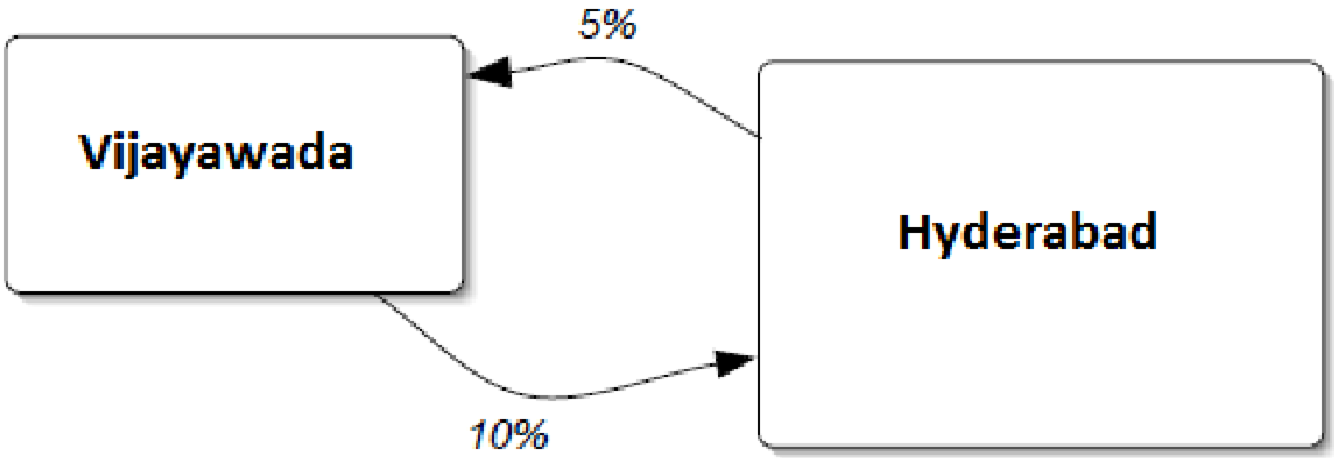
Yearly Migration Pattern between two Cities



- What will be population in two cities after n years, given the initial population vector?
- What will be long term behavior of the above system?
- If such a long term behavior of system can be determined, does it depend on the initial population vector?

Part - I

$$v_{n+1} = .9\,v_n + .05\,r_n$$
$$r_{n+1} = .1\,v_n + .95\,r_n$$

$$\begin{pmatrix} v_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} .9\,v_n + .05\,r_n \\ .1\,v_n + .95\,r_n \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v_n \\ r_n \end{pmatrix}$$

Part - I

$$A = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix}$$

We multiply population vectors by the matrix A to go from one year to the next.

$$\begin{pmatrix} v_{n+1} \\ r_{n+1} \end{pmatrix} = A \begin{pmatrix} v_n \\ r_n \end{pmatrix}$$

If we write $\vec{p} = \binom{v}{r}$ we can write this even shorter as

$$\vec{p}_{n+1} = A\vec{p}_n$$

Part - I

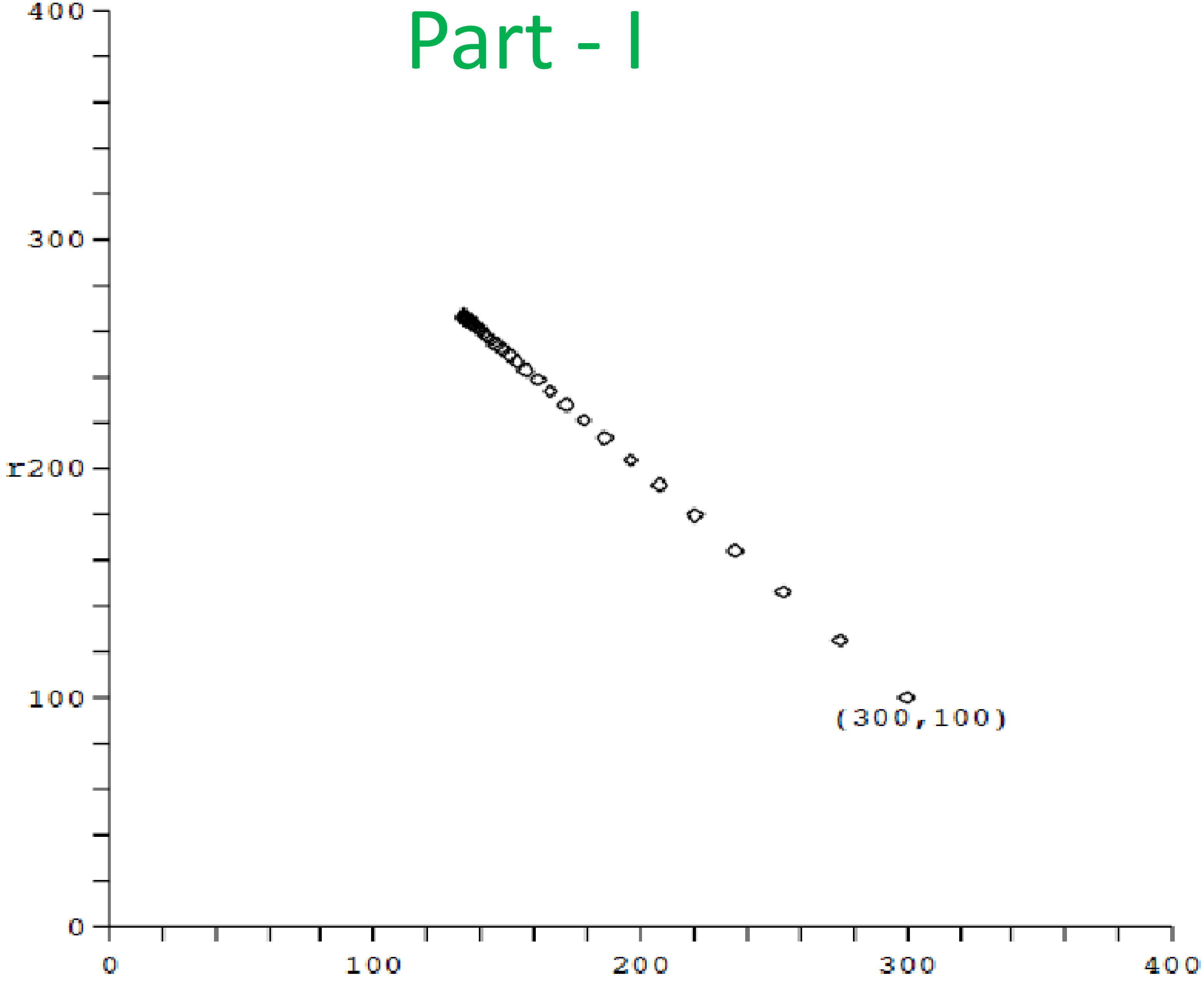
$$\begin{pmatrix} v_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} 300 \\ 100 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ r_1 \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} 300 \\ 100 \end{pmatrix} = \begin{pmatrix} 275 \\ 125 \end{pmatrix}$$

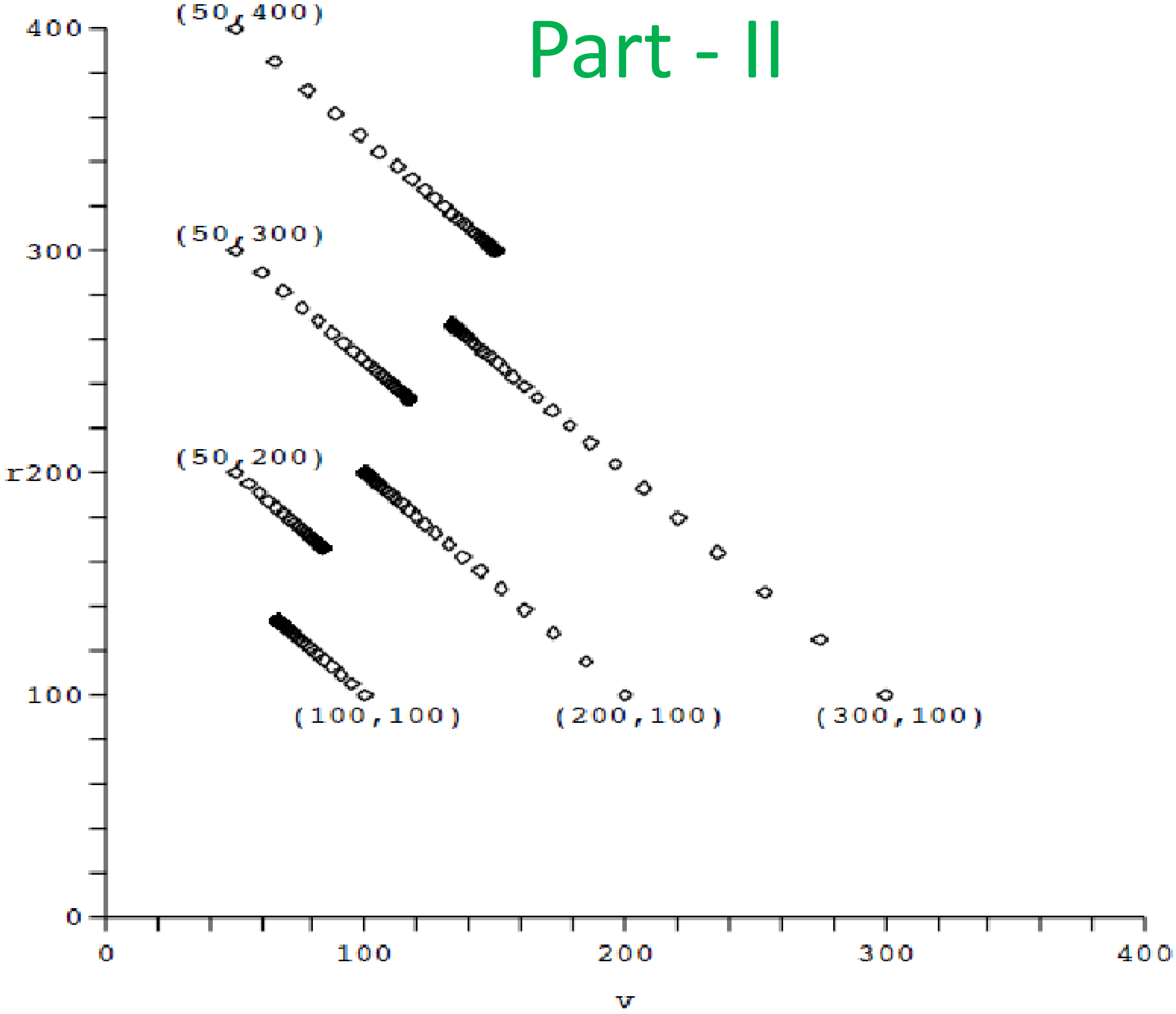
$$\begin{pmatrix} v_2 \\ r_2 \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} 275 \\ 125 \end{pmatrix} = \begin{pmatrix} 254 \\ 146 \end{pmatrix}$$

$$\begin{pmatrix} v_3 \\ r_3 \end{pmatrix} = \begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} 254 \\ 146 \end{pmatrix} = \begin{pmatrix} 236 \\ 164 \end{pmatrix}$$

Part - I



Part - II



(50,400)
(50,300)
(50,200)
(100,100)
(200,100)
(300,100)
400
300
r200
100
0
0
100
200
300
400
v

Part - III

As number of iterations becomes more and more, the smaller the difference between P_{n+1} (=A * P_n)and P_n. It seems to reach a fixed point i.e.,$A*p_n = P_n$

Part - III

$$\begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} v \\ r \end{pmatrix}$$

$$\begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix}$$

$$\begin{pmatrix} .9 & .05 \\ .1 & .95 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} .9-1 & .05-0 \\ .1-0 & .95-1 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -.1 & .05 \\ .1 & -.05 \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is a homogeneous system of equations for the indeterminants v and r, which we easily solve using Gaussian elimination:

$$\begin{pmatrix} -.1 & .05 \\ .1 & -.05 \end{pmatrix}_{R_1+R_2} \rightarrow \begin{pmatrix} -.1 & .05 \\ 0 & 0 \end{pmatrix}^{\cdot(-10)} \rightarrow \begin{pmatrix} 1 & -.5 \\ 0 & 0 \end{pmatrix}$$

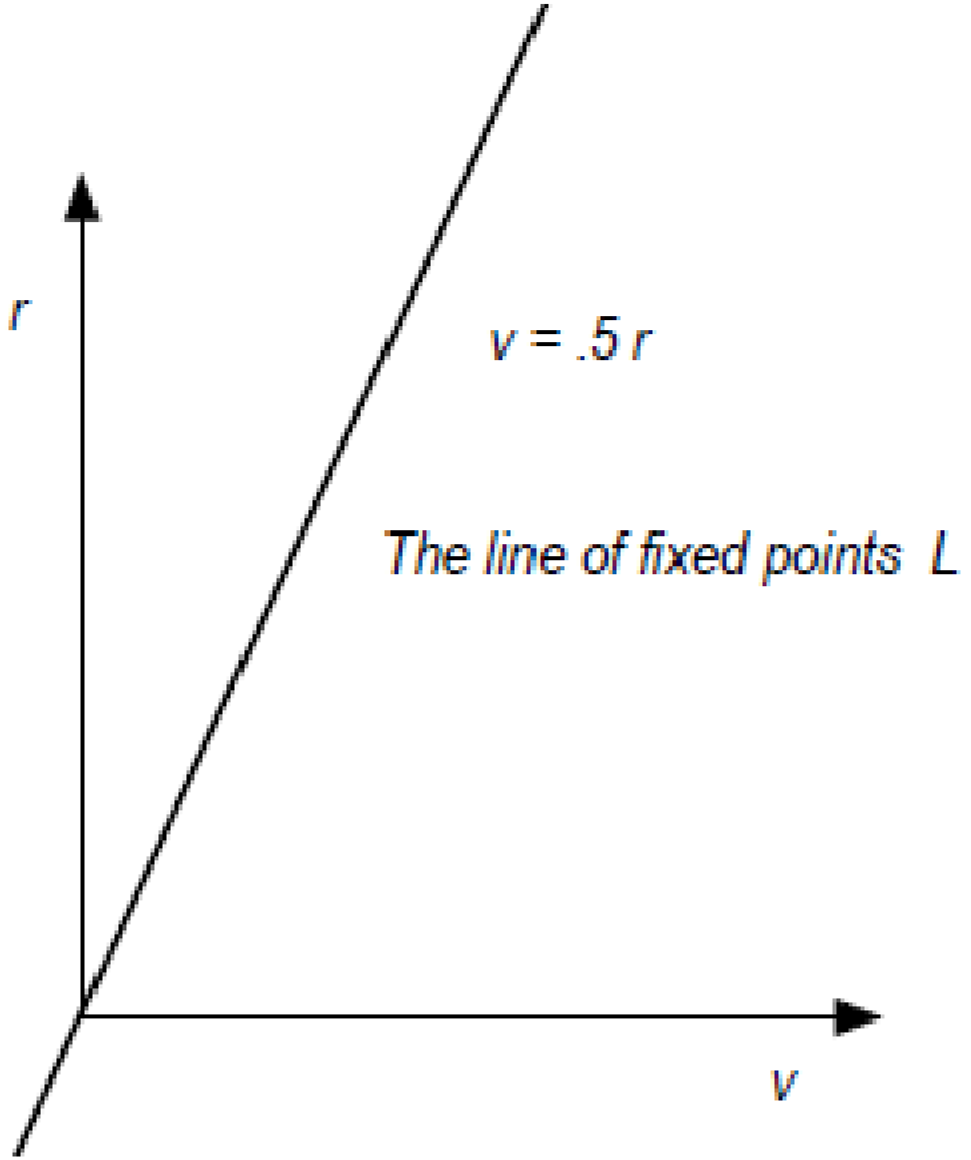
Part - III

$$\begin{aligned} v &- .5r &= 0 \\ v &+ r &= 400 \end{aligned}$$

This system has the unique solution

$$\begin{pmatrix} v \\ r \end{pmatrix} = \begin{pmatrix} 133 \\ 267 \end{pmatrix}$$

Part - III



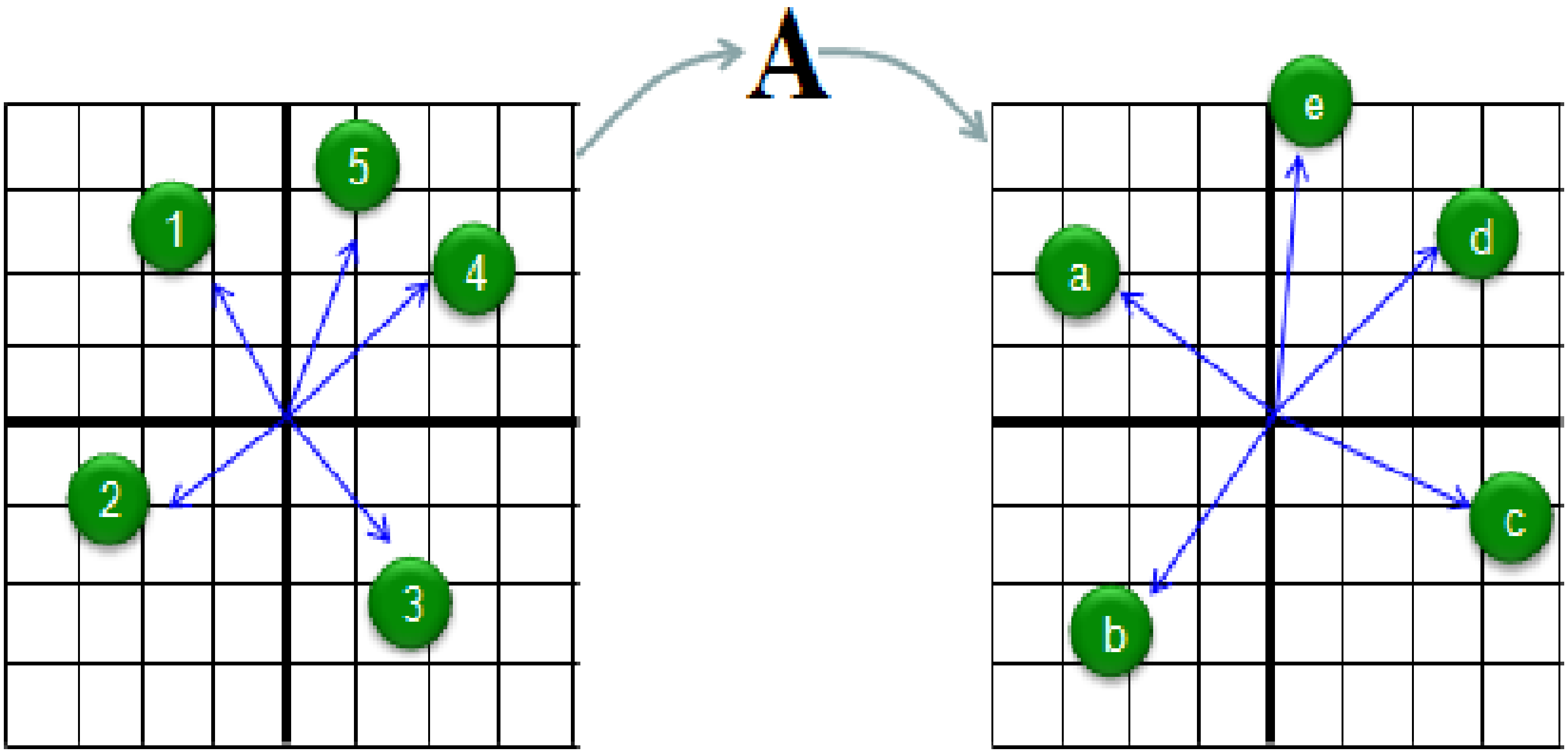
Eigen Vectors and Eigen Values

What is Eigen Vector?

In general, a matrix acts on a vector by changing both its magnitude and its direction.

However, a matrix may act on certain vectors by changing only their magnitude, and leaving their direction unchanged (or possibly reversing it). These vectors are the **eigenvectors** of the matrix.

What is Eigen Vector?



What is Eigen Vector?

1 ⇒ a : Magnitude changes, Direction Changes

2 ⇒ b : Magnitude changes, Direction Changes

3 ⇒ c : Magnitude changes, Direction Changes

4 ⇒ d : Magnitude changes, Direction does NOT Changes

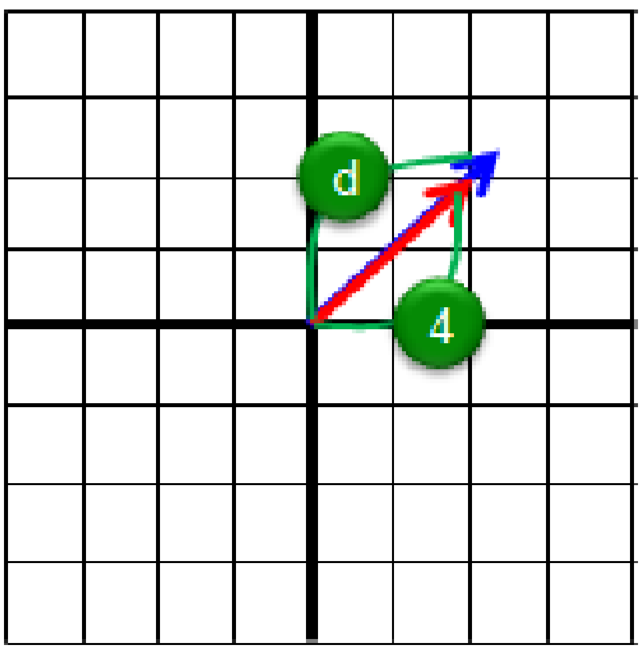
5 ⇒ e : Magnitude changes, Direction Changes

Vector (4) changes only Magnitude and does not change direction when it is transformed by the Matrix A. So this is an Eigenvector of the matrix A

What is Eigen Value?

A matrix acts on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is the **eigenvalue** associated with that eigenvector.

What is Eigen Value?



$$\frac{\text{Magnitude of vector } d}{\text{Magnitude of vector } 4} = \lambda$$

Mathematical perspective

A nonzero vector **x** is an ***eigenvector*** (or *characteristic vector*) of a square matrix **A** if there exists a scalar **λ** such that **Ax = λx**. Then **λ** is an ***eigenvalue*** (or *characteristic value*) of **A**.

Eigenvalue Computing Process

- Let x be an eigenvector of the matrix **A.** Then there must exist an eigenvalue λ such that $\mathbf{Ax = \lambda x}$ or, equivalently,

$$\mathbf{Ax - \lambda x = 0} \quad \text{or}$$

$$\mathbf{(A - \lambda I)x = 0}$$

- If we define a new matrix $\mathbf{B = A - \lambda I}$, then B$\mathbf{x = 0}$. If **B** has an inverse then $\mathbf{x = B^{-1}0 = 0}$. But an eigenvector cannot be zero.
- Thus, it follows that x will be an eigenvector of **A** if and only if **B** does not have an inverse, or equivalently $\mathbf{det(B)=0}$, or

$$\mathbf{det(A - \lambda I) = 0}$$

- This is called the **characteristic equation** of **A**. Its roots determine the eigenvalues of **A**.

Computing Eigenvalues: Examples

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: $-1, -2$

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Eigenvector Computing Process

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If $\boldsymbol{\lambda}_i$ is an eigenvalue then the corresponding eigenvector $\mathbf{x}_i$ is the solution of $(\mathbf{A} - \boldsymbol{\lambda}_i\mathbf{I})\mathbf{x}_i = \mathbf{0}$

Note: The zero vector can not be an eigenvector even though $\mathbf{A0} = \boldsymbol{\lambda}\mathbf{0}$. But $\boldsymbol{\lambda} = 0$ can be an eigenvalue.

Computing Eigenvectors: Examples

Example 1 (cont.):

$$\lambda = -1 : (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2 : (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

Example 2 (cont.): Find the eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Recall that λ = 2 is an eigenvector of multiplicity 3.

Solve the homogeneous linear system represented by

$$(2I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_1 = s,\ x_3 = t$. The eigenvectors of λ = 2 are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

s and *t* not both zero.