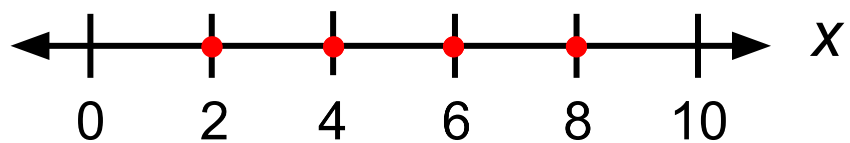
Discrete and Continuous Probability Distributions

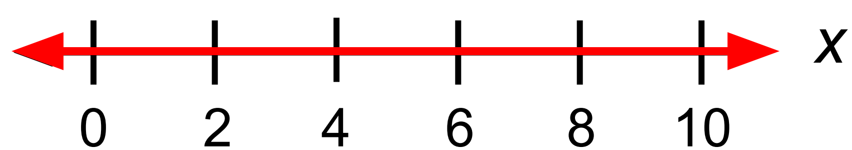
Random Variables

A **random variable *x*** represents a numerical value associated with each outcome of a probability distribution.

A random variable is **discrete** if it has a finite or countable number of possible outcomes that can be listed.



A random variable is **continuous** if it has an uncountable number or possible outcomes, represented by the intervals on a number line.



Discrete Probability Distributions

- A discrete random variable is a variable that can assume only a countable number of values

Many possible outcomes:

- number of complaints per day
- number of TV's in a household
- number of rings before the phone is answered

Only two possible outcomes:

- gender: male or female
- defective: yes or no
- spreads peanut butter first vs. spreads jelly first

Continuous Probability Distributions

- A continuous random variable is a variable that can assume any value on a continuum (can assume an uncountable number of values)
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches
- These can potentially take on any value, depending only on the ability to measure accurately.

Probability functions

- A probability function maps the possible values of *x* against their respective probabilities of occurrence, *p(x)*
- *p(x)* is a number from 0 to 1.0.
- The area under a probability function is always 1.

Probability Distributions

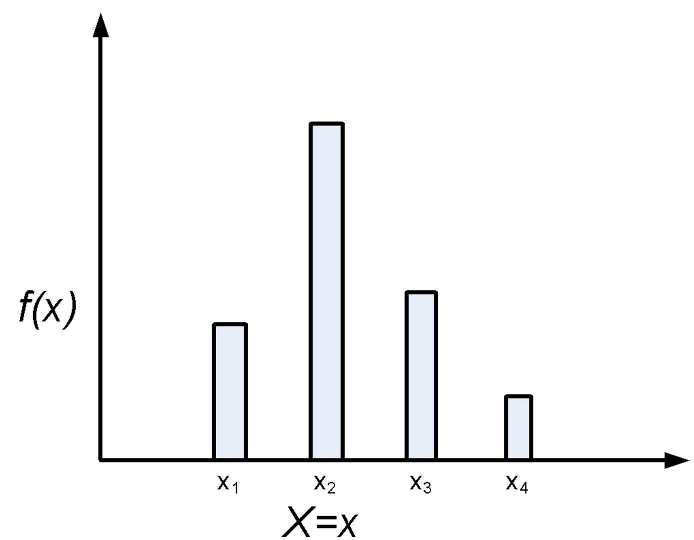
Probability Distributions

- **Discrete** Probability Distributions
  - Binomial
  - Poisson
  - Hypergeometric
- **Continuous** Probability Distributions
  - Normal
  - Uniform
  - Exponential

The Binomial Distribution

Probability Distributions

- **Discrete** Probability Distributions
  - Binomial
  - Poisson
  - Hypergeometric



Discrete Probability distribution

Constructing a Discrete Probability Distribution

Guidelines

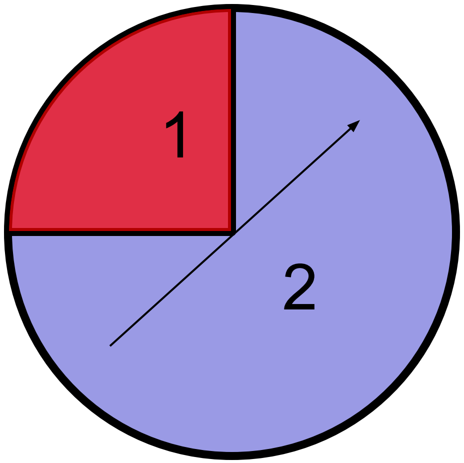
Let x be a discrete random variable with possible outcomes $x_1, x_2, \ldots, x_n$.

1. Make a frequency distribution for the possible outcomes.
2. Find the sum of the frequencies.
3. Find the probability of each possible outcome by dividing its frequency by the sum of the frequencies.
4. Check that each probability is between 0 and 1 and that the sum is 1.

Constructing a Discrete Probability Distribution

Example:

The spinner below is divided into two sections. The probability of landing on the 1 is 0.25. The probability of landing on the 2 is 0.75. Let *x* be the number the spinner lands on. Construct a probability distribution for the random variable *x*.



x	$P(x)$
1	0.25
2	0.75

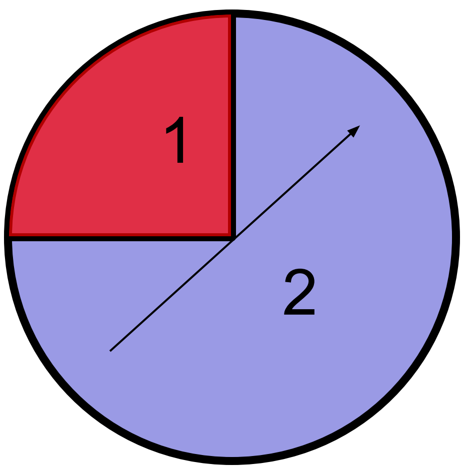
Each probability is between 0 and 1.

The sum of the probabilities is 1.

Constructing a Discrete Probability Distribution

Example:

The spinner below is spun two times. The probability of landing on the 1 is 0.25. The probability of landing on the 2 is 0.75. Let *x* be the <u>sum</u> of the two spins. Construct a probability distribution for the random variable *x*.



The possible sums are 2, 3, and 4.

P (sum of 2) = 0.25 × 0.25 = 0.0625

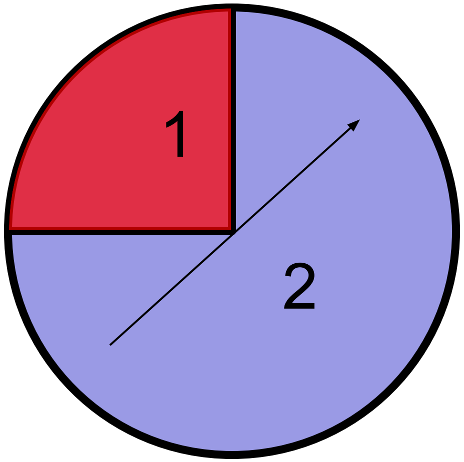
Spin a 1 on the first spin.

"and"

Spin a 1 on the second spin.

Constructing a Discrete Probability Distribution

Example continued:



P (sum of 3) = 0.25 × 0.75 = 0.1875

Spin a 1 on the first spin. "and" Spin a 2 on the second spin.

"or"

P (sum of 3) = 0.75 × 0.25 = 0.1875

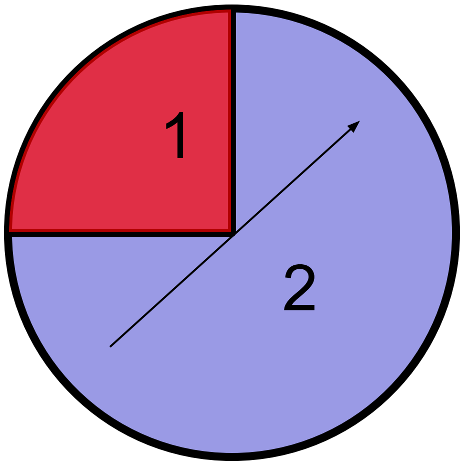
Spin a 2 on the first spin. "and" Spin a 1 on the second spin.

Sum of spins, x	$P(x)$
2	0.0625
3	0.375
4	

0.1875 + 0.1875

Constructing a Discrete Probability Distribution

Example continued:



P (sum of 4) = 0.75 × 0.75 = 0.5625

Spin a 2 on the first spin.

"and"

Spin a 2 on the second spin.

Sum of spins, x	$P(x)$
2	0.0625
3	0.375
4	0.5625

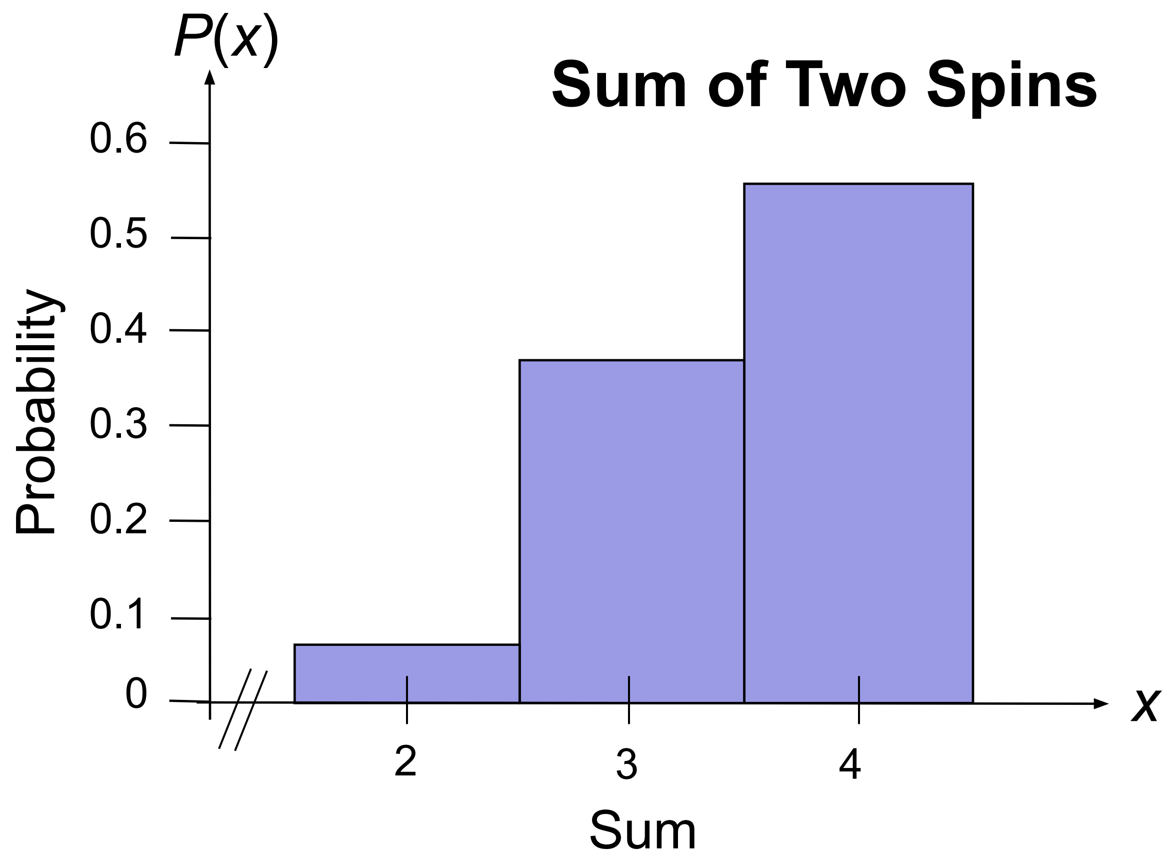
Each probability is between 0 and 1, and the sum of the probabilities is 1.

Graphing a Discrete Probability Distribution

Example:

Graph the following probability distribution using a histogram.

Sum of spins, x	$P(x)$
2	0.0625
3	0.375
4	0.5625



Mean

The **mean** of a discrete random variable is given by

$$\mu = \Sigma x P(x).$$

Each value of x is multiplied by its corresponding probability and the products are added.

Example:

Find the mean of the probability distribution for the sum of the two spins.

x	$P(x)$	$xP(x)$
2	0.0625	$2(0.0625) = 0.125$
3	0.375	$3(0.375) = 1.125$
4	0.5625	$4(0.5625) = 2.25$

$\Sigma x P(x) = 3.5$

The mean for the two spins is 3.5.

Variance

The **variance** of a discrete random variable is given by

$$\sigma^2 = \Sigma(x - \mu)^2 P(x).$$

Example:

Find the variance of the probability distribution for the sum of the two spins. The mean is 3.5.

x	$P(x)$	$x - \mu$	$(x-\mu)^2$	$P(x)(x-\mu)^2$
2	0.0625	–1.5	2.25	≈ 0.141
3	0.375	–0.5	0.25	≈ 0.094
4	0.5625	0.5	0.25	≈ 0.141

$\Sigma P(x)(x - 2)^2$

≈ 0.376

The variance for the two spins is approximately 0.376

Standard Deviation

The **standard deviation** of a discrete random variable is given by

$$\sigma = \sqrt{\sigma^2}.$$

Example:

Find the standard deviation of the probability distribution for the sum of the two spins. The variance is 0.376.

$$\sigma = \sqrt{\sigma^2}$$

x	$P(x)$	$x-\mu$	$(x-\mu)^2$	$P(x)(x-\mu)^2$
2	0.0625	–1.5	2.25	0.141
3	0.375	–0.5	0.25	0.094
4	0.5625	0.5	0.25	0.141

$$= \sqrt{0.376} \approx 0.613$$

Most of the sums differ from the mean by no more than 0.6 points.

Expected Value

The **expected value** of a discrete random variable is equal to the mean of the random variable.

Expected Value = $E(x) = \mu = \Sigma x P(x)$.

Example:

At a raffle, 500 tickets are sold for $1 each for two prizes of $100 and $50. What is the expected value of your gain?

Your gain for the $100 prize is $100 – $1 = $99.

Your gain for the $50 prize is $50 – $1 = $49.

Write a probability distribution for the possible gains (or outcomes).

Expected Value

Example continued:

At a raffle, 500 tickets are sold for $1 each for two prizes of $100 and $50. What is the expected value of your gain?

Gain, x	$P(x)$
\$99	$\frac{1}{500}$
\$49	$\frac{1}{500}$
–\$1	$\frac{498}{500}$

Winning no prize

$E(x) = \Sigma xP(x).$

$$= \$99 \cdot \frac{1}{500} + \$49 \cdot \frac{1}{500} + (-\$1) \cdot \frac{498}{500}$$

$$= -\$0.70$$

Because the expected value is negative, you can expect to lose \$0.70 for each ticket you buy.

Probability Mass Functions (PMFs)
compute the probability that a discrete random variable equals a specific value.

$$p(x) = P(X = x)$$

Cumulative Distribution Functions (CDFs)
we only define for the possible values of the random variable.

$$F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}.$$

The Binomial Distribution

- Characteristics of the Binomial Distribution:
  - A trial has only two possible outcomes – “success” or “failure”
  - There is a fixed number, n, of identical trials
  - The trials of the experiment are independent of each other
  - The probability of a success, p, remains constant from trial to trial
  - If p represents the probability of a success, then (1-p) = q is the probability of a failure

Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for a contract will either get the contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it

Counting Rule for Combinations

- A combination is an outcome of an experiment where x objects are selected from a group of n objects

$$C_x^n = \frac{n!}{x!(n-x)!}$$

where:

$n! = n(n - 1)(n - 2) \dots (2)(1)$

$x! = x(x - 1)(x - 2) \dots (2)(1)$

$0! = 1$ (by definition)

Binomial Distribution Formula

$$P(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

P(x) = probability of **x** successes in **n** trials,
with probability of success **p** on each trial

x = number of 'successes' in sample,
(x = 0, 1, 2, ..., n)

p = probability of "success" per trial

q = probability of "failure" = (1 – p)

n = number of trials (sample size)

Example: Flip a coin four times, let x = # heads:

n = 4

p = 0.5

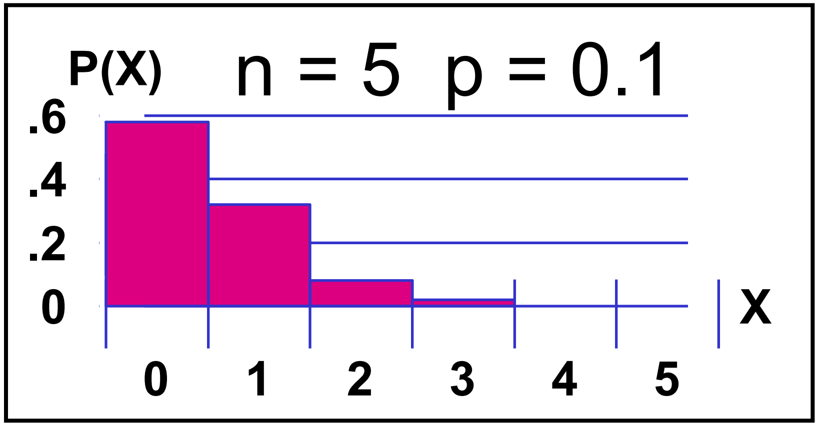
q = (1 - .5) = .5

x = 0, 1, 2, 3, 4

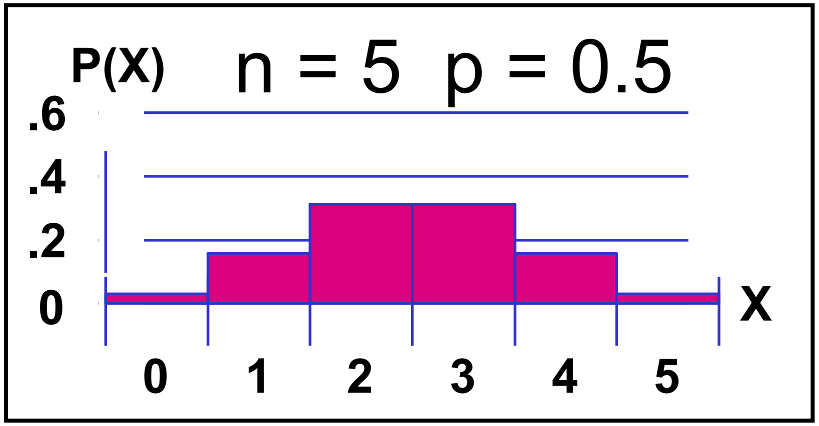
Binomial Distribution

- The shape of the binomial distribution depends on the values of p and n

  - Here, n = 5 and p = .1



  - Here, n = 5 and p = .5



Binomial Distribution Characteristics

- Mean

$$\boxed{\mu = E(x) = np}$$

- Variance and Standard Deviation

$$\boxed{\sigma^2 = npq}$$

$$\boxed{\sigma = \sqrt{npq}}$$

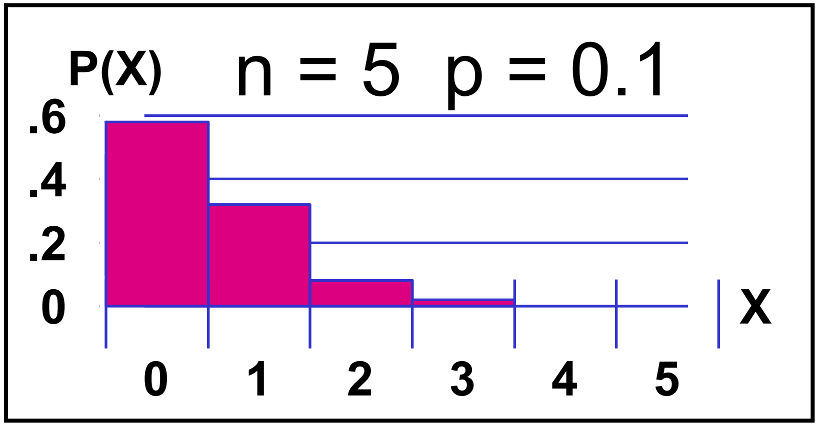
Where n = sample size
p = probability of success
$q = (1 – p)$ = probability of failure

Binomial Characteristics

Examples

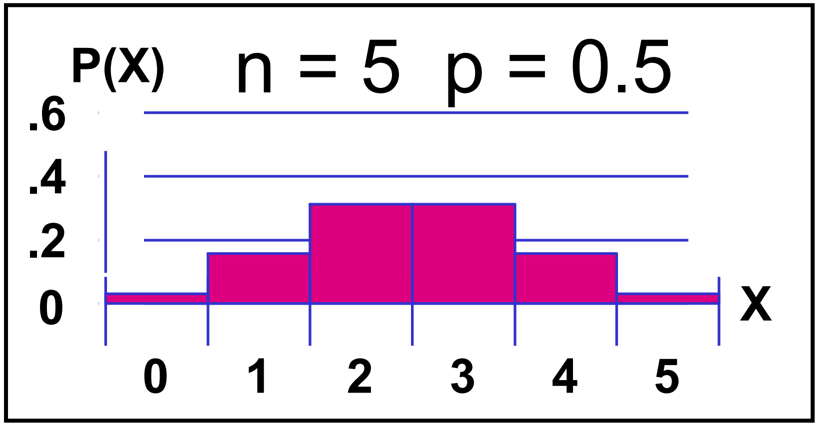
$$\mu = np = (5)(.1) = 0.5$$

$$\sigma = \sqrt{npq} = \sqrt{(5)(.1)(1-.1)} = 0.6708$$



$$\mu = np = (5)(.5) = 2.5$$

$$\sigma = \sqrt{npq} = \sqrt{(5)(.5)(1-.5)} = 1.118$$



Using Binomial Tables

n = 10									
x	p=.15	p=.20	p=.25	p=.30	**p=.35**	p=.40	p=.45	p=.50	
0	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010	10
1	0.3474	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098	9
2	0.2759	0.3020	0.2816	0.2335	0.1757	0.1209	0.0763	0.0439	8
3	0.1298	0.2013	0.2503	0.2668	**0.2522**	0.2150	0.1665	0.1172	7
4	0.0401	0.0881	0.1460	0.2001	0.2377	0.2508	0.2384	0.2051	6
5	0.0085	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461	5
6	0.0012	0.0055	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051	4
7	0.0001	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172	3
8	0.0000	0.0001	**0.0004**	0.0014	0.0043	0.0106	0.0229	0.0439	**2**
9	0.0000	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098	1
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0
	p=.85	p=.80	**p=.75**	p=.70	p=.65	p=.60	p=.55	p=.50	**x**

Examples:

n = 10, p = .35, x = 3: $P(x = 3|n = 10, p = .35) = .2522$

n = 10, p = .75, x = 2: $P(x = 2|n = 10, p = .75) = .0004$

The Poisson Distribution

There are some experiments, which involve the occurring of the number of outcomes during a given time interval (or in a region of space).

Such a process is called Poisson process.

Number of clients visiting a ticket selling counter in a metro station.

The Poisson Distribution

Properties of Poisson process

The number of outcomes in one time interval is independent of the number that occurs in any other disjoint interval [Poisson process has no memory]

The probability that a single outcome will occur during a very short interval is proportional to the length of the time interval and does not depend on the number of outcomes occurring outside this time interval.

The probability that more than one outcome will occur in such a short time interval is negligible.

The Poisson Distribution

The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval t, is

$$f(x, \lambda t) = P(X = x) = \frac{e^{-\lambda t}.(\lambda t)^x}{x!}, x = 0, 1, \dots \dots$$

where λ is the average number of outcomes per unit time and $e = 2.71828 \dots$

Poisson Distribution

Example:

The mean number of power outages in the city of Brunswick is 4 per year. Find the probability that in a given year,

a.) there are exactly 3 outages,b.) there are more than 3 outages.

a.) $\mu = 4, \ \ x = 3$

$$P(3) = \frac{4^3(2.71828)^{-4}}{3!}$$

$$\approx 0.195$$

b.) $P(\text{more than } 3)$

$$= 1 - P(x \leq 3)$$

$$= 1 - [P(3) + P(2) + P(1) + P(0)]$$

$$= 1 - (0.195 + 0.147 + 0.073 + 0.018)$$

$$\approx 0.567$$

Binomial problems with large sample sizes and small values of p, which then generate rare events, are potential candidates for use of the Poisson distribution.

As a rule of thumb, if $n > 20$ and $n\ p < 7$, the approximation is close enough to use the Poisson distribution for binomial problems.

The Hypergeometric Distribution

- The hypergeometric distribution has the following characteristics:
- It is discrete distribution.
- Each outcome consists of either a success or a failure.
- Sampling is done without replacement.
- The population, *N, is finite and known.*
- The number of successes in the population, *A, is known*

HYPERGEOMETRIC FORMULA

$$P(x) = \frac{{}_{A}C_{x} \cdot {}_{N-A}C_{n-x}}{{}_{N}C_{n}}$$

where

N = size of the population
n = sample size
A = number of successes in the population
x = number of successes in the sample; sampling is done *without* replacement

As an application of the hypergeometric distribution, consider the following problem.Twenty-four people, of whom eight are women, apply for a job. If five of the applicants are sampled randomly, what is the probability that exactly three of those sampled are women?

In summary, the hypergeometric distribution should be used instead of the binomial distribution when the following conditions are present:

1. Sampling is being done without replacement.

2. $n > 5\% \ N$.

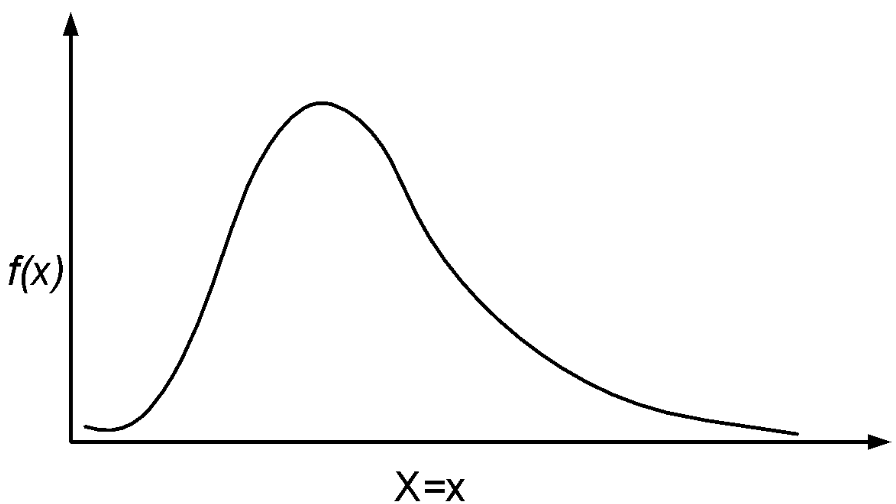
The mean =is nk/N

variance =is $nk(N - k)(N - n)/N^2(N - 1)$.

Continuous Probability Distributions

Probability Distributions

- **Continuous** Probability Distributions
  - Normal
  - Uniform
  - Exponential

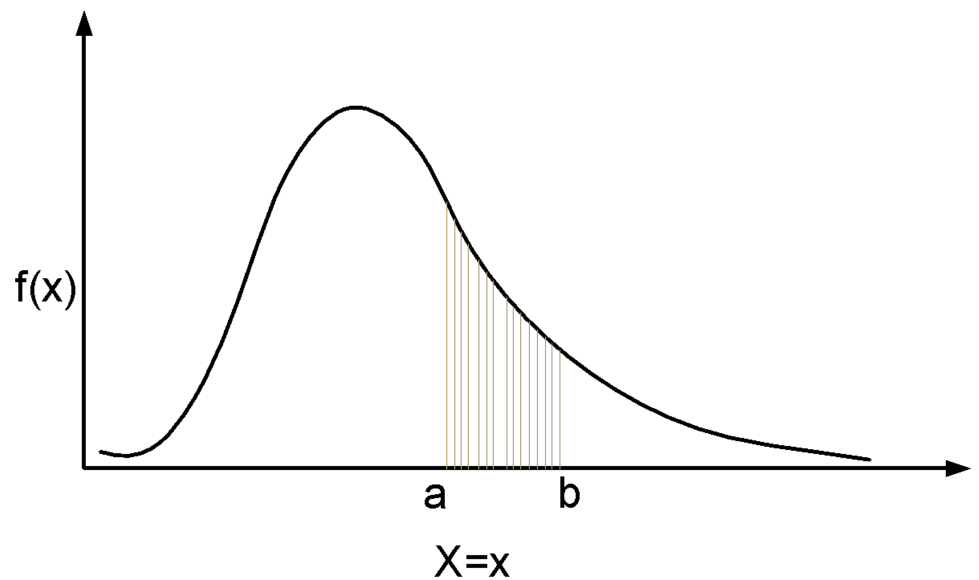


Continuous Probability Distribution

Properties of Probability Density Function

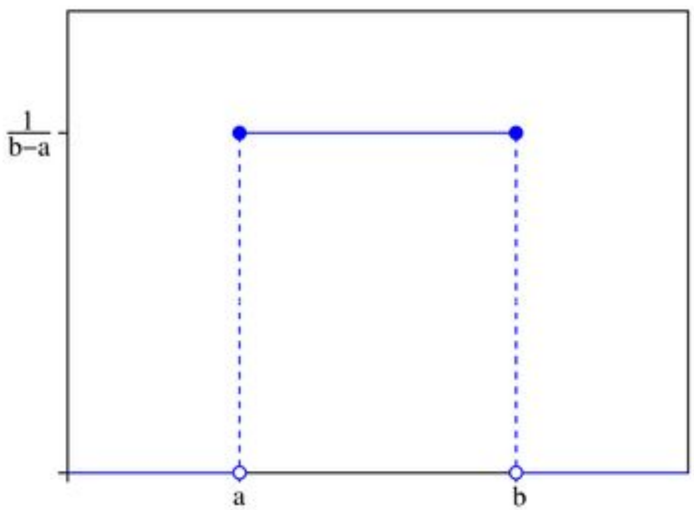
The function $f(x)$ is a probability density function for the continuous random variable X, defined over the set of real numbers R, if

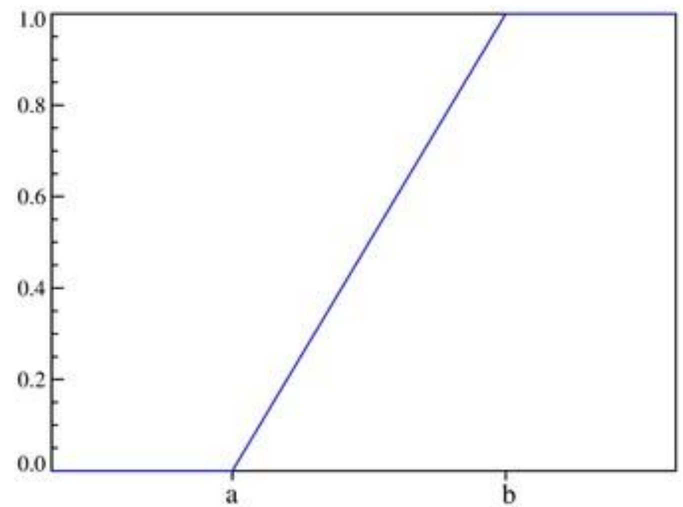
1. $f(x) \geq 0$, for all $x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$
4. $\mu = \int_{-\infty}^{\infty} xf(x)\, dx$
5. $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$



Uniform distribution

Uniform distribution pdf and cdf examples





The uniform distribution is used to model situations in which the probability is equal for all possible outcomes. There are two parameters for the uniform distribution, the lower bound and upper bound of possibilities, often denoted as *a* and *b.*

Consider that your company is having a strike, and at any point the strike is just as likely to end as at any other point. Therefore, it can be modeled using a uniform distribution over the next month of 30 days. What is the chance the strike is over in the first week (7 days)?

- The probability density function for a uniform distribution is:

$$P(X = x) = \frac{1}{b-a}; \quad x = [a, b]$$

- The expected value for a uniform distribution is:

$$E[X] = \frac{1}{2}(a + b)$$

- The variance for a uniform distribution is:

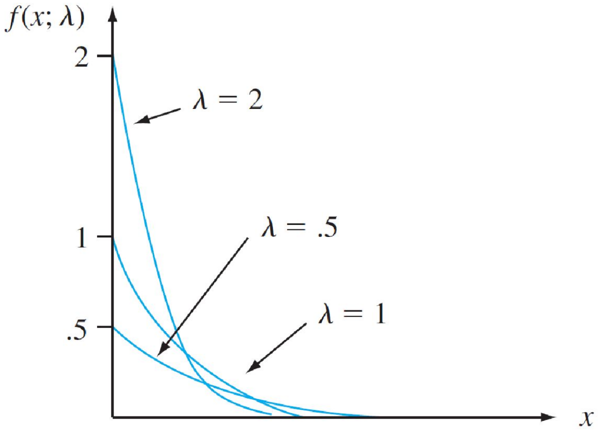
$$Var[X] = \frac{1}{12}(b - a)^2$$

The Exponential Distributions

- **Definition**
- *X* is said to have an **exponential distribution** with parameter λ (λ > 0) if the pdf of *X* is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{1}{\lambda} \qquad \sigma^2 = \frac{1}{\lambda^2}$$



The Exponential Distributions

- The expected value of an exponentially distributed random variable *X* is

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}\, dx$$

- Obtaining this expected value necessitates doing an integration by parts. The variance of *X* can be computed using the fact that $V(X) = E(X^2) - [E(X)]^2$.

The Exponential Distributions

- The exponential pdf is easily integrated to obtain

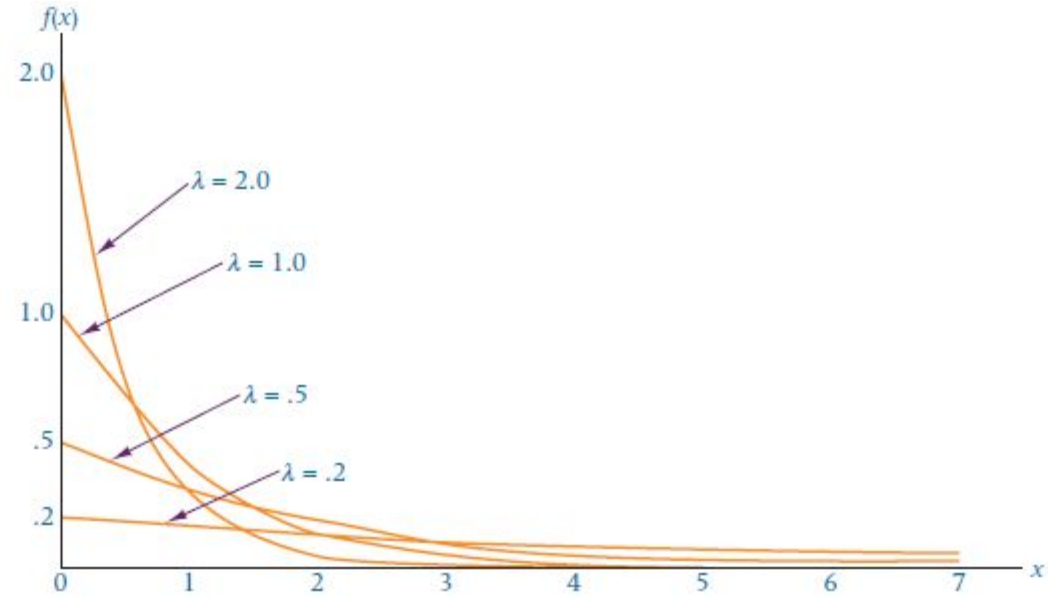
$$F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

The exponential distribution is frequently used as a model for the distribution of times between the occurrence of successive events, such as customers arriving at a service facility or calls coming in to a switchboard.

The Exponential Distributions

It is a continuous distribution.

- It is a family of distributions.
- It is skewed to the right.
- The *x values range from zero to infinity.*
- Its apex is always at *x = 0.*
- The curve steadily decreases as *x gets larger.*



Example

- Suppose that calls are received at a 24-hour "suicide hotline" according to a Poisson process with rate $\alpha = .5$ call per day.
- Then the number of days X between successive calls has an exponential distribution with parameter value .5, so the probability that more than 2 days elapse between calls is
- $P(X > 2) = 1 - P(X \leq 2)$
- $= 1 - F(2; .5)$
- $= e^{-(.5)(2)}$

$= .368$

The expected time between successive calls is $1/.5 = 2$ days.

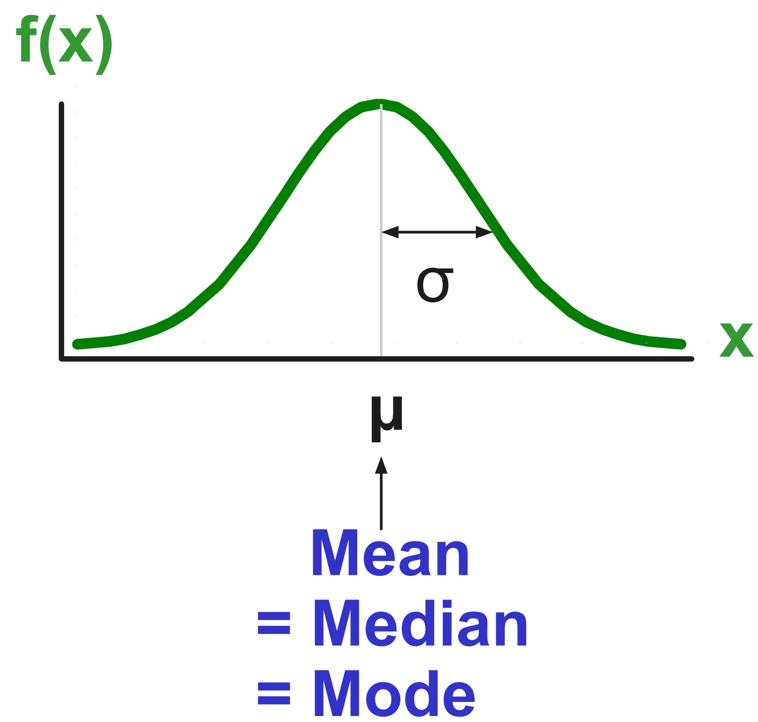
The Normal Distribution

- Bell Shaped'
- Symmetrical
- Mean, Median and Mode are Equal

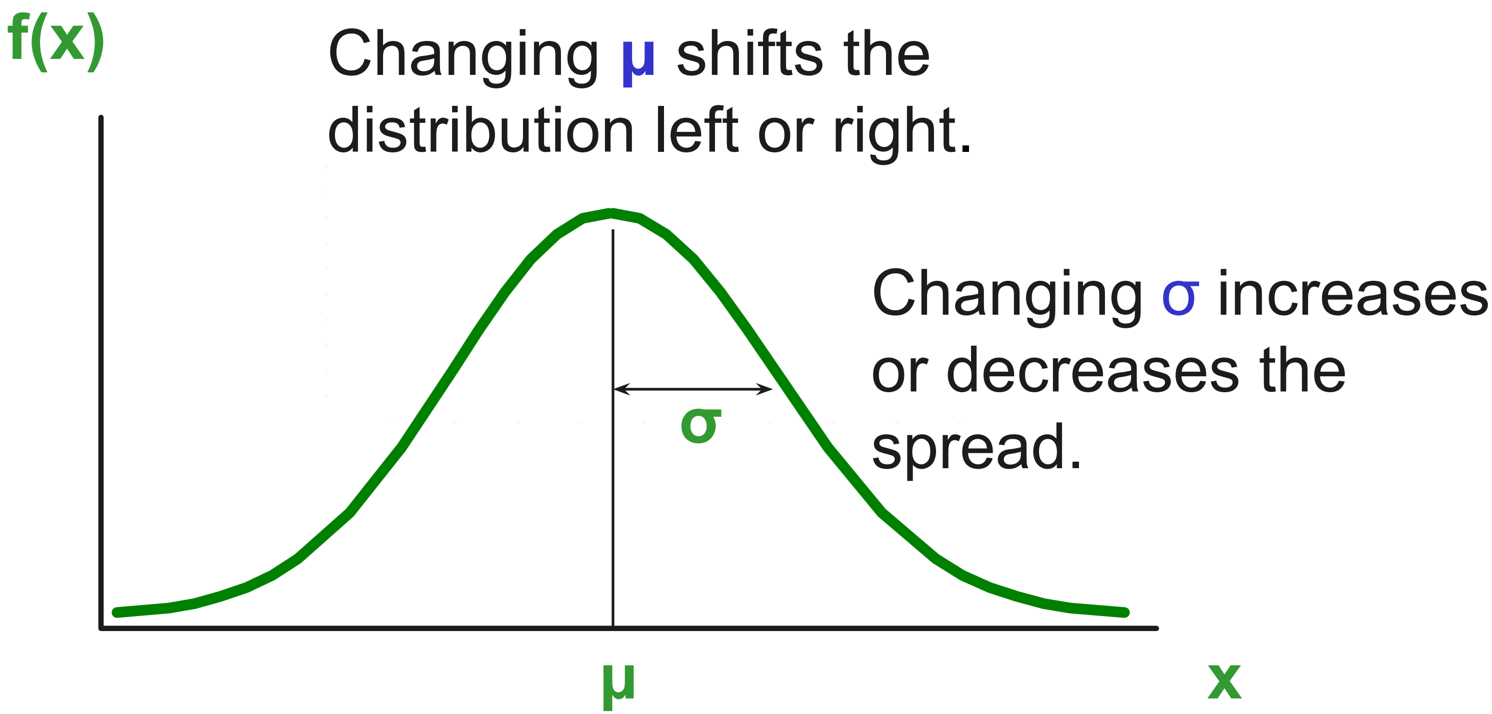
Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range:
$+\infty$ to $-\infty$



The Normal Distribution Shape

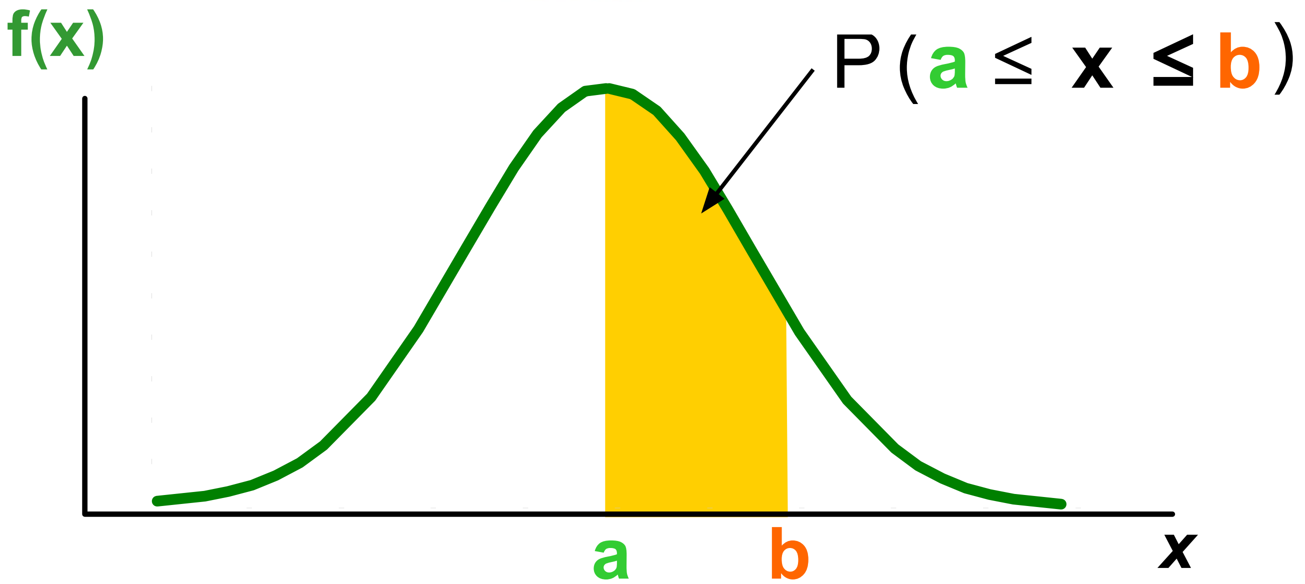


By varying the parameters μ and σ, we obtain different normal distributions

Finding Normal Probabilities

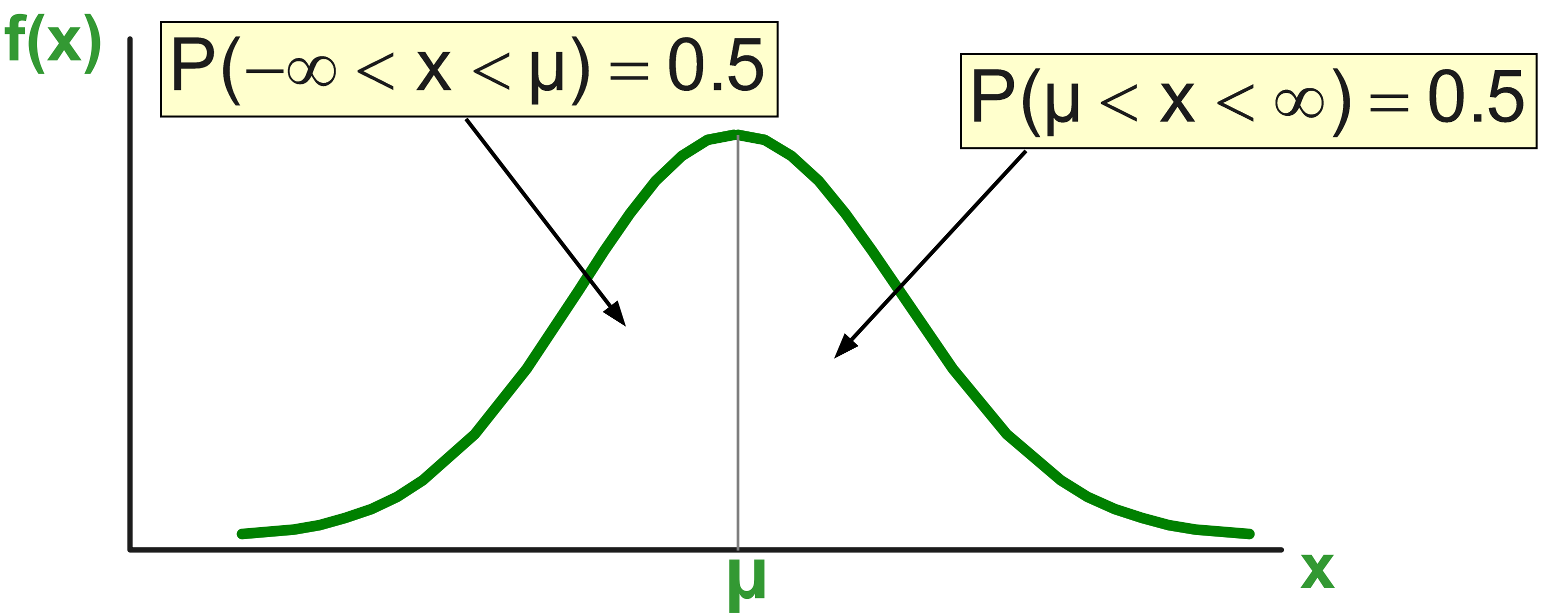
Probability is measured by the area under the curve

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2[(x-\mu)/\sigma)]^2}$$



Probability as Area Under the Curve

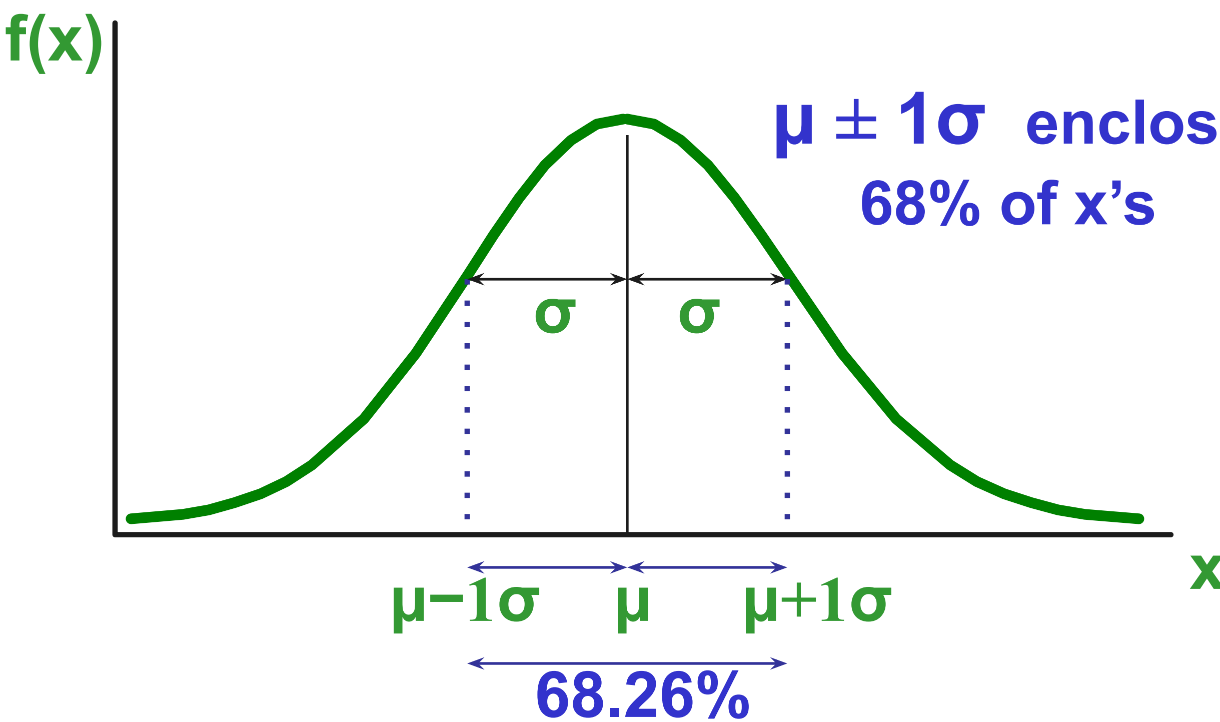
The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



$$P(-\infty < x < \infty) = 1.0$$

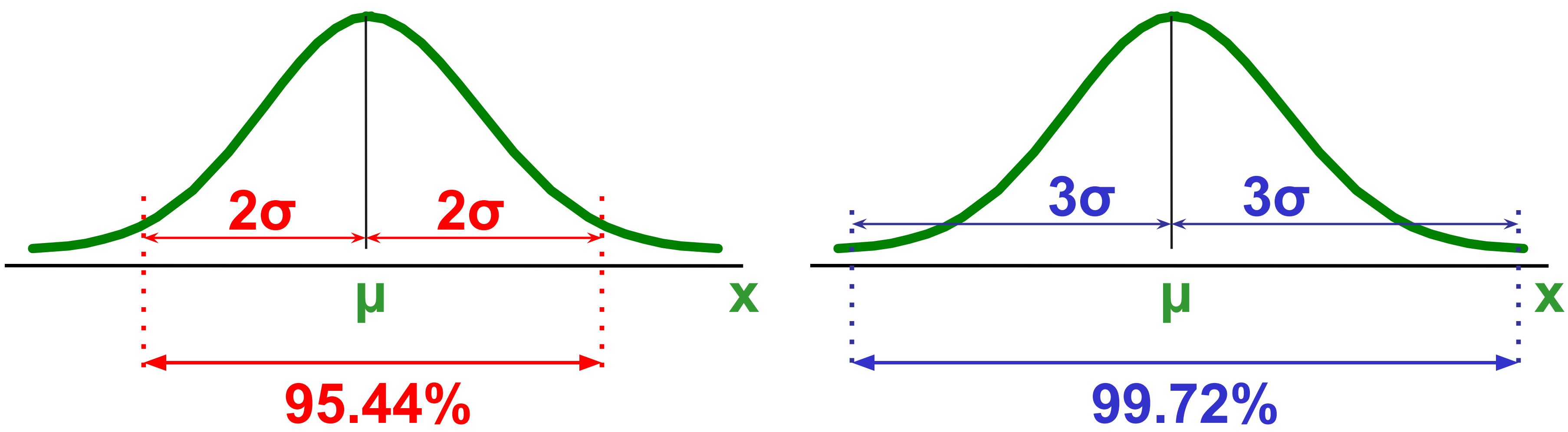
Empirical Rules

What can we say about the distribution of values around the mean? There are some general rules:



The Empirical Rule

- **$\mu \pm 2\sigma$ covers about 95% of x's**
- **$\mu \pm 3\sigma$ covers about 99.7% of x's**

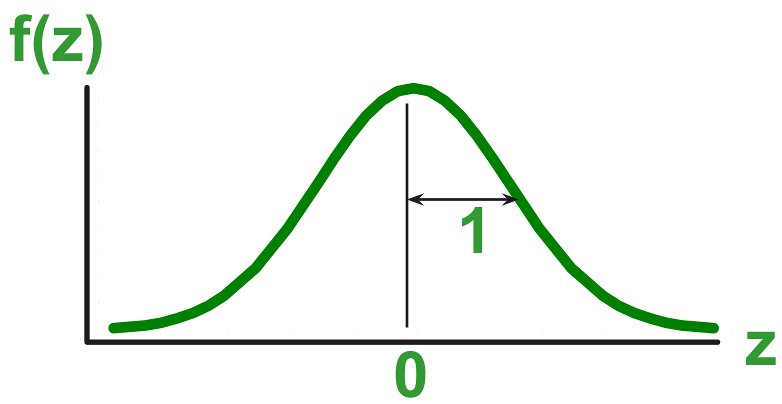


Importance of the Rule

- If a value is about 2 or more standard deviations away from the mean in a normal distribution, then it is far from the mean
- The chance that a value that far or farther away from the mean is highly unlikely, given that particular mean and standard deviation

The Standard Normal Distribution

- Also known as the “z” distribution
- Mean is defined to be 0
- Standard Deviation is 1



Values above the mean have positive z-values, values below the mean have negative z-values

The Standard Normal

- Any normal distribution (with any mean and standard deviation combination) can be transformed into the standard normal distribution (z)

- Need to transform x units into z units

Translation to the Standard Normal Distribution

- Translate from x to the standard normal (the “z” distribution) by subtracting the mean of x and dividing by its standard deviation:

$$z = \frac{x - \mu}{\sigma}$$

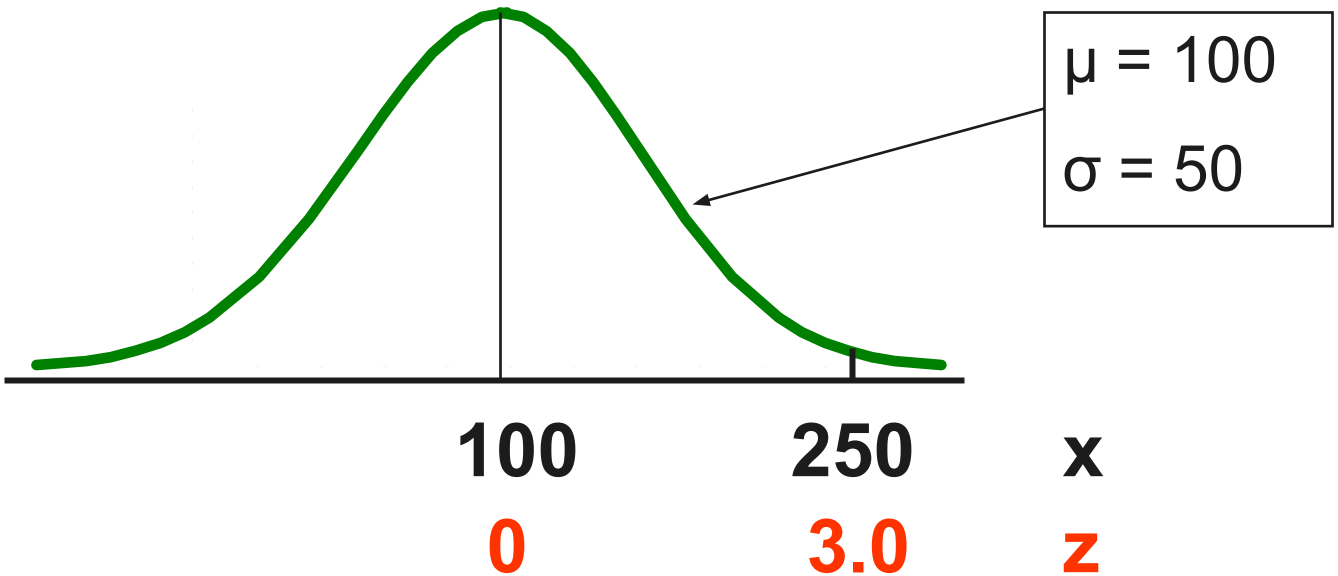
Example

- If x is distributed normally with mean of 100 and standard deviation of 50, the z value for $x = 250$ is

$$z = \frac{x - \mu}{\sigma} = \frac{250 - 100}{50} = 3.0$$

- This says that $x = 250$ is three standard deviations (3 increments of 50 units) above the mean of 100.

Comparing x and z units



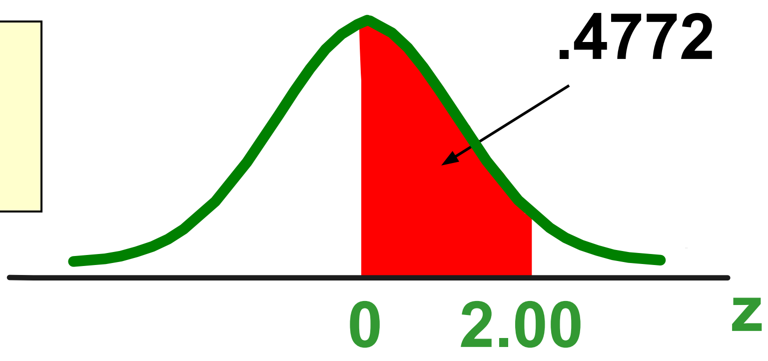
Note that the distribution is the same, only the scale has changed. We can express the problem in original units (x) or in standardized units (z)

The Standard Normal Table

- The Standard Normal table gives the probability from the mean (zero) up to a desired value for z

Example:

$P(0 < z < 2.00) = .4772$



The Standard Normal Table

The column gives the value of z to the second decimal point

z	0.00	0.01	0.02	…
0.1				
0.2				
.				
.				
.				
2.0	.4772			

The row shows the value of z to the first decimal point

The value within the table gives the probability from $z = 0$ up to the desired z value

$P(0 < z < 2.00) = .4772$

General Procedure for Finding Probabilities

To find $P(a < x < b)$ when x is distributed normally:

- Draw the normal curve for the problem in terms of x
- Translate x-values to z-values
- Use the Standard Normal Table

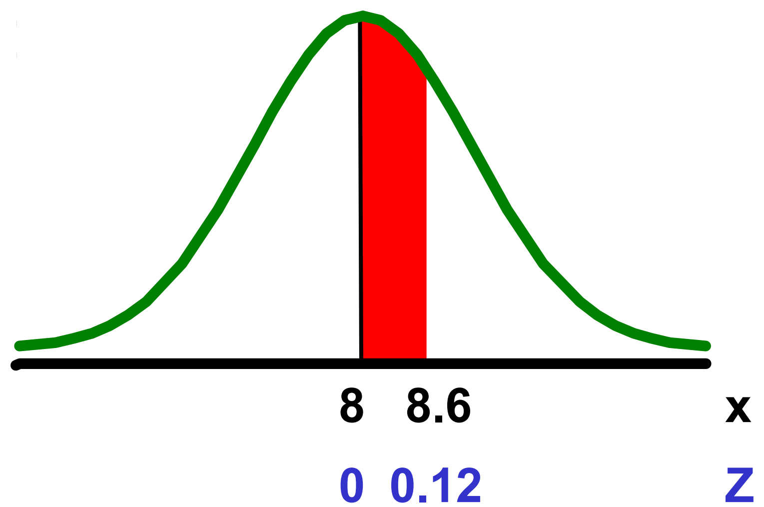
Z Table example

- Suppose x is normal with mean 8.0 and standard deviation 5.0. Find P(8 < x < 8.6)

Calculate z-values:

$$z = \frac{x - \mu}{\sigma} = \frac{8 - 8}{5} = 0$$

$$z = \frac{x - \mu}{\sigma} = \frac{8.6 - 8}{5} = 0.12$$

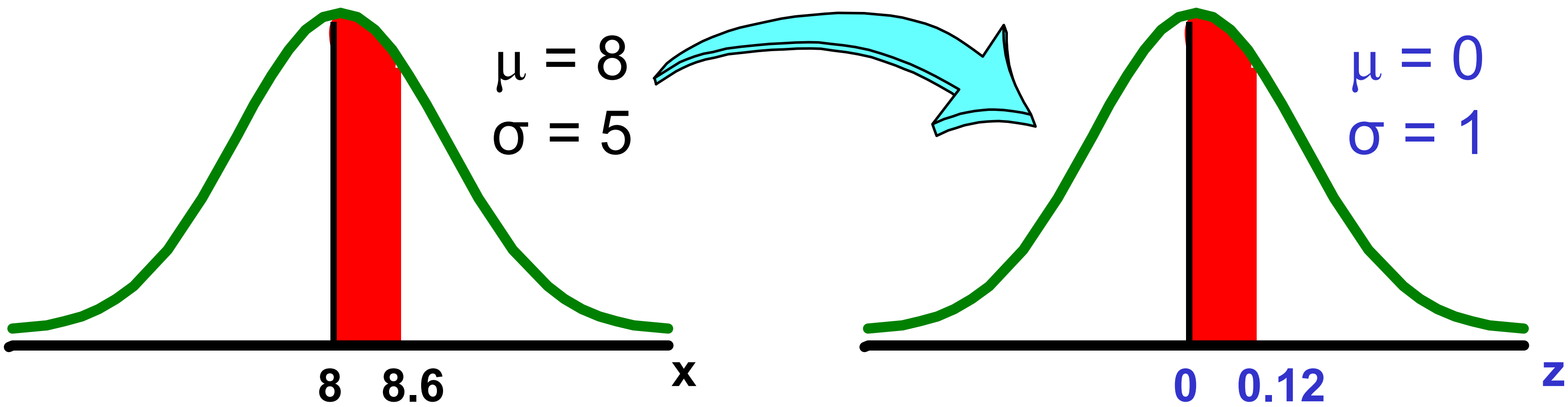


P(8 < x < 8.6)

= P(0 < z < 0.12)

Z Table example

- Suppose x is normal with mean 8.0 and standard deviation 5.0. Find $P(8 < x < 8.6)$



$P(8 < x < 8.6)$

$P(0 < z < 0.12)$

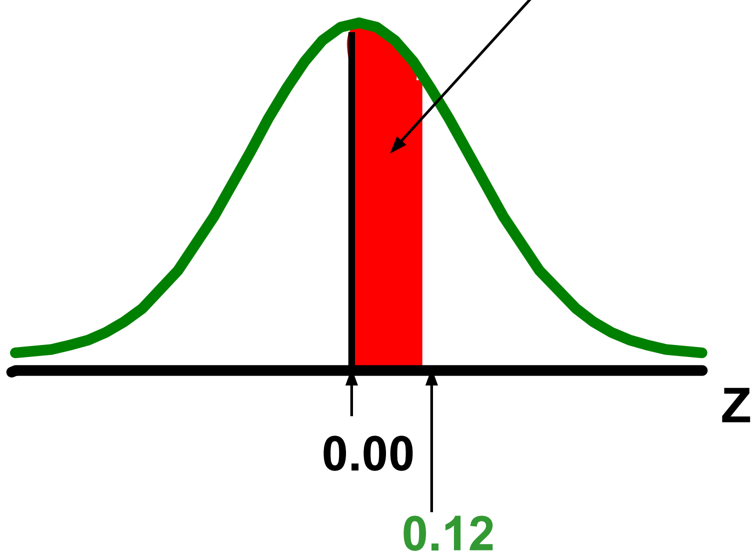
Solution: Finding P(0 < z < 0.12)

Standard Normal Probability Table (Portion)

z	.00	.01	.02
0.0	.0000	.0040	.0080
0.1	.0398	.0438	.0478
0.2	.0793	.0832	.0871
0.3	.1179	.1217	.1255

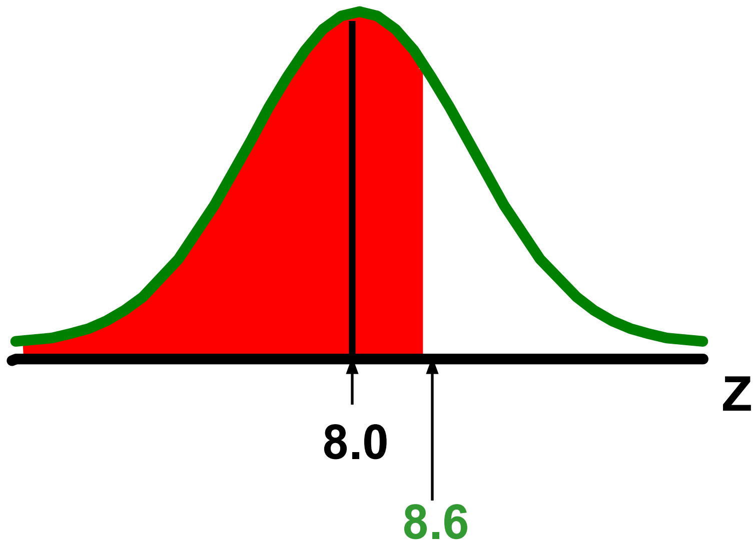
$P(8 < x < 8.6)$
$= P(0 < z < 0.12)$

.0478



Finding Normal Probabilities

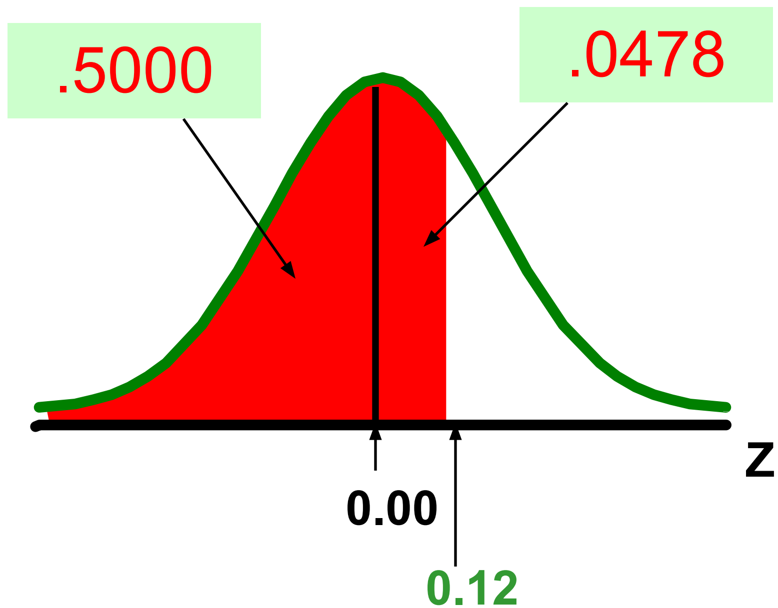
- Suppose x is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(x < 8.6)



Finding Normal Probabilities

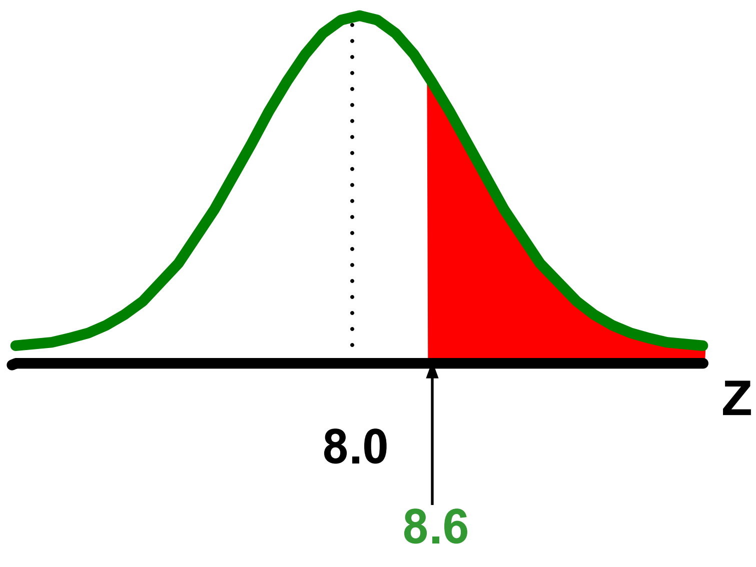
- Suppose x is normal with mean 8.0 and standard deviation 5.0.
- Now Find $P(x < 8.6)$

$$\begin{aligned} & P(x < 8.6) \\ &= P(z < 0.12) \\ &= P(z < 0) + P(0 < z < 0.12) \\ &= .5 + .0478 = \boxed{.5478} \end{aligned}$$



Upper Tail Probabilities

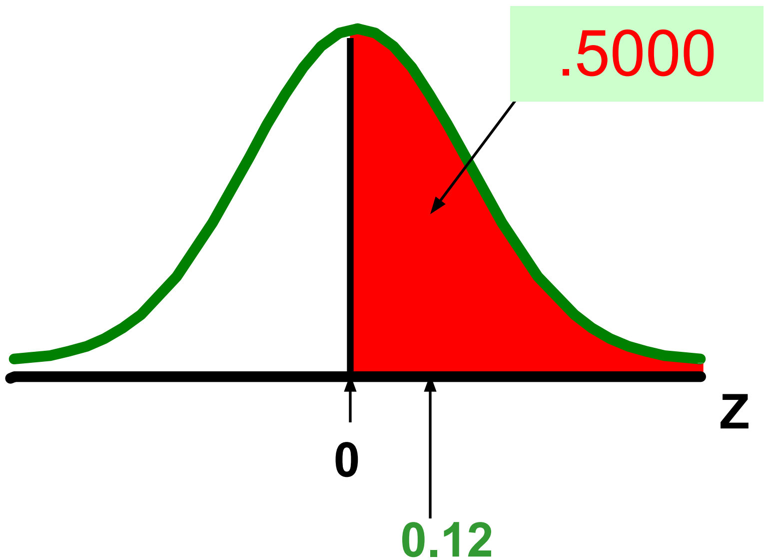
- Suppose x is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(x > 8.6)

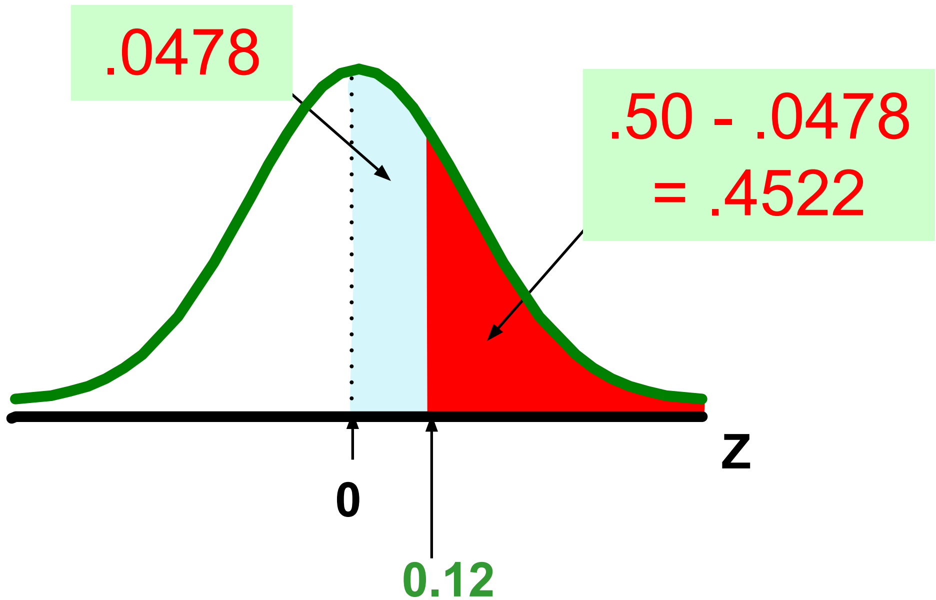


Upper Tail Probabilities

- Now Find $P(x > 8.6)$…

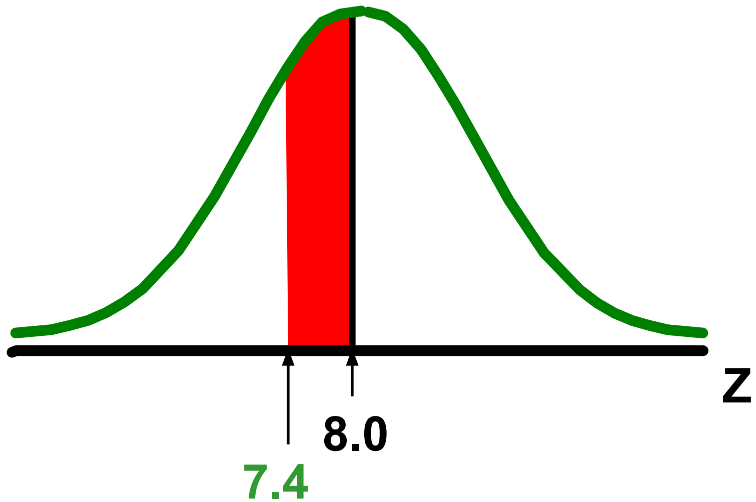
$$P(x > 8.6) = P(z > 0.12) = P(z > 0) - P(0 < z < 0.12)$$
$$= .5 - .0478 = \boxed{.4522}$$





Lower Tail Probabilities

- Suppose x is normal with mean 8.0 and standard deviation 5.0.
- Now Find $P(7.4 < x < 8)$



Lower Tail Probabilities

Now Find P(7.4 < x < 8)…

The Normal distribution is symmetric, so we use the same table even if z-values are negative:

$P(7.4 < x < 8)$

$= P(-0.12 < z < 0)$

$= \boxed{.0478}$