Probability

Why do we need probability?

Probability theory allows us to

- Analyze uncertain or random processes that takes place in real world.
- Make decisions in Uncertain scenarios.

Examples

- Gambling: How do we estimate the returns?
- Model building: How do we derive stochastic component of model?
- Lotteries
- Simulation

What do you mean by probability?

Frequentist - probabilities are the numbers that are derived from outcome of a large number of experiments or trials.

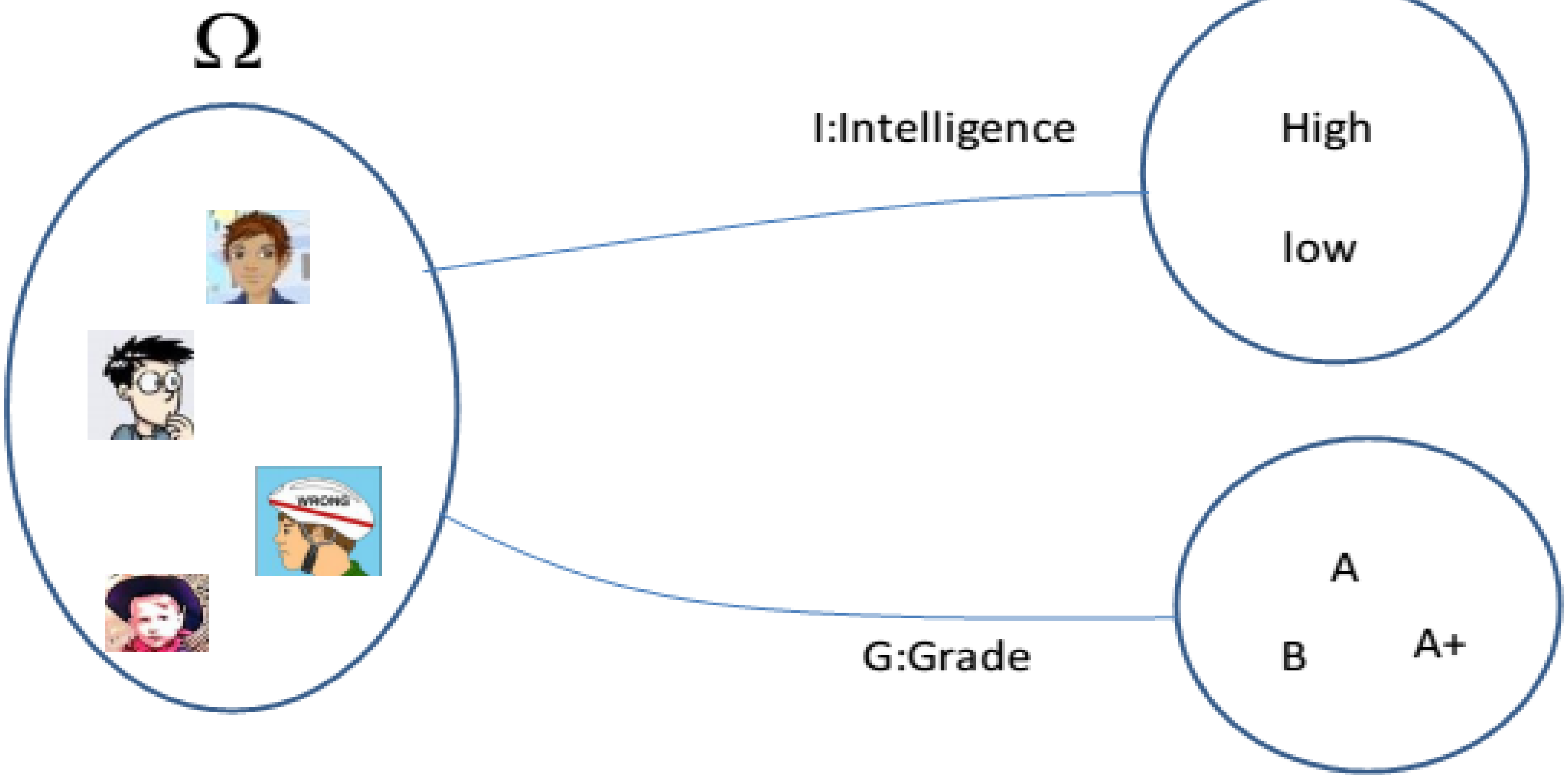
Bayesian - probabilities are numbers that express a degree of belief in something given any relevant evidence.

Random Experiments & Random Variables

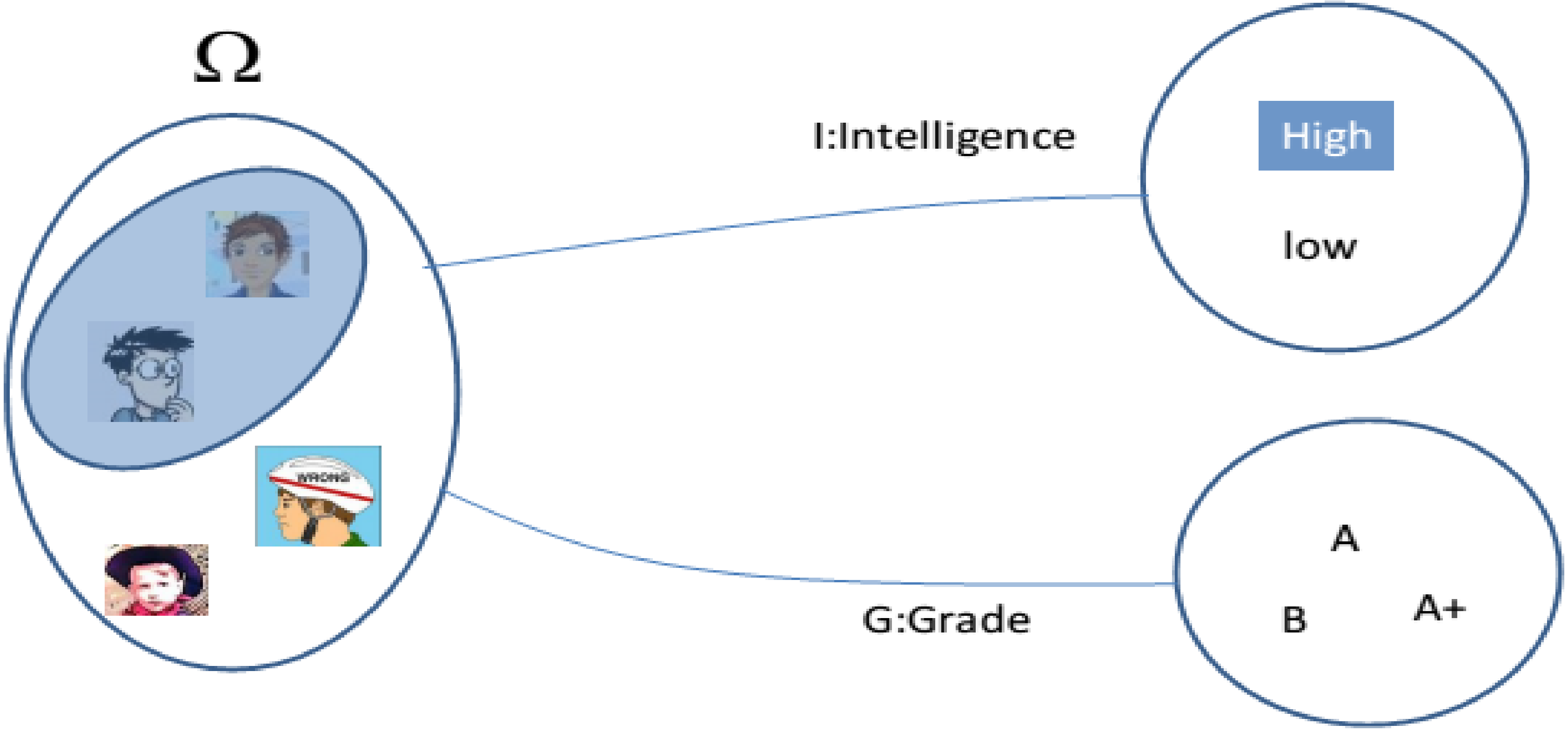
Random Variable

A variable is said to be a **random variable** when it comes from a random experiment

Random Variables



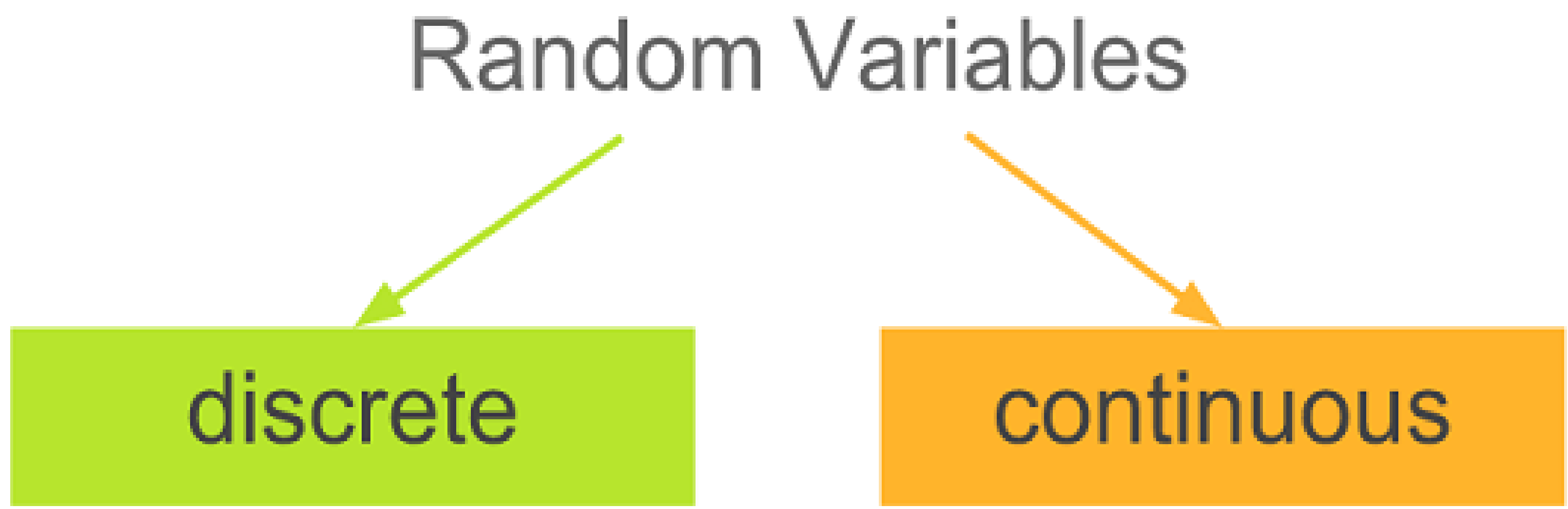
Random Variables



P(I = high) = P({all students whose intelligence is high})

P(G = A) = P({all students whose grade is A})

Types of random variables



Discrete RV Example

Experiment: toss a die and observe the value that appears on the top face

Random Variable: number on the top face

Continuous RV Example

Measurement

X: the height of an individual

Basic Probability

Probabilities

- What is the probability for getting heads if we toss a coin?

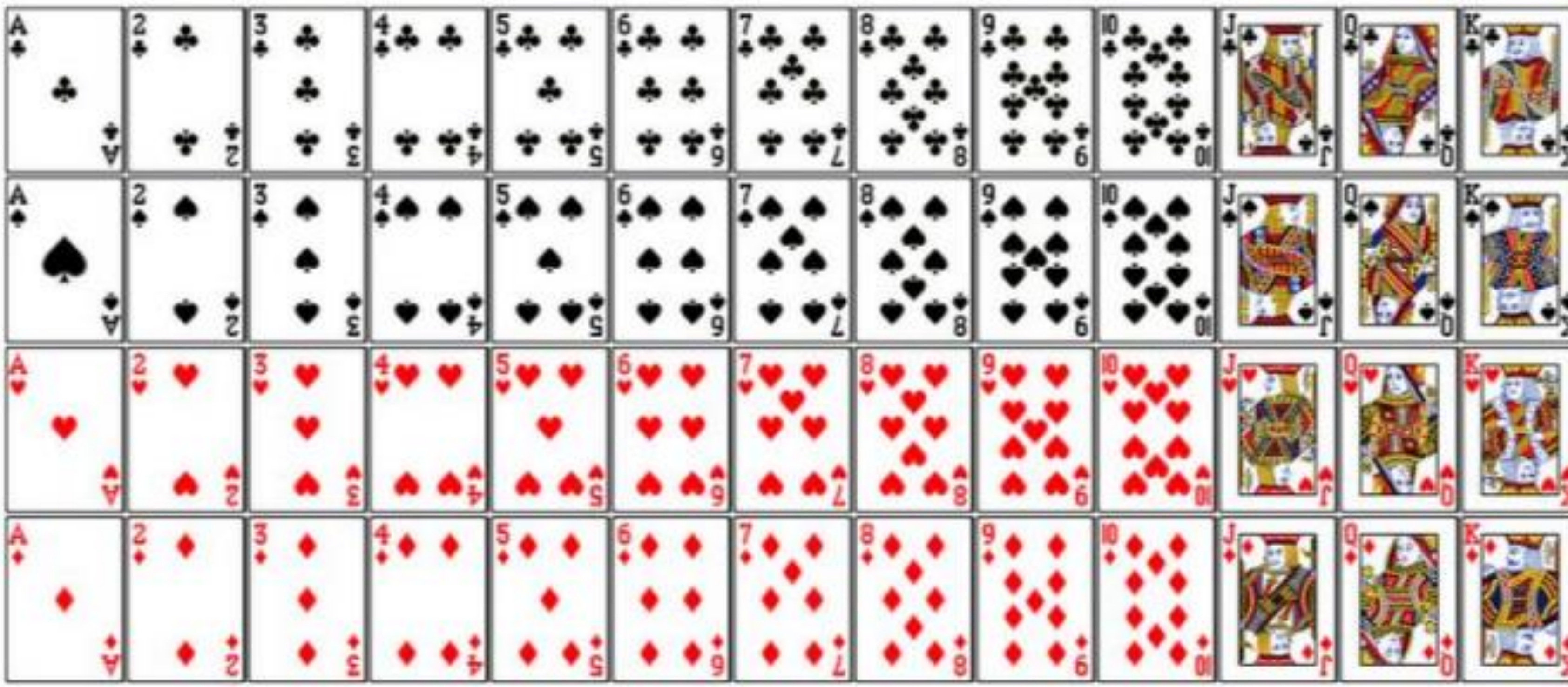
- What is the probability that we get 6 if we throw a dice?

Classical Approach

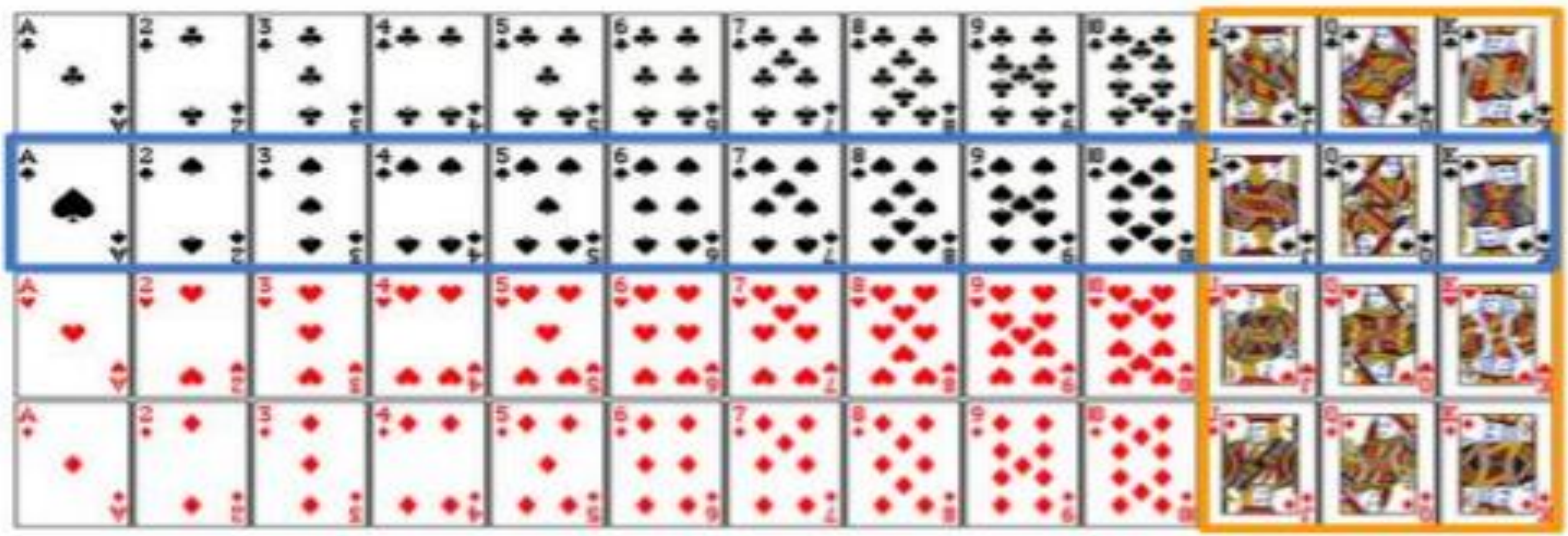
$$\text{Prob} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Addition Rule for Probabilities

Find the probability with which either spade or face card is selected?



Addition Rule



A = event a spade card is selected

B = event a face card is selected

P(A) = 13 / 52

P(B) = 12 / 52

P(A *and* B) = 3 / 52

P(A *or* B) = 13 / 52 + 12 / 52 - 3 / 52 = 22 / 52

Addition Rule

Probability of “A or B”

The probability of a composite event “A or B” is expressed as

$$P(A \textit{ or } B) = P(A) + P(B) - P(A \textit{ and } B)$$

Addition rule: disjoint events

If A and B are disjoint (i.e. mutually exclusive)

$$P(A \textit{ or } B) = P(A) + P(B) - \cancel{P(A \textit{ and } B)}$$

$$P(A \textit{ or } B) = P(A) + P(B)$$

Conditional Probabilities

Conditional Probabilities

	Gender	
Age	Females	Males
Under 25 years	48	50
25 years & over	74	66

a) If we pick random person, what is the probability for being male?
b) If we pick random person, what is the probability for being under 25 years?
c) If we pick random female, what is the probability for being above 25 years?
d) If we pick random person under 25 years, what is the probability for being female?

Conditional Probability

- The probability that event A occurs if **we know for certain that event B** will occur is called **conditional probability**.

- The conditional probability of A given B is denoted: **P(A/B)** [Read as the probability of A given B]

Definition of Conditional Probability

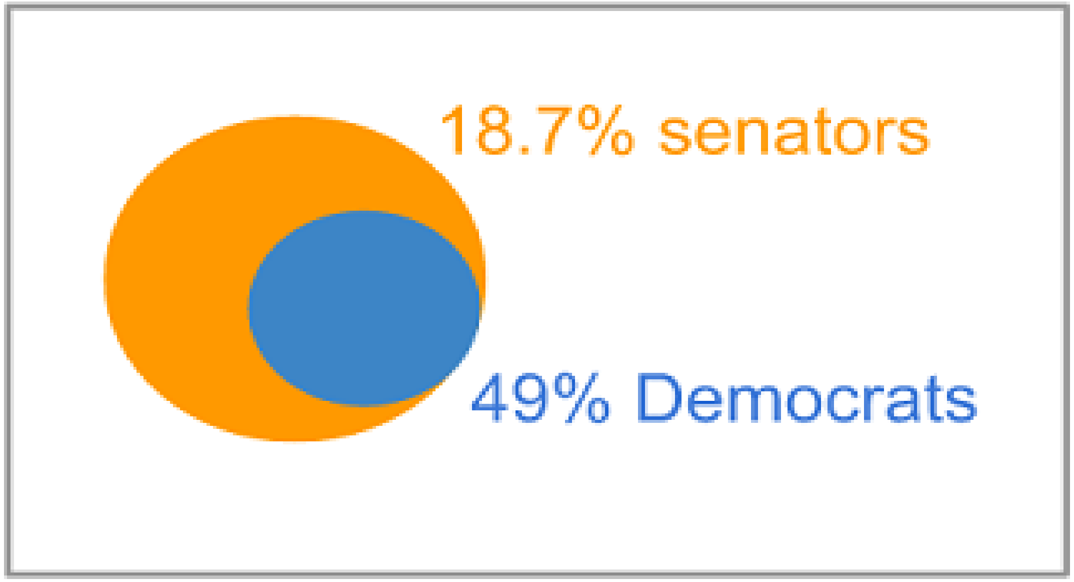
$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Corollary: The Chain Rule

$$P(A \text{ and } B) = P(A|B)\, P(B)$$

Multiplication Rule for Probability

110th Congress



Probability that a randomly selected member
is a **Democratic** **Senator**?

Multiplication Rule

$$P(Dem \textit{ and } Sen) = P(Sen)\, P(Dem \mid Sen)$$

$$P(D \textit{ and } S) = 0.187 \times 0.49 = 0.092$$

Multiplication Rule (general case)

For any events A and B:

$$P(A \textit{ and } B) = P(A)\,P(B \mid A)$$

or

$$P(B \textit{ and } A) = P(B)\,P(A \mid B)$$

Chain Rule

For 2 variables:

$$P(A,B) = P(A|B)\,P(B)$$

For 3 variables:

$$P(A,B,C) = P(A|B,C)\,P(B,C)$$
$$= P(A|B,C)\,P(B|C)\,P(C)$$

For n variables:

$P(A1, A2, \ldots, An) = P(A1|A2, \ldots, An)\,P(A2|A3, \ldots, An)\,P(An\text{-}1|An)\,P(An)$

Multiplication Rule (independent events)

If events A and B are independent:

$$P(A \textit{ and } B) = P(A)\,P(B)$$

or

$$P(B \textit{ and } A) = P(B)\,P(A)$$

B is independent of A means that knowing B does not change our belief about A.

What is the probability of the following outcomes when a fair coin is independently tossed 6 times? (H: heads, T: tails)

- H H H H H H

Relating Conditional Probabilities

Problem1: Cancer Diagnosis

- Let's suppose we want to diagnose patients who have taken a test for cancer
- We know the test is 90% reliable (90% of people with the cancer get a positive test result, 10% of people without the cancer also get a positive result)
- 2% of the total population have this cancer
- Suppose we have a positive test result
- What is the probability our patient has cancer?

Problems2: Fair Coin or Not?

We have a coin that can be either fair or double headed. We had prior belief of it being fair with 0.9 probability.

- If we get heads after tossing coin, what is the probability for coin being fair?
- If we get heads again after second toss, what is the probability for coin being fair?
- Does the prior probability has any impact on our analysis?

Bayes' Rule

- The most important rule in all of probability
- Lets you "flip round" conditional probabilities
- Updates our belief about *a* in light of new information *b*

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$\underbrace{p(C \mid \mathbf{x})}_{\text{posterior probability}} = \frac{\overbrace{p(\mathbf{x} \mid C)}^{\text{likelihood}}\,\overbrace{p(C)}^{\text{prior probability}}}{\underbrace{p(\mathbf{x})}_{\text{evidence}}}$$

Probability Rules

Description (Rule)	Expression
Range	$0 \leq P(A) \leq 1$
Complementary probability	$P(A^c) = 1 - P(A)$
Decomposition	$P(A) = P(A \textit{ and } B) + P(A \textit{ and } B^c)$
Conditional probability	$P(B \mid A) = P(A \textit{ and } B) / P(A)$
Independence	$P(B \mid A) = P(B)$; $P(A \mid B) = P(A)$
Multiplication rule	$P(A \textit{ and } B) = P(B \mid A)\, P(A)$
Addition rule	$P(A \textit{ or } B) = P(A) + P(B) - P(A \textit{ and } B)$
Baye's rule	$P(B \mid A) = P(A \mid B)\, P(B) / P(A)$

Expected Value, Variance & SD

A certain lottery works by picking 6 numbers from 1 to 49. It costs $1.00 to play the lottery, and if you win, you win $2 million after taxes. Answer the following questions:

a. If you play the lottery once, what are your expected winnings or losses?
b. If you play the lottery every week for 10 years, what are your expected winnings or losses?

Expectation or Expected Value

$$E(\mathbf{X}) = \mu$$

Expected Value (*Mu*)

The expected value of X is a weighted average of the possible values of X

Each value of X is weighted by the assigned probability

Expected Value Formula

$$\mu = p_1x_1 + p_2x_2 + \dots + p_nx_n$$

$$\mu = \sum_{i=1}^{n} p_i x_i$$

The probability function

x$	*p(x)*
-1	.999999928
+ 2 million	7.2×10^{-8}

Calculate the probability of winning in 1 try:

$$\frac{1}{\binom{49}{6}} = \frac{1}{\frac{49!}{43!6!}} = \frac{1}{13{,}983{,}816} = 7.2 \times 10^{-8}$$

"49 choose 6"

Out of 49 numbers, this is the number of distinct combinations of 6.

Expected Value

The probability function

x$	*p(x)*
-1	.999999928
+ 2 million	7.2×10^{-8}

Expected Value

$$
\begin{aligned}
E(X) &= P(win)*\$2{,}000{,}000 \;+\; P(lose)*-\$1.00 \\
&= 2.0 \times 10^{6} * 7.2 \times 10^{-8} + .999999928\,(-1) \\
&= .144 - .999999928 = -\$.86
\end{aligned}
$$

Conclusion for 1 time play:

Negative expected value is never good!

You shouldn't play if you expect to lose money!

Expected Value

Conclusion for playing lottery every week for 10years:

$520 \times (-.86) = -\$447.20$

Negative expected value is never good!

You shouldn't play if you expect to lose money!

Infomercial example

X: # of times buyers of a product watched a TV informercial before purchasing it

# times watched TV infomercial	*1*	*2*	*3*	*4*	*5*
Percentage of buyers	27%	31%	18%	9%	15%

- Find the expected number of times buyers of a product watched TV commercial?

- Find the standard deviation of number of viewing behavior?

Variance of a random variable

$$\mathrm{Var}(\mathbf{X}) = \sigma^2 \qquad \sigma^2 = \sum_{i=1}^{n} p_i (x_i - \mu)^2$$

Standard deviation of an RV

$$\sigma = \sqrt{\sum_{i=1}^{n} p_i (x_i - \mu)^2}$$