that’, the principle that all necessary truths are true is captured by:

(**T**) □ ϕ → ϕ ,

Adding its converse that all truths are necessary:

(**V**) ϕ → □ϕ

collapses the notions of truth and necessary truth.

Another modal principle, the (**K**) axiom (named after Kripke) that  distributes over →:

(**K**) □(ϕ → ψ) → (□ϕ → □ψ) .

Another way of expressing (**K**) is to state that necessary truths imply only necessary truths.

Medieval philosophers, concerned with such theological issues as articulating the nature of the Trinity, appealed to such modal notions as essence and accident, contingency and necessity in their labyrinthine theological reflections.[[3]](#footnote-3) Akin to the problem is logical fatalism is the problem of *theological fatalism*: the problem of reconciling divine foreknowledge and human freedom. Saint Augustine (354 - 430) in his treatise *On the Free Choice of Will* considers an argument for *theological fatalism* proposed by Evodius. Evodius argued that “God foreknew that man would sin, that which God foreknew must necessarily come to pass.” We may explicate this argument for theological fatalism for an instance central to theology:

If God knew that Adam would sin, then, *necessarily* Adam sinned.

God knew that Adam would sin (because God is omniscient).

Therefore, Adam *necessarily* sinned.

St. Thomas Aquinas (1225 - 1274) in his *Summa Contra Gentiles* (part I, chapter 67) criticized this kind of fatalistic argument as resting on an ambiguity. The critical first premise “if God knew Adam would sin, then, *necessarily*, Adam sinned” has two distinct readings:

(1a) It is *necessarily* the case that if God knows that Adam will sin then Adam will sin.

(1b) If God knows that Adam will sin, then it is *necessarily* the case that Adam will sin.

Aquinas distinguished between the conditional in (1a) the *necessity of the consequence* from (1b) the *necessity of the consequent*. Using the ‘□’ to abbreviate ‘it is necessary that’, the difference between these two propositions can be made more perspicuous in terms of the scope of the □:

(1a’) □(P → Q)

(1b’) (P → □Q)

In the former, the scope of □ is the conditional; in latter, the scope of □ is the consequent.

According to Aquinas, the Aristotle’s problem of fatalism, as well as the theological problem reconciling divine foreknowledge with human freedom, turn on exposing ambiguities of this sort.

Let’s use a metrical temporal logic to represent Aristotle’s dilemma. Let the parameter *tn* stand for “today”, *tn*-1 stand for “yesterday”, and *tn*+1 stand for “tomorrow”. Let ϕt stand for “ϕ is true at day *t*.” Then Aristotle’s Paradox of Future Contingents can be represented as follows:

(A1) □(ϕ*t-1* ∨ ~ϕ *t-1*)

(A2) (ϕ*t-1* → □ϕ *t+1*) (A2’) □(ϕ*t-1* → ϕ *t+1*)

(A3) (~ϕ*t-1* → □~ϕ *t+1*) (A3’) □(~ϕ*t-1* → ~ϕ *t+1*)

∴ □ϕ*t+1* ∨ □~ϕ *t+1*

Aquinas’s diagnosis is that the Aristotle’s argument for fatalism depends on equivocating between the necessity of the consequent as in (A2) and (A3) with the necessity of the consequence as in (A2’) and (A3’).

However, given the principle that that past is no longer open to change and so is necessarily true, then (A1) is equivalent to:

(A1’) □ϕ*t-1* ∨ □~ϕ *t-1*

Then assuming that the (**K**) axiom holds in this case, we have

(A1’) □ϕ*t-1* ∨ □~ϕ *t-1*0

(A2’) □(ϕ*t-1* → ϕ *t+1*) and □(ϕ*t-1* → ϕ *t+1*)→ (□ϕ*t-1* → □ϕ *t+1*)

(A3’) □(~ϕ*t-1* → ~ϕ *t+1*) and □(~ϕ*t-1* → ~ϕ *t+1*) → (□~ϕ*t-1* → □~ϕ *t+1*)

∴ □ϕ*t+1* ∨ □~ϕ *t+1*

which again leads to the necessity of the future. Aquinas’s distinction does not, by itself, resolve Aristotle’s problem of future contingents or the problem of logical fatalism.

The medieval counterpart to Aristotle’s problem of future contingents is the problem of reconciling God’s *foreknowledge* with human *freedom*. Christianity, as well as other theistic religions that hold a doctrine of divine omniscience and to doctrine of human free will, are faced with the problem of reconciling three essential theological doctrines:

1. Human beings have libertarian free will;
2. Future events are determined by Divine providence;
3. God is essentially omniscient.

For example, *Calvinism* replaces (I) libertarian free will with compatibilist free will affirming both (II) and (III); *open theism* affirms (I) while denying (II) while severely curtailing (III); and *Molinism* affirms (I) and softens (II) while attempting to deliver (III) by means of God’s *middle knowledge* in terms of what are now called “counterfactuals of freedom.” Later we will encounter some of the logical problems posed by counterfactuals.

Perhaps the most famous theological application of modal logic is Saint Anselm’s modal ontological argument. According to Saint Anselm (1033 - 1109), it follows from God’s nature that it is necessary that God exists if God exists at all. Moreover, this conditional itself, being a conceptual truth, is itself necessarily true. We then have the following argument:

Necessarily, if God exists, then God necessarily exists.

It is possible that God exists.

Therefore, God necessarily exists.

Using ‘◊’ for ‘it is possible that’, the above argument can be symbolized:

(2) □(G → □G) . G ∴ □G

The question of whether Anselm’s argument is valid became a precise question when various systems of modal logic were proposed and developed in the 1960s.

The emergence of modal logics is connected to implication, in particular, what are called the paradoxes of material implication. One of the most puzzling consequences for beginning logic students, is that a *contradiction implies anything*. This problem is known as *Lewis’s Dilemma* or as the *explosion* *problem.* In classical logic, the following is a theorem:

P  ~P  Q ,

This, intuitively outrageous, implication is justified by the, intuitively obvious, inference rules of simplification, addition, and *modus tollendo ponens*:

1. *~~Show~~* P  ~P  Q 6, CD

2. P  ~P Assume (CD)

3. P 2, S

4. ~P 2, S

5. P  Q 3, ADD

6. Q 5, 4 MTP

Thus any attempt to rescue implication from this unacceptable classical consequence must reject conditional derivation or at least one of the intuitively acceptable inference rules. There is a branch of logic known a *relevance* *logics* which explores these possibilities.

C. I. Lewis investigated modal logic in order to find a conditional—*strict implication*—which did not result in the paradoxes of material implication. Lewis defined strict implication P ⇒ Q (read “P *strictly implies* Q”) by combining modality with the truth-functional conditional:

P ⇒ Q :=◇(P  ~Q) ,

or alternatively,

P ⇒ Q :=□(P  Q) .

Lewis proposed axioms for different systems of strict impliations:

(**S4**) □ϕ → □□ϕ

and

(**S5**) ◇ϕ → □◇ϕ .

The philosopher Leibniz (1646 - 1716) explicitly invoked that language of possible worlds to explain the difference between necessary and contingent truths. What is logically or necessarily true are those truths truth in *all* possible worlds, whereas a contingent truth is one that is true in *some* possible world. Drawing this logical connection between universal and existential quantification with the modal notions of necessity and possibility implies that modal notions also obey the classical Aristotelian Square of Opposition expressing the laws of duality or modal negation.

∀*x*~F*x*: *Nothing is* F

🞎~P: *It is impossible that* P

∀*x*F*x*: *Everything is* F

🞎P: *It is necessary that* P

Contraries

Contradictories

Subcontraries

∃*x*~F*x*: *Something is not* F

~P: *It is possible that not* P

∃*x*F*x*: *Something is* F

P: *It is possible that* P

~∀*x*F*x*: *Not everything is* F

~□P: *It is unnecessary that* P

In the modern period, logicians noticed that other phenomena—such as obligation and permissibility, knowledge and belief, and future tenses such as “it was once true that” and “it has always been the case that”, as well as, past tenses such as “it will be the case that” and “it will always be the case that”—have modal structure. This led to the development of deontic, temporal, and epistemic logics.

In *deontic logic*, □ is interpreted as moral necessity or as “it is obligatory that” and ◊ is read “it is permissible that”. Notice that Axiom (**T**) is too restrictive for deontic logic since not everyone fulfills their moral obligations. Kant’s maxim that “*ought implies can*” implies that whatever is obligatory is possible. A related implication is that whatever is morally obligatory is morally permissible, which is captured in Axiom (**D**):

(**D**) □P → P .

*Epistemic logic* has been developed to model what philosophers have called propositional attitudes in which □ is read as “*s* knows that P” and ◊ is read as “*s* believes that P”. Axioms for epistemic logic include such axioms as:

(**K**) □(P → Q) → (□P → □Q) *logical omniscience*

(**T**) □P → P *veridicality*

(**4**) □P → □□P *positive introspection*

(**E**) ~□P → □~□P *negative introspection*

The axiom (**K**) of logical omniscience is the assumption that an agent knows that his or her beliefs are closed under *modus ponens*—that, he knows all the logical consequences of his beliefs. Axiom (**T**) states the truism that whatever is known is true. Axiom (**4**), sometimes known as the KK thesis, presupposes a high degree of *positive introspective knowledge*: it is the claim that if someone knows P, then she knows that she knows that P. Axiom (**E**) presupposes a high degree of *negative* introspective knowledge: it is the claim that if someone doesn’t know that P, then he knows he doesn’t know P.

The Browersche (**B**) axiom was named by the logician Carl Becker after the Dutch mathematician L. E. J. Brouwer who championed a philosophy of mathematics known as *intuitionism*. The (**B**) axiom expresses the acceptable form of double negation introduction allowed in intuitionistic logic:

(**B**) ϕ → ~◇~◇ϕ

The idea behind intuitionistic logic is that the connectives are reinterpreted as involving a kind of provability.

Diodorian temporal logic was studied by the logician A. N. Prior. To model temporal language, we introduce a pair of modal operators for the *future* and a pair of modal operators for the *past*.

□G It *is always going* [i.e., in *all* futures] to be the case that

◇F It *will* [i.e., in *some* future] be the case that

🞖H It *has always been* [i.e., in *all* pasts] the case that

🞛P It *was once* [i.e., in *some* past] the case that

These temporal operators can express a surprising number of tenses:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *I see Rose.* | R | *Present* |  | *I had seen Rose* | 🞛P🞛PR | *Past Perfect* |
| *I saw Rose* | 🞛PR | *Past* |  | *I will have seen Rose* | ◇F 🞛PR | *Future Perfect* |
| *I will see Rose* | ◇FR | *Future* |  | *I was going to see Rose* | 🞛P ◇FR | *Past Future Imperfect* |

Tense logic contains two axioms concerning the interaction of the past and future that have the form of the so-called Brouwersche axiom:

(B) ϕ → □G 🞛P ϕ What is, will always have been

(B) ϕ → 🞖H ◇F ϕ What is, has always been going to happen

Around the 1970s it was noticed that the famous incompleteness theorems of Gödel (1931) were propositional in character and that their logic could be captured in propositional modal logic. These modal provability logics added to Axiom (**K**) the following axiom known as the Gödel-Löb axiom or also the well-ordering axiom.

(**G**) □(□ϕ → ϕ) → □ϕ .

Modal Provability Logics developed in the 1970s, but the genesis of the idea goes back to a short note of Gödel’s [1933] in which he noted that intuitionistic truth defined in terms of proof and provability was a kind of necessity. The above axiom can be read as a kind of soundness theorem. If it is provable that ϕ’s being provable implies its truth, then ϕ is provable.

Later we will show have modal provability logics exhibit the propositional logic of key parts of Gödel’s First and Second Incompleteness Theorems.

In contemporary logic, modal logic has grown beyond these philosophical origins and is at the interface of a number of disciplines including the studies information flow and dynamics, game-theory, and computability.

### Exercises

1. Symbolize the following modal arguments and claims.
2. It is impossible to measure both the position and momentum of particle P. The position of P is measured. Therefore, it is not possible to measure the momentum of P.
3. It is conceivable that I am having experiences qualitatively identical to those I am having now about the external world on the supposition that I am being deceived by an evil genius. Therefore, I cannot have knowledge of the external world.
4. Johan van Bentham ([2010], 12) was once asked to symbolize the philosophical claim that “*nothing is absolutely relative*”. How would you symbolize this claim?[[4]](#footnote-4)
5. *“If it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic.”*
6. If George Bailey had not existed, then his baby brother Harry Bailey would have drowned as a child. If Harry Bailey had drowned as a child, then he would not have become a war hero saving the lives of 200 hundred men on the transport. Therefore, if George Bailey had not existed, then 200 men on the transport would not have been saved. (Is this *transitivity* of counterfactuals intuitively valid?)

2. Match the following modal symbolic sentences with their English translations:

|  |  |
| --- | --- |
| 🞛P 🞖H ϕ | Whatever will always be, will sometime be ϕ |
| □Gϕ → ◇F ϕ | Once upon a time, it was always the case that ϕ |
| 🞖H ◇F ϕ | It was always the case that it will sometime be the case that ϕ |
| ◇F 🞛P ϕ | It will sometime be the case that it was once the case that ϕ |

**2. Modal Propositional Logic.**

# 3. Axioms for Modal Logics.

The logical relations in the previous section capture what is common to the many of the various interpretations for necessity. The distinctive notions of necessity discussed initially can be characterized by formulating distinctive modal axioms. For example, the axiom

□P → P

says that whatever is necessary is actually the case. This axiom characterizes what some philosophers have called ‘tautological necessity.’ This axiom would hold for temporal logics since whatever is, was, or will be the case, is the case. This axiom, known as the (**T**) axiom, is named after a system of modal logic studied by Gödel, Feys, and von Wright in which *tautologies* are necessary truths.

Notice that the (**T**) axiom is too strong for deontic logics. George Washington, should he fail to live up to his reputation for veracity, might be morally obligated to tell the truth but fail to do so. It is a sad reality of moral life that obligations remain unfulfilled. However, whatever is morally obligatory is at least morally permissible. The (**D**) axiom (named for *deontic logic*) weakens the (**T**) axiom and states that what is necessary is possible.

□P → ◇P .

Let’s consider another axiom, known as the (**4**) axiom (named after the strict implication system S4 studied by C. I. Lewis). Jaakko Hintikka developed an epistemic interpretation using the (**4**) axiom:

□P → □□P .

This interpretation says that if an individual knows that P, then that individual knows that s/he knows that P (sometimes called the KK thesis). Axiom (**4)** as we have seen above, has been used to characterize some systems of temporal logic, for what is always true is always always true.

Another axiom has been championed by philosophers such as Alvin Plantinga, who claims that the notion of “broadly logical necessity” is best captured by axiom (**5)** (named after C. I. Lewis’ fifth axiom for strict implication):

◇P → □◇P .

Whereas Axiom **4** states that necessary truths are necessary, Axiom **5** extends this necessity to the modality of possibility: if P is possible, then it is necessary that P is possible. Intuitively, what is logically necessary is logically necessary in all possible worlds and what is logically possible is also logically possible in all possible worlds. Modal propositions are the same in all possible worlds.

Another modal axiom is axiom (**B)** was named by Oskar Becker (1930) after Luitzen Egbertus Jan Brouwer (1881-1966), the founder of intuitionism, a philosophy of mathematics in which the logical connectives are given a proof-theoretic interpretation.[[5]](#footnote-5) The (**B**) axiom is syntactically similar to the form of double negation valid in intuitionistic logic, the law of double negation introduction:

P → ~◇~◇P .

By the logical equivalences noted earlier, this axiom can also be expressed by the more familiar form:

P → □◇P .

|  |  |  |  |
| --- | --- | --- | --- |
| System | Key Axiom | Named After | Accessibility |
| **D** | □P → ◇P | **D**eontic Logic | *Serial* |
| **T** | □P → P | Tautology logic of Gödel, Feys, von Wright | *Reflexive* |
| **B** | P → □◇P | **B**rouwerian Intuitionistic Double Negation | *Symmetric* |
| **S4** | □P → □□P | Lewis’s Axiom **4** for Strict Implication | *Transitive* |
| **S5** | ◇P → □◇P | Lewis’s Axiom **5** for Strict Implication | *Euclidean* |

Ancient philosophers like Aristotle discussed modality and medieval logicians such as Jean Buridan discussed modal notions with great subtlety. However, the modern treatment of modal logic began around 1912 when C. I. Lewis, after reading Russell and Whitehead’s *Principia* *Mathematica*, proposed various axioms (such as **4** and **5**) to find a connective more suitable than the material conditional to express our informal concept of entailment. In the 1930s Gödel discussed an interpretation of the modal operator as ‘it is provable in system M that,’ and in the 1950s *epistemic*, *doxastic*, *deontic* and *tense-logic* interpretations were intensively investigated.

Around the 1960s modal logic was being investigated by numerous logicians—including, among others, Rudolf Carnap, Jaakko Hintikka, Richard Montague [1955], and Arthur Prior and Carew Meredith [1955].[[6]](#footnote-6) However, it was Saul Kripke’s remarkable paper *A Completeness Theorem in Modal Logic* (1961) (published while he was still a high school student in Lincoln, Nebraska) who first presented possible world semantics in an elegant way that ignited logical and philosophical research for decades.

Kripke’s *possible world semantics* gave formal expression to the Leibnizian idea that a proposition is *necessary* if it is true in *all* possible worlds, possible if it is true in *some* possible world, and *contingent* if it is neither necessary nor impossible. The above axioms turned out to be valid in modal systems with very natural conditions on the relation of modal accessibility or relative possibility:

Normal systems of modal logic are those in which the laws of logic, such as *modus ponens*, are assumed to hold. Axiom (**K**), which characterizes such normal systems, is named after Kripke:

□(P → Q) → (□P → □Q) .

In this chapter we will develop natural deduction systems for propositional modal logic and set forth the possible world semantics for them. In a subsequent chapter we’ll set forth these further axioms and the Provability Logic that enables us to capture the propositional modal logic structure of Gödel’s Incompleteness Theorems.

Exercises

1. Construct a diagram to chart these notions in logical space and give English examples of each.

(A) The realm of the *necessary*, i.e., propositions are necessarily true or necessarily false).

(B) The realm of the *contingent*, i.e., propositions which are neither necessary nor impossible, or alternatively, propositions that are either contingently true or contingently false).

(C) The realm of the *possible*, i.e., propositions are not impossible, or alternatively, necessarily true or contingent).

(D) The realm of the *true*, i.e., propositions which are necessarily true or contingently true.

(E) The realm of the *false*, i.e., propositions that are either impossible or contingently false.

2. Find succinct expressions using only ‘□’ and “~’ for each of the following notions:

|  |  |
| --- | --- |
| P is *contingently true* |  |
| P is *contingently false* |  |
| P is *contingent* |  |
| P and Q are *incompatible* |  |
| P and Q are *compatible* |  |
| P *strictly implies* Q |  |

3. Find succinct equivalent expressions using only ‘◇’ and ‘~’ and the relevant sentence letters.

|  |  |  |
| --- | --- | --- |
| D | □P → ◇P |  |
| **T** | □P → P |  |
| **B** | P → □◇P |  |
| **4** | □P → □□P |  |
| **5** | ◇P → □◇P |  |

4. Discuss whether the following pairs of axioms are intuitively acceptable in the modalities specified.

(A) ◇P ∧ ◇Q → ◇(P ∧ Q); ◇(P ∨ Q) → (◇P ∨ ◇Q) physical possibility

(B) □ϕ → ϕ ; □(□ϕ → ϕ) obligatoriness

(C) ϕ → □G🞛P ϕ ; ϕ → 🞖H◇ϕ temporality

(D) □ϕ → □□ϕ ; □ϕ → ϕ epistemology

(E) ϕ → □◇ϕ ; ◇◇ϕ → ◇ϕ conceivability

1. Translate the axioms into temporal propositions of English:

(4) □G P → □G □G P (*transitivity*)

(4) ◇FP →◇F ◇FP (*density*)

(Hamblin’s Axiom) P ∧ 🞖H P → □G🞖H P (*discreteness*)

5. Here ‘∇P’ is read “it is contingently true that ….” Translate the following equivalences into English.

∇P ↔ ~ (□P ∨ ~◊P)

∇P ↔ ∇~P

□P ↔ (P ∧ ~∇P)

# 4. Modal Inference Rules.

The new inference rules are intuitively analogous to inference rules from quantifier logic.

🞎P

*It is necessary that P*

🞎~P

*It is impossible that P*

Contraries

`Subaltern

~P

*It is actually not the case that P*

P

*It is actually the case that P*

Subaltern

Contradictories

Subcontraries

P

*It is possible that P*

~P

*It is possible that not P*

Figure 1 A Modal Diamond of Opposition

The modal sentences at opposite ends of the diagonals of the square are *contradictories*. These logical relations are captured in the rules of *modal negation* [MN].

~□ϕ ◇~ϕ ~◇ϕ □~ϕ

◇~ϕ ~□ϕ □~ϕ ~◇ϕ

The modal sentences at the top of the square are not contradictories, but *contraries*, a sentence cannot be both necessary and impossible, but it may be neither.

* A sentence that is neither necessary nor impossible is *contingent*.
* A sentence is *contingently true* if it happens to be true but could have been false.
* A sentence is *contingently false* if it happens to be false but could have been true.

The realm of the possible is everything other than the impossible, and so includes what is contingently true, contingently false, and necessarily true.

The analogy to quantification also holds for the corresponding laws of *modal distribution*:

□(P ∧ Q) ↔ (□P ∧ □Q)

◇(P ∨ Q) ↔ (◇P ∨ ◇Q)

The rule of *necessity instantiation* [NI] is analogous to an application of the rule of universal instantiation.[[7]](#footnote-7)

∀α ϕ □ϕ

ϕ ϕ

Here the rule of NI allows us to drop an initial box. Intuitively, this rule says that whatever is necessary is also the case. The rule of *Possibility Generalization* [PG] is a derived rule analogous to existential generalization:

ϕ ϕ

∃αϕ ◇ϕ

Here the rule of PG allows us to infer ‘ϕ’ from ϕ. Intuitively, this rule says that whatever is the case is also possibly the case. PG follows from NI and the third form of MN.

### 

### Exercises

1. Which of the following inferences come by an application of NI, PG, or MN? Briefly explain.

(A) □(P → Q) □(P → □Q) □P ∧ □Q □◇P → □Q ◇□◇ P

P → Q P → Q P ∧ □Q □◇P → Q ◇◇ P

(B) □(P → Q) □(P → □Q) □P ∧ □Q □◇P → □Q □◇P. \_

◇□(P → Q) □◇(P → □Q) ◇□P ∧◇□Q ◇□◇P → ◇□Q □◇◇P

(C) ~ □(P ∨ Q) ~ ◇(P → Q) □(P → ~◇~P) □ ~(P ↔ Q) ~ □◇□P

◇ ~ (P ∨ Q) □P → Q □(P → □P) ~ ◇(P ↔Q) ◇□◇~P

1. Show that PG can be derived from NI and the third form of MN.

# 5. Natural Deduction for (T), (B), (S4), and (S5).

In addition to the inference rules NI and MN, we add two form of strict derivation.

The strict form of derivation known as *necessity derivation* [ND] appears as follows:

*Show* □ϕ

χ1

.

.

.

χm

where the scope ϕ occurs unboxed among χ1 through χm.

In a *necessity derivation* for *Show* □ϕ, one derives ϕ using premises or antecedent lines that are themselves necessary. To ensure that this form of strict derivation only allows us to derive necessary truths, we must restrict the lines that may be imported into the strict derivation.

To understand these restrictions, we first supplement the notion of an *antecedent* line from propositional logic to obtain the notion of an *accessible* line for modal logic. An *antecedent* *line*, as it is defined for propositional logic, is a preceding line that is neither boxed nor contains uncancelled ‘*Show’*. An *accessible line* is defined as an antecedent line such that there is at most one intervening modal ‘*Show’* line, that is, a ‘*Show’* line of one of the forms

*Show* □ϕ

or

*Show* ◇ϕ .

The basic restrictions on for a strict derivation are that none of the lines χ1 through χn are (*i*) premises or (*ii*) comes by an application of an inference rule to an inaccessible line, or (*iii*) comes by an application of an inference rule other than an admissible *strict importation rule* to an accessible line.

Next we formulate the *admissible* *strict importation rules* for the various systems of modal logic. These strict importation rules can be elegantly formulated as restrictions on the rule of repetition.

1. The characteristic strict importation rule for **T** is NI: when importing an accessible sentence of the form ‘□ϕ’ into a strict derivation, we must drop the initial ‘□’.
2. The characteristic strict importation rule for **B** is PG: when importing an accessible line into a strict derivation, we must prefix a ‘◇’.
3. The characteristic strict importation rule for **S4** is □R: when importing an antecedent line into a strict derivation, you may apply the rule of repetition R to an accessible line of the form ‘□ϕ’.
4. The characteristic strict importation rule for **S5** is ◇R: when importing an antecedent line into a strict derivation, you may apply the rule of repetition R to an accessible line of the form ‘◇ϕ’.

Each of the modal systems **B**, **S4**, and **S5** has NI as a strict importation rule in addition to its characteristic strict importation rule. In addition, **S5** also has the strict importation rules for **B** and for **S4**, namely, PG and □R. (The annotation for these strict importation rules will be the line number of the sentence involved and either ‘NI**’**, ‘PG**’**, ‘□R**’**, or ‘◇R**’**.)

The second form of strict derivation is *possibility derivation* [Annotation: PD]. If we have a ‘*Show’* line of the form

*Show* ◇ϕ ,

and there is some accessible line ◊ψ, then on the next line immediately after th*e ‘Show’* line one may enter

ψ ,

as an assumption for *possibility derivation* [Annotation: Assume(PD)]. The restrictions for PD are the same as those for the first form of strict derivation: none of the lines χ1 through χn are premises or comes by an application of an inference rule to an inaccessible line, or comes by an application of an inference rule other than an admissible strict importation rule to an accessible line.

Intuitively, *possibility derivation* allows us to *Show* ◇ψ by deriving ψ using necessary truths from the assumption that ϕ, where ◊ϕ is an accessible line. The subsidiary derivation can be thought of as confined to a particular possible world in which ϕ is assumed to be true. The intuitive basis for this form of strict derivation is captured by T502:

□(P → Q) → (◇P → ◇Q) ,

which states that whatever is logically entailed by something possible it itself possible.

The foregoing remarks on derivations constitute only an informal introduction. The following is an explicit set of directions for constructing a derivation.

1. *If ϕ is a symbolic sentence, then*

*Show* ϕ

*may occur as a line.*

1. *Any one of the premises may occur as a line.* [*Annotation:* ‘Premise’]
2. *If* ϕ *and* ψ *are symbolic sentences such that*

*Show* (ϕ → ψ)

*occurs as a line, then* ϕ *may occur as the next line.* [*Annotation:* ‘Assume (CD)’*.*]

1. *If ϕ is a symbolic sentence such that*

*Show* ϕ

*occurs as a line, then*

~ϕ

*may occur as the next line; if ϕ is a symbolic sentence such that*

*Show* ~ϕ

*occurs as a line, then*

ϕ

*may occur as the next line.* [*Annotation:* ‘Assume (ID)’*.*]

1. *If ϕ is a symbolic sentence such that*

*Show* ◇ϕ

*occurs as a line, then*

ψ

*may occur as the next line provided that* ◊ψ *is an accessible line, that is, a preceding line that is neither boxed nor contains uncancelled ‘Show’ and is such that there is at most one intervening ‘Show’ line of the form*

*Show* □ϕ

*or*

*Show* ◇ϕ *.*

[*Annotation: the line number of the modal sentence involved and ‘*Assume (PD)*’.*]

1. *A symbolic sentence may occur as a line if it follows by an inference rule* 
   * 1. *of sentential logic applied to antecedent lines, that is, preceding lines which neither are boxed nor contain uncancelled ‘Show’* [*here the annotation should refer to line numbers of the preceding lines involved and the inference rule employed*]; *or*
     2. *of modal logic applied to accessible lines that is, antecedent lines for which there is at most one intervening modal ‘Show’ line* [*here the annotation should refer to line numbers of the preceding lines involved and the inference rule employed*].
2. *When the following arrangement of lines has appeared:*

*Show* ϕ

χ*1*

.

.

.

χ*n*

*where none of the* χ1*through* χ*n contains uncancelled ‘Show’ and either*

1. *the conclusion to be shown* ϕ *occurs unboxed among* χ1 *through* χ*n,and neither* χ1 *nor* ϕ *was introduced by an assumption for Possibility Derivation,*
2. ϕ *is of the form*

(ψ*1* → ψ*2*)

*and the consequent* ψ*2 occurs unboxed among occurs unboxed among* χ*1 through* χ*n and* χ*1 but was not introduced by an assumption for Possibility Derivation,*

1. *for some sentence* χ*, both* χ *and its negation occur unboxed among* χ1 *through* χ*n and neither of the contradictory sentences was introduced by an assumption for Possibility Derivation,*
2. ϕ *is of the form*

□ψ

*or*

◇ψ

*and* ψ *occurs unboxed among* χ1 *through* χ*n  and none of* χ1 *through* χ*n are premises or comes by an application of an inference rule to an inaccessible line, or by an application of an inference rule other than an admissible strict importation rule to an accessible line antecedent to the displayed occurrence of*

*Show* ϕ *,*

*then one may simultaneously cancel the displayed occurrence of ‘Show’ and box all subsequent lines.* [*When we say that a symbolic sentence* ϕ *occurs among certain lines, we mean that one of those lines is either* ϕ *or* ϕ *preceded by ‘Show’. Furthermore, annotations for clause (7), parts (i), (ii), (iii),and (iv) are, respectively,* ‘DD’, ‘CD, ‘ID’*, and either* ‘ND’ or ‘PD’, *to be entered together with the relevant line numbers justifying the cancelled ‘Show’.*]

A derivation is said to be *complete* if each of its lines is either boxed or contains cancelled ‘*Show’*. A symbolic sentence ϕ is said to be *derivable* from given symbolic sentences in *modal system* M if, by using only clauses (1) - (6) and the strict importation rules for M, a complete derivation from those premises can be constructed in which

*~~Show~~* ϕ

occurs as an unboxed line.

We can illustrate clause 6, part (ii), by verifying the claim made above that *Possibility Generalization* [PG] can be derived from Necessity Instantiation and the third form of Modal Negation.

P ∴ ◇P

1. *~~Show~~* ◇P 3, 5 ID
2. ~◇P Assume (ID)
3. P Premise
4. □~P 2, MN
5. ~P 4, NI

An annotated derivation in modal system **T** will illustrate a strict derivation corresponding to the form of derivation in quantificational logic known as *universal derivation*. We will construct a derivation for the (**K**) axiom (named after Kripke):

□(P → Q) → (□P → □Q) .

To use the modal premises properly one must be careful to employ the accompanying strict importation rules associated with the various modal systems.

1. *~~Show~~* □(P → Q) → (□P → □Q)
2. □(P → Q)
3. *~~Show~~* □P → □Q
4. □P
5. *~~Show~~* □Q

To complete the strict (necessity) derivation began in line 5, we use two applications of NI, the strict importation rule for **T**, and then apply MP.

1. *~~Show~~* □(P → Q) → (□P → □Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* □P → □Q 5, CD
4. □P Assume (CD)
5. *~~Show~~* □Q 8, ND
6. P → Q 2, NI
7. P 4, NI
8. Q 6, 7 MP

We may box and cancel line 4 by strict derivation because we have derived the *scope* of the modal sentence in line 5 and the lines 6 - 8 have only come from accessible lines by the strict importation rule for **T**.

Intuitively, the first form of strict derivation *necessity derivation* amounts to showing ‘□Q’ by deriving ‘Q’ in an arbitrary possible world in the box surrounding lines 6 - 8. The restriction on strict derivation here is that we are only allowed to enter the scope ϕ of accessible lines of the form ‘□ϕ’, that is, the lines entered into the strict derivation in lines 5-7 are sentences stripped of an initial ‘□’. If we can derive ‘Q’ for this arbitrary possible world, we are justified in concluding that ‘Q’ holds in all possible worlds.

The second form of strict derivation is *possibility derivation* [Annotation: PD]. The initial ‘*Show’* line is

*Show* ◇ϕ .

With possibility derivation, if ◊ψ occurs as an accessible line, then on the next line one may enter

ψ

as an assumption for *possibility derivation* [Annotation: *Assume* (PD)]. The restrictions are the same as those for the first form of strict derivation: none of the lines χ1 through χn are premises or comes by an application of an inference rule to an inaccessible line, or comes by an application of an inference rule other than an admissible strict importation rule to an accessible line. Intuitively, this form of strict derivation allows us to show a sentence is possibly true by deriving it using necessary truths from a sentence that is possible.

The intuitive basis for PD is modal theorem T502 □(P → Q) → (P → Q), which states that whatever is logically entailed by what is possible is possible. We may prove T502 to illustrate PD.

1. *~~Show~~* □(P → Q) → (◇P → ◇Q)
2. □(P → Q) Assume (CD)
3. *~~Show~~* ◇P → ◇Q
4. ◇P Assume (CD)
5. *~~Show~~* ◇Q
6. P Assume (PD)

Line 6 comes from ‘◇P’ in the modally accessible line 4 by possibility instantiation, which is analogous to existential instantiation in quantifier logic in which we have the restriction that which we must instantiate to a variable that is completely *new* to the derivation. Here, in contrast to the previous example, the derivation of ‘Q’ is in a *particular*, rather than an *arbitrary*, possible world, namely, a world in which ‘P’ is assumed to be true.

Therefore, unlike the necessity derivation in the previous example, since we begin with an assumption for PD for a *particular* possible world, we are only justified in concluding ‘◊Q’ by a strict *possibility* derivation.

1. *~~Show~~* □(P → Q) → (◇P → ◇Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* ◇P → ◇Q 5, CD
4. ◇P Assume (CD)
5. *~~Show~~* ◇Q 8, PD
6. P 4, Assume (PD)
7. P → Q 2, NI
8. Q 7, 6 MP

Readers are encouraged to check their grasp of the ideas in this example by annotating and boxing and cancelling the derivation for

◇(P ∨ Q) ∴ ~◇P → ◇Q .

1. *Show* ~◇P → ◇Q

2. ~◇P

3. □~P

4. ◇(P ∨ Q)

5. *Show* ◇Q

6. (P  Q)

7. ~P

8. Q

Next we illustrate the new strict importation rule PG which supplements the strict importation rule NI in the modal system **B**. We shall construct a derivation for the Anselmian modal ontological argument discussed in the introductory section above. Letting ‘G’ abbreviate the sentence ‘God exists’, we may symbolize the Anselmian argument as follows:

□(G → □G) ◇G ∴ □G

T570 P → □◇P Axiom (B)

1. *~~Show~~* P → □◇P 3, CD

1. P Assume (CD)
2. *~~Show~~* □◇P 4, ND
3. ◇P 2, PG

T571 ◇□P → P

Notice that T571, the dual form of Axiom (B), comes from T570 by contraposition and modal negation.

1. *~~Show~~* □G 11, DD
2. □(G → □G) Premise
3. ◇G Premise
4. *~~Show~~* ◇□G 7, PD
5. G 3, Assume (PD)
6. G → □G 2, NI
7. □G 5, 6 MP
8. ◇□G → G T571
9. G 4, 8, MP
10. G → □G 2, NI
11. □G 9, 10 MP

We suggest that the reader test her knowledge of the modal system B by annotating the following version of the above Anselmian argument:

□(G → ~◇~G) . ◇G ∴ G

To illustrate the strict importation rule for the modal system **S4**, we construct a derivation for

∴ □(P → Q) → □(□P → □Q) .

To show the theorem we assume its antecedent and show its consequent, which is a modal sentence. To show the consequent, therefore, we commence a strict derivation. In order to maintain the accessibility of the initial assumption, which is a necessity, we apply the rule of □R obtain line 4 before entering the next ‘*Show’* line in line 5 for the scope of modal sentence in line 3.

* 1. *Show* □(P → Q) → □(□P → □Q)
  2. □(P → Q) Assume (CD)
  3. *Show* □(□P → □Q)
  4. □(P → Q) 2, □R
  5. *Show* □P → □Q

We then proceed by conditional derivation until we reach the consequent.

* 1. *Show* □(P → Q) → □(□P → □Q)
  2. □(P → Q) Assume (CD)
  3. *Show* □(□P → □Q)
  4. □(P → Q) 2, 🞏R
  5. *Show* □P → □Q
  6. □P Assume (CD)
  7. *Show* □Q

We can now complete the derivation by applying NI to lines 4 and 6. Note that the ‘*Show’* line in 7 is the only intervening modal ‘*Show’* after line 2 since the ‘*Show’* line in line 5 is for conditional derivation.

1. *~~Show~~* □(P → Q) → □(□P → □Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* □(□P → □Q) 5, ND
4. □(P → Q) 2, □R
5. *~~Show~~* □P → □Q 8, CD
6. □P Assume (CD)
7. □(P → Q) 4, □R
8. *~~Show~~* □Q 11, ND
9. P → Q 7, NI
10. P 6, NI
11. Q 9, 10 MP

Technically, in applying □R we are allowed to pass over only one uncancelled modal ‘*Show’* at a time—that is, a ‘*Show’* of the form ‘*Show* □ϕ’ or ‘*Show* ◊ϕ’. Notice, however, with repeated applications of □R, we can, in effect, pass over any number of modal ‘*Show’* lines for strict derivations. This follows from the fact that the characteristic axiom for **S4** allows us to prefix any finite number of ‘□’s to a necessary sentence. In order to eliminate repetitious use of □R, we shall allow □R to be applied across *n*-nested strict derivations (annotation □R*n*). We may use this short-cut in our previous example,

1. *~~Show~~* □(P → Q) → □(□P → □Q) 3, CD
2. □(P → Q) Assume (CD)
3. *~~Show~~* □(□P → □Q) 4, ND
4. *~~Show~~* □P → □Q 6, CD
5. □P Assume (CD)
6. *~~Show~~* □Q 11, ND
7. □(P → Q) 2, □R2
8. P 5, NI
9. P → Q 7, NI
10. Q 9, 8 MP

Intuitively, we may derive necessary truths only from premises which are themselves necessary. Syntactically, this condition is satisfied if we require that the lines used to show a necessary statement are themselves either necessary or follow from necessary statements by the laws of propositional logic and modal negation. This requirement is met by a restriction on importation: whenever we import a sentence into a strict derivation by crossing over an uncancelled ‘*Show*’ for a strict derivation, we must drop an initial ‘□’. When we have a series of nested strict derivations, we may apply the rule of □R across one strict derivation ‘*Show*’ line at a time, or we may generalize the rule of □R to allow for importation across multiple nested strict derivations as illustrated above.

Lastly, we wish to construct a derivation in **S5** that illustrates the strict importation rule ◊R. Alvin Plantinga’s modal version of an ontological argument is based on the intuition that God exists if and only if God is the greatest possible being. If God is the greatest possible being, Plantinga argues, then God has the greatest form of existence possible, namely, necessary existence. Using the resources of propositional modal logic, we shall formulate God’s having necessary existence as the proposition that ‘it is necessary that God exists’. Then Plantinga’s modal ontological argument can be stated as follows:

If it is possible that God exists, then it is possible that it is necessary that God exists.

It is possible that God exists. Therefore, God exists.

◇G → ◇□G . ◇G ∴ G .

We shall construct an indirect derivation.

1. *~~Show~~* G
2. ~G Assume (ID)
3. ◇G → ◇□G Premise
4. ◇G Premise
5. ◇□G 3, 4 MP

Notice that applying PG to the indirect assumption yields ◊~G. Since modalities in **S5** are necessarily true, we may import this modal statement into a strict derivation by the characteristic strict importation rule for **S5**, ◊R. This allows us to construct a strict derivation for □~□G, which is, by modal negation, the contradictory of line 5.

1. *~~Show~~* G 5, 10 ID
2. ~G Assume (ID)
3. ◇G → ◇□G Premise
4. ◇G Premise
5. ◇□G 3, 4 MP
6. ◇~G 2, PG
7. *~~Show~~* □~□G 9, ND
8. ◇~G 6, ◇R
9. ~□G 9, MN

10. ~◇□G 7, MN

Now in **S5** we also have all the previous strict importation rules including NI and PG, and in particular both □R and ◊R. Technically, in applying □R and ◊R, we can pass over only one uncancelled modal ‘*Show’* at a time—that is, a ‘*Show’* of the form ‘*Show* □ϕ’ or ‘*Show* ◊ϕ’. Notice, however, in **S5** we can, in effect, prefix any finite number of ‘□’s to any modal sentence, and so, in effect, pass over any number of modal ‘*Show’* lines with any modal sentence. In order to eliminate repetitious uses of □R and ◊R, we shall allow □R and ◊R to be applied across *n*-nested strict derivations (annotations □R*n* and ◊R*n*).

Plantinga’s modal ontological argument uses these strong modal assumptions of **S5** in which the modal status of any modal proposition is the same is all possible worlds. Hence, if it is possible that God exists, then it is necessarily possible that God exists (and, inversely, if it is possible that God does *not* exist, then it is *impossible* that God exists). Even if Plantinga’s argument is valid (and, indeed, *sound* if there is *at least one* sound arguments for the existence of God), it need not be modally persuasive: the way the argument goes—either theistically or atheistically—depends on the loaded possibility premise.

These derivations in the various modal systems can be tedious due to the various restrictions. We can make the modal natural deduction system less restricted by proving modal theorems. It is particularly useful to generalize MN for a string of modal operators, and to be able to employ the rule of Interchange of Equivalence using previously proven biconditional theorems. Modal theorems for the various modal systems are listed in a subsequent section. However, first, we shall explain why the various restrictions as set forth in the formal characterization of a modal derivation are required.

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### Exercises

*In constructing modal derivations, the following strategic hints may be useful.*

*Hint 1: After entering the ‘Show’ line for the conclusion and taking any useful assumptions, enter the premises.*

*Hint 2: If your initial ‘Show’ line is a modal statement, you may need to reenter the ‘Show’ line to commence a strict derivation.*

*Hint 3: To show a sentence of the form* ‘□ϕ’*, use strict derivation in the form of a necessity derivation, that is derive* ϕ *from other necessary truths.*

*Hint4. To show a sentence of the form ‘*◇ϕ*’, use strict derivation in the form of possibility derivation, i.e., judiciously choose from the accessible lines of the form* ‘◇ψ’ *to get an assumption for possibility derivation that will enable you to derive* ϕ*.*

*Hint 5. If you have an iterated modal statement as an accessible line, you may need to get a contradiction for indirect derivation by modal negation.*

1. Construct derivations for theorems in the different modal systems.

* + 1. T553◇P ∨ ◇~P in **D**
    2. T550 P→ ◇P in **T**
    3. T571 ◇□P → P in **B**
    4. T561 ◇◇P → ◇P in **S4**
    5. T571 ◇□P → □P in **S5**

2. Construct **T**-derivations for the following.

(A) □(P → Q) . □(Q → R) . ~R ∴ □~P

(B) □(P ∨ Q) . ~◇P ∴ □Q

(C) □P ∨ ◇Q ∴ □(P ∨ ◇Q)

(D) □(P ∨ Q) . □(P → R) . □(Q → S) ∴ □(R ∨ S)

(E) □P ↔ □Q ∴ ~◇~P → ~◇~Q

3. Construct **B**-derivations for the following.

(A) ◇□P ∴ □◇P

(B) □(P → ~◇~P) ∴ ◇P → □P

(C) □(P ∧ □Q) ∴ ◇P ∧ ◇Q

(D) □(P ∨ ◇Q) ∴ □P ∨ ◇Q

(E) □(◇P → Q) ∴ (P → □Q)

4. Construct **S4**-derivations for the following.

(A) □P ∴ □(Q → □P)

(B) P → □Q ∴ P → □(R → □Q)

(C) □(P → Q) ∴ □(R → □(P → Q))

(D) ∴ □P ↔ □□P

(E) ∴ ◇P ↔ ◇◇P

5. Construct **S5**-derivations for the following.

(A) □(P ↔ □P) . ◇P ∴ □P

(B) □(P ↔ ◇□P) ∴ ◇P → □P

(C) ◇~P . □(~P → ~◇P) ∴ ~P

(D) ∴ □P ↔ ◇□P

(E) ∴ ◇P ↔ □◇P

6. In this exercise, we construct modal derivations of the following Anselmian modal arguments. For the purpose of this exercise, we allow ourselves to use the letter ‘G’ for the proposition ‘God exists.’

(A) Construct a **T**-derivation for the following argument**:**

It is necessary that if God exists then it is necessary that God exists. It is possible that God does not exist. Therefore, God does not exist.

(B) Construct a **B**-derivation for the following argument:

*“No one...doubts that if it [that than which a greater cannot be thought] did exist, its nonexistence, either in actuality or in the understanding, would be impossible. For otherwise it would not be that than which a greater cannot be thought.”*

□(G → ~◇~G)

*“Therefore, if that than which a greater cannot be thought can even be thought, it cannot be nonexistent.”*

∴ ◇G → G

(C) Construct an **S4**-derivation for the following argument taken from Anselm’s *Reply to Gaunilo* where he writes: *“But as to whatever can be conceived, but does not exist—if there were such a being, its non-existence, either in reality or in the understanding, would be possible.”*

This first premise is actually a schema since Anselm substitutes for the statement denying that God has existence, statements denying that God has various perfections:

□(◇G ∧ ~G → ◇~P)

If, however, God is a being than which none greater can be conceived, then God necessarily has all perfections. Hence, we have as a second premise:

□P .

From these two premises, derive the conclusion that it is necessary that if God’s existence is possible then God exists:

□(◇G → G) .

(D) Taking the conclusion of exercise (C) as a premise, namely ‘□(◇G → G)’, construct an **S5**-derivation for the conclusion that it is necessary that God exists if it is possible that God exists:

◇G → □G .

(E) Construct an **S5**-derivation for the argument:

It is necessary that if God exists then it is necessary that God exists. It is possible that God exists. Therefore, it is necessary that God exists.

# 6. Fallacies.

In the directions for constructing a modal derivation, a number of restrictions appear whose significance is not perhaps immediately obvious. Recall that a *fallacy* is a procedure that permits the validation of a *false English argument*, that is, an argument whose premises are true sentences of English and whose conclusion is a false sentence of English (or at least are regarded so for the sake of argument). Here we show that the neglect of the various new restrictions on modal derivation would lead to fallacies.

Consider the following argument:

If Descartes is thinking, then Descartes exists. Descartes is thinking. Therefore, it is necessary that Descartes exists.

Descartes is a continent being and so the conclusion of the argument is false. Let us take it for granted that Descartes exists if he is thinking and even that Descartes is thinking. It does not follow from these premises that it is necessary that Descartes exists.

P → Q . P ∴ □Q

Where is the fallacy in the following attempted derivation?

1. *~~Show~~* □Q 4, ND

2. P → Q Premise

3. P Premise

4. Q 2, 3 MP

The fallacy occurs in line 1, where one attempts to box and cancel by strict derivation. One of the restrictions on strict derivation is that premises cannot be among the lines that are boxed. Intuitively, premises are true in the actual world but may not be true in every possible world, and therefore, cannot enter into strict derivations.

Consider the argument:

Roses could be green, but they aren’t. Therefore, is possible that the moon is made of green cheese.

◇P ∧ ~P ∴ ◇Q

Here the premise does not assert an actual contradiction. Instead the premise asserts that it is contingently false that roses are green. The conclusion asserts something that is not only false but clearly irrelevant to the premises. Where is the fallacy in the following attempted derivation?

1. *~~Show~~* ◇Q 6, PD

2. ◇P ∧ ~P Premise

3. ◇P 2, S

4. *~~Show~~* P → ~P 5, CD

5. ~P 2, S

6. *~~Show~~* ◇Q 7, PD

7. *~~Show~~* Q 8, 9 ID

8. P 3, Assume (PD)

9. ~P 8, 4 MP

An assumption for possibility derivation can only occur immediately after a sentence of the form ‘*Show* ◊ϕ’. Therefore a fallacy occurs in line 8. Moreover, one cannot box and cancel by indirect derivation if the derivation begins with an assumption for possibility derivation. Consequently, a fallacy also occurs in line 7 when one attempts to box and cancel by indirect derivation.

The following argument is clearly invalid. In fact, assuming it is only contingently true that roses are red, it is a *false* English argument because its premise would be true while the conclusion is false.

Roses are red and violets are blue. Therefore, it is necessary that roses are red.

Where is the fallacy in the following attempted derivation?

P ∧ Q ∴ □P

1. *~~Show~~* □P 3, DD

2. P ∧ Q Premise

3. *~~Show~~* □P 4, ND

4. P 2, S

One of the restrictions on strict derivation is that inference rules cannot be applied to inaccessible lines. In this example, the premise in line 2 is not accessible from line 4. The rule of simplification cannot be applied to line 2 to obtain line 4 in the strict derivation. This line prohibits the boxing and cancelling in line 3. Intuitively, premises are true in the actual world but may not be true in every possible world.

The following English argument is not only *invalid* but *false*, assuming that the premise is a true future contingent proposition.

The sea battle will happen. Therefore, it is necessary that the sea battle will happen.

P ∴ □P

Where is the fallacy in the following attempted derivation?

* 1. *~~Show~~* □P 4, DD
  2. P Premise
  3. ◊P 2, PG
  4. *~~Show~~* □P 5, ND
  5. P 3, Assume (PD)

Here the assumption for possibility derivation in line 5 is incorrect. Such an assumption must occur immediately after a ‘*Show’* line of the form ‘*Show* ◊ϕ’, whereas in this case, the ‘*Show’* line is of the form ‘□ϕ’, a necessary statement. Furthermore, one may not box and cancel by strict derivation in line 4 even though we have obtained the scope of the sentence in the ‘*Show’* line 4, namely, P, in line 5. The reason is not that line 5 is a premise, or comes by an application of an inference rule to an inaccessible line, but because it was *obtained by a means other than an admissible strict importation rule to an accessible antecedent line.*

Consider the following two arguments. The first is intuitively valid:

It is possible that Schödinger’s cat is dead, or it is possible that Schrödinger’s cat is alive. Therefore, it is possible that Schrödinger’s cat is dead or alive.

Conversely, the following argument is also intuitively valid:

It is possible that Schrödinger’s cat is dead or alive. Therefore, it is possible that Schödinger’s cat is dead, or it is possible that Schrödinger’s cat is alive.

However, the argument obtained by *conjoining* the two separate possibility statements under a single possibility operator is clearly invalid:

It is possible that Schrödinger’s cat is dead. It is possible Schrödinger’s cat is alive.

Therefore, it is possible that Schrödinger’s cat is both dead *and* alive.

◇P . ◇Q ∴ ◇(P ∧ Q)

Where is the fallacy in the following attempted derivation?

1. *~~Show~~* ◇(P ∧ Q) 4, DD

2. ◇P Premise

3. ◇Q Premise

4. *~~Show~~* ◇(P ∧ Q) 7, PD

5. P 2, Assume (PD)

6. Q 3, Assume (PD)

7. P ∧ Q 5, 6 ADJ

The restriction on entering as assumption for possibility derivation states that such an assumption must occur *immediately after* the uncancelled ‘*Show*’ line. Line 5 is legitimate since occurs immediately after line 4, which is of the form ‘*Show* ◊ϕ’. However, line 6, the second application of entering as assumption for PD is fallacious since it does not occur immediately after the uncancelled ‘*Show’* line 4.

Consider next the following more intricate fallacious argument which is a variation on Ivan’s famous argument in *The Brother Karamazov*.

If God is dead, then everything is permitted. It is possible that God is dead.

Therefore, it is possible that everything is permitted.

P → Q . ◇P ∴ ◇Q

Where is the fallacy in the following attempted derivation?

1. *~~Show~~* ◇Q 4, DD

2. P → Q Premise

3. ◇P Premise

4. *~~Show~~* ◇Q 6, PD

5. P 3, Assume (PD)

6. Q 2, 5 MP

Here the fallacy occurs in line 4 in the attempt to box and cancel by strict derivation. In a strict derivation, none of the boxed lines may come from an application of an inference rule to an inaccessible line, or by an application of an inference rule other than an admissible strict importation rule to an accessible line. Notice that line 6 comes from an application of an inference rules to the premise in line 2 and so violates the above restriction.

Sometimes when we say that P is possible, what we mean is not that P is *metaphysically* possible, but only that “for all we know” P is the case. This is an *epistemic* interpretation of possibility. In 1637 Fermat read Diophantus’ *Arithmetic*, a 3rd century treatise and noted in the margins “dividing a cube into two cubes, or in general an *n*th power of two *n*th powers, is impossible if *n* is larger than 2. I have found a remarkable proof of this fact, but the margin is too narrow to contain it.” This enigmatic remark led to the centuries of mathematicians trying to prove “Fermat’s Last Theorem”. The status of Fermat’s conjecture was not established until 1995 when Andrew Wiles, together with his graduate student Richard Taylor, published a proof of Fermat’s Last Theorem.

Now consider the following English argument (given before 1995):

It is possible (*for all we know*) that Fermat’s Last Theorem is a theorem (and hence, necessarily true). Therefore, if it is possible (*for all we know*) that Fermat was right in believing he had a proof of his theorem, then Fermat’s Last Theorem is a true.

This argument is intuitively fallacious. Here’s an attempted symbolization:

◇□P ∴ ◇Q → P

(Here it could be argued that the premise is not properly symbolized because it mixes metaphysical and epistemic modalities.) Nevertheless, let’s consider the following attempted derivation of the argument as symbolized. Where is the fallacy?

1. *~~Show~~* ◇Q → P 3, DD

2. ◇□P Premise

3. *~~Show~~* ◇Q → P 5, CD

4. □P 2, Assume (PD)

5. P 4, NI

One may enter an assumption for possibility derivation only after a ‘*Show’* line of the form ‘*Show* ◊ϕ’. However, the sentence in line 3 is conditional, not a modal sentence. One may enter an assumption for possibility derivation only after a ‘*Show’* line of the form ‘*Show* ◊ϕ’. Therefore, a fallacy occurs in line 4. It happens that if we interpret the modal operators as uniformly metaphysically and assume the modal system **S5**, then we can construct a derivation of the conclusion from the premise. The verification of this claim is left as an exercise for the reader.

Consider one final intuitively fallacious English argument. Let’s suppose that the universe is “fine-tuned”, namely, that it is physically impossible for this universe to exist and for the fundamental constants of physics to be other than they actually are. We could ask the further speculative question of whether this universe could have existed if it operated according to a different system of physical laws together with a different set of constants. Suppose the answer to this question is ‘yes’. Then the following argument is intuitively fallacious:

It is possible that it is possible that the laws of physics could have been different than they are.

Therefore, the law of physics could have been different than they are.

One way to symbolize this argument is as follows:

◇◇P ∴ ◇P

Can you identify the fallacy in the following attempted derivation?

1. *~~Show~~* ◇P 3, DD

2. ◇◇P Premise

3. *~~Show~~* ◇P 4, DD

4. ◇P 2, Assume (PD)

One cannot box and cancel the direct derivation in line 3 because line 4 was introduced by an assumption for possibility derivation. One of the new restrictions on a direct derivation for ‘*Show* ϕ’ in modal propositional logic is that ϕoccurs unboxed among subsequent lines χ1 through χ*n*and χ1 but was *not introduced by an assumption for Possibility Derivation*. Therefore, a fallacy occurs in the boxing and cancelling of line 3. It turns out, however, that this argument can be validated in **S4**. The verification of this claim is left as an exercise for the reader.

The above examples justify, in part, the new restrictions set forth in our official syntactical characterization of the natural deduction systems for modal propositional logic. In the next section, we shall see how these restrictions—in particular, the rules of strict importation—are related to the characteristic axioms of the various modal systems. The reason for these *syntactical* restrictions on derivations can be more clearly understood in light of the *semantics* for propositional modal logic, which we set forth in the next section.

### Exercises

1. Identify all the fallacies steps in the following attempted derivations for intuitively fallacious (i.e., “false”) English arguments. Accurately state the restriction or restrictions that are violated and explain why how the restriction has been violated in the particular case.

(A) Clearly the following argument involves a fallacious wishful thinking.

It is possible that I will win the lottery. Therefore, I will win the lottery.

◇P ∴ P

Where are the fallacies committed in the following three attempted derivations?

1. *~~Show~~* P 7,8, ID

2. ~ P Assume (ID)

3. *~~Show~~* □~P 4, ND

4. ~P 2, R

5. ~◇P 3, MN

6. ◇P Premise

1. *~~Show~~* P 3, DD

2. ◇P Premise

3. *~~Show~~* P 4, 5 ID

4. ~ P Assume (ID)

5. P Assume (PD)

1. *~~Show~~* P 5, DD

2. ◇P Premise

3. *~~Show~~* ~P → P 4, CD

4. P 3, Assume (PD)

5. *~~Show~~* P 6, 7, ID

6. ~P Assume (ID)

7. P 6, 3, MP

(B) Evil exists and it is necessary that God exists. Therefore, necessarily, both that evil exists and that God exists.

P ∧ □Q ∴ □(P ∧ Q)

1. *~~Show~~* □(P ∧ Q) 7,8, ID
2. P ∧ □Q Premise
3. P 2, S
4. □Q 2, S
5. *~~Show~~* □(P ∧ Q) 8, ND
6. P 3, R
7. Q 4, NI
8. P ∧ Q 6, 7 ADJ

(C) It is possible that Theatetus is sitting. Therefore, if it is possible that Theatetus is standing, then it is possible that Theatetus is both sitting and standing.

◇P ∴ ◇Q → ◇(P∧ Q)

1. *~~Show~~* ◇Q → ◇(P ∧ Q) 4, CD
2. ◇Q Assume (CD)
3. ◇P Premise
4. *~~Sho~~w* ◇(P ∧ Q) 7, PD
5. P 3, Assume (PD)
6. Q 2, Assume (PD)
7. P ∧ Q 5, 6 ADJ

2. Show that allowing the following violations of the restrictions would result in the validation of a fallacious (i.e., ‘false’) English argument.

(A) Allowing the result of applying inference rules to premises into strict derivations.

(B) Allowing the application of a strict importation law to an inaccessible line.

(C) Allowing boxing and canceling by CD when the derivation begins with an assumption for possibility derivation.

(D) Allowing a necessity derivation to commence with an assumption for possibility derivation.

(E) Allowing an assumption for possibility derivation to be entered into a strict derivation but not immediately after the relevant uncancelled ‘*Show’* line.

# 7. Possible World Semantics.

The Leibnizian idea of characterizing necessity and possibility in terms of truth in all or some possible worlds was given an elegant formalization by Saul Kripke (1959, 1963) when he was only a teenager. According to Leibniz, a sentence is *necessary* if it is true in *every* possible world, and a sentence is *possible* if it is true is *some* possible world. Kripke showed that by placing very natural conditions on a relation of *relatively possibility* or *accessibility* on a set of possible world, the various systems of modal logic could be validated.

Intuitively, a possible world tells us for each sentence letter whether it is true or false in that world. Stripping away inessentials, we can represent a possible world by a subset of sentence letters. A *modal structure* **M** is an ordered triple <*W*, R, α>, where *W* is a set of possible worlds, R is a relation on *W* x *W* known as the *accessibility relation* or the *relative possibility relation*, and α is a distinguished element of *W* known as the *actual world*.

We can exhaustively characterize the notion of the *truth of a sentence in a possible world ββ*

⊨β ϕ .

(1) If ϕ is a sentence letter S, then

⊨ β S ⇔ S ∈ β (i.e., S is a member of β) .

(2) If ϕ is a ~ψ, then

⊨ β ~ψ ⇔ ⊭β ψ (i.e., it is not the case that ⊨ β ψ) .

(3) If ϕ is (ψ ∧ χ), then

⊨ β (ψ ∧ χ) ⇔ ⊨ β ψ  ⊨ β χ (i.e., both ⊨ β ψ and ⊨ β χ) .

If ϕ is (ψ ∨ χ), then

⊨ β (ψ ∨ χ) ⇔ (⊨ β ψ V ⊨ β χ ) (i.e., either ⊨ β ψ or ⊨ β χ, or both) .

If ϕ is (ψ → χ), then

⊨ β (ψ → χ) ⇔ (⊨ β ψ ⇒ ⊨ β χ) (i.e., if ⊨ β ψ then ⊨ β χ , or either |≠ β ψ or ⊨ β χ) .

If ϕ is (ψ ↔ χ), then

⊨ β (ψ ↔ χ) ⇔ (⊨ β ψ ⇔ ⊨ β χ) (i.e., ⊨ β ψ if, and only if, ⊨ β χ) .

Finally, the law clause gives the Leibnizian truth conditions for necessity and possibility:

(4) If ϕ is □ψ, then

⊨ β □ψ ⇔ ∀γ ∈ W(Rβγ ⇒⊨ γ ψ) (i.e., ψ is true in *all* possible worlds γ possible relative to β) .

If ϕ is ◇ψ, then

⊨ β ◇ψ ⇔ ∃γ ∈ W(Rβγ ⊨γ ψ) (i.e., ψ is true in *some* possible world γ possible relative to β) .

This completes the definition of truth in a model for modal propositional logic. Using this definition of truth, we can now defined what it means for a sentence to be *semantic valid.*

⊨ ϕ (i.e., ϕ is *semantically valid*) if and only if ∀α ∈ W, ⊨αϕ .

Next we obtain different systems of modal logic when various conditions are placed on the accessibility or relative possibility relation R. We say that a relation R is a *series* if R is serial. R is a *reflexivity* if R is totally reflexive. R is a *similarity* if R is totally reflexive and symmetric. R is a *partial ordering* if R is totally reflexive and transitive. R is an *equivalence relation* if R is totally reflexive and euclidean. It turns out that the axioms of modal logic discussed above are validated when natural conditions are imposed on the accessibility or relative possibility relation R.

|  |  |  |  |
| --- | --- | --- | --- |
| **D** | □ϕ → ◇ϕ | *serial* | ∀α∃β Rαβ |
| **T** | □ϕ → ϕ | *reflexive* | ∀α Rαα |
| **B** | ϕ → □◇ϕ | *symmetric* | ∀α∀β(Rαβ ⇒ Rβα) |
| **4** | □ϕ → □□ϕ | *transitive* | ∀α∀β∀γ( Rαβ & Rβγ ⇒ Rαγ) |
| **5** | ◇ϕ → □◇ϕ | *euclidean* | ∀α∀β∀γ (Rαβ & Rαγ ⇒ Rβγ) |

**Properties of Accessibility**

Systems of modal logic are *normal* when everything derivable from necessary truths are themselves necessary. This will be the case if the rule of *modus ponens* and axiom **K** are valid:

□(ϕ → ψ) → (□ϕ → □ψ) .

Axiom **K** expresses the intuition that *necessary truths imply only necessary truths*.

We can conveniently summarize the above systems of modal logic in a chart. The various modal systems can be characterized by the axioms that are valid in them. The smallest normal modal logic system **K** contains axiom **K**. The four most famous modal logics are system (T) named after the Gödel-Feyes-von Wright modal logic to model tautologies, system **S4** and **S5**, named after C. I. Lewis’s axioms for strict implication, and the unduly neglected Brouwersche system **B,** named by Becker after the intuitionist L. E. J. Brouwer due to the characteristic axiom’s similarity to intuitionistic double negation. All these systems contain **K** and **T**. The system (**D**) which is weaker system than (**T**), containing axioms K and **D**, isnamed for deontic logic.

Notice that relationships of containment among the modal systems follow from the logic of relations. Axiom **B** requires that R be symmetric and **4** requires that R be transitive. System **S5** with axioms **T** and **E** require R be an equivalence relation (i.e., reflexive, symmetric, and transitive); hence, **S5** could also be specified by requiring axioms **T**, **4**, and **B** to be valid. Therefore, **S5** contains **S4** and **B**, neither of which contains the other. Systems **S5**, **S4** and **B** all contain system **T**, which contains **D**.

A convenient way of describing these modal logics is by their *Lemmon code* listing the axioms valid in them. For example, **S5** = **KTE** = **KT4B** = **KD4B**. We can represent these containment relations in a diagram in which downward paths represent containment.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| System | Named For |  | Axiom | Code | Accessibility |
| **D** | Deontic Logic | D | □P → ◇P | KD | *seriality* |
| **T** | Gödel-Feys-von Wright Tautology | **T** | □P → P | **KT** | *reflexivity* |
| **B** | Brouwersche System | **B** | P → □◇P | **KTB** | *similarity* |
| **S4** | Lewis’ Strict Implication System 4 | **4** | □P → □□P | **KT4** | *partial ordering* |
| **S5** | Lewis’ Strict Implication System 5 | **E** | ◇P → □◇P | **KTE** | *uivalence relation* |

Figure 2 Modal Axioms and Accessibility

**S5**

**B S4 .**

**T .**

**D .**

Logical Containment of the Modal Systems

Various deontic modal systems can also be characterized by their axioms (*Lemmon code*).

|  |  |  |  |
| --- | --- | --- | --- |
| System **T** | **KT** | Deontic **D** | **KD** |
| System **B** | **KTB** |  |  |
| System **S4** | **KT4** | Deontic **S4** | **KD4** |
| System **S5** | **KTE = KT4B = KD4B** | Deontic **S5** | **KD4E** |

Figure 3 Lemmon Codes for Deontic Modal Systems

The logical relationships among the above systems of modal logic can be set forth in a diagram (due to Krister Segerberg who omits **KD5** and **K45**). As before, a modal logic is included in another if it is connected to it, directly or indirectly, by an upward path.

S5

**S4**

**dS5**

**T dS4**

**K5**

**K4**

**D**

**K**

**Picasso's Chair**

Hhere is a more elaborate “Picassos’ Electric Chair” that includes deontic modal logics.



One way to visualize how the conditions on the accessibility relation validate their respective axioms is use the definitions of ‘□’ and ‘’ in terms of possible worlds and use directed graphs from chapter IV to represent the accessibility relation R. Here the accessibility relation Rαβ (read “β is possible relative to α” or “β is accessible to α”) is represented by an arrow from a circle representing possible world α to a circle representing possible world β.

Rαβ

α β

**Accessibility Represented by Directed Graphs**

We can, using the directed graphs from the theory of relations, translate properties of accessibility relations into geometric properties of directed graphs. Symmetry, for example, requires that all accessibility arrows are double arrows. Reflexivity requires that every world be accessible to itself and so every world has a loop, which is a special case of a double arrow. Seriality requires that every world is a tail of an arrow. Transitivity requires that for every indirect path of accessibility from α to β and from β to γ, there is a direct path from α to γ. Being euclidean and serial is equivalent to being an equivalence relation, that is, to being reflexive, symmetric and transitive. Expressing R in terms of love and worlds in terms of persons, we have the following intuitive translations:

|  |  |  |
| --- | --- | --- |
| serial | ∀α∃β Rαβ | *Everyone is a lover.* |
| reflexive | ∀α Rαα | *Everyone is a self-lover.* |
| symmetric | ∀α∀β( Rαβ ⇒ Rβα) | *All love is requited; all love is mutual.* |
| transitive | ∀α∀β∀γ( Rαβ & Rβγ ⇒ Rαγ) | *Love is transitive.* |
| euclidean | ∀α∀β∀γ (Rαβ & Rαγ ⇒ Rβγ) | *All beloveds of the same lover, love each other and themselves.* |

**Relational Properties of Directed Graphs**

Using the definition of truth in a modal system set forth above, we can rigorously demonstrate that if R is transitive, then axiom **4** is valid. This demonstration is carried out in the meta-language. We use the use the symbols ‘∀’, ‘∃’, ‘&’, ‘⇒’ and ‘∈’ in the meta-language for ‘all’, ‘some’, ‘and’, ‘if… then’, and ‘is an element of’, respectively. Once the truth clauses are unpacked, the logical demonstration is no more complicated than a derivation in the theory of relations.

A semantic derivation shows that the transitivity of R implies the validity of axiom (4):

1. *~~Show~~* R is *transitive* ⇒ ⊨ (□ϕ → □□ϕ) 3, CD
2. ∀α∀β∀γ(Rαβ & Rβγ ⇒ Rαγ) Assume (CD), Def. *transitive*
3. *~~Show~~* ⊨ (□ϕ  □□ϕ) 24, DD
4. *~~Show~~* ∀α ∈ *W* (⊨α □ϕ ⇒ ⊨α □□ϕ) 5, UD
5. *~~Show~~* ⊨α □ϕ ⇒ ⊨α □□ϕ 8, CD
6. ⊨α □ϕ Assume (CD)
7. ∀β ∈*W* (Rαβ ⇒ ⊨β ϕ) 6, Truth-Def. □
8. *~~Show~~* ⊨α □□ϕ 23, DD
9. *~~Show~~* ∀β∈*W* (Rαβ ⇒ ⊨β □ϕ) 10, UD
10. *~~Show~~* Rαβ ⇒ ⊨β □ϕ 12, CD
11. Rαβ Assume (CD)
12. *~~Show~~* ⊨ β □ϕ 22, DD
13. *~~Show~~* ∀γ ∈ *W* (Rβγ ⇒ ⊨γ ϕ) 14, UD
14. *~~Show~~* Rβγ ⇒ ⊨γ ϕ 16, CD
15. Rβγ Assume (CD)
16. *~~Show~~* ⊨γ ϕ 22, DD
17. Rαβ & Rβγ 11, 15 ADJ
18. Rαβ & Rβγ ⇒ Rαγ 2, UI3 (transitivity!)
19. Rαγ 18, 17 MP
20. Rαγ ⇒ ⊨γ ϕ 7, UI
21. ⊨γ ϕ 20, 19 MP
22. ⊨ β □ϕ 12, Truth-Def. 🞎
23. ⊨α □□ϕ 9, Truth-Def. □
24. ⊨ (□ϕ  □□ϕ) 4, Def. ⊨ (*logical consequence*)

The above semantic derivation can be visualized graphically to prove the above result contrapositively. We will show that if

□ϕ → □□ ϕ

is false, then the accessibility relation cannot be transitive. Axiom (4) fails when its antecedent ‘□ϕ’ is true but its consequent ‘□□ ϕ’ is false in α. By modal negation, ~j is equivalent to ~j.

□ϕ

So both ‘□ϕ’ and ‘~ϕ’ are true in some possible world α. Eliminating the first , we have that in some possible world β accessible to α, ~j is true. Applying the definition of truth for  again, we have that for some world γ accessible to β, ~ϕ is true in γ. Notice that the transitivity of the accessibility relation R would require that γ be accessible to α, which would contradict that j. The invalidity of axiom (**4**) implies the failure of the transitivity of the accessibility relation R. Stating this result contrapositively, if the accessibility relation R is transitive, then axiom (**4**) is valid.

#

□ϕ ϕ ~ϕ

~ □□ϕ ~ϕ

~ϕ

γ

β

α

**Transitivity Validates Axiom (4)**

The semantic demonstration together with the graphic proof using directed graphs support one another in building our intuitions for modal logic. The graphic proofs are satisfying because of their simplicity and ability to make the critical step visually perspicuous. On the other hand, the semantic proof is satisfying insofar as the explicit definitions of , , ⊨ are shown to work elegantly and precisely with our natural deduction systems using the quantifier logic with UD, UI, EI, EG and the theory of relations. We will use the directed graphs in the context of discovering new axioms and the semantic proofs in the context of justifying that these axioms correspond to imposing requirements on the accessibility relation.

### Exercises

1. R is the relative possibility or accessibility relation in a modal system. First construct a graphic demonstration to discover the logical key that unlocks the relationship between the validity of an axiom and its corresponding accessibility relation, and then provide rigorous semantic derivation to justify your discovery.
   1. If R is a serial, then axiom (**D)** is valid.

ϕ

~j

□ϕ

~ϕ

~j

#

1. *~~Show~~* Ris *serial* ⇒ ⊨ (□ϕ → ϕ) 3, CD

2. \_\_\_\_\_\_\_\_\_\_\_\_\_ Assume (CD), Def. *serial*

3. *~~Show~~* ⊨ (□ϕ → ϕ) 18, DD

4. *~~Show~~* ∀α∈*W* ⊨α (□ϕ → ϕ) 5, UD

5. *~~Show~~* ⊨α (□ϕ  ϕ) 17, DD

6. *~~Show~~* ⊨α □ϕ ⇒ ⊨αϕ 16, CD

7. ⊨α □ϕ Assume (CD)

8. ∀β∈*W* (Rαβ ⇒ \_\_\_\_\_ ) 7, Truth-Def. □

9. *~~Show~~* ⊨αϕ 16, DD

10. *~~Show~~* ∃β∈*W* (Rαβ & \_\_\_\_) 15, DD

11. Rαγ 2, \_\_\_, \_\_\_

12. Rαγ ⇒ ⊨γ ϕ 8, \_\_\_

13. ⊨γ ϕ 11, 12 \_\_\_

14. Rαγ & ⊨γ ϕ 11, 13 \_\_\_

15. ∃β∈W (\_\_\_\_\_\_\_\_\_\_\_\_) 14, EG

16. \_\_\_\_\_\_ 10, Truth-Def. 

17. \_\_\_\_\_\_\_\_\_\_ 6, Def. \_\_\_\_

18. ⊨□ϕ → ϕ 4, Def. ⊨ (*logical consequence*)

* 1. If R is reflexive, then axiom (**T)** is valid.

#

1. *~~Show~~* Ris *reflexive* ⇒ ⊨ (□ϕ → ϕ) 3,CD

2. ∀αRαα Assume (CD), Def. *Reflexive*

3. *~~Show~~* ⊨ (□ϕ → ϕ) 14, DD

4. *~~Show~~* ∀α∈ *W* ⊨α (□ϕ → ϕ) 5, UD

5. *~~Show~~* ⊨α (□ϕ  ϕ) 13, DD

6. *~~Show~~* ⊨α □ϕ ⇒ ⊨αϕ 9, CD

7. ⊨α □ϕ Assume (CD)

8. ∀β∈*W* (Rαβ ⇒ ⊨β ϕ ) 7, Def. Truth 🞎

9. *~~Show~~* ⊨α ϕ 12, DD

10. Rαα ⇒ ⊨αϕ 8, UI

11. Rαα 2, UI

12. ⊨α ϕ 10, 11 MP

13. ⊨α (□ϕ  ϕ) 6, Def. Truth 

14. ⊨ (□ϕ  ϕ) 5, Def. |= (*logical consequence*)

An intriguing philosophical application of R is *symmetric* ⇒ ⊨ (ϕ → □ϕ) is a modal ontological argument, which we state in a weaker form that is possible.

(G → G) . G ∴ G

Let’s read  is “it is conceivable that” and  as “it is conceptually necessary that”. Then the first premise can be translated: “It is conceptually necessary that if God exists, then it is necessary that God exists.” The intuitive philosophical idea is that God’s existence, like the existence of numbers, couldn’t be accidental: if God exists at all, then God’s existence is not contingent but necessary. Moreover, this conditional itself is a conceptual truth. Then if it is even conceivable, i.e., conceptually possible that God exists, then God actually exists!

How can it be that purely conceptual premises logically imply the actual existence of God? Is there an “intuitionistic” sleight of mind going on here? It happens that modal axiom (**B**) is what is required to validate this argument. Use a semantic diagram to see why this is the case!

1. *~~Show~~* R is *symmetric* ⇒ ⊨ (ϕ → □ϕ)
2. ` `
3. *~~Show~~* ⊨(ϕ → □ϕ)
4. *~~Show~~*
5. *~~Show~~*
6. *~~Show~~* ⊨α ϕ ⇒ ⊨α □ϕ
8. *~~Show~~*

1. *~~Show~~*
2. *~~Show~~* Rαβ ⇒ ⊨β ϕ
4. *~~Show~~*
5. *~~Show~~*

8. Rβα & ⊨α ϕ
9. ∃γ ∈*W* (Rβγ & ⊨γ ϕ)
10. ⊨β ϕ
11. ⊨α □ϕ
12. ⊨α (ϕ → □ϕ)
13. ⊨ ϕ → □ϕ

* 1. If R is euclidean, then **E** is valid.

1. *~~Show~~* R is *euclidean* ⇒ ⊨ (ϕ → □ϕ)
3. *~~Show~~*
4. *~~Show~~* ∀α∈W⊨ α (ϕ → □ϕ)

5. *~~Show~~* ⊨ αϕ → □ϕ

6. *~~Show~~* ⊨= αϕ ⇒ ⊨ α □ϕ

7.

8. ∃β∈W(Rαβ & \_\_\_\_\_)

9. *~~Show~~* ⊨ α □ϕ

10. *~~Show~~* ∀β(Rαβ ⇒ \_\_\_\_\_)

11. *~~Show~~* Rαβ ⇒ ⊨ β ϕ

12.

13. *~~Show~~* ⊨ β ϕ

14. *~~Show~~* ∃γ∈W(Rβγ & ⊨γ ϕ)

15. Rαγ′ & ⊨ γ′ ϕ

16. Rαβ & Rαγ′

17.

18.

19. Rβγ′ & ⊨ γ′ ϕ

20. ∃γ∈W (Rβγ & ⊨γ ϕ)

21. ⊨ β ϕ

22. ⊨ α □ϕ

23. ⊨ αϕ → □ϕ

 ⊨ϕ → □ϕ

2. There are different axiomatizations for different philosophical conception of time. According to the *tensed theory of time* championed by the philosopher C. D. Broad, the past is real and the present moves up the tree of possible futures turning the unreal future into the real past. Another theory, due to A. N. Prior, known as *presentism*, holds that only the present is real and the past and the future are unreal. What do the following axioms express about the structure of time?

(D1) ◇F ◇F ϕ → ◇F ϕ *transitivity* ∀*x* < *t* ∀*y* < *x*: *y* < *t*

(D2) ◇F ϕ → □G(🞛P ϕ ∨ ϕ ∨ ◇Fϕ) *future-linearity* ∀*x*∀*y*(*y* < *x* → ∀*z*(R*xz* → *y* < *z*))

(D3) 🞛P **⊤** *regress into past* ∀*t* ∃*x* *x* < *t*

(D4) 🞛P ϕ → □🞛P ϕ *necessity of the past*

Premise D2 corresponds to the condition of *future-linearity*, namely, that all times future to a moment of time are comparable to a past moment of that time. The argument is valid because this premise implies the reduction of modality to temporality:

ϕ → (🞛P ϕ ∨ ϕ ∨ ◇F ϕ) .

What properties of time to the following axioms express?

◇F **⊤**

🞛P **⊤**

◇F ◇F ϕ → ◇F ϕ

🞛P 🞛P ϕ → 🞛P ϕ

◇F ϕ → ◇F ◇F ϕ

🞛P ϕ → 🞛P 🞛P ϕ

ϕ → ◇F 🞖H(ϕ∨ ◇F ϕ)

◇F ϕ ∧ ◇F ψ → ◇F (ϕ ∧ ψ) ∨ ◇F (ϕ ∧ ◇F ψ) ∨ ◇F (◇F ϕ ∧ ψ)

🞛P ϕ ∧ 🞛P ψ → 🞛P (ϕ ∧ ψ) ∨ 🞛P (ϕ ∧ 🞛P ψ) ∨ (🞛P (🞛P ϕ ∧ ψ)

*unending future* ∀*x*∃*y* *x* < *y*

*past successors*∀*x*∃*y* *y* > *x*

*future transitivity* ∀*x*∀*y*∀*z* (*x* < *y* < *z* → *x* < *z*)

*past transitivity* ∀*x*∀*y*∀*z* (*x* > *y* > *z* → *x* > *z*)

*future* *density* ∀*x*∀*y*(*x* < *y* → ∃*z*(*x* < *z* ∧ *z* < *y*))

*past density* ∀*x*∀*y*(*x* > *y* → ∃*z*(*x* > *z* ∧ *z* > *y*))

*future discreteness* ∃*y*(*x* < *y* ∧ ∀*z*(*z* < *y* → *z* = *y* ∨ *z* < *x*))

*linearity* *with respect to the future* ∀*y*∀*z*(*x* < *y* ∧ *x* < *z*) → (*y* < *z* ∨ *z* < *y* ∨ *z* = *y*))

*linearity* *with respect to the past* ∀*y*∀*z*(*x* > *y* ∧ *x* > *z*) → (*y* > *z* ∨ *z* > *y* ∨ *z* = *y*))

3. Prior’s axiomatization of temporal logic known as the *Diodorean System* consists of **KT4** and the following axiom **Dum** (named after the philosopher Michael Dummett, a champion of intuitionism). Give a reading of his axiom in temporal logic:

□(□(P→□P)→P)→ (□P →P) .

4. Demonstrate that the following axioms require the corresponding conditions on R.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| DC | Converse **D** |  | ∀α∀β∀γ (Rαβ & Rαγ ⇒ β = γ) | R is *functional* |
| **□T** | Necessitation **T** |  | ∀α∀β(Rαβ ⇒ Rββ) | R is *point reflexive* |
| **4C** | Converse **4** |  | ∀α∀β[Rαβ ⇒ ∃γ (Rαγ & Rγβ)] | R is *dense* |
| **C** | Convergent |  | ∀α∀β∀γ[R αβ & Rαγ ⇒ ∃η(Rαη & R γη)] | R is *convergent* |

Formulate each of the conditions on R in terms of a directed graph. Then demonstrate that if each of the axioms is assumed to be invalid, then the corresponding condition on R must fails to hold.

5. An elegant result due to Lemmon and Scott (1977) states a correspondence between conditions on R and axioms of the generalized form:

*h*□*i*ϕ → □*j**k*ϕ .

Here ◊*m* and □*n* represents *m* diamonds and *n* boxes in a row, respectively. The corresponding condition on R imposed by the generalized axiom is given by

R*h*αβ & R*j*αγ ⇒ ∃η(R*i*βη & R*k*γη) .

Here the *composition* R ° S of two relations R and S (in symbols, R ° S) can be defined as

R ° S αβ =df ∃γ(Rαγ & Sγβ) .

For example, if R is the relation *being a sister of* and S is the relation *being a parent of*, then the composition of the relations R ° S is the relation *being* *an aunt of*, that is

R ° S αβ ↔ ∃γ(Rαγ & Sγβ) .

If R is the relation *being a parent of*, then R ° R = R 2 is the relation *being a grandparent of*, and R ° R ° R = R3 is the relation *being a great-grandparent of*. R0 is the *identity relation*, i.e., R0αβ if and only if α = β, and R 1 = R.

Verify that the values for *h, i, j, k* as indicated in the chart yield the standard axioms, and that the standard condition on R for each of the axioms is equivalent to the one specified by the Lemmon-Scott Generalized Correspondence theorem:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | *h* | *i* | *j* | *K* | Condition on R | |
| **B** | 0 | 0 | 1 | 1 | *Symmetric* | ∀α∀β(Rαβ ⇒ Rβα) |
| **4** | 0 | 1 | 2 | 0 | *Transitive* | ∀α∀β∀γ(Rαβ & Rβγ ⇒ Rαγ) |
| **5** | 1 | 0 | 1 | 1 | *Euclidean* | ∀α∀β∀γ (Rαβ & Rαγ ⇒ Rβγ) |
| **C** | 1 | 1 | 1 | 1 | *Convergent* | Rαβ & Rαγ ⇒ ∃η(Rβη & R γη) |

5. Another interpretation of the ‘□’ is provability. The *Gödel-Löb Provability System* is the system **K4W,** wherethe **(W)** axiom (here ‘W’ stands for “well-ordering”) is as follows:

(**W**) □(□ϕ → ϕ) → □ϕ .

Give an informal argument to show that (**W**) requires the accessibility relation to be transitive, irreflexive, and *finite* (i.e., indirect paths of R chains terminate in a *finite* number of steps.) It is interesting to note that being ‘finite’ cannot be expressed in a sentence of first-order logic. This follows from the compactness meta-theorem that states that any infinite set of sentences that is finitely satisfiable (i.e., any finite subset has a model) must also be satisfiable (i.e., there is a model that satisfies the entire infinite set).

# 8. Theorems.

First we list the *characteristic axioms* for various modal systems.

T538 □P → P Axiom (**D**)

T560 □P → P Axiom (**T**)

T570 P → □P Axiom (**B**)

T580 □P → □□P Axiom (**4**)

T590 P → □P Axiom (**5**)

Next we list some theorem common to all the modal systems we have been discussing. T501 and T502 state *modal derivation* *principles*. T501, known as axiom (K**)** (after Kripke), expresses the principle that justifies strict derivation in the form of necessity derivation: whatever is entailed by a necessary truth is necessary. T502 is the basis for possibility derivation: whatever is logically entailed by something possible is itself possible.

T501 □(P → Q) → (□P → □Q)

T502 □(P → Q) → (P → Q)

Theorems T503-T506 express the *laws of modal negation*:

T503 ~ □P ↔ ◊~ P

T504 ~ P ↔ □~ P

T505 □P ↔ ~~P

T506 ◊P ↔ ~□~ P

In addition to T501 and T502 above, T507-T520 state *laws of modal distribution* that are valid in all systems of normal modal logic.

T507 (P ∨ Q) ↔ P ∨ Q

T508 □(P ∧ Q) ↔ □P ∧ □Q

T509 (P ∧ Q) → P ∧ Q

T510 □P ∨ □Q → □(P ∨ Q)

T511 (P → Q) → (P → Q)

T512 (□P → □Q) → (P → Q)

T513 □(P ↔ Q) → (□P ↔ □Q)

T514 □(P ↔ Q) → (P ↔ Q)

T515 ~ P → ~ □P

T516 □(P ∨ Q) → P ∨ Q

T517 ~P → ~P

T518 (P → □Q) → 🞎(P → Q)

T519 (P → □Q) → (□P → □Q)

T520 (P → □Q) → (P → Q)

T521 (~P ∧ Q) → (~P ∧ Q)

The following theorems are modal counterparts to *consequentia mirabilis*:

T522 (~□P → P) ↔ P

T523 (~P → P) ↔ P

T524 (P → ~P) ↔ ~P

T525 (P → □~P) ↔ ~P

T526 (P → ~P) ↔ ~P

T527 (~P → □P) ↔ P

Analogues to theorems T229 and T230 in monadic quantifier logic (which we may call the “Drinking Theorems” after a joke by Smullyan) are (**T)** theorems despite the nested modal operators:

T529 (P → P)

T530 (P → □P)

Corresponding to quantifier theorems T234 − T247 are the modal theorems T534 − T547. Theorems T533 and T535 state principles—*transitivity* and *strengthening*—that hold for strict implications but which do not hold for *counterfactual conditionals* (that is, conditionals of the form, *if it were the case that* P*, then it would be the case that* Q).

T533 □(P → Q) → □(P ∧ R → Q)

T534 □[(P → Q) ∧ (Q→ R) → (P → R)]

T535 □(P → Q) ∧ □(Q → R) → □(P → R)

T536 □(P ↔ Q) ∧ □(Q ↔ R) → □(P ↔ R)

T537 □(P → Q) ∧ □(P → R) → □(P → Q ∧ R)

T538 □P → P

T539 (□P ∧ Q) → (P ∧ Q)

T540 □(P → Q) ∧ (P ∧ R) → (Q ∧ R)

T541 □(P → Q ∨ R) → □(P → Q) ∨ (P ∧ R)

Theorems T542 − T547 concern *strict implication*. T544 states the equivalence of two characterizations of strict implication.

T542 ~□(P → Q) ↔ (P ∧ ~Q)

T543 ~(P ∧ Q) ↔ □(P → ~ Q)

T544 □(P → Q) ↔ ~(P ∧ ~Q)

T545 ~P ↔ □(P → Q) ∧ □(P → ~Q)

T546 ~P ∧ ~Q → □(P ↔ Q)

T547 (P → Q) ↔ (□P → Q)

Although C. I. Lewis’s axioms of *strict implication* were introduced to overcome the paradoxes of material implication, theorems T548-T552 are corresponding *paradoxes of strict implication*:

T548 ~□(P → Q) → P

T549 ~□(P → Q) → ~□Q

T550 □Q → □(P → Q)

T551 ~P → □(P → Q)

T552 (P → Q) ∨ □(Q → P)

Next we list theorems for the various modal systems. T538, the characteristic axiom for the modal system (**D),** is followed by other theorems of that system**.**

T538 □P → P

T553 P ∨ ~P

T554 ~(□P ∧ □~P)

T555 □(P → ~Q) → (□P → ~□Q)

T560, the characteristic axiom for system (**T),** is followed by its logical dual T561, other theorems of that system.

T560 □P → P

T561 P → P

T562 □(□P → P)

T563 □(P → P)

T564 □□P → □P

T565 P → P

T566 □P → P

T567 □P → □P

T570, the characteristic axiom for system (**B),** is followed by its logical dual T571, other theorems of that system. Theorems T573 and T574 are valid in **KB**, that is, their proofs do not require NI as an inference rule, but only NI and ◊R as admissible strict importation rules. Theorems T576 and T577 are related to Anselmian arguments discussed in the introduction that can be validated using (**B)**.

T570 P → □P

T571 □P → P

T572 P → ~~P

T573 □(P → Q) → (P → □Q)

T574 □(P → □Q) → (P → Q)

T575 □P → □P

T576 □(P → □P) ∧ ◊P → P

T577 □(P → □P) ∧ P → □P

T580, the characteristic theorem of system **S4**, is followed by its logical dual T581 and other theorems of system **S4** (also known as **KT4**). Theorems T582 − T584 are valid in **K4**, that is, they do not require NI as an inference rule but only NI and □R as admissible strict importation rules. Theorems T585 and T586 are the *modal reductions laws* for (**S4)**: one may collapse any string of iterated uniform operators to a single occurrence of that operator.

T580 □P → □□P

T581 P → P

T582 □(P → Q) → □(□P → □Q)

T583 □(P ∨ Q) → □(P ∨ □Q)

T584 □(□(P → Q) → R) → □(□(P→Q) → □R)

T585 □P ↔ □□P

T586 P ↔ P

Theorem T587 established that the (**G)** schema is valid in any normal **KB** system. T588 is an alternative basis for (**S4)**.

T587 ◊□P → □◊P

T588 □(P → Q) → □(□P → □Q)

Theorems T590−T595 are theorems of the system (**S5)** (also known as **KT5**). T590 is the characteristic theorem of the system, and T571 is its logical dual. Theorems T592−T598 are valid in **K5**, that is, they do not require NI as an inference rule, but they may employ NI, 🞏R, and ◊R as admissible strict importation rules. Theorems T599−T590 are valid in any normal **K5** system.

T590 P → □P

T591 □P → □P

T592 □P → □P

T593 □P → P

T594 □P → □□P

T595 □P → □P

T596 P → □P

T597 □(□P ∨ Q) → □P ∨ □Q

T598 P ∧ Q → (P ∧ Q)

T599 □(□P ↔ □□P)

T600 □(P ↔ P)

T601 □(□P ↔ □P)

T602 □(↔P ↔ □P)

T603 □□P ↔ □□□P

T604 □□P ↔ □□P

T605 □P ↔ □□P

T606 □P ↔ □P

T607 P ↔ P

T608 P ↔ □P

T609 □P ↔ □P

T610 □P ↔ □□P

Theorems T611−T612 are the additional *modal reduction laws* for **S5**. These modal reduction laws together with the modal reduction laws T585 and T586 for (**S4)**, which also hold in **(S5)**, imply that iterated modalities collapse to the innermost operator.

T611 □P ↔ □P

T612 P ↔ □P

### Exercises

1. Verify the following claims.

* 1. **K** is derivable in **T**.
  2. PGisderivablein each of the modal systems **T**, **B**, **S4**, and **S5**.

(C) Show that in each of the modal systems if ϕ is a theorem of classical sentential logic, then □ϕ is derivable (this is known as the rule of necessitation **Nec**).

2. Prove by induction that Interchange of Equivalents (see pp. 362-363 of KMM) is derivable in each of the modal systems.

3. Construct **T**-derivations of the following theorems:

(A) T501 □(P → Q) → (□P → □Q)

(B) T502 □(P → Q) → (P → Q)

(C) T541 P → P

(D) T542 □(□P → P)

(E) T523 (P → Q) ∨ □(Q → P)

4. Construct **B**-derivations for the following theorems:

(A) T550 □P → P

(B) T551 P → □P

(C) T552 □(P → Q) → (P → □Q)

(D) T553 □(P → □Q) → (P → Q)

(E) T554 □P → □P

5. Construct **S4**-derivations for the following theorems:

(A) T560 □P → □□P

(B) T561 P → P

(C) T562 □(P → Q) → □(□P → □Q)

(D) T563 □(P ∨ Q) → □(□P ∨ □Q)

(E) T564 □(□(P → Q) → R) → □(□(P→Q) → □R)

6. Construct **S5**-derivations for the following theorems:

(A) T571 □P → □P

(B) T573 P ∧ Q → (P ∧ Q)

(C) T574 □P → □P

(D) T594 □P ↔ □□P

(E) T597 P ↔ □P

7. Prove that if a relation R is reflexive, then R is euclidean if and only if R is both symmetric and transitive, that is prove the following theorem:

∀*x*R*xx* ⇒ (∀*u*∀*t*∀*v*(R*ut* & R*uv* ⇒ R*tv*) ⇔ [∀*x*∀*y*(R*xy* ⇒ R*yx*) & ∀*x*∀*y*∀*z*(R*xy* & R*yz* ⇒ R*xz*)]) .

8. For each of the following pairs of statements, one is a theorem in **T**, but the other is not. If a statement is a theorem, construct a **T**-derivation of its conclusion from its premises. If statement is not a theorem, provide an invalidating semantic diagram.

(A) □(P → Q) → (P → Q) ; (P → Q) → □(P → Q)

(B) (P ∧ Q) → (P ∧ Q) ; (□P ∧ Q) → (P ∧ Q)

9. Construct a semantic diagram and a semantic derivation in our metalanguage to show that the axiom

□(□P → Q) ∨ □(□Q → P)

is valid in any modal structure in which the accessibility relation R is *weakly connected*, i.e.,

∀*x*∀*y*∀*z*(R*xy*  R*xz* ⇒ R*yz* **∨** R*zy*) .

10. Prove the following theorems about strict implication and strict equivalence in system (**T**) to which we add the following deductive rules:

*Strict Implication* [SI]: From P ⇒ Q to infer □(P → Q), and conversely.

*Strict Equivalence* [SE]: From P ⇔ Q to infer □(P ↔ Q), and conversely.

(A) ~(P ∧ ~Q) ↔ (P ⇒ Q)

(B) (P ⇒ Q) ∧ (P ⇒ ~Q) ⇒ 🞎Q

(C) (~P ∧ ~Q) ⇒ (P ⇔ Q)

(D) □Q ∴ P ⇒ Q

(E) ~P ∴ P ⇒ Q

11. A statement P is *contingent* if it is neither necessary nor impossible. We may define the *contingency operator* ∇ as follows:

∇P ↔ ~(□P  □~P) .

Translate the following axioms about contingency into English, and then, using the above definition, provide derivations of each of them in **S5** in which you may use both Interchange of Equivalences (IE) and biconditional derivation (BD).

(A) ∇P ↔ ∇~P

(B) ~∇(P ↔ Q) → (∇P → ∇Q)

(C) ∇(P → Q) → (~∇P → P)

(D) ~∇(P → ∇Q)

(E) □P ↔ (P ∧ ~∇P)

# 9. Counterfactuals “Would” and “Could”

Frege argued that if the conditional is truth-functional then it must have the values it is assigned in the traditional truth table.

|  |  |  |  |
| --- | --- | --- | --- |
|  | P | Q | P  Q |
| 1 | T | T | T |
| 2 | T | F | F |
| 3 | F | T | T |
| 4 | F | F | T |

Recall that an argument is *semantically valid* if and only if its logical form is such that it is impossible for its premises to be true while its conclusion is false. Conditional arguments occur with enough frequency to have acquired traditional names for its valid, and fallacious, inferences.

|  |  |  |
| --- | --- | --- |
| Valid Inferences | P Q  P  ∴ Q | P  Q  ~Q  ∴ ~P |
|  | *Modus Ponens* | *Modus Tollens* |
| Common Fallacies | P  Q  Q  ∴ P | P  Q  ~P  ∴ ~Q |
|  | *Fallacy of Affirming the Consequent* | *Fallacy of Denying the Antecedent* |

Notice that row 3 of the truth-table for the conditional must be T if the fallacies of affirming the consequent or denying the antecedent are invalid. By the definition of invalidity, it must be possible for the following argument to have true premises and a false conclusion (steps 1-3 and 3’) However, given that a negation must have the opposite value of the sentence it negative, P must be **F** (step 4 and 4’).

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| P |  | Q |  | ~ | P |  | ∴ | Q |  |
| **F** | **T** | **F** |  | **T** | **F** |  |  | **F** |  |
| 4’ | 1 | 3’ |  | 2 | 4 |  |  | 3 |  |

Therefore, if the argument is to be invalidated, the conditional must be true when its antecedent is **T** and its consequent is **F**. The value of the conditional in row 3 must therefore be **T**.

Next, we note that we must be able to distinguish a *conditional* (P  Q) from its *converse* (Q P). Therefore, the value of the second row of the truth table for the conditional must be *different* from its value in the third row. The value of the conditional in row 2 must therefore be **F**.

Beside the common fallacies noted above, there are some uncommon, or deviant, fallacies, which we shall call *modus tortoise* *ponens*, and *modus tortoise tollens*.

|  |  |  |
| --- | --- | --- |
| Uncommon Fallacies | P  Q  P  ∴ ~Q | P  Q  ~Q  ∴P |
|  | *Modus Tortoise Ponens* | *Modus Tortoise Tollens* |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| P |  | Q |  |  | P |  | ∴ | ~ | Q |
| **T** | **T** | **T** |  |  | **T** |  |  | **F** | **T** |
| 2’ | 1 | 4’ |  |  | 2 |  |  | 3 | 4 |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| P |  | Q |  | ~ | Q |  | ∴ |  | P |
| **F** | **T** | **F** |  | **T** | **F** |  |  |  | **F** |
| 3’ | 1 | 2’’ |  |  | 2 |  |  |  | 3 |

To invalidate *modus tortoise ponens*, row 1 of the truth table for the conditional must be T, and to invalidate *modus tortoise tollens*, row 4 of the truth table for the conditional must also be T.

Given this standard truth-functional definition of the conditional, other validities also follow. Some of these have a paradoxical quality if the conditional is interpreted “implies” and so these are known as the *paradoxes of material implication*.

One of the most puzzling validities is known as *Lewis’s Dilemma*:

P  ~P  Q ,

which states *“a contradiction implies anything”*. This implication follows from the inference rules of simplification, addition, and *modus tollendo ponens*, which are themselves not particularly puzzling.

1. *~~Show~~* P  ~P  Q 6, CD

2. P  ~P Assume (CD)

3. P 2, S

4. ~P 2, S

5. P  Q 3, ADD

6. Q 5, 4 MTP

Therefore, the justification for the truth-table for the conditional, the definition of semantic validity, and the derivation of other theorems using conditional derivation and standard rules of inference are all interconnected. To question one of these elements requires revisions of all the other concepts to form a coherent whole.

The following theorems (and tautologies) are known as the paradoxes of material implications because of their variance from the natural language uses of the conditional.

C. I. Lewis investigated modal logic in order to find a conditional—strict implication—which did not results in these paradoxes. Lewis defined strict implication P ⇒ Q (read “P *strictly implies* Q”) by combining modality with the truth-functional conditional:

P ⇒ Q := (P  ~Q) ,

or alternatively,

P ⇒ Q :=□(P  Q) .

Lewis then added axioms such as axiom **4** and axiom **5** to characterize different implication relations.

Nevertheless, strict implication inherited some of the paradoxical qualities of the material implication.

T551 ~P → □(P → Q) Law of Impossible Antecedent for Implication

T550 □Q → □(P → Q) Law of Necessary Consequent for Implication

T552 (P → Q) ∨ □(Q → P) Law of Implication Excluded Middle

The *counterfactual conditional* (or subjunctive condition) states what would be the case if its antecedent were true (although it is not true). In contrast, the material (or indicative) conditional indicates what is (in fact) the case if its antecedent is (in fact) true (which may or may not be true).

The difference between indicative and subjective conditionals can be illustrated with respect to a past time:

(1) If Oswald *did* not shoot Kennedy, then someone else *did*. Indicative conditional

(2) If Oswald *had* not shot Kennedy, then someone else *would* *have*. Subjunctive conditional

In the first sentence, the antecedent may or may not be true, and the consequent may or may not be true, but what the speaker asserts (in the past tense *indicative* mood) is what would have had to have been the case if the antecedent were in fact true. It is a conditional statement about the actual past.

In the second sentence, the speaker takes it for granted that the antecedent (formulated in the past perfect subjunctive, or what is called the *pluperfect* subjunctive, mood) is contrary to fact and asserts what would have been the case if under that supposition.

Neither the truth-functional conditional nor the strict conditional represents the logic of the counterfactual conditional. The antecedents of counterfactuals are false, but that does not make them all true, as in the case of the material conditional. Neither are counterfactual conditionals strict conditionals.

*Strengthening*: P ⇒ Q ∴ P  R ⇒ Q

*Transitivity*: P ⇒ Q . Q ⇒ R ∴ P ⇒R

Counterexample to Strengthening the Antecedent:

1. If the Nazis had built an atomic bomb before the U.S., Germany would have won WWII.

2. If the Nazis had built an atomic bomb before the U.S. and detonated it in Berlin, then Germany would have won WWII.

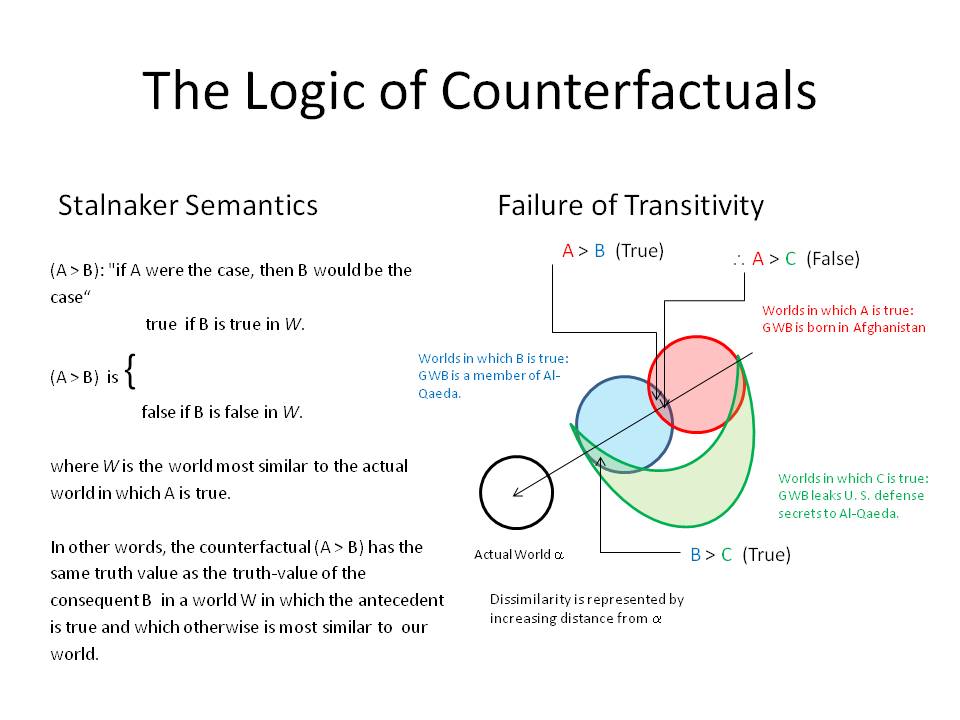
Counterexample to Transitivity:

1. If George W. Bush had been born in Afghanistan, then he would have been a member of Al-Qaeda.

2. If George W. Bush had been a member of Al-Qaeda, then he would have been leaking U. S. defense secrets to Al-Qaeda prior to 9-11.

3. So, if George W. Bush has been born in Afghanistan, then he would have been leaking U. S. defense secrets to Al-Qaeda prior to 9-11.

Why does transitivity fail for counterfactuals? The antecedents of the counterfactuals don’t relate to the same states of affairs. The first premise has to do with states of affairs (or possible worlds) in which--other things being as close as possible to the actual world—GWB was born in Afghanistan. The second premise, however, has to do with states of affairs (or possible worlds) in which--other things being as close as possible to the actual world—in which GWB as President of the U. S. was secretly an Al-Qaeda operative. The states of affairs that the first premise has to do with need not coincide with those of the second: if GWB had been born in Afghanistan, he wouldn't be President of the U. S. and so presumably wouldn't have access to U. S. defense secrets.



### Exercises

1. The counterfactual “would’ and ‘might’.

P □Q If P had been the case, then Q *would* have been the case.

P Q If P had been the case, then Q *might* have been the case.

Negation of a Stalnaker Counterfactual [NSCF]:

~(ϕ □ ψ) ∴ ϕ □ ~ψ

Stalnaker Counterfactual Excluded Middle: (ϕ □ ψ)  (ϕ □ ~ψ)

Lewis’s “*Might*” Counterfactual: (P Q) := ~(P □ ~Q)

Notice that given the Stalnaker Counterfactual Excluded Middle, Lewis’s definition of “*might*” would collapse “*would*” and “*might*” counterfactuals.

How can Stalnaker define the “*might*” counterfactuals in terms of the “*would*” counterfactual?

P ◊Q := (P □Q)

P □Q is true in a world α iff Q is true is *all* the worlds that make P true and are closest to Q.

P Q is true in a world α iff Q is true is *at least one* of the worlds that make P true and are closest to Q.

# 10. Quantified Modal Logic

Such a logic gives us the ability to distinguish between what happens to be *universally* true about all members of a kind K and what is *essentially* true of membership in kind K. When Aristotle defined human beings as *rational* animals, he was attempting to state an *essential* characteristic of being a member of the kind *human being*. What makes a human being essentially different from other animals, according to Aristotle, is the characteristic of rationality. Suppose that all and only human beings happened to be featherless bipeds. Still such a definitions would not be a good *essential definition* of a human being, since a plucked chicken would be a possible counterexample. Benjamin Franklin (and before him Daniel Defoe) bemoaned the universality of death and taxes, but this common plight is not essential to being human. On the other hand, being *risible* (i.e., being capable of laughter or being able to see the point of a joke) does seem to be essentially related to rationality.[[8]](#footnote-8)

Descartes in *Meditation VI* claims that bodies are essentially divisible while souls are essentially indivisible. Can modal logic be used to explicate the modal features of Descartes’ claims?

The propositional modal logic we have developed treats modal operators as applying to sentences as unanalyzed units. This allowed us to distinguish between the *necessity of the consequence* and the *necessity of the consequent*, in terms of the scope of the modal operator:

□(P → Q)

(P → □Q)

And to formulate the premise, combining the two, required for an Anselmian ontological argument:

□(G → □G)

Another important modal distinction is that between *de dicto* and *de re* attributions. This distinction can also be explained in terms of modal scopal ambiguity, but not over unanalyzed propositions, but over open formulas with variables. This leads to the development of *quantified modal logic.*

Consider the following set of modal claims:

(1) □∀*x*(*x* is a body  *x* is divisible);

(2) ∀*x*(*x* is a body  □ *x* is divisible);

(3) □∀*x*(*x* is a body  □ *x* is divisible);

(4) ∀*x*□(*x* is a body  *x* is divisible);

(5) ∀*x*□(*x* is a body  □*x* is divisible).

These symbolizations can be translated as follows:

(1) Necessarily, whatever is a body is divisible;

(2) Whatever is a body is necessarily or essentially divisible;

(3) It is necessary that whatever is a body is essentially divisible;

(4) It is universally true that it is necessary that whatever is a body is divisible;

(5) It is universally true that it is necessary that whatever is a body is essentially divisible.

Statement (1) is the *de dicto* claim that it is necessary that all bodies are divisible. There is an essential connection between being a body and being divisible. Statement (1) expresses the necessity of the universal over the conditional or consequence.

Statement (2) is the *de re* claim that everything that is (actually) a body is essentially divisible. This statement does not require any necessary connection between being a body in the actual world and being divisible. It is compatible, for example, with there being something that is a body in some possible world—say, an absolutely indivisible atom—but which is not essentially divisible. The scope of the universal quantifier in (2) is the actual world. Statement (2) expresses the universality of a conditional or consequence with a necessary consequent.

Statement (3) combines the *de dicto* necessity of (1) and the *de re* necessity of (2). It asserts that it is a necessary truth that all bodies are essentially divisible. It rules out the possibility of the possible, but non-actual, absolutely indivisible atom mentioned above—of there being something that is a body in some possible world that isn’t essentially divisible. Statement (3) expresses the necessity of the universality of a conditional or consequence with a necessary consequent.

Statement (4) is also a *de re* claim. The *de re* statement (2) attributes the property of being *essentially divisible* to all bodies. Statement (4) attributes a hypothetical and essential property to all actual things—namely, the property *essentially* having the hypothetical property of *being divisible if a body*. Statement (4) says of any actual thing that it is essentially such that if it is a body then it is divisible. This statement does not preclude the possibility that there might have been something that is a body and indivisible since the quantifier ranges over actual things. Statement (4) expresses the universality of the consequent or conditional with a necessary consequent.

Statement (5) adds the feature that the hypothetical and essential property in (4) holds, not only for all actual things, but also for everything that could be. Statement (5) states that it is a necessary truth that all things are such that they essentially have the hypothetical property of being divisible if a body. Statement (5) expresses the universality of the necessity of the consequence or conditional in which the consequent is also necessary.

What are the logical relationships among these modal statements? Statement (3) is the strongest statement. Statement (3) is stronger than either (1) or (2). Statement (3) implies each of (2) and (1), but the converses do not hold. Statement (3) is also stronger than either (4) or (5). Statement (3) implies (5), which, in turn, implies (4).

(3)

(1) (2) (5)

(4)

The connection between the *de dicto* necessity in (1) and the *de re* necessity in (2) is a matter of controversy involving the so-called Barcan formula and its converse. We will look at these technical issues after a brief philosophical discussion of essential properties and possible world semantics.

Quine in “Three Grades of Modal Involvement” (1966) argued that “…necessity resides in the way we say things, and not in the things we talk about.” Quine didn’t want to get too modally involved. With regard to modality—and expanding to all non-extensional contexts—Quine’s position was to tolerate *de dicto* statements, but to banish *de re* statements. In a famous passage about mathematical cyclists, Quine claims that distinguishing between necessary and contingent properties of an object is “baffling”:

Perhaps I can evoke the appropriate sense of bewilderment as follows. Mathematicians may conceivably be said to be necessarily rational and not necessarily two-legged; and cyclists necessarily two-legged and not necessarily rational. But what of an individual who counts among his eccentricities both mathematics and cycling. Is this concrete individual necessarily rational and contingently two-legged or vice versa? Just insofar as we are talking referentially of the object, with no special bias towards a background grouping of mathematicians as against cyclists or vice versa, there is no semblance of sense in rating some of his attributes as necessary and others as contingent. Some of his attributes count as important and others as unimportant, yes, some as enduring and others as fleeting; but none as necessary or contingent.

These latter statements *de re* were “semantically incoherent” (in the case of belief), “conceived in sin” (in the case of modalities) and were logically backward because presupposed the coherence of “Aristotelian essentialism.” This latter term of abuse seems to have entered into the philosophical lexicon with Quine’s preemptive articles and books. Quine’s rhetorical skills and intellectual influence set back the development of quantified modal logics for generations.

Quine’s dogma began to be questioned around 1961 when *de dicto* and *de re* necessity and essentialism became central topics of philosophical debate. Here, for example, is Quine’s taunt seems to be tantamount to confusing modality *de dicto* and *de re*.

Consider Quine’s argument to show that modality *de re* is incoherent in the case of numbers.

□ 9 > 8 (*true*)

The number of the planets = 9. (*true*, at least before 2006)

∴ □ The number of the planets > 8. (*false*, it’s a contingent fact)

Any good Russellian can distinguish two readings for the conclusion—one in which the definite description has primary scope and one in which it has secondary scope.

(1) ∃*y*(*y* = ⎤ ℩*x*N*x*  □ *y* > 8)

(2) 🞎 ∃*y*(*y* =  ℩*x*N*x*  *y* > 8)

Statement (1) with the primary occurrence of the definite description is the *de re* reading of necessity. It asserts that the actual number of the planets (i.e., the number 9 prior to 2006, and now, officially 8) [[9]](#footnote-9) is such that that number is necessarily greater than 8.

Statement (2) with the secondary occurrence of the definite description is the *de dicto* reading of necessity. It asserts that it is necessary that the number of the planets is greater than 8. This statement of necessity is false. It’s clearly a contingent fact that *there are the number of planets that there are* in our solar system. It could have been the case—due to some cataclysmic cosmic event—that there were more or less planets in our solar system.

Notice that the former rendering presupposes the coherence of *de re* necessity, and the second explanation of why statement (2) is false uses the coherence of *de re* necessity (*italicized in blue*).

Russell’s distinction between primary and secondary scope readings of the definite description can explain the apparent failure of the law of identity in Quine’s counterexample.

We have been thinking of modality in terms of possible worlds. Using this heuristic idea, *de dicto* necessity is the *truth* of a proposition in all possible worlds and essential *de re* attribution is an attribute being *true of* an individual in all possible worlds.

To say that Socrates is *essentially rational* is to say that he has the property of being rational in every possible world. Socrates in fact was snub-nosed, but this property was *accidental* rather than essential. Socrates might not have been snub-nosed. What does it mean to say that Socrates *possibly has an elegant Roman* nose? It is to say that Socrates, although he is snub-nosed in the actual world, has an elegant Roman nose in some possible world. In other words, Socrates is *accidentally* snub-nosed, *possibly* Roman-nosed, but *essentially* rational. To say that Socrates is a contingent being is to say that Socrates might not have existed. If Socrates might not have existed, then how could he be essentially rational—in the sense of being rational in every possible world? One option, which is counterintuitive, is to say that Socrates although he doesn’t exist is some possible world, nevertheless has the property of being rational is that world. An alternative is to say that Socrates’ being essentially rational means that he has the property of being rational in every world in which he exists.

But what are possible worlds? One way to think of them, proposed by David Lewis, is to think of them as some kind of concrete object out there in the universe. Another way to think about possible worlds is to regard them as shadowy worlds containing the ghosts of possible but non-actual objects. Alvin Plantinga has proposed thinking of possible worlds as abstract objects with an existence similar to numbers or other mathematical objects. He proposed of thinking of a possible world as a maximally consistent set of statements.

On this way of thinking about possible worlds, *Socrates exists in* those *possible worlds* in which it is impossible for those worlds to be actual and Socrates not exist. So Socrates is a *contingent* being provided that there are possible worlds in which Socrates does not exist, and God is a *necessary* being provided that it is impossible for any world to be actual and God not exist. And to say that Socrates is *essentially rational* then is to say that it is not possible that Socrates exist and not be rational.

Whether individual exists in more than one possible world is a matter of philosophical dispute. David Lewis, and Leibniz, held that an individual could exist in only one possible world. Individuals are “world bound” or exist in one and only one possible world. Lewis talked about our counterparts in other possible worlds.

It is natural to think that some possible worlds lack individuals who exist in the actual world (e.g., contingent beings like Socrates) and that some worlds contain individuals who do not exist in the actual world (e.g., possible beings such as Santa Claus or Superman). Are we committed to Meinongian entities? One way to avoid such a commitment is to model possible but not actual beings as sets of properties that would have been *exemplified* if some other world has been actual. *Actualism* is an attempt to avoid the excesses associated with Meinongian realism.

*Actualism*: there neither are nor could have been objects that do not exist.

According to *actualism*, propositions, properties, and essential properties all exist, but there are no non-existent objects, or could there be any. According to the actualist, there neither are nor could have been non-existent objects—although there could have been objects different form each of the objects that in fact there are.

This terminology, Plantinga cautions, perpetuates the confusion between actuality and existence. One who claims there are no non-actual objects could mean

(1) there is nothing that does not actually exists (*true*)

or

(2) there is nothing that is possible but not actual (*false*) .

What the actualist holds is not that whatever is, is *actual* (since there are merely possible states of affairs exist), but whatever there is *exists*. *Actualism* should be called *existentialism*.

According to Plantinga’s playful terminology, *existentialism* is the view that singular propositions, states of affairs, and possible worlds do not exist if the objects involved in them do not exist. According to Plantinga, existentialism is false.

What Plantinga called the “*serious actualist*” holds that objects do not have properties in worlds in which they do not exist.

*Serious actualism*: the view that no object *x* has any property in any world in which *x* does not exist.

Here are some desiderata with regard to quantified modal logic:

(1) There is one and only one *actual world*. The *actually existing* individuals are those who exist in the actual world.

(2) Individuals are *not world-bound*—they exist in more than one possible world. There are *contingent* beings—beings who exist in the actual world but fail to exist on other possible worlds. There are *possible*, but non-actual, beings—beings who exist in some possible worlds but not the actual world.

(3) Some proper names are *rigid designators*—they refer to the same individual in every world in which that individual exists.

(4) Individuals do not have properties in worlds in which they do not exist.

(5) Individuals have some properties *accidentally* and other properties *essentially*. An individual has a property *essentially* provided that that individual has that property in every possible world in which that individual exists. Otherwise the individual has the property *accidentally*.

The connection between the *de dicto* necessity and the *de re* necessity is a matter of controversy involving the so-called Barcan formula and its converse.

|  |  |
| --- | --- |
| Barcan Formula | ∀*x*□F*x*  □∀*x*F*x* |
| Converse Barcan | □∀*x*F*x* ∀*x*□F*x* |
| Buridan Formula | ∀*x*F*x *∀*x*F*x* |
| Converse Buridan | ∀*x*F*x* ∀*x*F*x* |

Consider the Barcan formula

∀*x*□F*x*  □∀*x*F*x .*

This formula states *“if everything is necessarily F, then it is necessary that everything is F.”*[[10]](#footnote-10) Should this be adopted as an axiom for quantified modal logic?

Arthur Prior argued that was derivable in a quantified version of **S5**.

1. *~~Show~~* ∀*x*□F*x*  □∀*x*F*x* 3, CD

*2.* ∀*x*□F*x* Assume (CD)

3. *~~Show~~* □∀*x*F*x* 4, D

4. *~~Show~~* ∀*x*F*x* 5, UD

5. *~~Show~~* F*x* 8, DD

6. ∀*x*□F*x* 2, □R

7. □F*x* 6, UI

8. F*x* 7, NI

Here the disputed step occurs in line 6. In **S4** (and hence **S5**) the intuitive justification for □R is that it is redundant to add a box to a sentence (using the axiom □P  □□P) and then remove the box as required by importation in a strict derivation. In this case, however, the symbolic sentence that is imported into the strict derivation by □R does not begin with □. Therefore, line 6 is illegitimate., and this derivation of the Barcan formula fails.

This is well and good since the Barcan formula is not intuitively valid. Consider a possible world, let’s call it Plato’s *Mathematica* which consists only of the abstract Platonic objects of mathematics. In such a possible world it is true of every mathematical object (*de re*) that it is abstract and, moreover, *essentially* abstract. Does it follow that it is necessary (*de dicto*) that everything is abstract? Not if we are allowed to add a non-abstract objects to other possible worlds. Suppose that in some possible world there also exists Newton’s apple, which, for the sake of argument, we shall take to be a particular physical apple and *not* essentially abstract. If this scenario is possible, then Barcan formula is not true. While it is true of every object (*de re*) in Plato’s *Mathematica* that it is *essentially* abstract, but it is not true (*de dicto*) that it is necessary that everything is abstract. The Barcan formula, in effect, is a principle that prohibits additions to the universes of possible worlds. The Barcan formula should not then be accepted as an axiom for quantified modal logic. It has the undesirable consequence that no world can have individuals in it that do not already exist in every possible world.

Next consider the Converse Barcan formula:

□∀*x*F*x* ∀*x*□F*x .*

This statement says that *if it is necessary that everything is F, then everything is essentially F*. Whereas the Barcan formula prohibits *adding* individuals to other possible worlds, the converse Barcan formula prohibits *omitting* individuals from other possible worlds.

For something to be *essentially* F means that an object has F in all possible worlds.

Here’s a purported derivation.

1. *~~Show~~* □∀*x*F*x*  ∀*x*□F*x* 3, CD

*2.* □∀*x*F*x* Assume (CD)

3. *~~Show~~* ∀*x*□F*x* 4, UD

4. *~~Show~~* 🞎F*x* 5, ND

5. □∀*x*F*x* 2, 🞎R

6. ∀*x*F*x* 5, NI

7. F*x* 6, UI

The problem is that the variable of generalization in line 3 ranges over objects in the actual world, whereas the free variable in line 7 ranges over the individuals in the possible world (introduced by NI in line 6. Suppose there are individuals in the *actual* world (call it α) which don’t exist in the *possible* world introduced in line 6 (call it β). Then the fact that an arbitrary ‘*x*’ is F in the diminished world β doesn’t prove that an arbitrary individual ‘*x*’ in α has the property F in every possible world in which it exists.

The converse of the Barcan formula is also counterintuitive:

□∀*x*F*x* ∀*x*□F*x*

Let’s grant that it’s true *de dicto* that everything exists, but it doesn’t follow that everything essentially exists. To accept both the Barcan and its converse would collapse the distinction between *de dicto* and *de re* modalities, which is also unacceptable.

Consider next the Buridan formula. In the 14th century, Jean Buridan, the medieval master of paradox and logic, gave the following counterexample to the formula that now bears his name.

Similarly, this is not valid: ‘It is possible that every being is God; therefore, every being can be God’, for the first is true—as this was the case before the creation of the world, and this would also be the case if God annihilated all creature--, and the second is false—for it is not possible for a creature to be God.[[11]](#footnote-11)

According to classical theology, God freely creates not out of necessity but love. It is therefore possible that God should not have created anything. In such a scenario, it is possible that nothing but God exists and so it is possible that everything is identical to God. However, according to Buridan’s medieval, decidedly non-Buddhist, worldview, it is certainly false that everything that does exist is possibly identical to God. To represent this formally, we have the following counterexample to the Buridan formula:

∀*x* *x* = God  ∀*x**x* = God .

Here the antecedent ‘It is possible that God alone exists, i.e., it is possible that nothing but God exists, but the consequent ‘everything is possibly identical with God’ is false.

The converse Buridan formula also should be rejected.

∀*x*F*x* ∀*x*F*x*

Suppose everything is *temporally contingent.* For everything that exists there is some time or other when it does not exist. Does it logically follow that there is some time at which nothing exists? No. Imagine an infinite chain of contingent beings such that the first begets the second and eventually ceases to exist, the second begats the third and eventually ceases to exist, the third begats a fourth, and so forth to infinity. In this scenario, each being does not exist at some time or other and so is contingent, but it is not the case that there is a time at which everything ceases to exist. From the assumption that each thing does not exist at some time it does not follow that there is a time when nothing exists.

## Exercises

1. Provide your own examples in which the following formulas hold true, and your own counter-examples, in which they fail.

|  |  |
| --- | --- |
| Barcan Formula | ∀*x*□F*x*  □∀*x*F*x* |
| Converse Barcan | □∀*x*F*x* ∀*x*□F*x* |
| Buridan Formula | ◊∀*x*F*x *∀*x*◊F*x* |
| Converse Buridan | ∀*x*◊F*x* ◊∀*x*F*x* |

For example, you may wish to consider the passage surrounding the one quoted from Buridan above. Before that passage, Burdian writes:

And this seems to be tough to swallow, although this opinion has certain appeal [*apparentia*], for from a composite modal [proposition] there does not appear to follow a divided [proposition]. For example, this is not valid: ‘It is necessary that every donkey is an animal; therefore, every donkey is necessarily an animal’, for Aristotle would concede the first, but would deny the second, for the reason that donkey may not be, and consequently, may not be an animal…

And after the passage quoted above, Buridan continues:

Therefore, likewise, it seems that the following should not be valid: ‘I know that every B is A; therefore, every B I know to be A’, or even ‘I know every triangle to have three etc.; therefore, every triangle I know to have three etc.’ But one might say that the case is not the same with all kinds of modal propositions; for although it may not be valid in all cases, nothing prevents it from being valid in some. For example, this is valid: ‘It is true that every B is A; therefore, every B is truly an A’; therefore, it seems probable that this is valid: ‘I know that every triangle has three etc.; therefore, of every triangle I know that it has three etc.; or even ‘therefore, every triangle I know to have three etc.

2. Among his extensive unpublished writings, Gödel formalized an intriguing ‘ontological proof’ for the existence of God. The argument is based on Leibniz’s notion of ‘positive’ and ‘negative’ properties, and defines a God-like being having as its essence maximally positive properties. Whether or not Gödel actually believed it to be a proof, it is an illustration of the power of the axiomatic method to clarify the logical relations among complex concepts.

The following reconstruction of Gödel’s ontological proof is an intriguing logical exercise that goes beyond propositional modal logic into second-order modal predicate logic, but one in which the steps can be logically verified. This particular formulation of Gödel’s argument is due to C. Anthony Anderson.

*Axiom* 0. If a property F is positive, then it is necessarily positive, i.e., Pos(F) → □Pos(F).

*Axiom* 1. If a property F is *positive* then its negation is not positive, i.e., Pos(F) → ~ Pos(~F).

Let F and H be properties, then property F *entails* property H (in symbols, F ⇒ H) if □∀*x*(F*x* → H*x*).

*Axiom* 2. Any property *entailed* by a positive property is positive, i.e., Pos(F) → [(F ⇒ H) → Pos(H)].

*Lemma* A. If it is impossible that F is exemplified, then F entails ~F, i.e., ~◊∃*x*F*x* → (F ⇒ ~F).

This derivation is left to the reader since it follows from MN and quantifier logic and so ‘~◊∃*x*F*x*’ implies ‘□∀*x*(F*x* → ~F*x*)’.

*Theorem* 1. Any positive property is possibly exemplified, i.e., Pos(F) → ◊∃*x*F*x*.

*Proof*.

1. *~~Show~~* Poss(F) → ◊∃*x*F*x* 3, CD

2. Pos(F) Assume (CD)

3. *~~Show~~* ◊∃*x*F*x* 6, 7 ID

4. ~◊∃*x*F*x* Assume (ID)

5. F ⇒ ~ F 4, Lemma A

6. Pos(~ F) 2, 5, Axiom 2

7. ~ Pos(~F) 2, Axiom 1

The next three definitions define being *God-like*, being an *essence* of an individual, and the property of *necessary existence*.

*Definition* 1. Something *x* is *God-like* if, and only if, every essential property of *x* is positive and *x* has every positive property as an essential property, i.e., G*x* := ∀F [□F*x* ↔ Pos(F)].

*Definition* 2. A property F is an *essence* of *x* (in symbols, F Ess *x*) if it is necessary that *x* has all properties entailed by F, i.e., F Ess *x* := ∀H[□H*x* ↔ (F ⇒ H)].

*Definition* 3. An individual *x* has *necessary existence* (in symbols, NE(*x*)) if every property that is an essence of *x* is necessarily exemplified in some individual, i.e., NE*x* := ∀F(F Ess *x* → □∃*y* F*y*).

*Axiom* 3. Being God-like is a positive property, i.e., Pos(G).

*Corollary* 1. ◊∃*x*G*x*, i.e., being God-like is possibly exemplified.

1. *~~Show~~* ∃*x*G*x*

2*.* Pos(G) Axiom 3

3. Pos (G) → ∃*x*G*x* Theorem 1

4. ∃*x*G*x* 3, 4 MP

*Axiom* 4. Having necessary existence is a positive property, i.e., Pos(NE).

*Lemma* B. If *x* is God-like, then any property that *x* has essentially is implied by *x* being God-like, i.e.,

G*x* → [□H*x* → (G ⇒ H)].

[*Hint*: Suppose that □H*x*, we have by Definition 1 that Pos(H). From Definition G1, we have Pos(H), but anything that is God-like must have all positive properties.]

*Lemma* C. If *x* is God-like, then any property entailed by *x* being God-like is an essential property of *x*, i.e., G*x* → [(G ⇒ H) → □H*x*].

[*Hint*: From Axioms 2 and 3, we have that Pos(H). It follows that a God-like individual *x* has property H necessarily by Definition 1.]

*Theorem* 2. If *x* is God-like, then being God-like is the essence of *x*, i.e. G*x* → G Ess *x.*

1. *~~Show~~* G*x* → G Ess *x* 3, CD

``

2. G*x* Assume (CD)

3. *~~Show~~* G Ess *x*

4*. ~~Show~~* ∀H[□H*x* ↔ (G ⇒ H)] 7, UD

5. □H*x* → (G ⇒ H) *Lemma* B

6. (G ⇒ H) → □H*x Lemma* C

7. □H*x* ↔ (G ⇒ H)

8. G Ess *x* 4, *Definition* of Ess

*Theorem* 1 states that any positive property is possibly exemplified. *Axiom* 3 states that being God-like is a positive property. From *Theorem* 1 it folows that being God-like is possibly exemplified. *Theorem* 2 states that being God-like is the essence of anything that is God-like. Thus, it is necessary there is a God-like being.

*Theorem* 3. It is necessary that there is something God-like: □∃*x*G*x*.

1. *~~Show~~* □∃*x*G*x* 20, DD

2. □∃*x*Gx → □∃*x*G*x* Axiom **5**◊

3. *~~Show~~* □∃*x*Gx 16, DD

4. *~~Show~~* □(∃*x*G*x* → □∃*x*G*x*) 5, ND

5. *~~Show~~* ∃*x*G*x* → □∃*x*G*x* 14, CD

6. ∃*x*Gx Assume(CD)

7. G*y* 6, EI

8. Pos(NE) Axiom 4

9. □NE*y* ↔ Pos(NE) Definition 1, UI, 7 SL

10. □NE*y* 9, BC, 8, MP

11. G*y* → G Ess *y* Theorem 2

12. G Ess *y* 10, 12 MP

13. G Ess *y* → □∃*x* G) 10, NI, Definition 3, UI

14. □∃*x* G*x* 12, 13, MP

15. ∃*x*G*x* Corollary 1

16. *~~Show~~* □∃*x*G*x* 19, PD

17. ∃*x*Gx 15, Assume (PD)

18. ∃*x*G*x* → □∃*x*G*x* 4, NI

19. □∃*x*G*x* 17, 18 MP

20. □∃*x*G*x* 2, 16 MP

Whether or not Gödel believed that this unpublished argument was a theological argument that proved the existence of God or whether he claimed, as reported by Oskar Morgenstern, to be investigating the ontological argument for its logical interest is a matter of historical dispute. Some cite Gödel’s philosophical views about on the ‘theological worldview’ in a letter to his mother:

We are of course far from being able to confirm scientifically the theological world picture… What I call the theological worldview is the idea, that the world and everything in it has meaning and reason, and in particular a good and indubitable meaning. It follows immediately that our worldly existence, since it has in itself at most a very dubious meaning, can only be the means to the end of another existence. The idea that everything has a cause is an exact analogue of the principle that everything has a cause, on which rests all of science. [GCW IV, XXX]

Others cite Oscar Morgenstern’s diary in which he stated that Gödel hesitated to publish his ontological proof for fear it would be thought “that he actually believes in God, whereas he is only engaged in a logical investigation (that is, in showing that such a proof with classical assumptions [completeness, etc.], correspondingly axiomatized, is possible.” On either account, it is clear the Gödel held strong convictions about the axiomatic method as a powerful tool for conceptual analysis. In the next section, we will examine abstract modal versions of Gödel’s celebrated Incompleteness Theorems, some of the most important results in twentieth century logic.

# 11. Modal Provability Systems and Gödel’s Incompleteness Theorems.

About the time of his famous Incompleteness Theorems (1931), Gödel wrote up a short note on the embedding intuitionistic logic in modal logic. The idea was that intuitionistic truth was defined in terms of proof and provability is a kind of necessity.

Gödel’s First Incompleteness Theorem (1931) is based on constructing a sentence in a formal system that intuitively says, “I am not provable.” In 1950 Leon Henkin asked if the sentence “I am provable” is provable, and in 1954 Martin Löb, answered this questioning the affirmative. In 1963 Richard Montague made the connection between provability as a box and self-reference, but it was not until the 1970s that logicians fully realized that Gödel’s Incompleteness Theorems were propositional in character and that the abstract logic of provability could be captured in propositional modal logic.

Russell, after discovering Russell’s paradox in Gottlob Frege’s monumental *Gründgesetze der Arithmetik I and II* (1893-1903), sought with Alfred North Whitehead in their monumental *Principia Mathematica* (1910-1913) to reduce mathematics to logic in a paradox-free form and so secure the certainty of mathematics. Russell and Whitehead, however, were not altogether clear about the concept of the meta-language in which this reduction was to take place. This defect was repaired by David Hilbert. Hilbert believed that the logical paradoxes that occupied Russell were merely due to the semantic content of language and so could be banished by making mathematical statements meaningless, that is, by emptying of their content and retaining only their formalistic structure.

Hilbert’s rigorous refinement of the axiomatic method enabled the central questions in the foundations of mathematics to be precisely and rigorously formulated. The general strategy for establishing the consistency of different axiomatic systems exploits a theorem about logic—a piece of *meta-mathematics*—namely, that a system being consistent is equivalent to some statement in the system being unprovable. In a lecture to the International Congress of Mathematics in Paris in 1900, Hilbert welcomed the new century by listing 23 open problems, which he challenged mathematicians to solve. The second on the list of these challenges was to demonstrate the consistency of the axioms of arithmetic.

Hilbert’s program lasted until 1931 when Kurt Gödel produced a 25 page technical paper that would make him the most famous logician of the 20th century. This remarkable paper drove a logical wedge between the notions of proof and truth conflated since Euclid and at the same time dashed Hilbert’s dreams of a finitary consistency proof for mathematics. Gödel’s celebrated Incompleteness Theorems were a shocking awakening from the enchantment with Euclidean reductionism.

Gödel’s First Incompleteness Theorem showed that even elementary arithmetic was essentially incomplete insofar as there will always be truths unattainable by proofs for any recursively enumerable set of axioms. Gödel’s Second Incompleteness Theorem showed that it is impossible to prove the consistency of arithmetic, assuming that arithmetic is consistent, using methods that are formalizable within the syntax of elementary arithmetic, thus dashing Hilbert’s hopes for a finitistic proof of the consistency of mathematics. It turns out that these results can be set forth in propositional modal logic in which the ‘□’ is interpreted as provability.

Modal logic is useful in clarifying our understanding of central results concerning provability in the foundations of mathematics (see, Boolos (1993)). Provability logics are systems where the propositional variables range over formulas of some mathematical system such as Peano Arithmetic. Gödel showed that arithmetic has strong expressive powers. Using Gödel numbering as a method for coding arithmetic sentences, he was able to demonstrate an effectively calculable correspondence between sentences of arithmetic and facts about which sentences are and are not provable in Peano Arithmetic.

In Provability Logics, ‘□P’ is interpreted as expressing that ‘P is provable’ in a system M such as Peano arithmetic. Using ‘⊥’ [known as ‘eet’ or ‘tac’] as a constant of provability logic denoting a contradiction, the sentence

~ □⊥

says that a contradiction is not provable in system M and so says that M is consistent. We define Consis(M) to be the above sentence. The best-known Provability logic adds an axiom to **K** known as axiom **W** (for ‘well-ordering’):

□(□ϕ → ϕ) → □ϕ .

Axiom **T** ‘□ϕ → ϕ’ says that Peano Arithmetic is *sound*, namely, that if ϕ were provable, then ϕ would be true. This claim is too strong if we wish to countenance the possibility that Peano Arithmetic might prove something that is false. Intuitively, **W** is the more modest claim that if Peano Arithmetic proves a sentence that claims soundness for a given sentence ϕ, then ϕ is already provable in Peano Arithmetic. Peano Arithmetic does not insist that a proof of ϕ entails ϕ is true unless it already has a proof of ϕ to back up that claim.

In other words, if Arithmetic is consistent, then it cannot prove the Gödel form of the Liar sentence. We therefore also have a form of Gödel’s Second Incompleteness Theorem: If |- G ↔ ~□G , then if our system is consistent, then |− ~□ 0 = 0.

A *provability system* consists minimal modal logic **K** with the additional axiom:

(**W**) □(□ϕ → ϕ) → □ϕ .

*Theorem* (Dick de Jongh, 1970s)[[12]](#footnote-12): Axiom (**4**) holds in the Gödel-Lob Provability Logic.

*Proof*:

Replace ϕ by (□ϕ ∧ ϕ) in Axiom (**W**) to obtain:

□(□(□ϕ ∧ (□ϕ ∧ ϕ)) → (□ϕ ∧ ϕ)) → □(□ϕ ∧ ϕ) .

It happens that the following lemma is derivable in minimal modal logic **K** (see exercises below):

□ϕ → □(□(□ϕ ∧ (□ϕ ∧ ϕ)) → (□ϕ ∧ ϕ))

It therefore follows that

□ϕ → □(□ϕ ∧ ϕ)

and hence

□ϕ → □□ϕ .

Next comes the Löb-Gödel Theorem, which we shall call the “Magical Modal Mystery Tour”:[[13]](#footnote-13)

if |- □ϕ → ϕ, then |- ϕ

1. |- □ϕ → ϕ Assumption (NI)
2. |- σ ↔ (□σ → ϕ) Gödel Fixed-Point Theorem
3. *~~Show~~* ϕ 20, DD
4. *~~Show~~* □σ 16, ND
5. *~~Show~~* □σ → ϕ 14, CD
6. □σ Assume (CD)
7. *~~Show~~* □[σ ↔ (□σ → ϕ)] 8, ND
8. σ ↔ (□σ → ϕ) 2, Gödel Fixed-Point Theorem
9. □σ → □(□σ → ϕ)] 7, T81, IE, MD, S, Axiom (K), MP
10. □(□σ → ϕ) 9, 6 MP
11. □□σ → □ϕ 10, Axiom (K), MP
12. □σ → □□σ Axiom (4)
13. □ϕ 12, 6 MP, 11 MP
14. ϕ 13, NI
15. (□σ → ϕ) → σ 2, BC
16. σ 15, 5 MP
17. σ → (□σ → ϕ) 2, BC
18. σ 4, NI
19. □σ → ϕ 17, 18 MP
20. ϕ 19, 4 MP

Now we may obtain a modal version of Gödel’s First Incompleteness Theorem. Gödel’s famous arithmetical version of the Liar G intuitively says, “I am not provable”:

G ↔ ~□G

Now that the consistency of Peano Arithmetic may be formulated as: ~□ 0 ≠ 0

*Gödel’s First Incompleteness Theorem*: If |- G ↔ ~□G, then |- ~□ 0 ≠ 0 → ~□G.

This theorem states that if the Gödel sentence G is provable, then if the system is consistent, then Gödel sentence cannot be proved. Now the Gödel sentence “says” it is unprovable and it is unprovable if the system is consistent. Therefore, if the system is consistent, it is incomplete, i.e., there is a truth which is unprovable.

1. |- G ↔ ~□G Gödel Liar from Fixed-Point Lemma
2. *~~Show~~* ~ □ 0 ≠ 0 → ~□G 15, CD
3. ~ □ 0 ≠ 0 Assume (CD)
4. *~~Show~~* □(□G → ~G) 7, ND
5. G ↔ ~□G 1, Fixed-Point Theorem
6. G → ~□G 5, BC
7. □G → ~G T14, 6, MP
8. □□G → □~G Axiom (K), 4, MP
9. *~~Show~~* □G → □ 0 ≠ 0 13, 14 ID
10. □G Assume (CD)
11. □G → □□G De Jongh’s Theorem
12. □~G 11, 10, MP, 8, MP
13. G 10, NI
14. ~G 12, NI
15. ~□G 9, 3 MT

Exercises

1. Annotate the following (**K**) derivation of the lemma used in the proof that (**4**) is a consequence of (**W**):

1. *~~Show~~* □ϕ → □(ϕ → (□(□ϕ ∧ ϕ) → □ϕ ∧ ϕ))

2. □ϕ

3. □(□(□ϕ ∧ ϕ) → □ϕ ∧ ϕ) → □(□ϕ ∧ ϕ)

4. *~~Show~~* □(ϕ → (□(□ϕ ∧ ϕ) → □ϕ ∧ ϕ))

5. *~~Show~~* ϕ → (□(□ϕ ∧ ϕ) → □ϕ ∧ ϕ)

6. ϕ

7. *~~Show~~* □(□ϕ ∧ ϕ) → □ϕ ∧ ϕ

8. □(□ϕ ∧ ϕ)

9. □□ϕ ∧ □ϕ

10. □ϕ

11. □ϕ ∧ ϕ

1. Show that the following are equivalent formulations of fixed-point properties:

* A system M is *self*-*reflexive* if ∀ϕ, there exists a sentence σ such that M ⊢ σ ↔ (□σ → ϕ).
* A system M has the *Löb property* if ∀ϕ, M ⊢ (□ϕ → ϕ) ⇒ M⊢ ϕ.
* A system M is *Gödelian* if ∀ ϕ, M ⊢ □(□ϕ → ϕ) ⇒ M ⊢ □ϕ .

# 12. Philosophical Remarks (*Nathan Salmon*).

Modal logic—the logic of what might have been and of what must be—raises difficult issues and questions that remain a matter of significant controversy. Some of these issues concern modal sentential logic, others the blending of the logic of quantification with that of modality.

The principal issue concerning modal sentential logic involves nested modality, wherein some modality is attributed to a state of affairs that is itself already modal. Consider the following argument:

It is mathematically necessary that 2 + 3 = 5 .

Therefore, that 2 + 3 = 5 *could not have been* merely accidental .

It is unclear whether the premise entails this nested modal conclusion, even if it is true, as opposed, say, to the unnested modal conclusion that 2 + 3 = 5is not accidental.

The sentential logic of un-nested modality is known as ‘**T**’. Let us say that the antecedent of a necessary conditional *strictly implies* the consequent. Then we may say that that **T** is motivated by four principles: (1) that necessity and possibility are logical duals (i.e., something is necessary just in case its denial is impossible and that something is impossible if its denial is necessary); (2) that whatever is necessary is true; (3) that the logical truths of classical and modal logic are necessary truths; and (4) that whatever is strictly implied by a necessary truth is itself a necessary truth (i.e., if the antecedent of a necessary conditional is itself necessary, then the consequent is also necessary). Any deductive apparatus for modal logic that respects these four principles is known as a *normal system*.

System **T** is the weakest normal system. It assigns no special logic to the nesting or iteration of modality. Whereas the sentential logic of un-nested modality is relatively uncontroversial, there exist numerous rival systems dealing with nested modality. The most important normal systems that impose a special logic on nested modality are **B** (named for the intuitionist, L. E. J. Brouwer) and **S4** and **S5** (named after C. I. Lewis’ systems for strict implication). Each of these three systems builds on **T** through the addition of a specific principle governing an iterated modality. Recall that system **B** adds the principle that whatever is the case is such that its possibility is necessary:

P → □P ,

(or equivalently in terms of its dual, whatever is possibly necessary is the case). System ***S4*** adds the principle that whatever is necessary is such that its necessity is itself necessary:

□P → □□P ,

(or equivalently in terms of its dual, whatever is possibly possible is possible). System **S5** adds the principle that whatever is possible is such that its possibility is necessary:

P → □P ,

(or equivalently in terms of its dual, whatever is possibly necessary is necessary). System **S5** is the strongest of the four systems under discussion. Recall that the **S5** principle, together with the basic sentential modal logic **T**, yields both the **B** and **S4** principles. Whereas **S5** includes **T**, **B**, and **S4**, neither of **B** and **S4** includes the other.

The study of modal logic underwent a quantum leap with the advent of a semantic analysis due to Saul Kripke [1963], among others, based on Leibniz’ notion of a *possible world*. A possible world, on one conception, is an entire world history that might have obtained. Socrates might have been a Spartan if and only if there is a possible world in (or, according to) which Socrates is a Spartan. In possible-world semantics, the classical extensional semantic attributes, like truth and denotation, are intensionalized by relativization to a possible world.

A sentence is deemed true with respect to a possible world just in case the proposition expressed is true in that world. Necessity of a proposition is identified with its truth in all possible worlds, possibility with truth in some possible worlds. More accurately, to say that a proposition *p* is *necessary* in a world *w*1 is to say that *p* is true in every world *w*2 that is a possible world in *w*1—i.e., *p* is true in every world *w*2 such that in *w*1,*w*2 is a genuine possibility (or, possible relative to) *w*1. To say that a proposition *p* is *possible* in a world *w*1 is to say that *p* is true in at least one world *w*2 that is possible in (or, relative to) *w*1. With these identifications, the sentential logic of modality is easily derived from the classical logic of ‘all’ and ‘some’, in much the same way that the logic of ‘all’ and ‘some’ may be based on the logic of ‘and’ and ‘or’. The four basic principles of ***T*** emerge as analogues, respectively, of Quantifier Negation, Universal Instantiation, Universal Derivation, and T201.

This conception of possible world semantics extends straightforwardly to the iteration of modality. To say that *p* is necessarily necessary in a world *w*1 is to say that for every world *w*2 that is possible in *w*1, for every world *w*3 that is possible in *w*2, *p* is true in *w*3, i.e., *p* is true in every world that is possible in any world that is possible in *w*1. [What is possible in that world is possible in any world.] To say that *p* is necessarily possible in *w*1 is to say that for every world *w*2 possible in *w*1, there is least one world *w*3 that is possible in *w*2 and in which *p* is true. In Kripke’s terminology, a world *w*2 is said to be *accessible to* a world *w*1 if *w*2 is possible in *w*1. The binary relation of accessibility is reflexive in any normal sentential modal logical system. Kripke demonstrated a close connection between each of the four main sentential modal logical systems and accessibility. If accessibility is symmetric, then the **B** principle is verified. If accessibility is transitive, the **S4** principle is verified. And if it is an equivalence relation⎯reflexive, symmetric, and transitive⎯the **S5** principle is verified.

There are many notions of necessity. Whereas it is *logically* necessary that either nine is odd or it is not, it is *mathematically* necessary that nine is indeed odd. It is *metaphysically* necessary, and arguably also logically necessary, that if *x* and *y* are one and the same object, then *x* and *y* are exactly alike in every respect. (As mentioned in chapter V, this Leibnizian principle of the *indiscernibility of identicals* is reflected in the inference rule of Leibniz’ Law.) Whereas it is *physically* necessary that the force acting on a physical object is equal to the product of the mass of the object with its acceleration, this is neither *logically* nor *mathematically* nor *metaphysically* necessary. Not all notions of necessity need share the same logic.

Physical necessity may be seen as a restricted notion of necessity: a proposition *p* is *physically necessary* in a world *w* just in case the laws of the physics of *w* strictly imply it, in the sense that *p* is true in every world that is accessible to *w* and in which the physical laws of *w* are true. Though any actual physical law is thus *physically* necessary, that same law may be *metaphysically* contingent; there may be a world that is accessible to the actual world and in which the law fails. Had there been an additional physical law over and above those that actually obtain (perhaps because of the physical possibility of some universal generalization that does not in fact obtain, or of the evolution of some species that does not in fact exist), the actual physical laws together with the additional law would all be physically necessary, and the actual state of affairs thus physically impossible. This would rule out **B**, hence also **S5**, as the logic of physical modality.

It is often maintained that metaphysical necessity is unrestricted necessity. A proposition is *metaphysically* necessary in a world *w*, it is observed, just in case it is true in every possible world accessible to *w*, without exception. This would yield the result that the sentential logic of metaphysical modality is ***S5***. The argument for the ***S5*** principle that whatever is metaphysically possible is metaphysically necessarily so is straightforward: assume that *p* is metaphysically possible. Then there is at least one possible world *w* (which may or may not bear any special relation to the actual world) in which *p* is true. In that case, for every possible world *w′*, without exception, there is at least one world (which may or may not bear any special relation to *w*′) in which *p* is true, *viz*., *w*. Though many philosophers have been persuaded by this argument, it is in fact fallacious.

Whether a given world is metaphysically possible or not might be a contingent fact. The same world that is metaphysically possible in (i.e., accessible to) one world might be metaphysically impossible in another. Even metaphysical modality thus involves one crucial restriction: *p* is metaphysically necessary in a world *w* just in case *p* is truein every world *that is metaphysically possible in* *w*; and *p* is metaphysically possible in *w* just in case *p* is true in at least one world *that is metaphysically possible in* *w*1. Suppose *p* is metaphysically possible in some world *w*1 (e.g., the actual world),so that there is a world *w*2 that is metaphysically possible in *w*1 and in which *p* is true. It does not follow that for every world *w*3 metaphysically possible in *w*1, there is a world that is metaphysically possible in *w*3 and in which *p* is true. What follows instead is that for every world *w*3 possible in *w*1, there is a world—*viz*., *w*2—that is possible *in a world in which w3 is possible* and in which *p* is true. But we have so far no reason to suppose that *w*2 is possible in *w*3, and hence no reason to suppose that there is a world that is possible in every world possible in *w*1 and in which *p* is true.

To presuppose that because *w*2 and *w*3 are each possible in *w*1, *w*2 is likewise possible in *w*3 is tantamount to the assumption that accessibility is *euclidean*:

∀*x*∀*y*∀*z*(R*xz*  R*yz* ⇒ R*xy*) .

A binary relation is both reflexive and euclidean if and only if it is an equivalence relation. But the argument provides no reason to suppose that accessibility is euclidean.

On the contrary, there is a compelling reason to suppose that metaphysical accessibility is *not* transitive (let alone an equivalence relation)—or at least, that logic does not preclude the prospect that accessibility is intransitive. Some metaphysicians (including the present writer) maintain that a typical material artifact, such as a bicycle or a ship, might have originated from at least slightly different matter (molecules) from its actual original matter, but could not have originated from altogether different matter. This suggests that for each material artifact, there is a threshold or limit such that the artifact might have originated from matter different from its actual original matter up to that limit, but could not have originated from matter that differs any more than that.

Consider the *Paradox of the Ship of Theseus* which goes back to a Greek legend reported by Plutarch:

The ship wherein Theseus and the youth of Athens returned [from Crete] had thirty oars, and was preserved by the Athenians down even to the time of Demetrius Phalereus, for they took away the old planks as they decayed, putting in new and stronger timber in their place, insomuch that this ship became a standing example among the philosophers, for the logical question of things that grow; one side holding that the ship remained the same, and the other contending that it was not the same.

There is also an additional perplexity: if the replaced parts were stored in a warehouse and later used to reconstruct the ship, which—if either—would be the original ship of Theseus?

Suppose, without any loss of generality, that in a world *w* (perhaps the actual world), a given ship *S* which originated from particular matter *m* might have originated instead from the same amount of matter but with as many as 10% different molecules, and could not have originated from the same amount of matter as *m* while differing from *m* by more than 10%. Then there is a world *w*′ that is metaphysically possible in (i.e., accessible to) *w* and in which *S* originated from particular matter *m*′ that is 90% of *m* together with 10% different matter. Now in *w*′, *S* might have originated from matter that differs from *m′* by as much as 10%. Hence, there is a world *w″* that is possible in *w*′ and in which ship *S* originated from particular matter *m*″ that is more than 90% the same as *m*′, but less than 90% of *m* with more than 10% different matter from *m*. But in *w*, *S* could not have originated from *w*″. Hence, though *w*″ is possible in *w*′, which is itself possible in *w*, *w″* is not possible in *w*. In *w*, *w*″ is an impossible (albeit possibly possible) world. It is only contingently necessary in *w* that ship *S* does not originate from matter *m*″. This yields a counter-instance to the ***S4*** principle that whatever is necessary is necessarily so.

On this metaphysics of materiality, metaphysical accessibility is intransitive. Opponents may argue that any such metaphysic is not only incorrect but necessarily so. Nevertheless, it cannot be plausibly argued that such a metaphysic is logically inconsistent. On the contrary, whether it is necessarily correct or necessarily incorrect, the theory in question is obviously coherent. Hence the logic of metaphysical modality is not as strong as ***S5***, or even ***S4***. No equally plausible counter-instance to ***B*** (as a deductive system governing metaphysical modality) is presently known. But even if the ***B*** principle is correct, here also it seems that its truth is not required by logic. As far as logic is concerned, the actual world might have been a metaphysically impossible world instead of a metaphysically possible one (even if this prospect is metaphysically impossible).

The logic of combining of modality with quantifiers is known as *quantified modal logic*. Quantifiers and Leibniz’ Law call for special attention in modal logic. According to the logician Willard van Orman Quine, identity is governed by Leibniz’s Law stated as a *principle of substitutivity*: the terms of a true identity statement are everywhere intersubstitutive, preserving truth (or *salva veritate)*. As soon at the principle is stated, however, one is confronted with counterexamples. For example, although there are (let us suppose) exactly nine planets in the solar system, this is surely a matter of contingent fact—there might have been eight planets instead of nine. Though it is mathematically necessary that nine is odd, it is not a necessary truth that the number of planets is odd. A puzzle, which we will call *Quine’s conundrum*, points to a needed restriction on Leibniz’ Law formulated as a principle of substitutivity. Relative to the scheme of abbreviation,

A : 9

F : *a* is odd

G : *a* numbers the planets,

The sentence

□FA

is true, whereas

□F ℩*y*G*y*

is false, even though

A =  ℩*y*G*y*

is true. Where a symbolic term ζ stands within a formula ϕζ within the scope of an occurrence of a modal operator, and either ζ or η is a definite description, Leibniz’ Law may be weakened as follows:

□ζ = η

ϕζ

ϕη

where ϕη is like ϕζ except for having free occurrences of η where ϕζ has free occurrences of ζ. The stronger form of Leibniz’s Law remains operative when neither of the symbolic terms ζ nor η is a definite description, and also when ϕη is like ϕζ except for having free occurrences of η where φζ has free occurrences of ζ not within the scope of an occurrence of a modal operator. A more general form of Leibniz’s Law is employed in this chapter, specifically:

□□…□ζ = η

ϕζ

ϕη

where ‘□□…□’ represents a string of occurrences of ‘□’ whose length is unrestricted if neither ζ nor η is a definite description, and is otherwise at least as great as the largest number of occurrences in φζ of symbolic formulas of the form

□ψ

or of the form

ψ

where ψ is a symbolic formula, such that there is a free occurrence of ζ that stands within each (i.e., the largest number of modal-operator occurrences having the same free occurrence of ζ in their scope).

A choice needs to be made concerning the range of quantification with respect to a given world. *Possibilist quantification* allows the quantifiers to range over all possible individuals with respect to all worlds. So-called *actualist* (or internal) quantification, which is the alternative employed in this chapter, allows each quantifier range with respect to a world *w* only over the individuals that exist in *w*. Universal derivation and the rules of existential instantiation, universal instantiation, and existential generalization are modified accordingly. This choice also excludes as theorems the *Barcan formula*

∀*x*□F*x* → □∀*x*F*x ,*

as well as the *Buridan formula*,

∃*x*□F*x* → □∃*x*F*x* ,

and its converse. Instead there is a *weakened version of the converse Barcan formula*,

□∀*x*F*x* → ∀*x*□(∃*y* *x* = *y* → F*x*) ,

as well as a *weakened version of the Buridan formula*,

∃*x*□(∃*y x* = *y* ∧ F*x*) → □∃*x*F*x .*

Each of these formulas involves quantification across a modal operator into a modal context. Quine argued that such constructions are problematic at best, perhaps even incoherent. Consider the number, which is both nine and the number of planets, apart from any particular manner of specification. Is it a necessary truth about this number that it is odd? Abstracting from any particular manner of specification (or what Frege [1892] called a *mode of presentation*) of the number, then the grounds for saying that it is a necessary feature of the number that it is odd—*viz*., that it is necessary that *nine* is odd—seem perfectly counterbalanced by equally good grounds for saying that the oddness is only a contingent feature: it is not necessary that *the number of planets* is odd. Yet Leibniz’ Law prohibits one from consistently saying that oddness is a necessary feature of nine and, at the same time, only an accidental feature of the number of planets. For the number of planets *is* nine; hence, any property of the number of planets (e.g., being contingently odd) is equally a property of nine, and vice versa.

More formally, there seems to be no coherent answer to the question of whether the number that is both nine and the number of planets satisfies the open formula

□F*x*

relative to the scheme of abbreviation displayed above. A symbolic formula of the form

□ϕ ,

where ϕ is a symbolic formula, is true with respect to a world *w*, and under an assignment *s* of values to variables, just in case the proposition expressed by ϕ under assignment *s* is a proposition that is a necessary truth in *w*—i.e., just in case ϕ is true, under *s*, with respect to every world accessible to *w*. In particular, a symbolic formula of the form

□Fα ,

where α is a symbolic term that denotes the number nine, is true on the scheme of abbreviation displayed above, and under an assignment *s* of values to variables, if and only if the proposition expressed by

Fα ,

on the same scheme of abbreviation and under assignment *s*, is a necessary truth. But which proposition is expressed depends on the manner in which the term α specifies nine. If α specifies nine as *the successor of eight* (or as *the predecessor of ten*, *the square of three*, etc.), a necessary truth is expressed. If α specifies nine as *the number of planets* (or as *Bill’s favorite odd number*, etc.), a contingent truth is expressed. When the variable ‘*x*’ occurring in (2) is assigned the number of planets as its value, it denotes nine (at least until the recent demotion of the status of Pluto) but does not specify nine in any particular manner. We are thus at a loss concerning whether the formula ‘F*x*’ expresses a necessary truth under that assignment. The result of substituting ‘A’ for ‘*x*’ in (2) is true, whereas the result of substituting ‘℩*y*G*y*’ is false, even though each of the two substituends—‘A’ and ℩*y*G*y*’—denotes nine. This situation appears to render the construction ‘∃*x*🞏F*x*’ without a coherent interpretation.

As Arthur Smullyan [1948] noted against Quine, Russell’s [1905] theory of descriptions, when applied to quantified modal logic, yields a straightforward solution to this conundrum. According to Russell’s theory, a definite description is not a term, and symbolic sentence (1) is subject to an ambiguity of scope. One reading of the sentence (the Russellian “*secondary occurrence*” reading) is given by:

□∃*x*[∀*y*(G*y* ↔ *x* = *y*) ∧ F*x*] ,

i.e., necessarily, something that uniquely numbers the planets is odd. This is false, since there might have been an even number of planets. The other reading of (1) (the “*primary occurrence*” reading) is given by:

∃*x*[∀*y*(G*y* ↔ *x* = *y*) ∧ □F*x*] ,

i.e., something that uniquely numbers the planets is necessarily odd. This is evidently true, since the number in question (nine, i.e., the number of planets) is such that it could not have been an even number instead of odd.

The matter of interpreting (2) is straightforward on a Russellian theory. (Notice, however, that Russell’s theory of descriptions plays no role in the interpretation.) The proposition expressed by ‘F*x*’ relative to the scheme of abbreviation displayed above, and under the assignment of the number of planets as value for the variable ‘*x*’, is the proposition about the number in question that it is odd. The latter is a *singular proposition*, one that is about a particular individual that occurs as a propositional component, representing itself in the proposition. The singular proposition about nine that it is odd is true in any possible world in which nine is an odd number (whether or not it numbers the planets in that world). Formula (2), when evaluated under the assignment in question, functions in a manner exactly analogous to (0).

Semantic theories—like those of Frege [1892], Church [1946], and Quine [1950]—that reject singular propositions may not avail themselves of this theoretically elegant and satisfying solution to Quine’s conundrum. Such theories fail to provide a proposition to serve as the content of an open formula like ‘F*x*’, leaving (2) in need of interpretation. The insistence that denotation is invariably determined by an accompanying manner of specification leads to a famous problem in modal metaphysics: the so-called problem of *cross-world identification*.

The open formula ‘F*x*’ is true relative to the scheme of abbreviation displayed above, and under the assignment of nine as value for the variable ‘*x*’, if and only if for every possible world *w* (i.e., for every world *w* accessible to the actual world), the individual denoted by the variable ‘*x*’ with respect to *w*, and under the same assignment of nine as value, is odd in *w*. The problem is that, though the variable ‘*x*’ is assigned nine (*qua* the number of planets) as its value with respect to the actual world, it is not assigned any particular manner of specification of nine (e.g., as *the number of planets*, as *the successor of eight*, or in some alternative manner). This fails to provide any fact of the matter concerning which item from *w*, if any, is the value of ‘*x*’ with respect to *w*.

There is a question concerning which object from *w* is to be identified with the actual number of planets? This alleged problem does not even arise once it is recognized that the variable ‘*x*’ is what Russell called a *logically proper name* of its value. That is, the proposition expressed by the open formula ‘F*x*’ relative to the scheme of abbreviation displayed above, and under the assignment of nine as value for ‘*x*’, is simply the singular proposition about nine that it is odd. One very significant benefit of countenancing singular propositions, in fact, is that the “problem” of cross-world identification is thereby exposed as a pseudo-problem.

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1. Johan van Bentham’s *Modal Logic for Open Minds* [2010] CSLI Lecture Notes Number 119, Stanford California, p. 1. [↑](#footnote-ref-1)
2. *Dear Carnap, Dear Van: The Quine-Carnap Correspondence and Related Work by W. V. Quine and Rudolf Carnap [1991]*, edited with an Introduction by Richard Creath, University of California Press. [↑](#footnote-ref-2)
3. The theological, if not the political, roots Great Schism of 1054, which can be traced back to a disagreement about the modalities of the Persons of the Trinity. The Nicean Creed (325) use the term *homoousios* (from the Greek *homo* = ‘same‘ and *ousios* = ‘essence’ or ‘substance’), in contrast to *homoiousios* (from the Greek *homoi* = ‘similar’) making the solitary *i* the jot and tittle of Nicean Creedal Orthodoxy. The Greek Church preferred the latter term since the former had been used by the Syrian Bishop of Antioch to espouse *modal monarchism*, the heresy thatthe Heavenly Father, Resurrected Son and Holy Spirit are not three distinct Persons, but are rather different *modes* or *aspects* of one monadic God perceived by believers as distinct persons. The Latin Church adopted the former siding with Athanasius against the heretic Arius, who denied that Jesus was co-equal and co-eternal with the Father. The Second Council of Nicea (381), among other changes, inserted the word *filioque* (from the Latin *filio* = the son, and *que* = “and”) into the Nicean-Constantinopolitan Creed and the Latin mass. In Latin theology, the three Persons of the Trinity are logically distinguished by the formal relations of “proceeding from” citing such proof texts as Phil. 1:9, Titus 3:6, Acts 2:33. Orthodox theology, citing the words of Jesus in proof texts such John 15:26, regarded the insertion of *filioque* into the Nicean-Constantipolitan creed, is as Semi-Sabellianism. The great church historian Jaroslav Pelikan (*The Melody of Theology*, Cambridge, Mass: Harvard University Press, 1988, p. 90) opined: “If there is a special circle of the inferno described by Dante reserved for historians of theology, the principal homework assigned to that subdivision of Hell for at least the first several eons of eternity may well be a thorough study of all the treatises… devoted to the inquiry: Does the Holy Spirit proceed from the Father only, as Eastern Christendom contends, or from both the Father and the Son as the Latin Church teaches?” In 1989 Pope John Paul II and Patriarch Demetrius knelt together in Rome and recited the Nicene Creed without the *filioque*. [↑](#footnote-ref-3)
4. Van Bentham came up with the following: ~□(ϕ ∧ ~ϕ). Using familiar equivalences from propositional logic and the modal negation laws to show that this symbolization is equivalent to *McKinsey’s Axiom*: □ϕ → □ϕ. [↑](#footnote-ref-4)
5. Oskar Becker, "Zur Logik der Modalitäten", *Jahrbuch für Philosophie und phänomenologische Forschung*, Bd. XI (1930), 497–548. [↑](#footnote-ref-5)
6. See Jack Copeland [2014}, Jack Copeland. Keynote address at the Arthur Prior Centenary Conference, Balliol College, Oxford, August 22, 2014. http://conference.prior.aau.dk http://www.priorstudies.org. [↑](#footnote-ref-6)
7. van Bentham [2010], p. 51, notes “this often-cited analogy is not quite right, if you think about it.” [↑](#footnote-ref-7)
8. Daniel Defoe, *The Political History of the Devil*, 1726 and Benjamin Franklin, Letter to Jean-Baptiste Leroy, 1789. [↑](#footnote-ref-8)
9. In fact, in 2006, Pluto was demoted from a full-fledged planet to a dwarf planet. According to the new definition, a planet is an object (1) that orbits the sun and is large enough to become round due the force of its own gravity, and (2) has the ability to dominate the neighborhood around its orbit. Pluto does not dominate its neighborhood. All true planets are larger than their moons, but Pluto’s large “moon” Charon is half the size of Pluto. [↑](#footnote-ref-9)
10. Ruth Barcan Marcus, “A Functional Calculus of the First Order Based on Strict Implication,” *Journal of Symbolic Logic* 11 (1946), 1-16. [↑](#footnote-ref-10)
11. John Buridan, *Summulae de Dialectica*, translated by Guyla Kima, Yale University Press, 2001, 9.4, *Thirteenth Sophism*, 899. [↑](#footnote-ref-11)
12. At the Tenth International Tbilisi Symposium on Language, Logic and Computation in 2013, one of the speakers referred to this theorem part of the folklore of the field not realizing that de Jongh was in the audience. [↑](#footnote-ref-12)
13. van Benthan [2010], p. 245 describes this theorem “as a piece of ‘magical’ modal reasoning that has delighted generations.” [↑](#footnote-ref-13)