A Fregean Foundational System

“If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts and by freeing thought from that with which only the means of expression of ordinary language, constituted as they are, saddle it, then my ideography, further developed for these purposes, can become a useful tool for the philosopher. To be sure, it too will fail to reproduce ideas in a pure form, and this is probably inevitable when ideas are represented by concrete means; but, on the one hand, we can restrict the discrepancies to those that are unavoidable and harmless, and, on the other hand, the fact that they are of a completely different kind from those peculiar to ordinary language already affords protection against the specific influence that a particular means of expression might exercise.”[[1]](#footnote-1)

—Gottlob Frege, *Begriffsschrift*, [1879]

Section 1. Frege’s Logicist Dream.

Frege’s *Begriffsschrift*, or conceptual ideography, was a partial fulfillment of Leibniz’s dream of a universal logical language in which philosophical and mathematical concepts could be expressed with great precisions. Once this language was discovered, Leibniz envisions, answers to philosophical questions could be resolved, as in mathematics, by a *calculus philosophicus* or *ratiocinator*. Unlike his predecessors such Boole and Schröder who used mathematical symbols and fo*r*mulae to calculate logical relations, Frege’s goal was more radical: to construct a fundamentally new and more powerful *symbolic logic* that could serve as the *foundation* for mathematics.

Frege dreamed of constructing the logical foundations for arithmetic. This dream was known as *logicism*, which can be expressed as the following thesis:

Arithmetic = Logic + Set Theory .

To fulfill this dream Frege realized he had to discover, or create, a more powerful conception of logic than Aristotle and to create a more rigorous concept of proof than Boole. Frege’s *Begriffsschrift* [1879] is now regarded as the “founding document” of modern logic.[[2]](#footnote-2) Jean van Heijenoort who edited, and set a high standards for scholarship in logic, wrote the following in his classic resource book in mathematical logic *From Frege to Gödel*:

This is the first work that Frege wrote in the field of logic, and, although a mere booklet of eighty-eight pages, it is perhaps the most important single work ever written in logic. Its fundamental contributions, among lesser points, are the truth-functional propositional calculus, the analysis of the proposition into function and argument(s) instead of subject and predicate, the theory of quantification, a system of logic in which derivations are carried out exclusively according to the form of the expressions, and a logical definition of the notion of mathematical sequence. Any single one of these achievements would suffice to secure the book a permanent place in the logician's library.

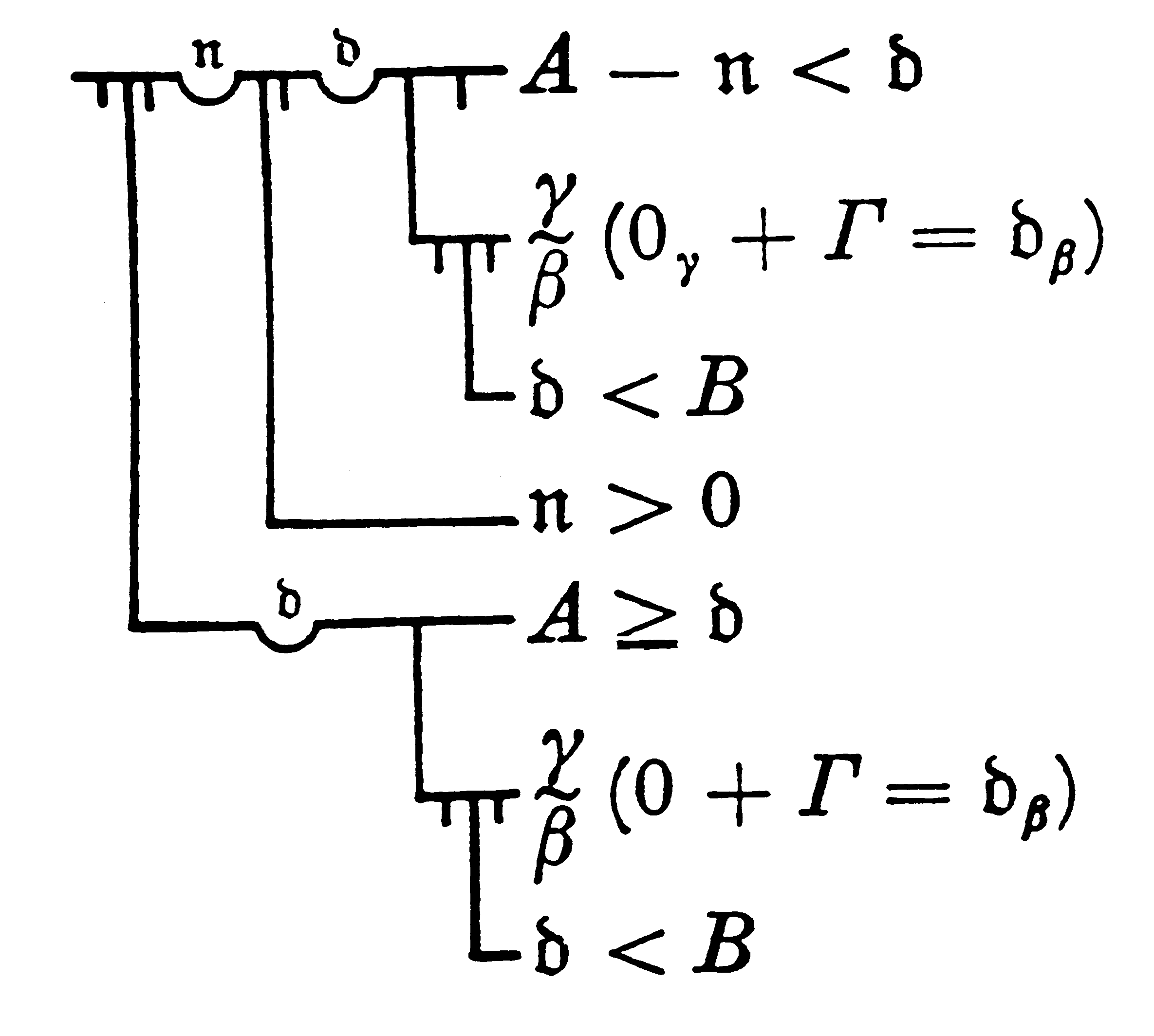
Frege compared his *Begriffsschrift* to a microscope (fittingly, Frege’s research was anonymously supported by a company that made precision optical equipment):

I believe that I can best make the relation of my ideography [*Begriffsschrift*] to ordinary language clear if I compare it to that which the microscope has to the eye. Because of the range of its possible uses and versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope.... But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals....

Like a microscope, Frege’s *Begriffsschrift* would resolve philosophical confusions with the clarity of logical precision:

If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts and by freeing thought from that with which only the means of expression of ordinary language, constituted as they are, saddle it, then my Concept Script, further developed for these purposes, can become a useful tool for the philosopher.

Frege’s discovery was that a new and more comprehensive logic could be based, not on the *subject* and *predicate* distinction of Aristotelian logic, but on the mathematical notion of *function* and *argument*. Frege predicted, “In particular, I believe that the replacement of the concepts of *subject* and *predicate* by *argument* and *function*, respectively, will stand the test of time.”

The reception of Frege’s work was hindered by the forbidding nature of his notation. Russell in his preface to *The Philosophy of Mathematics* said he had seen Frege’s *Grundgesetze der Arithmetik* [vol. 1, 1893] but admitted that he had “failed to grasp its importance or to understand its contents” because of “the great difficulty of his symbolism” [p. xvi]. A sample of Frege’s notation is displayed (*right*).[[3]](#footnote-3)

Frege’s writings contain two conceptions of generality.In his *Begrifftsschrift*, Frege explained the notion of generality [*Die Allgemeinheit*] *semantically* by stating that each instance of a generality is a fact, but later added the “illuminating” observation that generality can be inferred from *provability* from an arbitrary instance. Translating his notation into modern symbols, Frege had formulated the rule:[[4]](#footnote-4)

It is even illuminating that one can derive P → ∀*x*F*x* from P → F*a*, if P is an expression in which *a* does not occur and if *a* stands in F*a* only in argument places.

These two notions of generality correspond to the *semantics* for universal quantification (UI) and the syntactical inference rule of universal generalization (UG) or, alternatively, the derivation form known as universal derivation (UD). In his *Grundgesetze der Arithmetik*, Frege introduced quantification as a *second-order concept*—or as a concept of concepts.

Frege believed that “Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician.”[[5]](#footnote-5) Here Frege’s relatively half-hearted, and reserved, claim can be constrained with Leibniz’s full-throated proclamation: “Without mathematics we cannot penetrate deeply into philosophy. Without philosophy we cannot penetrate deeply into mathematics. Without both we cannot penetrate deeply into anything.”[[6]](#footnote-6)

As a philosopher of language, Frege’s work set much of the agenda of 20th century analytic philosophy.[[7]](#footnote-7) As a logician and mathematician, Frege set a standard of rigor[[8]](#footnote-8) that was not equaled in Russell and Whitehead’s *Principia Mathematica* [1910 – 1913]. In Quine’s (1908 – 2000) famous assessment: “Logic is an old subject, and since 1879 has been a great one.”

In this chapter, we want to get some sense of the grandeur of Frege’s vision, how Frege’s more power symbolic logic was able to subsume earlier developments of logic such as Aristotle’s syllogism and Boole’s algebra of logic, and how Frege’s ill-fated conception of sets (or “courses of values”) licensed the fatal derivation of Russell’s paradox leading Frege to dred that his life’s work had been a star-crossed dream.

Section 2. Fregean versus Cantorian Set Theory.

How does Fregean set theory d*iffer from Cantorian set theory? First let’s state a principle upon which both conceptions agree, namely, the axiom* *of* *extensionality*, which states that set identity depends only on membership.

*Axiom of Extensionality.* Sets are identical if (and only if) they have the same members:

x = y ↔ ∀z(z ∈ x ↔ z ∈ y) ,

where the e*x*pression ‘x ∈ y’ is read ‘x is a member of y’. (The *“only if”* part of the biconditional follows by the logic of identity.)

The second axiom of sets, known as the *Naïve Comprehension* *axiom*, distinguishes Fregean from Cantorian set theory. Given a formal logical language in which ϕ(*x*) is a well-formed formula containing the free variable *x,* then there exists a set

{*x*|ϕ(*x*)} ,

consisting of the objects which have the property expressed by ϕ(*x*). The e*x*pression ‘{*x*|ϕ(*x*)}’ is read ‘the set of all *x* such that ϕ(*x*)’.

The axiom is denigrated as “naïve” insofar it now seems obvious that treating sets as *extensions* of concepts leads to contradiction. According to Cantorian set theory, sets are collections built up iteratively by constructing all possible subsets of previous sets beginning with the empty set or a primitive set of ur-elements.

The conceptual parting of the ways between Fregean and Cantorian set theory can be illustrated by conundrum posed to children: what happens when an *irresistible* force meets an *immovable* object? The problem is that an irresistible force cannot co-exist with an immovable object. Stories with disguised self-contradictory properties have been told in many ancient cultures and serve as cultural metaphors for logical contradictions. [[9]](#footnote-9)

What Russell’s paradox has to do with the non-existence of a universe of a set all sets—the tension between Cantor’s powerset theorem and Frege’s postulation of the existence of the universe of all sets—we’ll come back to later. Our goal for the present is to formulate a first-order Fregean foundational system and to derive the Basic Laws of Arithmetic, also known as Peano Postulates, as theorems of that system. Only then shall we revisit this question why paradoxes arise.

Exercises:

Dear Alfred,

Do not despise the pieces I have written. Even if all is not gold, there is gold in them. I believe there are things here which will one day be prized much more highly than they are now. Take care that nothing gets lost.

Your loving father,

It is a large part of myself which I bequeath to you herewith.

1. Much of what Frege wrote on philosophical logic remained unpublished. Frege and his wife Margaret had several children, all of whom died young, and so they adopted a son, Alfred, who became an engineer. When Frege made his will in January 1925, he left his unpublished papers to Alfred, with the following note:

Six months later, Frege died. He did not know that he would come to be regarded as a founding father of the most influential movement in 20th century philosophy.

Symbolize the following sentences using the dictionary provided using modern quantifier-predicate logic, which is due to Frege.

*Dictionary*:

W*xy*: *x* is written by *y* J*x*: *x* is published S*xy*: *x* is a spouse of *y*

E: Frege G*x*: *x* is (a piece of) gold C*xy*: *x* is a child of *y*

B: *Begriffsschrift* H*xy*: *x* has *y* K*xy*: *x* was adopted by *y*

A: Alfred M*x*: *x* is male L*x*: *x* was no longer alive

D: Margaret F*x*: *x* is female

G: *Die Grundlagen der Arithmetik* [1884]

G1: *Die Grundgesetze der Arithmetik, vol. I* [1893].

G2: *Die Grundgesetze der Arithmetik, vol. 2* [1903].

1. All of Frege’s unpublished writings have some gold (*or other*).

*Paraphrase*: For all *x*, if *x* is written by Frege and *x* is unpublished, then there is a *y*, *y* is gold and *x* has *y.*

1. Some (particular) gold (one and the same) is contained in each of Frege’s unpublished writing.

*Paraphrase*: There is a *y*, *y* is gold and for all *x*, if *x* is written by Frege and *x* is unpublished, then *x* has *y.*

1. It is not the case that none of Frege’s unpublished writings contain gold, but it is the case that not all of Frege’s unpublished writings contain gold.

*Paraphrase*: It is not the case that no F is G, but it is the case not all F is G, where F = *x* is written by Frege and *x* is unpublished.

Symbolize all possible, if not always plausible, ambiguous readings of:

1. There are two (elements of) gold contained in all of Frege’s unpublished writings.
2. There are two (elements of) gold in each of Frege’s unpublished writings.

#2. Frege developed his own theory of definite descriptions.

The *x* such that F is symbolized by ℩*x*F*x*.

According to Frege’s theory, this term denotes the unique thing that satisfies F*x*, if there is such an *x.* Otherwise*,* the definite description is *improper*, and the term denotes the object chosen to be the denotation of all improper descriptions. This chosen object is always denoted by the logically improper definite description ℩*x x* ≠ *x*.

1. There was one and only one author of the *Begriffsschrift* and that individual is identical to the one and only author of the *Die Grundlagen.*
2. The author of the *Begriffsschrift* is the author of *Die Grundlagen*.
3. The largest prime number is identical to the non-self-identical thing.
4. Margaret was the wife of Gottlob Frege.
5. Alfred was the adopted son of Gottlob and his wife Margaret.

Frege’s logic was a second-order logic, which is more powerful than the first-order formulations of logic that are typically used today to express the mathematical theories in Zermelo-Fraenkel Set Theory. Our reconstruction of a Fregean foundational system will be a first-order formal system. However, we will examine Frege’s second-order laws temporarily to derive two first-order principles concerning sets and their extensions, which are consequences of his Basic Law V.

*Frege’s Basic Law V*: For any concepts F and G,

{*x*|F*x*} = {*x*|G*x*} ↔ ∀*x*(F*x* ↔ G*x*)

There is an important principle that can be derived from Basic Law V—a second order versio of the Axioms of Comprehension:

*Axiom of Abstraction*: given any concept or property F, there is a set consisting of all and only those things that “fall under” the concept or have the property F, i.e.,

∀*F*∀*x*(*x* ∈ {*y*|F*y*} ↔ F*x*} .

In other words, any concept determines an object, namely, the *extension* of the concept or the set of all objects falling under the concept. It will turn out that seemingly innocuous principle is the faulty pillar of Frege’s foundational system. One way to see that it will violate a fundamental principle of “Cantorian Set Theory” is that this principle implies that there exists an object that corresponds to every concept, namely the set that is the extension of the concept.

Section 3. A First-Order Fregean Foundational System

*Alphabet*:

*x*, *y*, *z*, ... the individual variables

~, →, ∧, ∨, ↔ the logical connectives for sentential logic

∀, ∃ the universal and e*x*istential quantifiers, respectively

∈ the set membership relation

{ | } the variable binding set forming operator

), ( the pair of parentheses

*Grammar*:

(F1) Every individual variable is a term.

(F2) If αand β are terms, then

α ∈ β

is a well-formed formula (wff).

(F3) If ϕ and ψ are wffs, then so are

~ϕ ,

(ϕ ∧ ψ) ,

(ϕ ∨ ψ) ,

(ϕ → ψ) ,

(ϕ ↔ ψ) .

(F4) If *x* is a variable and ϕ is a wff, then

∀*x*ϕ

and

∃*x*ϕ

are wffs, in which the variable *x* is bound.

(F5) If ϕ(*x*) is a wff containing the free variable *x,* then

{*x*|ϕ(*x*)}

is a term, in which the variable *x* is bound.

Propositional Logic: Here we use the KM2 system of natural deduction with the three forms of derivation Direct Derivation [DD], Conditional Derivation [CD], and Indirect Derivation [ID], as well as the derived form of Biconditional Derivation [BD]. We shall freely cite sentential inference rules, sequences of inference rules, and theorems.

Quantifier Logic: Here we use the KM2 rules universal instantiation [UI], existential instantiation [EI], existential generalization [EG], Universal Derivation [UD], and the derived rules of the derived rules quantifier negation [QN].

Our Fregean set theory, discussed above, can be formulated as a pair of axioms:

Axiom of Extensionality: α = β ↔ ∀*z*(*z* ∈ α ↔ *z* ∈ β), where *z* is the first variable that does not occur in either of the terms α or β.

First-Order Axiom Schema of Abstraction: ∀*x*[*x*∈{*y*|ϕ(*y*)} ↔ ϕ(*x*)], where the variable *y* occurs free in ϕ(*y*), and ϕ(*x*) comes from ϕ(*y*) by replacing all the free occurrences of *y* in ϕ(*y*) with free occurrences of the variable *x* in ϕ(*x*).

Notation: Square brackets will be interchanged with a pair of matching parentheses for readability. We use the abbreviations ‘α ≠ β’ for ‘~ α = β’ and ‘α ∉ β’ for ‘~ α ∈ β’.

Section 4. Theorems of the Fregean Foundational System

We begin by establishing the existence of the universal set. The first step is to show that every set is equal to itself. According to propositional logic, we have the theorem T91 P ↔ P, which has as an instance,

*y* ∈ *x* ↔ *y* ∈ *x ,*

which implies by UD and extensionality that every set is equal to itself.

***Theorem*** 1. ∀*x* *x* = *x* (Every set is equal to itself.)

1. *~~Show~~* ∀*x* *x* = *x* 5, UD

2. *~~Show~~* ∀*y*(*y* ∈ *x* ↔ *y* ∈ *x*) 3, UD

3. *y* ∈ *x* ↔ *y* ∈ *x* **T**91

4. *x* = *x* ↔ ∀*y*(*y* ∈ *x* ↔ *y* ∈ *x*) *Extensionality*

5. *x* = *x* 4, BC, 2, MP

Having established that every set is self-identical, we can define the universal set.

***Definition* 1**. V := {*x*|*x* = *x*} (V is the *universal set*)

Everything is a member of the universal set V, including V itself.

***Theorem*** 2. ∀*x* *x* ∈ V

1. *~~Show~~* ∀*x* *x* ∈ V 4, UD

2. *~~Show~~* ∀*x* *x* ∈ {*x*|*x* = *x*} 5, UD

3. ∀*x*[*x* ∈ {*x*|*x* = *x*} ↔ *x* = *x*] *Abstraction*

4. *x* ∈ {*x*|*x* = *x*} ↔ *x* = *x* 3, UI

5. *x* = *x* Theorem 1, UI

6. *x* ∈ {*x*|*x* = *x*} 4, BC, 5, MP

7. ∀*x* *x* ∈ V 2, Def. V

***Corolla***ry 2.1. **V** ∈ **V** (The universal set V contains itself.)

Next we construct the empty set ∅ using the universal negative property of being non-self-identical.

***Definition*** 2. ∅ = {*x*|*x* ≠ *x*}

Since nothing has the property of being non-self-identical, the empty set has no members.

***Theorem*** 3. ∀*x* *x* ∉ ∅

Does the empty set really exist? Bas van Fraassen offered the following “ontological proof” for the existence of the empty set for skeptics: “The fool hath said in his heart “there is no empty set.” But then consider the set of all empty sets. This set would be empty. Contradiction. Hence, the empty set exists.”

By extensionality, the empty set is unique.

***Theorem*** 4. ∀*y* (∀*x* *x* ∉ *y* ↔ *y* = ∅)

Next we define some set-theoretic relations—subsets, power sets, complements, intersections, unions—that enable us to derive the principles of *Boolean algebra* as theorems.

***Definition*** 3. A ⊆ B := ∀*z*(*z* ∈ A → *z* ∈ B), where the variable *z* does not occur in A or B.

***Theorem* 5**. A = B ↔ A ⊆ B ∧ B ⊆ A

The *complement* of a set A is defined as the set of all things *not* in A.

***Definition*** 4. Ac := {*x* | *x* ∉ A}, where A is any set.

The *intersection* of sets A and B is defined as the set of elements in both A and B.

***Definition*** 5. (A ∩ B) := {*x* | *x* ∈ A ∧ *x* ∈ B}, where A and B are sets.

The *union* of sets A and B is defined as the set of all things that are elements of A or B.

***Definition*** 6. (A ∪ B) := {*x* | *x* ∈ A ∨ *x* ∈ B}, where A and B are sets.

The *difference* of sets A and B is defined as the set of all things that are elements of A but not of B.

***Definition*** 7. (A ─ B) := {*x* |*x* ∈ A ∧ *x* ∉ B}, where A and B are sets.

Exercises.

#1. Construct proofs for Theorems 3, 4, and 5.

#2. Prove each of the following conditions are equivalent. You may use the corresponding biconditional propositional theorems and the rule of Interchange of Equivalence [IE]:

1. A ⊆ B
2. A ∩ B = A T72 (P → Q) ↔ [(P ∧ Q) ↔ P]
3. A ∪ B = B T73 (P → Q) ↔ [(P ∨ Q) ↔ Q]
4. Bc ⊆ Ac T13 (P → Q) ↔ (~Q → ~P)
5. B – (B – A) = A

For illustration, we construct a biconditional derivation to establish the equivalence of (1) and (2).

1. *~~Show~~* (A ⊆ B) ↔ A ∩ B = A 6, BD
2. A ⊆ B Assume (BD)
3. ∀*x*(*x* ∈ A → *x* ∈ B) 2, Def. ⊆
4. ∀*x*(*x* ∈ A ∧ *x* ∈ B ↔ *x* ∈ A) 3, T72, IE
5. ∀*x*(*x* ∈ A ∩ B ↔ *x* ∈ A) 4, Def. ∩
6. A ∩ B = A 5, Ext.

In a *biconditional derivation* you may assume one constituent of the biconditional and then each succeeding line must be obtained by an Interchange of Equivalents (IE), i.e., by a previously proven biconditional theorem, a definition, or an axiom—until the other constituent is obtained. Then the subsequent lines can be boxed and cancelled by biconditional derivation [BD].

This exercise establishes the following family of theorems:

Theorem B-1. A ⊆ B ↔ A ∩ B = A ↔ A ∪ B = B ↔ Bc ⊆ Ac (Subset Theorems)

#3. Prove the following theorems of Boolean Algebra.

Theorem B-2 A ∩ A = A; A ∪ A = A (Idempotent Laws)

Theorem B-3. A ∩ ∅ = ∅; A ∪ V = V (Identity Laws)

Theorem B-4. (A ∩ B) ⊆ A ⊆ (A ∪ B) (Max and Min Laws)

Theorem B-5. (A ⊆ C) ∧ (B ⊆ C) → (A ∪ B) ⊆ C

(C ⊆ A) ∧ (C ⊆ B) → C ⊆ (A ∩ B)

Theorem B-6. A ∩ (A ∪ B) = A; A ∪ (A ∩ B) = A (Absorption Laws)

Theorem B-7. A ∩ B = B ∩ A; A ∪ B = B ∪ A (Commutative Laws)

Theorem B-8. A ∩ (B ∩ C) = (A ∩ B) ∩ C; A ∪ (B ∪ C) = (A ∪ B) ∪ C (Associative Laws)

Theorem B-9. A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C); A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C) (Distributive Laws)

Theorem B-10. [(A ∪ B = ∅ ∧ A ∪ C = ∅) ∧ (A ∩ B = V ∧ A ∩ C = V)] → B=C (Complement Unique)

Theorem B-11. (A ∩ B)c = (Ac ∪ Bc); (A ∪ B)c= (Ac ∩ Bc) (DeMorgan’s Laws)

Theorem B-12. A ⊆ B → (C ─ A) ⊆ (C ─ B) (difference Laws)

A ─ B = A ─ (A ∩ B)

B ─ (B ─ A) = A

A ∪ (B ─ A) = B

C ─ (A ∪ B) = (C ─ A) ∩ (C ─ B)

For illustration, we construct a formal derivation of the first form of DeMorgan’s Laws.

|  |  |  |  |
| --- | --- | --- | --- |
| 1. | *~~Show~~* (A ∩ B)c = (Ac ∪ B c) | 11, DD |  |
|  |  |  |  |
| 2. | *~~Show~~* ∀*x*[*x* ∈ (A ∩ B)c ↔ *x* ∈ (Ac ∪ B c))] | 3, UD |  |
|  |  |  |  |
| 3. | *~~Show~~* [*x* ∈ (A ∩ B)c ↔ *x* ∈ (Ac ∪ B c))] | 10, BD |  |
|  |  |  |  |
| 4. | *x* ∈ (A ∩ B) c | Assume (BCD) |  |
| 5. | *x* ∉ (A ∩ B) | Def. Ac |  |
| 6. | *x* ∉ {*x* | *x* ∈ A ∧ *x* ∈ B} | Def. ∩ |  |
| 7. | ~[*x* ∈ A ∧ *x* ∈ B] | Abstraction |  |
| 8. | [*x* ∉ A ∨ *x* ∉ B] | T65 DeMorgan’s Law |  |
| 9. | *x* ∈ Ac ∨ *x* ∈ B c | Def. Ac |  |
| 10. | *x* ∈ Ac ∪ B c | Def. ∪ |  |
|  |  |  |  |
|  |  |  |  |
| 11. | (A ∩ B)c = (Ac ∪ B c) | 2, Def. = |  |

Section 4. A Digression: Boole’s Life and Logic.

“The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind.”

— George Boole, *An Investigation of the Laws of Thought* (1854)

Logic, according to Boole, deals with the “laws of thought”—those “fundamental laws of those operations of the mind by which reasoning is performed.” These laws can be expressed mathematically as a calculus or algebra. Boole regarded his discovery of the laws of thought as discovering the workings of the mind just as Newton’s laws of physics had discovered the laws of nature.

(B1) Logic deals with the “laws of thought”;

(B2) The laws of thought can be expressed mathematically as algebraic laws;

(B3) The laws of thought mirror the physics of the mind.

Contemporary philosophers would be careful to distinguish between the *descriptive* or *empirical* laws of thought and the *prescriptive* or *normative* laws of the science of correct reasoning. Nevertheless, Boole’s conception of his project was an inspiring one.

In the digital world today, one often hears about “Boolean variables”—0 and 1, true or false. One might think, what a simple idea! Why did someone have to invent it? What sense can be made of such peculiar equations as 1 + 1 = 0? In this chapter, we shall look at the backstory of Boole’s work in logic in which Boolean variables are simply a beginning.[[10]](#footnote-10)

George Boole (1815-1864) was the son of a poor English shoemaker who had a strong interest in science and mathematics, in Lincoln, England. George was a self-taught prodigy. At the age of 14, young George translated a Greek poem that was published in the local paper. At the age of 16, he was hired as a teacher and learned calculus on his own. Supporting himself by starting his own elementary school at the age of 19, Boole developed educational theories about the importance of discovery, emphasizing the importance of deep understanding as opposed to rote memorization. By the age of 23, he began publishing papers on the mathematics of his time, but his interests soon turned towards expressing the laws of logic in a mathematical form, which would become to be known as the Boolean algebra of logic. A talented lecturer Boole explored way of bending algebra so it could be used to express laws of logic so that algebraic operations would corresponding to valid logical inferences.

The result was a concise monograph, 86 pages in length, entitled *The Mathematical Analysis of Logic* [1847]. As a result of this book, Boole began corresponding with leading British mathematicians of his day. Boole considered going to Cambridge to become a “university person” but was put off then told he would have to start with a standard undergraduate course and stop doing his own research. Boole’s friend, the logician Augustus De Morgan had become involved in controversies, and this caused Boole to shy away from controversy and to continue developing his own thoughts on the matter.

Approaching logic in terms of mathematics was significant because it brought together two streams of thought. Since ancient times, logic and mathematics has developed on different trajectories. Although Aristotle regarded logic as propaedeutic to science and mathematics, Aristotle’s syllogistic logic was inadequate for explaining the mathematical reasoning involved in Euclid’s *Elements*. The philosopher Gottfried Wilhelm Leibniz (1646 – 1716) had dreamed of a universal alphabet of thought (*characteristic universalis)* in which ideas could be represented to that disputes among philosophers could be resolved by a symbolic logic (*calculus ratiocinator*) in the same way as disputes among mathematicians:

The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate, without further ado, to see who is right.

Boole’s work was revolutionary because it brought these two streams of thought that has been developing independently—logic and mathematics—together. In 1849 Boole was hired by Queen’s College, Cork (now University College Cork) as its first professor of mathematics (which was considered a less prestigious than classical subjects like Latin or Greek).[[11]](#footnote-11)

As a university professor, Boole began writing his famous book *An Investigation of the Laws of Thought: the Mathematical Theories of Logic and Probabilities* [1854]*.* The first part of the book expresses the laws of thought as algebraic laws in which the variables take on the Boolean values of 0 or 1. In the second part of the book, the algebraic variables are interpreted as probabilities also taking on values between 0 and 1. In the final chapter of his book, Boole discusses such philosophical questions as whether free will is compatible with definite laws of thought and whether there are truths that go beyond what mathematical laws can explain.

Eager to apply his methods to philosophical arguments such as proofs for the existence of God’s existence, Boole illustrated his logic by applying it to the proofs given by Samuel Clarke. (Clark was a follower of Newton, who was embroiled in a bitter dispute with Leibniz about priority in the discovery the calculus.) Here Boole was careful to distinguish between proving the truth of the conclusion and proving that the conclusion of an argument followed logically from its premises:

Herein too may be felt the powerlessness of mere Logic, the insufficiently of the profoundest knowledge of the laws of the understanding, to resolve those problems which lie nearer to our hearts, as progressive years strip away from our life the illusions of its golden dawn.

|  |  |  |
| --- | --- | --- |
|  | Logic | Algebra |
| Law of Non-Contradiction | ~(P ∧ ~P) | *x* ⊗ (1 − *x*) = 0 |
| Law of Excluded Middle | (P ∨ ~P) | *x* ⊕ (1 − *x*) = 1 |

Logic can determined whether a conclusion such as “God exists” follows from a set of premises, but it cannot determined exactly what the premises state or, given they have some definite meaning, whether they are true. “The chief practical difficulty of this inquiry,” Boole noted, “will consist, not in the application of the method to the premises once determined, but in ascertaining what the premises are.”

While at Queen’s College George Boole met in 1850 his future Mary Everest, the niece of a John Ryall, a professor of Greek, and they were married give year later. Although single for most of his life, Boole married at the age of in 1855. Although single for most of his life Boole at the age of 40 married Mary Everest Boole, who was 17 years younger. Boole died seven years later after walking in the rain to give his lecture while soaking wet. His wife, following the medical advice of the day, doused him with cold water when he got home. Boole caught pneumonia and died. His wife outlived him by 56 years and published books herself with such intriguing titles as *Philosophy and the Fun of Algebra*, *Logic Taught by Love*, and *The Message of Psychic Science to the World*.

Boole’s work is based upon observing an intriguing symmetry between the laws of logic and algebra. Imagine Boole’s excitement at discovering that Aristotle’s Laws of Thought could be formulated in a Boolean algebra in which the values of the variables are either 1 or 0, corresponding to true and false! Propositions in logic have one of the two truth-values, true or false. This is analogous to an algebra in which the values of the variables are one of two distinct values 1 or 0. If a variable *x* has one of those two values, then its *complement* (1 − *x*) has the opposite value, thus corresponding to *negation* in logic. The *product* of *x* and *y*, *x* ⊗ *y*, has the value 1 if and only if both *x* and *y* have the value 1, which corresponds to the truth rule for “*and*”: a *conjunction* (P ∧ Q) is true if and only if both P and Q are true.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x* ⊗ *y* | 0 | 1 |  | (P ∧ Q) | F | T |
| 0 | 0 | 0 |  | F | F | F |
| 1 | 0 | 1 |  | T | F | T |

In defining the operation of addition ⊕, the above correspondences still leave open two choices. These choices correspondence to two interpretations for the disjunction—the *inclusive* “*or*” (symbolized by ‘∨’ for the Latin *vel*) or the *exclusive* ‘*or’* (in Latin *aut* or in computer science *xor* and symbolized by ⊽ or the negation of a biconditional ↮):

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x* ⊕ *y* | 0 | 1 |  | (P ∨ ⊽ Q) | F | T |
| 0 | 0 | 1 |  | F | F | T |
| 1 | 1 | 1/0 |  | T | T | T/F |

The latter interpretation leads to the equation 1 + 1 = 0, which looks suspicious. At least it did to the English logician William Stanley Jevons (1835 -1882), who, in an 1863 letter, criticized Boole harshly:

It is surely obvious, however, that X + X is equivalent only to X, … Professor Boole’s notation … is inconsistent with a self-evident law. If my view be right, his system will come to be regarded as a most remarkable combination of truth and error.

Evidently Jevons was thinking of + in terms of an operation like the *union* of set in which we have the law *x* ∪ *x* = *x*. However, Boole’s conception of + was in terms of the *disjoint* union of X and Y, so that

X + Y = (X – Y) ∪ (Y – X) , whereupon substituting X for Y yields *x* + *x* = 0. When Jevons continued to press his criticism, Boole in exasperation replied:

To be explicit, I now, however, reply that it is not true that in Logic *x* + *x* = *x*, though it is true that *x* + *x* = 0 is equivalent to *x* = 0. If I do not write more it is not from any unwillingness to discuss the subject with you, but simply because if we differ on this fundamental point it is impossible that we should agree in others.

Obviously how will formulate the laws of logic will depend on how + is to be interpreted. For example, here are two forms of disjunction syllogism—one is formulated using the exclusive ‘*or’* and the other using with the *inclusive* ‘*or’*. The Boolean equations on the right correspond to the law of disjunctive syllogism (*inclusive* or *exclusive* ‘*or’*) on the left:

|  |  |  |
| --- | --- | --- |
|  | Disjunctive Syllogisms | |
| *Inclusive* ‘*or’* | P ∨ Q  ~ P  ∴ Q | *x* + *y* − *xy* = 1  *x* = 0  ∴ *y* = 1 |
| *Exclusive* ‘*or’* | P ⊽ Q  P  ∴ ~ Q | *x* + *y* − 2*xy* = 1  *x* = 1  *∴ y* = 0 |

Rather than challenging individual laws in isolation, it is more help to examine how the proposed laws functioned within Boole’s *system* of algebraic logic.

Why was Frege’s conception of logic an advance over Boole’s? Out the outset it is important to distinguish, as Boole did not, between the *descriptive* *empirical* laws “concerning the nature and constitution of the human mind” and the *prescriptive* *normative* laws of the science of logic. Boole’s conception of logic as the laws of thought confused descriptive and prescriptive questions: do the laws of logic describe the physics of thought or do they prescribe what is required of correct thinking?

Secondly, Boole failed to give a formal account of when a sequence of equations constituted a proof. Our Fregean Foundational system has been build on the foundation of a system of logic in which the rules of proof are formally characterized.

Thirdly, Boole’s logic exploited isomorphisms between Aristotle’s Syllogistic Logic, Boolean Set Theory, and Propositional Logic; however, Boole’s logic was still inadequate for the formalizing of richness of mathematical thinking.

Boolean algebras are useful because many different applications within logic, mathematics and linguistics share this kind of structure—the structure of partially order sets or *Posets*—with the operations of *join* and *meet*.

* Consider all the subsets of {*a*, *b*, *c*} ordered by the subset relation with the operations of intersection ‘∩’ (*meet* or ↓ in the lattice), union ‘∪’ (*joint* or ↑ in the lattice), and complementation ‘Ac’.
* Consider the set of numbers that are the set of factors of 30: {1, 2, 3, 5, 6, 10, 15, 30}. Interpret *meet* (or *infimum*) as the greatest common divisor (↓ in the lattice), *join* (or *supremum*) as the least common multiple (↑ in the lattice), and *inverse* of a number *x* as the number 30/*x*.

These two Boolean algebras have exactly the same structure. Indeed, the isomorphic mapping between the two lattices is easy to see when we replace each of the composite numbers on the right-hand lattice with set of their prime factors. (By convention 1 is not prime in order to ensure that every number as a unique prime factorization. Since 1 has no prime factors, it corresponds to the empty set.)

{*a, b, c*}

{*a, b*} {*b, c*} {*a, c*}

{*b*} {*a*} {*c*} {*a,c*}

{}

30

10

6 15

1

2 3 5

The rules and the law for manipulating algebraic expressions that Boole formulated were sufficiently mechanical that the English logician W. S. Jevons was able to use them to build a mechanical reasoning machine, which he demonstrated, to the Royal Society in 1870.

The relationships between Boolean Algebra and propositional logic can be seen in the following lattice, which happens to be equivalent to a tesseract or hyper-cube:



What can be said in assessment of Boole’s logic? Boole developed a rich and suggested algebra of logic. Boole’s algebra has certain similarities with ordinary arithmetic, but also some striking differences. For example, the distributive laws of × over + holds:

*x* × (*y* + *z*) = (*x* × *y*) + (*x* × *z*);

However, the law of the distribution of + over × does not have a counterpart in ordinary arithmetic:

*x* + (*y* × *z*) = (*x* + *y*) × (*x* + *z*) .

Working with algebraic analogies, Boole did not provide a precise formal language, which leads to our first criticism: Boole failed to give a formal account of when a sequence of equations constituted a proof. Our second criticism is the Boole’s logic is *expressively inadequate*: Boole’s logic is still inadequate for the formalizing of richness of mathematical thinking. In particular, Boole’s logic is missing a calculus of relations. Thirdly, Boole failed to distinguish the descriptive from a prescriptive account of reasoning: does his laws of thought describe the physics of thought or do they prescribe the laws required for correct logical reasoning?

Section 4. Peano’s Postulates as Theorems

Next we commence a Fregean logical construction of the natural numbers.

In order to answer the question “What is number?” correctly, one must be clear on, what Russell, influenced by Wittgenstein, called, the *grammar* of our inquiry. What we are seeking is not an understanding of particular numbers, like 0 and 1, but the concept of number itself. No particular collection having a certain number of objects, for example, will be identical with that number. Moreover, we must distinguish the concept of two collections *having the same number of elements*, which is logically prior to, but an essential step towards, logically constructing *the concept of number* in general.

There is no better way to introduce puzzles about the concept number and the important of securing the foundations of arithmetic, than to quote the opening of Frege’s *Grundlagen*:

“When we ask someone what the number one is, or what the symbol 1 means, we get as a rule the answer “Why, a thing”. And if we go on to point out that the proposition

“the number one is a thing”

is not a definition, because it has the definite article on one side and the indefinite on the other, or that is only assigns the number one to the class of things, without stating which thing it is, then we shall very likely be invited to select something for ourselves—anything we please—to call one. Yet, if everyone had the right to understand by this name whatever he pleased, then the same proposition about one would mean different things for different people, —such propositions would have no common content.

Some, perhaps, will decline to answer the question, pointing out that it is impossible to state, either, what is meant by the letter *a*, as it is used in arithmetic; and if we were to say ‘*a* means a number,” this would be open to the same objection as the definition “one is a thing.” Now I the case of *a* it is quite right to decline to answer: *a* does not mean some one definite number which can be specified, but serves to express the generality of general propositions. If, in *a* + *a* – *a* = *a*, we put for *a* some number, any we please but the same throughout, we get always a true identity. This is the sense in which the letter *a* is used.

With one, however, the position is essentially different. Can we, in the identity 1 + 1 = 2 put for 1 in both places some one and the same object, say the Moon? On the contrary, it looks as though, whatever we put for the first 1, we must put something different for the second. Why it is that we have to do here precisely what would have been wrong in the other case?...”

“Questions like these catch even mathematicians for that matter, or most of them, unprepared with any satisfactory answer. Yet is it not a scandal that our science should be so unclear about the first and foremost among its objects, and one which is apparently so simple? Small hope, then, that we shall be able to say what number is. If a concept fundamental to a mighty science gives rise to difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome; especially as we shall hardly succeed in finally clearing up negative numbers, or fractional or complex numbers, so long as our insight into the foundation of the whole of arithmetic is still defective.”

According to Bertrand Russell, the reason why it took so long to discover a definition of natural numbers was that the natural numbers was thought of as the result of counting:

The philosophy of arithmetic was wrongly conceived by every writer before Frege. The mistake that all of them made was a very natural one. They thought of numbers as resulting from counting, and got into hopeless puzzles because things that are counted as one can equally well be counted as many. p. 53

Consider Frege’s *relativity argument* from his *Grundlagen* [1884]:

“What number is represented by a standard deck of cards?” Excluding jokers, there are 52 playing cards in a complete deck. However, aren’t there other numbers that the deck can represent? There are 13 ordinal card rankings—Aces, deuces, threes, fours, …., tens, Jacks, Queens, Kings. There are 4 suits of cards—hearts, spades, diamonds, and clubs. There are 2 decks one consisting of all the black cards and the other consisting of all the red cards. Or why can’t I simply say there is just 1 complete deck?

So what’s the point? Bertrand Russell explains:

It is obvious, therefore, that what makes anything one from a point of view of counting is not its physical constitution, but the question, ‘Of what is this an instance?’ The number that you arrive at by counting is the number of some collection, and that collection has whatever number it does have before you count it. It is only *qua* many instances of something that a collection is many. The collection itself will be an instance of something else, and *qua* instance counts as one in enumeration. We are thus forced to face the question, ‘What is a collection?’ and ‘What is an instance?’….

Let’s consider more carefully how children learn to count. One way to begin is by teaching the child begins to count the fingers on one hand with the numbers 1, 2, 3, 4, 5. At first the child may practice enumerating the fingers in exactly the same way, perhaps beginning with the pointer finger as 1, the middle finger as 2, and ending with the thumb as 5. (There are some interesting cultural differences in the way children are taught to count. For example, American children are taught to point their fingers while counting, whereas Chinese children are typically taught to close their fingers while counting.)

Now ask the child if you’ll get the same result if you count the fingers in a *different order*. Does it make a difference whether you count the fingers from left to right or right to left? Will you get to the same number if you *count* your fingers in any *order* so long as you don’t count the same finger twice?

Next the child is taught how to count a set of objects such as the following set of symbols:

⭘ 🞤 △ □ ⬟

One way to do this is to pair them one-to-one with the fingers on one hand. Here we have the beginnings of the distinction between *order* and *size*, between what mathematicians call the *ordinal* numbers (*first, second, third*, ….) and what they call the *cardinal* numbers (one, two, three, … ). The child discovers that if you can pair the symbols in a one-to-one fashion with the fingers on one hand, then there are the *same number* of fingers and as are symbols. And as we have already discovered there are five fingers on one hand. Hence, there are five symbols.

The child may then begin to recognize geometric properties of the symbols. The circle can be drawn with s single line. The cross is made using two line segments. The triangle has three sides. The square has four sides, and the pentagon has five sides. There is a natural way to pair the numbers with the shapes, but it turns out that no matter what order you count the shapes, so long as each shapes is counted once and only once, you will end up counting that there are five shapes.

From this simple example, we can see that numbers are used in two different ways—to list the symbols in some *order* and to describe the *size* of the set. This is the intuitive difference between *ordinals* (numbers used to *order* the collection) and *cardinals* (numbers used to assign a *size* to the collection). An *ordinal* number relates to a thing’s *position* in a series or sequence—whether it occurs first, second, or third in our counting sequence. A *cardinal* number, on the other hand, describes the *size* of a set. When we’re done counting or enumerating the members of the above set, the last *ordinal* number gives us the *cardinal* number of that set of objects.

According to Russell, when you say that a set has three elements, you are using “3” as an *adjective* to describe the set as having a certain property. In contrast, when you assert that 3 + 5 = 8, you are using “3” as a *noun* to refer to a mathematical object, “5” as a noun to refer to another mathematical object, and asserting that combining these two objects with the operation of addition results in another mathematical object, the number 8.

Must the arithmetic for ordinals obey that same laws as those for cardinals? Consider, for example, Henry VIII (1491 – 1547), who ruled England from 1509 until his death. Did we arrive at Henry VII by taking three Henrys and then adding five more:

3 + 5 = 8 .

No, Henry VIII is just one man (although he had six marriages)[[12]](#footnote-12). What we can say using this equation is that Henry VIII was the fifth king called Henry after Henry III. You can list your Henrys in the row: if you advance three steps along your list and then advance five more that the say at going from the first to the eighth position. When you think about it this way, it is not immediately evident that addition must be commutative. In fact, the arithmetic for infinite ordinals fails to follow the rules for finite ordinals.

Addition for ordinals α + β is the result of α followed by β then recounting. For example:

2 1 0 4 3 2 1 0

3 + 5 | | | | | | | | = 8

0 1 2 3 4 5 6 7

For finite numbers this operation is commutative:

4 3 2 1 0 2 1 0

5 + 3 | | | | | | | | = 8

0 1 2 3 4 5 6 7

However, this law does not hold for infinite ordinal numbers.

*Digression*: Can infinity be a number? According to David Hilbert’s formalist conception of mathematics, a mathematical object exists if we can construct a consistent formal system in which infinity is treated as a number. Hence, mathematical existence depends, in this sense, on logic.

Here’s how to construct an ordinal arithmetic. The first infinite ordinal is ω. It is the next ordinal after all of {0, 1, 2, 3, ….}. How can we represent this ordinal? One way is to picture them as an infinite array of telephone poles that appear shorter and shorter as they approach the horizon:



Let’s see what we get when we compare 1 + ω with ω + 1:

1 + ω = ω ≠ ω + 1

When we renumber the telephone poles on the left, we simply get ω again, but we get something structurally different on the right. As we shall see, Cantor’s arithmetic of ordinals and cardinals leads to some quite beautiful arithmetic truths:

ω + 1 ≠ 1 + ω, 2 × ω = 2ω < ω × 2 = ω + ω, = 1 + ,

= 1 + + …

End of disgression.

Kronecker famously said that God created the natural numbers, and mankind created the rest. If Kronecker was willing to leave the creation of the natural numbers to God, Frege was not. Frege was intent on taking the foundation project even further back. Frege’s idea is to construct a set-theoretical object that provably satisfies the Peano A*x*ioms. If this construction can be accomplished and the Peano Axioms derived as theorems, then all the arithmetical consequences of the Peano Axioms will have been deduced from, and reduced to, logic together with set theory.

The Italian mathematician Giuseppe Peano (1858 – 1932) advanced the algebraic period of mathematical logic through his axiomatization of the natural numbers including the method of mathematical induction (which are known, in his honor, as Peano’s postulates) through his *Formulario* Project (1891) and his in *Notations de Logique* (1894). Boole’s and Peano’s work, along with that of the American pragmatist C. S. Pierce (1839 – 1914) and the German mathematician and logician Ernst Schröder (1841 – 1902), advanced logic from is adolescent algebraic period to the more mature period of modern symbolic logic at the dawn of 20th century.

At the dawn of the 20th century in 1900, Bertrand Russell (1872 - 1970) attended the first International Congress of Philosophy in Paris where he became acquainted with the notation of Peano. In *My Philosophical Development* [1959] Russell writes about the revelation inspired Peano’s notation:

It was at the International Congress of Philosophy in Paris in the year 1900 that I became aware of the importance of logical reform for the philosophy of mathematics. It was through hearing discussion between Peano and Turin and the other assembled philosophers that I became aware of this. I had not previously known his work, but I was impressed by the fact that, in every discussion, he showed more precision and more logical rigour than was shown by anybody else. I went to him and said, ‘I wish to read all your works. Have you got copies with you?’ He had, and I immediately read them all. It was they that gave the impetus to my own views on the principles of mathematics….

The enlightenment that I derived from Peano came mainly from two purely technical advances of which it is very difficult to appreciate the importance unless one had (as I had) spent years in trying to understand arithmetic. Both these advances had been made at an earlier date by Frege, but I doubt whether Peano knew this.… [pp. 51-2]

*Peano’s Planets.*Once upon a time, there was a universe of planets called *Omega* ω. Each of the planets in *Omega* gives birth to another planet called the child of that planet. Symbolize the following planetary postulates using the dictionary provided:

P1. The *null planet* is in ω.

P2. Every planet in ω has a (unique) *child*.

P3. The null planet is not the *child* of any planet.

P4. No two planets have the same child.

P5. For every set of planets Σ, if

1. the null planet is in Σ and
2. for every planet in Σ its child is in Σ,

then every planet is in Σ (i.e., ω ⊆ Σ).

Dictionary:

0 : the null planet

ω : the universe of planets

*x* ∈ ω : *x* is a planet in Omega

S(*x*) : the child of *x*

*x* = *y*: *x* is identical as *y, i.e., x* and *y* are the same planet

*x* ≠ *y*: *x* is not identical to *y, i.e., x* and *y* are different planets

A ⊆ B: A is a subset of B

As you have probably guessed, these planetary postulates are Peano’s postulate for the natural numbers:

P1: 0 is a natural number

P2: If *x* is a natural number, so is the successor of *x*.

P3. For any natural number *x*, 0 is not the successor of *x*.

P4. No two distinct natural numbers have the same successor.

P5. *Principle of Mathematical Induction*: For any set Σ, if

1. 0 is a member of Σ,
2. for every natural number *x*, *x* is a member of Σ then the successor of *x* is a member of Σ,

then all natural numbers are in Σ.

Our next task is to derive the Peano’s Postulates as *theorems* of our Fregean foundational system. Here is a list of the Peano’s Postulates (we use “numbers” to mean “natural numbers”).

|  |  |  |
| --- | --- | --- |
| P1. Zero is a number. | 0 ∈ **N** | T122 |
| P2. The successor of a number is a number. | ∀*x*(*x* ∈ **N**→ S(*x*) ∈ **N**) | T123 |
| P3. Zero is not the successor of any number. | ∀*x*(*x* ∈ **N** → 0 ≠ S(*x*)) | T125 |
| P4. No two numbers have the same successors. | ∀*x*∀*y*[*x* ∈ **N**∧*y* ∈ **N**∧*x* ≠ *y* → S(*x*)≠S(*y*)] | T124 |
| P5. If zero has F and the successor of *x* has F whenever *x* has F, then all numbers have F. | [F0 ∧ ∀*x*(F*x*→FS(*x*))] → ∀*x*(*x* ∈ **N**→F*x*) | TS121 |

Peano’s postulates were first published in 1889, but an instructive discussion of the postulates was discovered in an 1890 letter by Dedekind. This letter was first translated and published in 1957[[13]](#footnote-13) by Hao Wang. Dedekind calls such anomalies as the Quixotic planets “alien intruders”:

The postulates “are still far from being adequate for completely characterizing the nature of the number sequence *N*… What, then, must we add to the facts above in order to cleanse our system … again of such alien intruders … as disturb every vestige of order and to restrict it to *N*?”

We can ask some questions about the kind of structure Peano’s postulates impose upon our universe.

1. Must Omega ω have an infinite number of planets?
2. According to ancient Omegan mythology, there is an alien planet Q (known as a *Quixotic* planet) that was *created after* (in symbols, >) the null planet or any of its *descendants* (we may define inductively the notion of a *descendant* of a planet to be either the child of that planet or any child of a descendant):

{Q > [0], Q > [1], Q > [2], Q > [3], …}

Is such a mythological *Quixotic* universe *consistent* with the Peano planetary postulates?

1. (Larson’s Question) Planets are *created before* (in symbols, α < β) and *created after* (in symbols, α > β) each other. Suppose we add a new postulate that says that any two planets in Omega are *comparable* with respect to creation: for any two planets α and β (in symbols, α ≠ β) either α was *created before* β or β was *created after* α. We can express comparability as follows:

α ≠ β → (α < β) ∨ (β > α) .

Does adding this postulate of *comparability* rule out the existence of *alien intruders*? (These alien intruders have the form of Z-chains or *n*-cycles).

Frege’s goal is to logically construct the natural numbers in set theory so that the Peano postulates are derivable as theorems. Kronecker famously said that God created the natural numbers, and man created the rest. If Kronecker was willing to leave the creation of the natural numbers to God, Frege was not. Frege was intent on taking the foundation projects even further back. Frege’s idea is to construct a set-theoretical object that provably satisfies the Peano A*x*ioms. If this construction can be accomplished and the Peano Axioms derived as theorems, then all the arithmetical consequences of the Peano Axioms will have been deduced from, and reduced to, logic together with set theory.

***Definition***: {ξ} := {*x* | *x* = ξ}, where ξ is any term and *x* is a variable not occurring in ξ.

(The *singleton* set {ξ} is defined as a set consisting of ξ alone.)

Next we define *zero* to be the empty set. Here rather than follow Frege’s and Russell’s logical construction, which defines the number 3 to be the *class* of all sets which have three elements, we shall follow von Neumann’s idea of defined the number 3 to be a particular three element *set*. It is more elegant to have numbers as sets within our Fregean foundational system.

***Definition***. 0 := ∅

Next we define the *successor*of a set α (in symbols, α+) to be the set consisting of α ∪ {α}.

***Definition***. For any set α, S(α) = α ∪ {α}.

Alonzo Church was one of the great logicians of the 20th century. When Church used to count the number of handouts to pass down a row with seven students, he could say, “zero, one, two, three, four, five, six and that makes seven” and then put down seven pieces of paper. When you count Church’s way, then the *number* of objects you’ve counted is the earliest number that you *didn’t* use. In Church’s system when you count an empty set of things you arrive at the number zero since it is the first number you didn’t use (since you didn’t use any). Although this procedure might seem odd, it is really very natural for some purposes. If you order 1000 lottery tickets, there usually numbered from 000 to 999.

A physicist, a biologist and a mathematician are sitting in a street café watching people entering and leaving the house on the other side of the street. First they see two people entering the house. Time passes. After a while they notice three people leaving the house. The physicist says, "The measurement wasn't accurate." The biologist says, "They must have reproduced." The mathematician says, "If one more person enters the house then it will be empty."[[14]](#footnote-14)

Many logicians prefer to count objects starting with zero rather than one since that is the natural way to logically construct the ordinal numbers. Von Neumann’s elegant idea was that each ordinal can be defined as the well-ordered set of all previous ordinals.

|  |  |  |
| --- | --- | --- |
| 0 |  | = Ø |
| 1 | = { 0 } | = {Ø} |
| 2 | = { 0, 1 } | = { Ø, {Ø} } |
| 3 | = { 0, 1, 2 } | = { Ø, {Ø} , {Ø, {Ø}} } |
| 4 | = { 0, 1, 2, 3 } | = { Ø, {Ø}, {Ø, {Ø}}, {Ø, {Ø}, {Ø, {Ø}}} } |

Unlike Zermelo’s construction in which 3 = {{{0}}}, von Neumann’s idea generalized to infinite ordinals.

We define an*inductive**set* to be a set that contains 0 and the successors of all its members.

***Definition***. I*y* ↔ [0 ∈ *y* ∧ ∀*z*(*z* ∈ *y* → S(*z*) ∈ *y*)], where *y* and *z* are distinct variables.

Next we defined the “natural numbers” to be in the intersection of every inductive set.

***Definition***. **N** := {*x*| ∀*y*(I*y* → *x* ∈ *y*)}.

V is an inductive set. To prove that zero is a natural number, we show that zero is a member of every inductive set, which is obvious by construction.

***Theorem*** 6. 0 ∈ **N** (Peano Postulate P1.)

1. *~~Show~~* 0 ∈ **N** 13, DD

2. ∀*x* [*x*  **N** ↔ *x*  {*x*|∀*y*(I*y* → *x* ∈ *y*)}] Abstraction, Def. **N**  
3. 0 ∈ **N** ↔ 0 ∈ {*x*|∀*y*(I*y* → *x* ∈ *y*)} 2, UI

4. *~~Show~~* 0 ∈ {*x*|∀*y*(I*y* → *x* ∈ *y*)} 12, DD

5. 0 ∈{*x*|∀*y*(I*y* → *x* ∈ *y*)} ↔ ∀*y*(I*y* → 0 ∈ *y*) Abstraction

6. *~~Show~~* ∀*y*(I*y* → 0 ∈ *y*) 7, UD

7. *~~Show~~* I*y* → 0 ∈ *y* 11, CD

8. I*y* Assume (CD)

9*.* I*y* ↔ 0 ∈ *y* ∧ ∀*z*(*z* ∈ *y* → S(*z*) ∈ *y*) Def, Inductive Sets

10. 0 ∈ *y* ∧ ∀*z*(*z* ∈ *y* → S(*z*) ∈ *y*) 9, BC, 8 MP

11. 0 ∈ *y* 10, S

12. 0 ∈{*x*|∀*y*(I*y* → *x* ∈ *y*)} 5, BC, 6 MP

13. 0 ∈ **N** 2, BC, 4 MP

Next we prove that the successor of a natural number is a natural number, which is also obvious.

***Theorem*** 7. ∀*x*(*x* ∈ **N**→ S(*x*) ∈ **N**) (Peano Postulate P2)

1. *~~Show~~* ∀*x*(*x*  **N**→ S(*x*)  **N**) 2, UD

2. *~~Show~~* *x* ∈ **N** → S(*x*)  **N** 20, CD

3. *x*  **N** Assume (CD)

4. ∀*x* [*x*  **N** ↔ *x*  {*x*|∀*y*(I*y* → *x* ∈ *y*)}] Abstraction, Def. **N**  
5. *x* ∈ **N** ↔ *x* ∈ {*x*|∀*y*(I*y* → *x* ∈ *y*)} 4, UI

6. *x* ∈ {*x*|∀*y*(I*y* → *x* ∈ *y*)} 5, BC, 3, MP

7. *x* ∈ {*x*|∀*y*(I*y* → *x* ∈ *y*)} ↔ ∀*y*(I*y* → *x* ∈ *y*) Abstraction

8. ∀*y*(I*y* → *x*  *y*) 7, BC, 6, MP

9. S(*x*)  **N**↔ ∀*y*(I*y* → S(*x*) ∈ *y*) 4, UI

10. *~~Show~~* ∀*y*(I*y* → S(*x*)  *y*) 11, UD

11. *~~Show~~* I*y* → S(*x*)  *y* 19, CD

12. I*y* Assume (CD)

13. I*y* → *x*  *y* 8, UI

14. *x*  *y* 13, 12 MP

15. I*y* ↔ [0 ∈ *y* ∧ ∀*z*(*z* ∈ *y* → S(*z*) ∈ *y*)] Def. I*y*

16. 0  *y* ∧ ∀*z*(*z*  *y* → S(*z*)  *y*) 15, BC, 12 MP

17. ∀*z*(*z*  *y* → S(*z*)  *y*) 16, S

18. *x*  *y* → S(*x*) *y* 17, UI

19. S(*x*)  *y* 18, 14, MP

20. S(*x*)  **N** 9, BC, 10 MP

Lemma: ∀*x*(*x* ∈ **N ∧** 0≠ *x* → 0 ∈ *x*)

The third Peano Postulate states that zero is not a successor. Suppose zero were the successor of some natural number *x*. Then zero would have *x* as a member and so would not be empty.

***Theorem*** 8. ∀*x*(*x*  **N** → 0 ≠ S(*x*)) (Peano Postulate P3)

1. *~~Show~~* ∀*x*(*x*  **N** → 0 ≠ S(*x*)) 2, UD

2. *~~Show~~* *x*  **N** → 0 ≠ S(*x*) 4, CD

3. *x*  **N** Assume (CD)

4. *~~Show~~* 0 ≠ S(*x*) 14, 17 ID

5. 0 = S(*x*) Assume (ID)

6. 0 = S(*x*) ↔ ∀*y*(*y* ∈ 0 ↔ *y* ∈ S(*x*)) Extensionality

7. ∀*y*(*y* ∈ 0 ↔ *y* ∈ S(*x*)) 6,BC, 5, MP

8. 0  0 ↔ 0 ∈ S(*x*) 7, UI

9. 0 ∉ 0 Thm 3, UI

10. 0 ∉ S(*x*) 8, BC, 9, MT

11. 0 ∉ *x* ∪ {*x*} 10, Def. *x*+

12. ~(0  *x* ∨ 0 = *x*) 11, Def. ∪

13. 0 ∉ *x* ∧ 0 ≠ *x* 12, DM

14. 0 ∉ *x* 13, S

15. ∀*x*(*x* ∈ **N ∧** 0≠ *x* → 0 ∈ *x*) Lemma

16. *x* ∈ **N ∧** 0≠ *x* → 0 ∈ *x* 15, UI

17. 0 ∈ *x* 3, 13, S, ADJ, MP

Our next theorem is the Peano Postulate stating the schema for mathematical deduction: if zero has the property F and whenever a natural number has the property F its successor has the property F, then all natural numbers have the property F. This property of the natural numbers is also obvious given that the natural numbers are a member of every inductive set. The von Neumann construction makes it easy to define < as set membership which allow us to state a strong form of induction. Strong induction is equivalent to the Least Number Principle and Fermat’s Method of Proof by Infinite Descent:

Mathematical Induction Chart.pdf

TS121 (Weak) Mathematical Induction: [F0 ∧ ∀*x*(F*x* → FS(*x*))] → ∀*x*(*x*  **N** → F*x*)

TS121 Strong Mathematical Induction: ∀*x*[*x*  **N** ∧ ∀*y* [*y*  **N** ∧ *y* < *x* → F(*y*)] → F(*x*)] → ∀*x*[*x*  **N** → F*x* ]

TS119 Least Number Principle: ∃*x*[F(*x*) ∧ ∀*y*[F(*y*) → N(*y*)] → ∃*z*(F(*z*) ∧ ∀*y*[F(*y*) → *y* < *z*])

Fermat’s Proof by Infinite Descent: ∀*x*[N(*x*) ∧ F(*x*) → ∃*y*[N(*y*) ∧ *y* < *x* ∧ F(*y*)] → ∀*x*[N(*x*) → ~F(*x*)]

Exercise: Show the equivalence of all these variations of the principle of induction.

***Theorem*** 10. [F0 ∧ ∀*x*(F*x* → FS(*x*))] → ∀*x*(*x*  **N** → F*x*) (P5 Mathematical Induction)

1. *~~Show~~* F0 ∧ ∀*x*(F*x* → FS(*x*)) → ∀*x*(*x*  **N** → F*x*) 5, CD

2. F0 ∧ ∀*x*(F*x* → FS(*x*)) Assume (CD)

3. F0 2, S

4. ∀*x*(F*x* → FS(*x*)) 2, S

5. *~~Show~~* ∀*x*(*x*  **N** → F*x*) 6, UD

6. *~~Show~~* *x*  **N** → F*x* 30, CD

7. *x*  **N** Assume(CD)

8. *x*  **N** ↔ ∀*y*(I*y* → *x*  *y*) Def. **N**

9.∀*y*(I*y* → *x*  *y*) 8, BC, 7 MP

10. *~~Show~~* F*x* 30, DD

11. *~~Show~~* I{*x*|F*x*} 26, DD

12. I{*x*|F*x*} ↔ [0 ∈ {*x*|F*x*} ∧ ∀*z*(*z* ∈ {*x*|F*x*} → S(*z*) ∈ {*x*|F*x*})] Def. I*y*

13. *~~Show~~* 0 ∈ {*x*|F*x*} 15, DD

14. F0 ↔ 0 ∈ {*x*|F*x*} Abstraction

15. 0 ∈ {*x*|F*x*} 14, BC, 3, MP

16. *~~Show~~* ∀*z*(*z* ∈ {*x*|F*x*} → S(*z*) ∈ {*x*|F*x*}) 17, UD

17. *~~Show~~* *z*  {*x*|F*x*} → S(*z*)  {*x*|F*x*} 24, CD

18. *z*  {*x*|F*x*} Assume (CD)

19. *z*  {*x*|F*x*} ↔ F*z* Abstraction

20. F*z* 19, BC, 18, MP

21. F*z* → FS(*z*) 4, UI

22. FS(*z*) 20, 21 MP

23. S(*z*)  {*x*|F*x*} ↔ FS(*z*) Abstraction*,* UI

24. S(*z*)  {*x*|F*x*} 23, BC, 22, MP

25. 0 ∈ {*x*|F*x*}  ∀*z*(*z* ∈ {*x*|F*x*} → S(*z*) ∈ {*x*|F*x*}) 13, 16, ADJ

26. I{*x*|F*x*} 12, BC, 25, MP

27. I{*x*|F*x*} → *x*  {*x*|F*x*} 9, UI

28. *x*  {*x*|F*x*} 27, 11 MP

29. *x*  {*x*|F*x*} ↔ F*x* Abstraction, UI

30. F*x* 29, BC, 28, MP

Exercise. Prove Peano Postulate IV, which states that no two natural numbers have the same successors.

***Theorem*** 9. ∀*x*∀*y*(*x* ∈ **N** ∧ *y* ∈ **N** ∧ *x* ≠ *y* → S(*x*) ≠ S(*y*)).

This theorem is equivalent to

∀*x*∀*y*(*x* ∈ **N** ∧ *y* ∈ **N** ∧ S(*x*) = S(*y*) → *x* = *y*).

***Definition***. A set or class A is *transitive* if every element of A is itself a class of elements of A

∀*x*∀*y*(*x* ∈ *y* ∧ *y* ∈ A → *x* ∈ A).

***Theorem***: The following conditions are equivalent:

1. ∪α ⊆ α
2. α ⊆ P(α)
3. α is transitive

Section 5. Numerals, Finite and Infinite Sets

The standard *numerals* can be introduced as successive applications the successor operation to zero. Here [*n*] is the numeral corresponding to the number *n*, and S*n* is an abbreviation for applying the successor operation *n* times.

***Definition***. [1] := S(0); [2] := S(1), [3] := S(2); [4] := S(3); [5] := S(4), …, [*n*] := S*n*(0)

We can now define finite and infinite sets. A set is finite if and only if it is an element of a natural number, and the set is infinite if and only if it is not finite.

***Definition***. Fin(α) := ∃*x*(*x* ∈ **N** ∧ α ∈ *x*), where α is any term in which the variable *x* does not occur.

***Definition***. Inf(α) := ~Fin(α), where α is any term.

What can be said for our Fregean foundational system that we have developed thus far? First, our foundational system has been expressed as a formal system and it has been adequate to prove a reasonably large portion of mathematics. We can subsume both Aristotle’s syllogistic logic and Boolean algebra. Secondly, the foundation has been derives from some intuitively natural principles. It seems intuitively true that a property can be used to define the set of all things having the given property, so long as that property is restricted to those that can be expressed in our formal language. Thirdly, our basic principles have been economical. We have only two basic postulates for set theory and our underlying system of natural deduction has allowed us to derive relatively complex theorems in a systematic and orderly manner.

There is however a fundamental problem: our Fregean foundational system is quite inconsistent.

## 

Section 6. Russell’s Paradox

Russell's paradox, perhaps the most famous of all the set-theoretical paradoxes, destroyed Frege’s dream of completing his logicist project. And it couldn’t have come as a more disappointing time. Just as the second volume of Frege’s *Gründgesetze* was in the press, Frege received a devastating letter from Bertrand Russell in which he derived a contradiction from his Basic Law V, a principle from which our Fregean abstraction principle can be derived.

Basic Law V: {*x*|*f*(*x*)} = {*x*|*g*(*x*)} ↔ ∀*x*(*f*(*x*) ↔ *g*(*x*))

Facing with the inconsistency posed by Russell’s paradox for his foundational system, Frege reacted with unflinching scientific integrity:

Frege’s conception of his work in logic is most grandly and eloquently expressed in an essay written at the end of academic career, “*The Thought: A Logical Inquiry*” (1918), in which he talks about the science of logic, not aiming at the laws of *thought*, but at the laws of *truth*. This vision is expressed in the quotation that introduces this chapter.

Although expressed eloquently, Frege’s hopes for his logicist project of reducing arithmetic to logic had foundered on Russell’s famous paradox. On the eve of the publication of the second volume of his *Grundgesetze* [1903], Frege received a fateful letter from Russell, which posed his famous paradox. Frege immediately reproduced Russell’s paradox in his notation and then hastily composed an appendix to his life’s work. It is clear that Frege had the intellectual honesty to acknowledge that Russell’s paradox threatened the consistency, and hence the validity and value, of his entire life’s work. Here are Frege’s words:

Hardly anything more unwelcome can befall a scientific writer than that one of the foundations of his edifice be shaken after the work is finished. I have been placed in this position by a letter of Mr. Bertrand Russell just as the printing of this [second] volume was nearing completion. It is a matter of my Basic Law V [i.e., the abstraction principle]. I have never concealed from myself its lack of the self-evidence which the others possess, and which must properly be demanded of a law of logic.

Frege died thinking that his life project had ended in failure. Anthony Kenny compares Frege’s career to that of Christopher Columbus.

Just as Columbus failed in his project of discovering a westward passage to India, but unknowingly make Europe acquainted with a whole new continent, so Frege failed in his task of deriving arithmetic from logic but made discoveries in logic and advances in philosophy which permanently changed the whole map of both subjects. Like Columbus, Frege failed to appreciate the full value of his own genuine discoveries, and he suffered discouragement and depression in consequence.

Frege devoted the best years of his life to proving the continuity between arithmetic and logic. All the other innovations of his fertile mind were intended to serve that overarching purpose and when the project aborted it was natural that he should be tempted to think that his life’s work had been a failure.[[15]](#footnote-15)

Frege’s tried, hastily, to try to repair is system, but his repairs failed. Why was the derivation of Russell’s paradox so devastating? As the Cambridge mathematician G. H. Hardy wrote in *A Mathematician’s Apology* [1967], a classic defense of the mathematician’s and logician’s life’s work, reasoning by *reductio ad absurdum* is a dangerous game to play.

“The proof [of the infinity of primes] is by *reductio ad absurdum*, and *reductio ad absurdum*, which Euclid loved so much, is one of the mathematician’s finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.”

The mathematician “offers the game” because a contradiction, in classical logic, allows anything whatsoever to be deduced. The derivation of Russell’s paradox, a contradiction, threatened the entire foundation for Frege’s mathematical work.

When asked to donate his correspondence with Frege to the now classic source book in mathematical logic *From Frege to Gödel* edited by Jean van Heijenoort, Russell wrote:

Dear Professor van Heijenoort,

I should be most pleased if you would publish the correspondence between Frege and myself, and I am grateful to you for suggesting this. As I think about acts of integrity and grace, I realize that there is nothing in my knowledge to compare with Frege’s dedication to the truth. His entire life’s work was on the verge of completion, much of his work has been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

Yours sincerely,

Bertrand Russell

What is Russell’s paradox? It is customary to introduce the paradox with logically similar pseudo-paradox of the barber. In the village of Alcala, there is a barber who shave all the only those who do not shave themselves. Who shaves the barber?

If suppose the barber shaves himself (or herself), then s/he does not *only* those who do not shave themselves. If we suppose the barber does not shave him or herself, then s/he does not shave *all* those who do not shave themselves. We can conclude there can be no such barber.

Russell’s paradox, unlike the pseudo-paradox of the barber, is considered to be an *antinomy* (from *anti* (against) + *nomos* (law)) because the existence of the Russell set seems to be guaranteed by intuitive laws for the existence of sets. As Gödel wrote in his essay on “Russell’s Mathematical Logic” [1944], the significance of Russell’s paradox is this:

“By analyzing the paradoxes to which Cantor’s set theory had led, he freed them from all mathematical technicalities, thus bringing to light the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as: truth, concept, being, class, etc. are self-contradictory.” (p. 452, Benacerraf & Putnam)

Russell’s paradox can be intuitively sketched as follows. As we have seen, some set contain themselves as elements, such as the universal set V. However, most sets do not contain themselves as elements, e.g., such as the set of natural numbers is not a natural number and so is not a member of itself. Color all the sets that do contain themselves as members green and all the sets that do not contain themselves as members red. Now consider the collection R of all and only red sets. If R colored red or green? If it is colored red, then since R contains all the red sets, R is a member of R and so must be colored green. However, if R is colored green, then it must be a member of itself, but then it must be red because R contains only red sets.

Here are the formal derivations in our Fregean foundational system.

***Theorem*** 11. {*x*| *x* ∉ *x*} ∉ {*x* | *x* ∉ *x*} (i.e., R = {*x*| *x* ∉ *x*} is not a member of itself).

1. *~~Show~~* {*x*| *x* ∉ *x*} ∉ {*x* | *x* ∉ *x*} 2, 4 ID

2. {*x*| *x* ∉ *x*} ∈ {*x*| *x*∉*x*} Assume (ID)

3. {*x*| *x* ∉ *x* }∈ {*x*|*x*∉*x*} ↔ {*x*|*x* ∉ *x*} ∉ {*x*|*x*∉*x*} Abstraction

4. {*x*| *x* ∉ *x*} ∉ {*x*| *x*∉*x*} 3, BC, 2, MP

***Theorem*** 12. {*x*|*x* ∉ *x*} ∈ {*x*| *x*∉*x*} (i.e., R = {*x*| *x* ∉ *x*} is a member of itself). We leave as an exercise.

Frege’s foundation for arithmetic was based on the faulty intuition was that for any property, there is the corresponding set of all and only those things that have the property. This is called the *naïve comprehension axiom* (naïve because we now know it to be too simple to be true).

We can express the logic that generates Russell’s paradox as well as Russell’s pseudo-paradox of the village barber of Alcala who shaves all and only those in the village who don’t shaves themselves in a theorem known as Russell’s Law.

T269 ~∃*y*∀*x*(R*xy* ↔ ~R*xx*)

For various interpretation of the relation R, we obtain a variety of paradoxes. When R is interpreted as “*x* shaves *y*” we obtain the Barber Paradox, when it is interpreted as “*x* ∈ *y*” we obtain Russell’s paradox, and when it is interpreted as “*x* is true of *y*” we obtain the heterological paradox.

We can even formulate a second-order version of Russell’s Law as follows:

T275'    ~∃F[F*y* ∧ ∀*x*(F*x*→ [R*xy* ↔ ~R*xx*])]

This generalization proves the non-existence of any subset formed using the Russell property and the *Aussorderungs* Axiom or Axiom of Specification which treatz V as a second-order variable for a universal derivation.

Paul Halmos in his classic *Naive Set Theory* (New York: Van Nostrand, 1960, pp. 6‐7) states the point of Russell’s paradox with flair:

We have proved, in other words, that *nothing contains everything,* or, more spectacularly,

*there is no universe* .

“Universe” here is used in the sense of “universe of discourse,” meaning in any particular discussion, a set that contains all the objects that enter into that discussion.

In older (pre‐axiomatic) approaches to set theory, the existence of a universe was taken for granted, and the argument in the preceding paragraph was known as Russell’s paradox. The moral is that it is impossible, especially in mathematics, to get something for nothing. To specify a set, it is not enough to pronounce some magic words (which may form a sentence such as “*x* ∉ *x*”): it is necessary also to have at hand a set to whose elements the magic words apply.

Halmos’s moral can be summarized in a theorem:

T270

This theorem states that if for any set, there is a Russellian set defined on that set, then there is no universal set.

An indirect disproof of the existence of a universal set V = {*x*| *x* = *x* } is as follow. Suppose V exists, then y the *Aussorderungs* Axiom, we have the set

R = {*x* ∈ V| *x* ∉ *x* } .

Then by the Axiom of Extensionality, we have that

∀*x* (*x* ∈ R ↔ *x* ∈ V ∧ *x* ∉ *x*) ,

from which we may derive Russell’s Paradox.

1. *~~Show~~* V = {*x*: *x* = *x*} does not exist. 5, 12 ID

2. V = {*x*| *x* = *x* } exists Assume (ID)

3. R = {*x* ∈ V| *x* ∉ *x*} exists Axiom of Specification

4. ∀*x* (*x* ∈ R ↔ *x* ∈ V ∧ *x* ∉ *x*) Axiom of Extensionality

5. *~~Show~~* R ∉ R 6, 8 ID

6. R ∈ R Assume (ID)

7. R ∈ V ∧ R ∉ R 4, UI, BC, 6, MP

8. R∉R 7 S

9. R = R Aristotle’s Law of Identity

10. R ∈ V 9, Axiom of Extensionality, BC, 9, MP

11. R ∈ V ∧ R ∉ R 10, 5, ADJ

12. R ∈ R 11, Axiom of Specification

Almost every conceivable step of Russell’s paradox has been questioned. Here is a representative list of five ways out of the contradiction.

1. *Revising Syntax*: Russell’s Simple Theory of Types assigns types to all the terms of the language. It makes expressions of the form

*x* ∈ *x*

not grammatically well-formed unless the type of the variable on the right is one higher than the type of the variable on the left:

*xn* ∈ *xn+1*

This restriction on the syntax makes it impossible to *formulate* Russell’s paradox, but does it *explain* the paradox?

1. *Revising Classical Logic*: Another tact is to cripple the logic. *Intuitionistic logic*, for example, restricts the structural rules of classical logic so that it is not possible to derive a contradiction using the two conditionals:

*x* ∈ *x* → *x* ∉ *x*

*x* ∉ *x* → *x* ∈ *x* .

This restriction on the logic of paradox makes it impossible to *deducee* Russell’s paradox, but does it *explain* the paradox? Another approach is to adopt a *relevance* logic in which a contradiction does not imply everything and hence produces a *para-consistent logic*. Does this approach satisfactorily explain away the paradox?

1. *Revising Postulates*: The most commonly adopted strategy for avoiding contraction is to replace the unrestricted, or naïve, comprehension action with a restricted *Separation* or *Aussonderungs* Axiom:

∀*y* ∃*z* ∀*x* [*x* ∈ *z* ↔ *x* ∈ *y* ∧ ϕ(*x*)] .

Given a previously existing set *y* and any first-order predicate expression ϕ(*x*) with the free variable *x*, we may “separate out” a subset *z* of *y* that consists of all and only those members *x* of *y* that satisfy ϕ(*x*). This approach requires specifying separate axioms for the existence of the empty set, a set of pairs, the union of set, etc.

1. *Drawing a Distinction*: Related to the above strategy is the Cantorian or Gödel-Bernays-von-Neumann set theory, in which there is a dual conception of set that distinguishes *consistent* and *inconsistent* multiplicities (or, alternatively, distinguishes *sets* from *classes*). On the iterative conception of set, in contrast to the Fregean conception of a set as the extension of a predicate, sets are constructed in stages based on the power set operation applied to a set at the previous iteration. Intuitively, the inconsistent multiplicities are those collections that are “too big” to be set in the sense that they have members from arbitrarily high level or iterations.
2. *Second-Order Arithmetic with Hume’s Principle*: In the 1970’s, the logician George Boolos and his students (i.e., Richard Heck) discovered that Frege’s inconsistent Basic Law V (which implied the unrestricted comprehension axioms) could be replaced with a weaker principle which states that the number of Fs is the same as the number of Gs if, and only if, the Fs can be put into one-to-one correspondence with the Gs.

(V*b*) ∀F∀G(#F = #G ↔∀*x*(F*x* ↔ G*x*)) ,

which they called *Hume’s principle* insofar as Frege cited a passage from Hume’s *Treatise* (Book I, Part III, Section I) in support of the principle:

(HP) ∀F∀G(#F = #G ↔ F ≈ G) .

It turns out that Hume’s principle is *consistent* assuming second order-arithmetic is consistent and this principle is sufficient for the derivation of arithmetic while avoiding the inconsistent Basic Law V. They derive “Frege’s Theorem”, the vindication of a Fregean development of arithmetic using Hume’s Principle.

This was a surprising, and deeply satisfying, reversal of Frege’s honest, but noble, logicist despair. Although he did not live to see his dream fulfilled, Frege was, in the end, vindicated by the history of logic. Subsequent generations of logic students, inspired by Frege’s visionary dream, eventually completed Frege’s life work by rigorously providing the logicist foundations of arithmetic. Due to his logical vision, Frege was forced to face the possibility that his life’s work was nothing but a failed dream. ON the other hand, without such dreams, no dream comes true.

In a subsequent chapter, we will return to Russell’s paradox and show it can be discovered from the proof of Cantor’s theorem. We will discuss the profound development of Zermelo-Fraenkel axiomatic set, and Gödel’s philosophical justification of the axioms based on the Iterative Notion of Cantorian Set Theory. We will address such questions as the following:

* What is the difference between Frege’s and Cantor’s conception of set? Is there a way to justify the Zermelo-Fraenkel-Skolem axioms for set theory as articulate a coherence conception of set?
* Are questions such as Cantor’s Continuum Hypothesis about an objectively existing reality or is set theory the result of human construction?
* What is Aczel Set Theory with the Anti-Foundation Axiom and why is it useful?

1. Van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Third Printing, [1977], 7. [↑](#footnote-ref-1)
2. Van Benthem, *Modal Logic for Open Minds* [2010], 1. [↑](#footnote-ref-2)
3. At UCLA David Kaplan had proposed a project for an independent study in which I formalized Frege’s *Grundgesetze*, which had been translated by Montgomery Furth (1933 - 1991). All first-year students had to take a four-hour Master’s Comprehensive Examination in Logic at the end of their first year. This exam was a “washout” exam and even a graduate student who had studied logic at U. C. Berkeley with Robert Vaught had failed the exam. When it came time for my class to take the exam, I was talking graduate courses in model theory from C. C. Chang, and so had permission to arrive late. When I walked into the Philosophy Department Reading Room, I saw the other graduate students in my cohort sweating out the exam. Typically, there was always a question about Russell’s paradox, and so I had toyed with the idea of reproducing Russell’s paradox in Frege’s original notation. I came an hour late and left an hour early. [↑](#footnote-ref-3)
4. Jan von Plato, “Generality and Existence: Quantificational Logic in Historical Perspective”, *The Bulletin of Symbolic Logic*, vol. 0, no. 4, Dec. 2014, p. 418. Von Plato notes that Van Heijenoort’s edition missed Frege’s emphasis translating Frege’s *Auch ist einleuchtend* by “it is clear also”. [↑](#footnote-ref-4)
5. Attributed to Frege in: A. A. B. Aspeitia (2000), *Mathematics as grammar: 'Grammar' in Wittgenstein's philosophy of mathematics during the Middle Period*, Indiana University, p. 25. [↑](#footnote-ref-5)
6. Reference [↑](#footnote-ref-6)
7. Today Frege is regarded, along with Bertrand Russell, as one of the founding fathers of the analytic tradition in philosophy. Clark Glymour, in *Thinking Things Through: An Introduction to Philosophical Issues and Achievements* (MIT, 1992) pays tribute to Frege: “Despite the obscurity of his writings during his lifetime, Frege’s efforts opened the way to much of modern mathematics, modern logic, modern approaches to topics in economics, and the modern theory of computation. Many people (including me) think that most work of value in twentieth-century philosophy is in some way indebted to Frege’s ideas…. Frege stands to logic roughly as Newton stands to physics” (p. 120)*.* [↑](#footnote-ref-7)
8. Gödel in “Russell’s Mathematical Logic” [1947] notes: “It is to be regretted that this first comprehensive and thorough going presentation of a mathematical logic and the derivation of Mathematics from it is so greatly lacking in formal precision in the foundation (contained in \*1 - \*21 of *Principia*), that it presents in this respect a considerable step backwards as compared with Frege.” Reprinted in Benacerraf & Putnam [1983], p. 448. [↑](#footnote-ref-8)
9. According to Greek mythology, Laelaps (Greek: Λαῖλαψ), the magical hound who never failed to catch her prey, was a gift from Zeus to Europe, and the Teumessian fox, the magical fox who was destined never to be caught, was sent by Dionysus to prey upon the children of Thebes as a punishment for the city’s crime. The Regent of Thebes Creon assigned Amphitryon the seemingly impossible task of destroying the Teumessian fox. Amphitryon decided to deploy Laelaps to catch Teumessian fox. The endless chase went on until Zeus, perplexed by their contradictory fates, decided to turn them both into stone and then cast them into the heavens as the constellations *Canis Major* (Laelaps) and *Canis Minor* (the Teumessian fox).

   The origin of the Chinese word for contradiction originates in a puzzle in the philosophical writings of Han Feizi (3rd century BCE) about a man trying to sell a spear and shield. Touting the virtues of the spear, the seller claimed it could pierce any shield, while touting the virtues of the shield claimed it could fen off all spears. When he is asked what would happen when the shield was struck by the spear, the seller had no answer being caught in a self-contradiction. The Chinese word for self-contradiction 矛盾, or in pinyin máodùn, literally means "Spear-Shield." [↑](#footnote-ref-9)
10. Stephen Wolfram [2016], *Idea Makers: Personal Perspectives in the Lives and Ideas of Some Notable People*, p. 43 notes that the mention of Boole increased dramatically from the 1980s – 2000. [↑](#footnote-ref-10)
11. The contemporary emphasis on STEM (Science, Technology, Engineering and Mathematics) to the detriment of the humanities is a misplaced emphasis in the opposite direction. What use is a stem without flowers? One part of that flower of the humanities is the PETAL of philosophy: Philosophy, Epistemology, *Techne*, Aesthetics, and Logic. [↑](#footnote-ref-11)
12. Henry VII is known for his efforts to have his first marriage to Catherine of Aragon annulled, which resulted in a disagreement with the Pope that led to separating the Church of England from papal authority. [↑](#footnote-ref-12)
13. *Journal of Symbolic Logic*, vol. 22 [1957], 145-157, reprinted in van Heijenoort [Third edition, 1977], 89-103. [↑](#footnote-ref-13)
14. Three logicians walk into a bar. The bartender asks, “Does everyone want a beer?” The first two logicians correctly answer, “I don’t know.” The third logician then correctly answers the bartender’s question based on *deduction* from this information. What did she answer? [↑](#footnote-ref-14)
15. Anthony Kenny, *Frege: An Introduction to the Founder of Analytic Philosophy* (Penguin Books, 1995), p. 207. Kenny’s heroic assessment of Frege is tempered by Hilary Putnam’s sober comments in “Peirce the Logician,” *Historia Mathematica* (1982. vol. 9, pp. 290-301, reprinted in *Realism with a Human Face* (Harvard University Press: 1990), pp. 252-60) which register some resistance to Frege’s domineering role compared to C. S. Peirce and Schröder in historical discussions of logic:

    “When I started to trace the later development of logic, the first thing I did was to look at Schröder's *Vorlesungen über die Algebra der Logik*, ...[whose] third volume is on the logic of relations (*Algebra und Logik der Relative*, 1895). The three volumes immediately became the best-known advanced logic text, and embody what any mathematician interested in the study of logic should have known, or at least have been acquainted with, in the 1890s.

    “While, to my knowledge, no one except Frege ever published a single paper in Frege's notation, many famous logicians adopted Peirce-Schröder notation, and famous results and systems were published in it. Löwenheim stated and proved the Löwenheim theorem (later reproved and strengthened by Thoralf Skolem, whose name became attached to it together with Löwenheim’s) in Peircian notation. In fact, there is no reference in Löwenheim's paper to any logic other than Peirce's. To cite another example, Zermelo presented his axioms for set theory in Peirce-Schröder notation, and not, as one might have expected, in Russell-Whitehead notation.

    “One can sum up these simple facts (which anyone can quickly verify) as follows: Frege certainly discovered the quantifier first (four years before Oscar Howard Mitchell, going by publication dates, which are all we have as far as I know). But Leif Ericson probably discovered America “first” (forgive me for not counting the native Americans, who of course really discovered it “first”). If the effective discoverer, from a European point of view, is Christopher Columbus, that is because he discovered it so that it stayed discovered (by Europeans, that is), so that the discovery became known (by Europeans). Frege did “discover” the quantifier in the sense of having the rightful claim to priority; but Peirce and his students discovered it in the effective sense. The fact is that until Russell appreciated what he had done, Frege was relatively obscure, and it was Peirce who seems to have been known to the entire world logical community. How many of the people who think that “Frege invented logic” are aware of these facts?” [↑](#footnote-ref-15)