# Probability

# Probability

Numerical measure of likelihood that an event will occur.

The probability of any event must be between 0 and 1, inclusive.

The sum of all probabilities across all events (or integral of the area  under a curve if continuous) must be 1.

# Approaches to Assigning Probabilities

There are three ways to assign a probability, $P(O_i)$, to an outcome, $O_i$, namely:

- Classical approach: based on equally likely events.
- Relative frequency: assigning probabilities based on experimentation or historical data.
- Subjective approach: Assigning probabilities based on the assignor's (subjective) judgment

# Classical Approach

- If an experiment has n possible outcomes, this method would assign a probability of 1/n to each outcome. It is necessary to determine the number of possible outcomes.

| | |
|---|---|
| Experiment: | Rolling a die |
| Outcomes | {1, 2, 3, 4, 5, 6} |
| Probabilities: | Each sample point has a 1/6 chance of occurring. |

# Relative Frequency Approach

<u>The experiment</u>

This spinner was spun 40 times and the results recorded in this table:

| Colour | Frequency |
|---|---|
| Blue | 20 |
| Yellow | 10 |
| Red | 5 |
| Green | 5 |

<u>Relative frequency</u>

$$\boxed{\frac{\text{frequency of event}}{\text{total number of trials}}}$$

**Event** means **one possible outcome**; here, one colour on the spinner.

There were 20 blues recorded...

...out of 40 spins.

$$P(\text{blue}) = \frac{20}{40}$$

Simplify: $P(\text{blue}) = \frac{20}{40} = \frac{2}{4} = \frac{1}{2}$

# Subjective Approach

"In the subjective approach we define probability as the degree of belief that we hold in the occurrence of an event"

E.g. weather forecasting's "P.O.P."

POP 60% – based on current conditions, there is a 60% chance of rain (say).

# Terminology

- Experiment
- Trial
- Event
- Elementary Event

# Sample space

- The set of all elementary events for an experiment is called a sample space.
- Suppose if you roll a die, there are you can get 1 2 3 4 5 6 these are the sample space there are different methods for describing the sample space one is listing tree diagram, set builder notation and Venn diagram, you will see what is that.

# Probabilities of events

- We study methods to determine probabilities of events that result from combining other events in various ways.

- There are several types of combinations and relationships between events:
- Complement event
- Intersection of events
- Union of events
- Mutually exclusive events
- Dependent and independent events

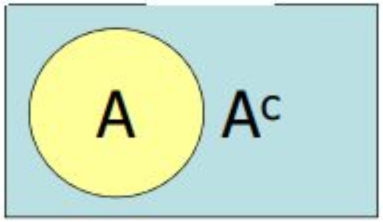
# Complement of an Event

The ***complement of event*** A is defined to be the event consisting of all sample points that are "not in A".

Complement of A is denoted by $A^c$

The Venn diagram below illustrates the concept of a complement.

$$P(A) + P(A^c) = 1$$




For example, the rectangle stores all the possible tosses of 2 dice {(1,1), 1,2),… (6,6)}

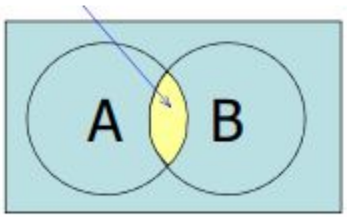
Let A = tosses totaling 7 {(1,6),(2, 5), (3,4), (4,3), (5,2), (6,1)}

P(Total = 7) + P(Total not equal to 7) = 1

# intersection of Two Events

The ***intersection of events*** A and B is the set of all sample points that are in both A and B.

The intersection is denoted: **A and B**




The joint probability of A and B is the probability of the intersection of A and B,i.e. P(A and B)

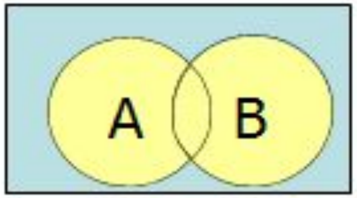
# Union of Two Events

The union of two events A and B, is the event containing all sample points that are in A or B or both: Union of A and B is denoted: A or B

For example, let A = tosses where first toss is 1 {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)} and

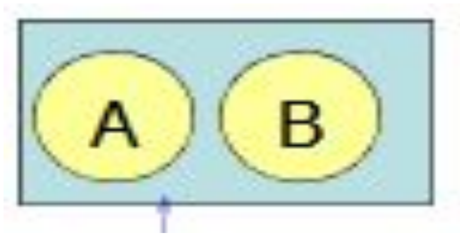
B is the tosses that the second toss is 5 {(1,5), (2,5), (3,5), (4,5),(5,5), (6,5)}

Union of A and B is {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),(2,5), (3,5), (4,5), (5,5), (6,5)}



# Mutually Exclusive Events

- When two events are mutually exclusive (that is the two events cannot occur together), their joint probability is 0, hence:




- Mutually exclusive; no points in common…
- For example A = tosses totaling 7 and
- B = tosses totaling 11

# Example

- **Why are some mutual fund managers more successful than others? One possible factor is where the manager earned his or her MBA. The following table compares mutual fund performance against the ranking of the school where the fund manager earned their MBA:**

| | Mutual fund outperforms the market | Mutual fund doesn't outperform the market |
|---|---|---|
| Top 20 MBA program | **.11** | **.29** |
| Not top 20 MBA program | **.06** | **.54** |

E.g. This is the probability that a mutual fund outperforms **AND** the manager was in a top-20 MBA program; it's a **joint probability**.

Alternatively, we could introduce shorthand notation to represent the events:

- A1 = Fund manager graduated from a top-20 MBA program
- A2 = Fund manager did not graduate from a top-20 MBA program
- B1 = Fund outperforms the market
- B2 = Fund does not outperform the market

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | .11 | .29 |
| $A_2$ | .06 | .54 |

E.g. P(A2 and B1) = .06 = the probability a fund outperforms the market and the manager isn't from a top-20 school.

# Marginal Probabilities

Marginal probabilities are computed by adding across rows and down columns; that is they are calculated in the margins of the table:

$P(A_2) = .06 + .54$
"what's the probability a fund manager isn't from a top school?"

| | $B_1$ | $B_2$ | $P(A_i)$ |
|---|---|---|---|
| $A_1$ | **.11** | **.29** | **.40** |
| $A_2$ | **.06** | **.54** | **.60** |
| $P(B_i)$ | **.17** | **.83** | **1.00** |

$P(B_1) = .11 + .06$
"what's the probability a fund outperforms the market?"

BOTH margins must add to 1
(useful error check)

# Conditional Probability

Conditional probability is used to determine how two events are related; that is, we can determine the probability of one event given the occurrence of another related event. Conditional probabilities are written as P(A | B) and read as “the probability of A given B” and is calculated as:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

Again, the probability of an event given that another event has occurred is called a conditional probability…

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example What's the probability that a fund will outperform the market given that the manager graduated from a top-20 MBA program?

Recall:

A1 = Fund manager graduated from a top-20 MBA program

A2 = Fund manager did not graduate from a top-20 MBA program

B1 = Fund outperforms the market

B2 = Fund does not outperform the market

Thus, we want to know "what is P(B1 | A1)

We want to calculate $\mathbf{P(B_1 \mid A_1)}$

| | $B_1$ | $B_2$ | $P(A_i)$ |
|---|---|---|---|
| $A_1$ | .11 | .29 | .40 |
| $A_2$ | .06 | .54 | .60 |
| $P(B_j)$ | .17 | .83 | 1.00 |

$$P(B_1 \mid A_1) = \frac{P(A_1 \text{ and } B_1)}{P(A_1)} = \frac{.11}{.40} = .275$$

Thus, there is a 27.5% chance that that a fund will outperform the market given that the manager graduated from a top-20 MBA program.

# Independence

One of the objectives of calculating conditional probability is to determine whether two events are related.
In particular, we would like to know whether they are independent, that is, if the probability of one event is not affected by the occurrence of the other event.
Two events A and B are said to be independent if
$P(A|B) = P(A)$ or $P(B|A) = P(B)$

For example, we saw that $P(B1 \mid A1) = .275$
The marginal probability for B1 is: $P(B1) = 0.17$
Since $P(B1|A1) \neq P(B1)$, B1 and A1 are not independent events.
Stated another way, they are dependent.
That is, the probability of one event (B1) is affected by the occurrence of the other event (A1).

# Union Example:

- **Determine the probability that a fund outperforms (B1) or the manager graduated from a top-20 MBA program (A1).**

$A_1$ or $B_1$ occurs whenever:

$A_1$ and $B_1$ occurs, $A_1$ and $B_2$ occurs, or $A_2$ and $B_1$ occurs…

| | $B_1$ | $B_2$ | $P(A_i)$ |
|---|---|---|---|
| $A_1$ | **.11** | **.29** | .40 |
| $A_2$ | **.06** | **.54** | .60 |
| $P(B_j)$ | .17 | .83 | 1.00 |

$P(A_1 \text{ or } B_1) = .$

i.e. at $A_2$ and $B_2$

| | $B_1$ | $B_2$ | $P(A_i)$ |
|---|---|---|---|
| $A_1$ | .11 | .29 | .40 |
| $A_2$ | .06 | **.54** | .60 |
| $P(B_j)$ | .17 | .83 | 1.00 |

$P(A_1 \text{ or } B_1) = 1 - P(A_2 \text{ and } B_2) = 1 - .54 = \mathbf{.46}$

# Probability Rules and Trees

We introduce three rules that enable us to calculate the probability of more complex events from the probability of simpler events…

The Complement Rule,
The Multiplication Rule, and

The Addition Rule

# Complement Rule

- As we saw earlier with the complement event, the complement rule gives us the probability of an event NOT occurring.
- That is: P(AC) = 1 – P(A)
- For example, in the simple roll of a die, the probability of the number “1” being rolled is 1/6
- The probability that some number other than “1” will be rolled is 1 – 1/6 = 5/6.

# Multiplication Rule

The multiplication rule is used to calculate the joint probability of two events. It is based on the formula for conditional probability defined earlier:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

If we multiply both sides of the equation by P(B) we have:

$P(A \text{ and } B) = P(A \mid B) \bullet P(B)$

Likewise, $P(A \text{ and } B) = P(B \mid A) \bullet P(A)$

If A and B are independent events, then $P(A \text{ and } B) = P(A) \bullet P(B)$

# Example

- **A graduate statistics course has seven male and three female students. The professor wants to select two students at random to help her conduct a research project. What is the probability that the two students chosen are female? probability that the two students chosen are female?**

•Let A represent the event that the first student is female $P(A) = 3/10 = .30$

•Let B represent the event that the second student is female $P(B \mid A) = 2/9 = .22$

•That is, the probability of choosing a female student ***given*** that the first student chosen is 2 (females) / 9 (remaining students) = 2/9

•Thus, we want to answer the question: what is **P(A and B)** ?

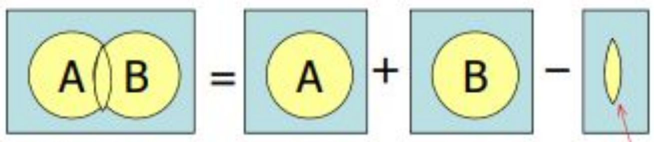
•$P(A \text{ and } B) = P(A) \cdot P(B|A) = (3/10)(2/9) = 6/90 = .067$

•"There is a 6.7% chance that the professor will choose two female students from her grad class of 10."

# Addition Rule

Recall: the addition rule is used to compute the probability of event A or B or both A and B occurring; i.e. the union of A and B.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$




$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

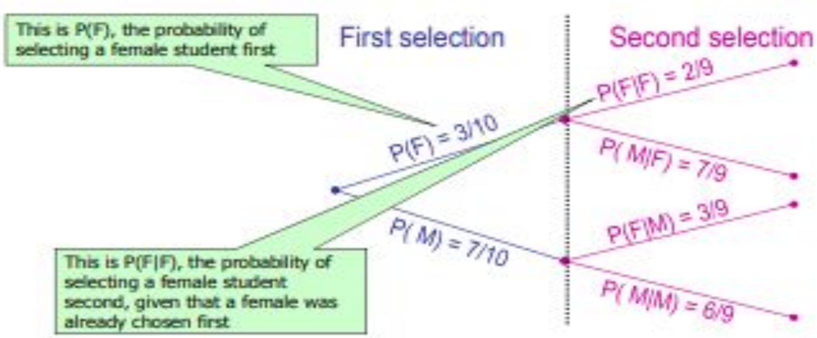
If A and B are mutually exclusive, then this term goes to zero

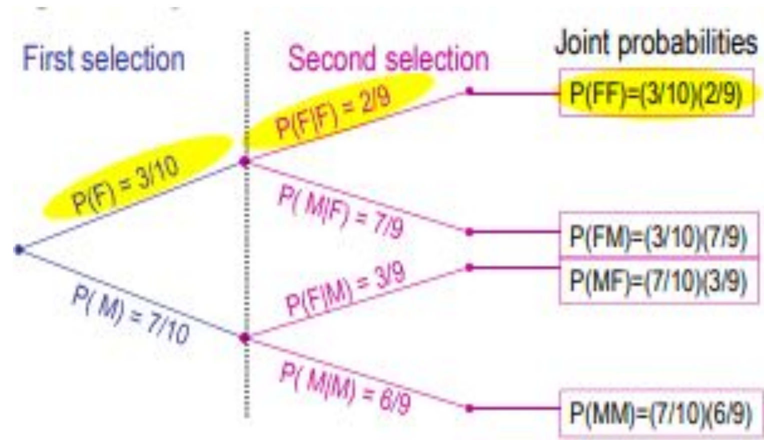
# Example

- **In a large city, two newspapers are published, the Sun and the Post. The circulation departments report that 22% of the city's households have a subscription to the Sun and 35% subscribe to the Post subscribe to the Post. A survey reveals that 6% of all households subscribe to both newspapers. What proportion of the city's households subscribe to either newspaper?**
- P(Sun or Post) = P(Sun) + P(Post) – P(Sun and Post)
- = .22 + .35 – .06 = .51
- "There is a 51% probability that a randomly selected household subscribes to one or the other or both papers"

# Probability Trees

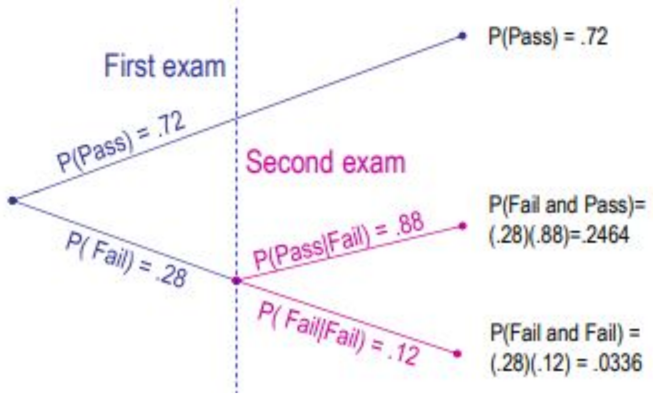
- An effective and simpler method of applying the probability rules is the probability tree, wherein the events in an experiment are represented by lines. The resulting figure resembles a tree, hence the name. We will illustrate the probability tree with several examples, including two that we addressed using the probability rules alone.



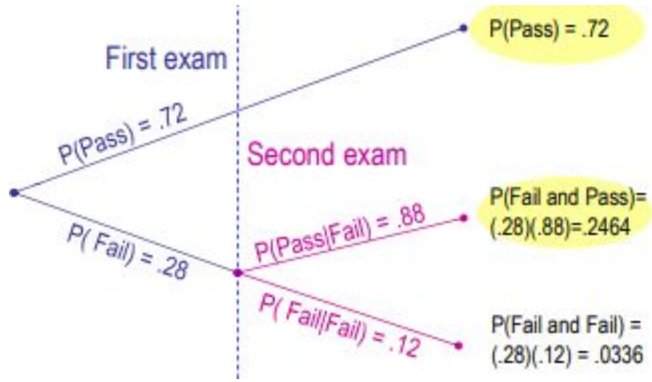
# Example

- **Law school grads must pass a bar exam. Suppose pass rate  for first-time test takers is 72%. They can re-write if they fail  and 88% pass their second attempt. What is the probability  that a randomly grad passes the bar?**



- What is the probability that a randomly grad passes the bar?
- ***"There is almost a 97% chance they will pass the bar"***
- P(Pass) = P(Pass $1^{st}$) + P(Fail $1^{st}$ and Pass $2^{nd}$)

$$= 0.7200 + 0.2464 = .9664$$



# Bayes' Law

- Bayes' Law is named for Thomas Bayes, an eighteenth century mathematician.
- In its most basic form, if we know P(B | A),
- we can apply Bayes' Law to determine P(A | B)

- The probabilities P(A) and P(AC) are called prior probabilities because they are determined prior to the decision about taking the preparatory course.

- The conditional probability P(A | B) is called a posterior probability (or revised probability), because the prior probability is revised after the decision about taking the preparatory course.