Some remarks on high-order finite-differences schemes. Consequences on the observability of the wave equation.

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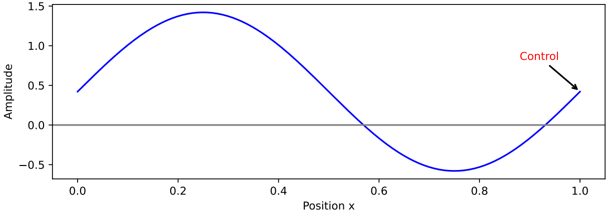
Table of contents

Motivation

$$\begin{cases} \ddot{u}(x,t) - \partial_x^2 u(x,t) = 0, & x \in (0,1),\ t \in (0,T), \\ u(0,t) = 0, \quad u(1,t) = v(t), & t \in (0,T), \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), & x \in (0,1), \end{cases} \tag{1}$$

where $u = u(x,t)$ denotes the system state, and $v = v(t)$ is the boundary control.



Control action applied at the boundary.

Null Controllability

We say that the equation (1) is **null controllable** in time $T \geq 2$ if for every initial data $(u_0, u_1) \in L^2(0,1) \times H^{-1}(0,1)$, there exists a control $v \in L^2(0,T)$ such that

$$u(x,T) = 0,\ \dot{u}(x,T) = 0,\ \forall x \in (0,1).$$

The **controllability problem** introduced previously can be reformulated in terms of the **adjoint wave equation**:

$$\begin{cases} \ddot{\varphi}(x,t) - \partial_x^2 \varphi(x,t) = 0, & x \in (0,1),\ t \in (0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), \quad \dot{\varphi}(x,0) = \varphi_1(x), & x \in (0,1). \end{cases} \tag{2}$$

The energy of the solutions is conserved in time

$$E(t) = \frac{1}{2}\int_0^1 \left(|\partial_x \varphi(x,t)|^2 + |\dot{\varphi}(x,t)|^2 \right) dx = E(0), \quad \forall\, 0 \leq t \leq T.$$

Observability inequality

The equation (2) is **observable** in time $T \geq 2$ if there exists a constant $C_{obs} > 0$ such that for every initial data $(\varphi_0, \varphi_1) \in H_0^1(0,1) \times L^2(0,1)$ the solution verifies

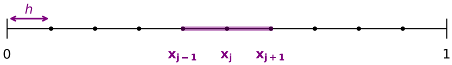
$$E(0) \leq C_{obs} \int_0^T |\partial_x \varphi(1,t)|^2 \, dt.$$

Remark

Exact controllability in time $T \Longleftrightarrow$ Observability of the adjoint system (via HUM).

Lions (1988), *Exact controllability, stabilization and perturbations, Lecture Notes in Math., 170, Springer*

Semi-discretization in space using second order finite-differences



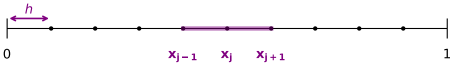
$$\partial_x^2 \varphi(x_j) \approx \frac{\varphi(x_{j-1}) - 2\varphi(x_j) + \varphi(x_{j+1})}{h^2}$$

Notations

$$\mathcal{U}_h = \{ w_h = (w_1, \dots, w_N) \in \mathbb{R}^N \}, \quad (w_h, z_h)_h = h \sum_{i=1}^{N} w_i z_i, \quad \|w_h\|_h^2 = (w_h, w_h)_h$$

$$A_h^2 = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad \left(\partial_R^1 w_h\right)_i = \frac{w_{i+1} - w_i}{h} \text{ (forward difference)}$$

Semi-discretization in space using second order finite-differences



$$\partial_x^2 \varphi(x_j) \approx \frac{\varphi(x_{j-1}) - 2\varphi(x_j) + \varphi(x_{j+1})}{h^2}$$

Semi-discretized wave equation

$$\begin{cases} \ddot{\varphi}_h(t) + A_h^2 \varphi_h(t) = 0, & t \in (0, T), \\ \varphi_h(0) = \varphi_{0,h}, \quad \dot{\varphi}_h(0) = \varphi_{1,h}, \end{cases} \tag{3}$$

where

$$\varphi_h = (\varphi_1, \ldots, \varphi_N)^T \in \mathbb{R}^N.$$

The energy of the solutions is conserved in time

$$E_h(t) = \frac{1}{2}\|\dot{\varphi}_h(t)\|_h^2 + \frac{1}{2}\|\partial_R^1 \varphi_h(t)\|_h^2, \quad t \in [0, T],$$

Discrete observability inequality

$$E_h(0) \leq C_h \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt.$$

Blow-up of the discrete observability constant

Blow-up of the estimate

Discrete observability fails for $T \geq 2$

Unbounded constant: $C_h \to \infty$ as $h \to 0$

Infante & Zuazua (1999), *Boundary observability for the space semi-discretizations of the 1-D wave equation.*

Blow-up of the discrete observability constant

Blow-up of the estimate

Spectral properties of A_h^2

$$\begin{aligned}
A_h^2\,\phi_k &= \lambda_{k,2}\,\phi_k,\\
\phi_k &= (\phi_{k,1},\ldots,\phi_{k,N})^T,\\
\phi_{k,i} &= \sin(i\pi kh), \quad i = 1,\ldots,N,\\
\lambda_{k,2} &= \frac{4}{h^2}\sin^2\left(\frac{\pi kh}{2}\right), \quad k = 1,\ldots,N.
\end{aligned}$$

Blow-up of the discrete observability constant

Blow-up of the estimate

Spectral properties of the discretization matrix

$$
\begin{aligned}
A_h^2\,\phi_k &= \lambda_{k,2}\,\phi_k,\\
\phi_k &= (\phi_{k,1},\dots,\phi_{k,N})^T,\\
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\end{aligned}
$$

Multiplier method

$$
\begin{aligned}
v_h(t) &= J\cdot\partial_C^1\varphi_h(t),\\
J &= (h,2h,\dots,Nh)^T,\\
(\ddot\varphi_h(t),v_h(t))_h &+ (A_h^2\varphi_h(t),v_h(t))_h = 0
\end{aligned}
$$

Blow-up of the discrete observability constant

Blow-up of the estimate

$C_h \to \infty$ as $h \to 0$

Spectral properties

$$A_h^2\,\phi_k = \lambda_{k,2}\,\phi_k,$$

$$\phi_k = (\phi_{k,1},\dots,\phi_{k,N})^T,$$

$$\phi_{k,i} = \sin(i\pi kh), \quad i = 1,\dots,N,$$

$$\lambda_{k,2} = \frac{4}{h^2}\,\sin^2\left(\frac{\pi kh}{2}\right), \quad k = 1,\dots,N.$$

Multiplier method

$$v_h(t) = J \cdot \partial_C^1 \varphi_h(t),$$

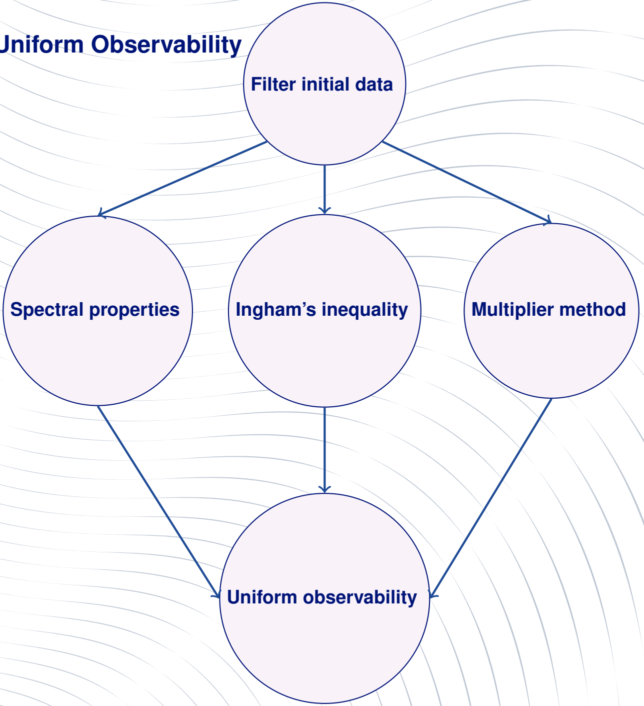
$$J = (h, 2h, \dots, Nh),$$

$$(\ddot\varphi_h(t), v_h(t))_h + (A_h^2\varphi_h(t), v_h(t))_h = 0.$$

Observability constant

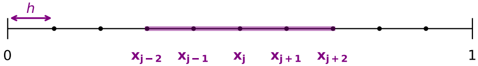
$$C_h = \frac{2}{T\,(4 - \lambda_{k,2}h^2)}$$

Restoring Uniform Observability



Higher-order finite differences approximation
Order n approximation of the d-th derivative

Let d be an even strictly positive integer and $m = \frac{d+n-2}{2}$.



$$\partial_x^d \varphi(x_j) \approx \frac{1}{h^d} \sum_{k=-m}^{m} a_k^{d,n} \varphi(x_{j+k})$$

Higher-order finite differences approximation
Order n approximation of the d-th derivative

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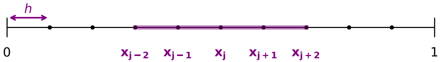

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Hildebrand (1987), *Introduction to numerical analysis*
Fornberg (1988), *Math. Comp.*
Khan & Ohba (1999), *J. Comput. Appl. Math.*
Mashreghi & Yazdi (2014), *J. Comput. Appl. Math.*
Sadiq & Viswanath (2014), *Math. Comp.*

$$\partial_x^d \varphi(x_j) \approx \frac{1}{h^d} \sum_{k=-m}^{m} a_k^{d,n} \varphi(x_{j+k})$$

Research on the coefficients $a_k^{d,n}$

Efficient computation for any derivative order and grid

Accuracy and truncation error analysis

Stability in high-order finite difference formulas

Adaptation to non-uniform grids

$$a_k^{d,n} = (-1)^{\frac{d}{2}+k} \frac{d!}{(d+n-2)!} \binom{d+n-2}{\frac{d+n-2}{2}-k} e_{\frac{n-2}{2}}\left(\mathrm{p}_k^{d,n}\right), \qquad 1 \leq k \leq m,$$

$$\mathrm{p}_k^{d,n} = \left(1^2, 2^2, \ldots, (k-1)^2, (k+1)^2, \ldots, m^2\right)$$

Computation of the coefficients $a_j^{d,n}$

Finite difference approximation

Let f be a sufficiently smooth function. The d-th derivative is approximated by:

$$D_h^{d,n} f(x) = \frac{1}{h^d} \sum_{j=-m}^{m} a_j^{d,n} f(x + jh), \quad m = \frac{d + n - 2}{2}.$$

Central coefficient conditions

To ensure consistency with the d-th derivative:

$$a_0^{d,n} = \sum_{j=-m, j \neq 0}^{m} a_j^{d,n}$$

Taylor expansion

For $-m \leq j \leq m, j \neq 0$:

$$f(x + jh) = \sum_{k=0}^{2m} \frac{(jh)^k}{k!} f^{(k)}(x) + \mathcal{O}(h^{d+n}).$$

Multiplying and summing over j

Multiplying the Taylor expansion by $a_j^{d,n}$ and summing over j gives:

$$h^d D_h^{d,n} f(x) = \sum_{j=-m, j \neq 0}^{m} a_j^{d,n} f(x) + \sum_{k=1}^{2m^{d,n}} \frac{h^k}{k!} f^k(x) \left(\sum_{j=-m, j \neq 0}^{m} j^k a_j^{d,n} \right) + \mathcal{O}(h^{d+n}).$$

Computation of the coefficients $a_j^{d,n}$

Taylor expansion of the finite difference

$$h^d D_h^{d,n} f(x) = \sum_{k=1}^{2m} \frac{h^k}{k!} f^{(k)}(x) \left(\sum_{j=-m, j\neq 0}^{m} j^k a_j^{d,n} \right) + \mathcal{O}(h^{d+n})$$

$$\sum_{j=-m, j\neq 0}^{m} j^k a_j^{d,n} = \begin{cases} 0, & 0 \leq k \leq 2m, k \neq d, \\ d!, & k = d. \end{cases}$$

Matrix formulation

We define the **invertible** matrix $M^{d,n} \in \mathcal{M}_{2m}(\mathbb{R})$:

$$M^{d,n} = \begin{pmatrix} -m & -m+1 & \cdots & -1 & 1 & 2 & \cdots & m \\ (-m)^2 & (-m+1)^2 & \cdots & (-1)^2 & 1^2 & 2^2 & \cdots & m^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (-m)^d & (-m+1)^d & \cdots & (-1)^d & 1^d & 2^d & \cdots & m^d \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (-m)^{2m} & (-m+1)^{2m} & \cdots & (-1)^{2m} & 1^{2m} & 2^{2m} & \cdots & m^{2m} \end{pmatrix} \in \mathcal{M}_{2m}(\mathbb{R})$$

Symmetry and reduced linear system for $a_j^{d,n}$

Symmetry of coefficients- this allows us to reduce the linear system to m unknowns instead of 2m

Due to the central stencil, we have

$$a_{-j}^{d,n} = a_j^{d,n}, \quad 1 \leq j \leq m.$$

Reduced linear system

The system with m unknowns can be written as:

$$\begin{pmatrix} 1^2 & 2^2 & \dots & m^2 \\ 1^4 & 2^4 & \dots & m^4 \\ \vdots & \vdots & & \vdots \\ 1^d & 2^d & \dots & m^d \\ \vdots & \vdots & & \vdots \\ 1^{2m} & 2^{2m} & \dots & m^{2m} \end{pmatrix} \begin{pmatrix} a_1^{d,n} \\ a_2^{d,n} \\ \vdots \\ a_{d/2}^{d,n} \\ \vdots \\ a_m^{d,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{d!}{2} \\ \vdots \\ 0 \end{pmatrix}$$

Remark

The solution of this system can be obtained in a nearly explicit form using Cramer's rule together with lacunary Vandermonde determinants.

Properties of the coefficients $a_j^{d,n}$

Notations

We define $a_j^{d,n}$ for every $j \in \mathbb{Z}$ as follows:

$$a_j^{d,n} = 0 \text{ for every } |j| > m,$$
$$\mathrm{a}^{d,n} = (a_j^{d,n})_{j\in\mathbb{Z}}.$$

Symmetry and recurrence properties

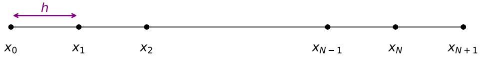
For every even integers $d \geq 2$ and $n \geq 2$, the coefficients satisfy:

$$\mathrm{a}^{d,n+2} = \mathrm{a}^{d,n} + a_{m+1}^{d,n+2}\ \mathrm{a}^{d+n,2},$$

$$\mathrm{a}^{d+2,2} = \mathrm{a}^{d,2} * \mathrm{a}^{2,2}\ .$$

Discrete differential operators

Spatial grid and boundary conditions



$(x_j)_{j\in\{0,1,\ldots,N+1\}}$

$$\partial_x^{2k} u(0,t) = \partial_x^{2k} u(1,t) = 0, \quad k \in \{0,\ldots,\tfrac{d}{2}-1\}$$

Matrix representation of the discrete differential operator ∂_x^d

$$A_h^{d,n} = \left(\alpha_{i,j}^{d,n}\right)_{1\le i,j\le N} \in \mathcal{M}_N(\mathbb{R})$$

Explicit form of $\alpha_{i,j}^{d,n}$

$$\alpha_{i,j}^{d,n} = \frac{1}{h^d}\left(a_{j-i}^{d,n} - a_{-(i+j)}^{d,n} - a_{2(N+1)-(i+j)}^{d,n}\right), \qquad 1 \le i,j \le N.$$

Matrix identities

Let $d, n > 2$ be even integers and $h > 0$ sufficiently small. The following identities hold:

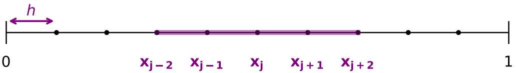
$$A_h^{d,n+2} = A_h^{d,n} + a_{m^{d,n}+1}^{d,n+2}\, h^n\, A_h^{d+n,\,2},$$

$$A_h^{d+2,\,2} = A_h^{2,2}\, A_h^{d,2}.$$

$$\begin{cases} \partial_t^2 \varphi(x,t) - \partial_x^2 \varphi(x,t) = 0, & x \in (0,1),\ t \in (0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), \quad \dot{\varphi}(x,0) = \varphi_1(x), & x \in (0,1). \end{cases}$$

$n = 4,\ d = 2,\ m = 2$

$$\partial_x^2 \varphi(x_j) \approx \frac{1}{h^2} \sum_{k=-2}^{2} a_k^{2,4} \varphi(x_{j+k})$$



k	-2	-1	0	1	2
$a_k^{2,4}$	$-\frac{1}{12}$	$\frac{4}{3}$	$-\frac{5}{2}$	$\frac{4}{3}$	$-\frac{1}{12}$

Wave equation
Fourth order semi-discretization

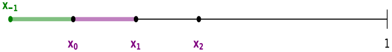
$n = 4, d = 2, m = 2$

$$\partial_x^2 \varphi(x_j) \approx \frac{1}{h^2} \sum_{k=-2}^{2} a_k^{2,4} \varphi(x_{j+k})$$

k	-2	-1	0	1	2
$a_k^{2,4}$	$-\frac{1}{12}$	$\frac{4}{3}$	$-\frac{5}{2}$	$\frac{4}{3}$	$-\frac{1}{12}$

- **Natural conditions to impose** $\varphi(x_0) = \varphi(x_{N+1}) = 0$
- **Ghost points** $\varphi(x_{-1}) = -\varphi(x_1), \varphi(x_{N+2}) = -\varphi(x_N)$

Ghost points computation



$$\varphi(-h,t) + \varphi(h,t) = 2\varphi(0,t) + h^2 \partial_x^2 \varphi(0,t) + \mathcal{O}(h^4)$$

$\Downarrow$ Conditions at $x = 0$

$$\varphi(0,t) = 0, \quad \partial_x^2 \varphi(0,t) = \partial_t^2 \varphi(0,t) = 0,$$

$$\Downarrow$$

$$\varphi(-h,t) \approx -\varphi(h,t) \to \varphi(x_{-1}) = -\varphi(x_1)$$

Wave equation
Fourth order semi-discretization

$$\begin{cases} \partial_t^2 \varphi(x,t) - \partial_x^2 \varphi(x,t) = 0, & x \in (0,1),\ t \in (0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), \quad \dot{\varphi}(x,0) = \varphi_1(x), & x \in (0,1). \end{cases}$$

$$\star \begin{cases} \partial_t^2 \varphi_h(t) + A_h^4 \varphi_h(t) = 0, \quad t \in (0,T), \\ \varphi_h(0) = \varphi_{0,h}, \quad \partial_t \varphi_h(0) = \varphi_{1,h}, \end{cases}$$

$$A_h^4 = \frac{1}{12h^2} \begin{pmatrix} 29 & -16 & 1 & 0 & \cdots & 0 \\ -16 & 30 & -16 & 1 & \ddots & \vdots \\ 1 & -16 & 30 & -16 & \ddots & 0 \\ 0 & 1 & -16 & 30 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -16 \\ 0 & \cdots & 0 & 1 & -16 & 29 \end{pmatrix}$$

- **Natural conditions to impose** $\varphi(x_0) = \varphi(x_{N+1}) = 0$
- **Ghost points** $\varphi(x_{-1}) = -\varphi(x_1), \varphi(x_{N+2}) = -\varphi(x_N)$

$$\partial_x^2 \varphi(x_1) \approx \frac{1}{h^2} \left[-\frac{1}{12}\, \varphi(x_{-1}) + \frac{4}{3}\, \varphi(x_0) - \frac{5}{2}\, \varphi(x_1) + \frac{4}{3}\, \varphi(x_2) - \frac{1}{12}\, \varphi(x_3) \right]$$

$$\partial_x^2 \varphi(x_1) \approx \frac{1}{12h^2} \left[-29\, \varphi(x_1) + 16\, \varphi(x_2) - \varphi(x_3) \right]$$

Wave equation
Fourth order semi-discretization

$$\begin{cases} \partial_t^2\varphi(x,t) - \partial_x^2\varphi(x,t) = 0, & x \in (0,1),\ t \in (0,T), \\ \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), \quad \partial_t\varphi(x,0) = \varphi_1(x), & x \in (0,1). \end{cases}$$

$$\star \begin{cases} \partial_t^2\varphi_h(t) + A_h^4\varphi_h(t) = 0, \quad t \in (0,T), \\ \varphi_h(0) = \varphi_{0,h}, \quad \dot{\varphi}_h(0) = \varphi_{1,h}, \end{cases}$$

$$A_h^4 = \frac{1}{12h^2}\begin{pmatrix} 29 & -16 & 1 & 0 & \cdots & 0 \\ -16 & 30 & -16 & 1 & \ddots & \vdots \\ 1 & -16 & 30 & -16 & \ddots & 0 \\ 0 & 1 & -16 & 30 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -16 \\ 0 & \cdots & 0 & 1 & -16 & 29 \end{pmatrix}$$

Properties of the matrix A_h^4

$$\mathbf{A_h^4} = \mathbf{A_h^2} + \frac{\mathbf{h}^2}{12}(\mathbf{A_h^2})^2$$

$$\lambda_{k,4} = \lambda_{k,2} + \frac{h^2}{12}\lambda_{k,2}^2$$

⋆**Eigenvectors** The matrices A_h^4 and A_h^2 share exactly the same eigenvectors.

⋆**Eigenvalues** The eigenvalues $\lambda_{k,4}$ of A_h^4 are related to those of A_h^2

Computation of the boundary terms

Discrete multipliers method

$$\bigstar \begin{cases} \partial_t^2 \varphi_h(t) + A_h^4 \varphi_h(t) = 0, & t \in (0, T) \\ \varphi_h(0) = \varphi_{0,h}, & \dot{\varphi}_h(0) = \varphi_{1,h}, \end{cases}$$

The energy E_h is conserved in time.

$$E_h(t) = \frac{1}{2}\|\dot{\varphi}_h(t)\|_h^2 + \frac{h^2}{24}\|A_h^2\varphi_h(t)\|_h^2 + \frac{1}{2}\|\partial_R^1\varphi_h(t)\|_h^2, \quad t \in [0, T].$$

Lemma - second order multiplier

Let $h > 0$ and let φ_h be a solution of the equation $\bigstar$. Then, the following identity holds:

$$TE_h(0) + \frac{h^2}{12}\int_0^T \left(\|A_h^2\varphi_h(t)\|_h^2 - 3\|\partial_R^1\dot{\varphi}_h(t)\|_h^2 - \frac{h^2}{4}\|\partial_R^1 A_h^2\varphi_h(t)\|_h^2 \right) dt$$

$$+X_h(T) - X_h(0) = -\int_0^T \frac{(\varphi_{N-1}(t) - 8\varphi_N(t))\,\varphi_N(t)}{12h^2}\, dt$$

where

$$X_h(t) = (\dot{\varphi}_h(t), J \cdot \partial_C^1\varphi_h(t))_h$$

and

$$J = (h, 2h, \ldots, Nh).$$

Computation of the boundary terms

Discrete multipliers method

Lemma - fourth order multiplier

Let $\mathbf{h} > \mathbf{0}$ and let φ_h be a solution of the equation ★. Then, the following identity holds:

$$TE_h(0) + \frac{h^4}{12}\int_0^T \left(\frac{1}{3}\|\partial_R^1 A_h^2 \varphi_h(t)\|_h^2 - \|A_h^2 \dot{\varphi}_h(t)\|_h^2\right) dt + Y_h(T) - Y_h(0)$$
$$= \frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2 dt + \frac{1}{4}\int_0^T \left(\left|\frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h}\right|^2 + \left|\frac{\varphi_N(t)}{h}\right|^2\right) dt$$

where

$$X_h(t) = (\dot{\varphi}_h(t), J \cdot \partial_{C,4}^1 \varphi_h(t))_h$$

and

$$J = (h, 2h, \ldots, Nh).$$

Remark

$$\partial_{C,4}^1 = \partial_C^1 + \frac{h^2}{6}\partial_C^1 A_h^2.$$

Blow-up of the discrete observability constant

Behavior of the highest modes

For each $T > 2$, we have that

$$\sup_{\varphi_h \text{ solution of } \star} \frac{E_h(0)}{\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2 \, dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt} \to \infty \text{ as } h \to 0.$$

Blow-up of the discrete observability constant

Behavior of the highest modes

For each $T > 2$, we have that

$$\sup_{\varphi_h \text{ solution of } \star} \frac{E_h(0)}{\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2 \, dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt} \to \infty \text{ as } h \to 0.$$

$$E_h(t) = \frac{1}{2}\|\dot{\varphi}_h(t)\|_h^2 + \frac{h^2}{24}\|A_h^2 u\varphi_h(t)\|_h^2 + \frac{1}{2}\|\partial_R^1 \varphi_h(t)\|_h^2, \quad t \in [0, T].$$

Steps

Analyzing the behavior of solutions associated with the N-th eigenvector of A_h^4

$$\lambda_{N,4} = \lambda_{N,2} + \tfrac{h^2}{12}\lambda_{N,2}^2$$

- $\varphi_h(t) = e^{i\sqrt{\lambda_{N,4}}t}\phi_N \qquad (t \in [0, T])$
- Initial energy:

$$E_h(0) = \frac{1}{2}\|i\sqrt{\lambda_{N,4}}e^{i\sqrt{\lambda_{N,4}}t}\phi_N\|_h^2 + \frac{h^2}{24}\|e^{i\sqrt{\lambda_{N,4}}t}A_h^2\phi_N\|_h^2 + \frac{1}{2}\|e^{i\sqrt{\lambda_{N,4}}t}\partial_R^1\phi_N\|_h^2$$

$$= \frac{\lambda_{N,4}}{2} + \frac{h^2\lambda_{N,2}^2}{24} + \frac{\lambda_{N,2}}{2} = \lambda_{N,4}.$$

Blow-up of the discrete observability constant

Behavior of the highest modes

For each $T > 2$, we have that

$$\sup_{\varphi_h \text{ solution of } \star} \frac{E_h(0)}{\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2 \, dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt} \to \infty \text{ as } h \to 0.$$

Steps

Analyzing the behavior of solutions associated with the N-th eigenvector of A_h^4

- $\varphi_h(t) = e^{i\sqrt{\lambda_{N,4}}t}\phi_N \qquad (t \in [0, T])$
- Initial energy: $E_h(0) = \lambda_{N,4}$.
- Contribution of the boundary terms
  - $\displaystyle\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt = T \left| \frac{\phi_{N,N}}{h} \right|^2,$
  - $\displaystyle\int_0^T \left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 dt = Tc_N^2 \left| \frac{2\cos(N\pi h) - 8}{6} \right|^2 \left| \frac{\phi_{N,N}}{h} \right|^2,$
  - $\displaystyle\int_0^T |\dot{\varphi}_N(t)|^2 \, dt = Th^2\lambda_{N,4} \left| \frac{\phi_{N,N}}{h} \right|^2.$

Blow-up of the discrete observability constant

Behavior of the highest modes

For each $T > 2$, we have that

$$\sup_{\varphi_h \text{ solution of } \star} \frac{E_h(0)}{\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2 \, dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt} \to \infty \text{ as } h \to 0.$$

Steps

Analyzing the behavior of solutions associated with the N-th eigenvector of A_h^4

- $\varphi_h(t) = e^{i\sqrt{\lambda_{N,4}}t}\phi_N \qquad (t \in [0, T])$
- Initial energy: $E_h(0) = \lambda_{N,4}$.
- Contribution of the boundary terms
  - $\displaystyle\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt = T \left| \frac{\phi_{N,N}}{h} \right|^2,$
  - $\displaystyle\int_0^T \left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 dt = Tc_N^2 \left| \frac{2\cos(N\pi h) - 8}{6} \right|^2 \left| \frac{\phi_{N,N}}{h} \right|^2,$
  - $\displaystyle\int_0^T |\dot{\varphi}_N(t)|^2 \, dt = Th^2\lambda_{N,4} \left| \frac{\phi_{N,N}}{h} \right|^2.$

$$\left| \frac{\phi_{N,N}}{h} \right|^2 = 2\lambda_{N,4} \left(1 - \frac{h^2\lambda_{N,2}}{4} \right) \left(1 + \frac{h^2\lambda_{N,2}}{12} \right)^{-1}$$

Blow-up of the discrete observability constant

Behavior of the highest modes

For each $T > 2$, we have that

$$\sup_{\varphi_h \text{ solution of } \star} \frac{E_h(0)}{\underbrace{\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2\,dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt}_{B}} \to \infty \text{ as } h \to 0.$$

Steps

- $E_h(0) = \lambda_{N,4}$
- $\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt = T \left| \frac{\phi_{N,N}}{h} \right|^2,$
- $\displaystyle\int_0^T \left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 dt = Tc_N^2 \left| \frac{2\cos(N\pi h) - 8}{6} \right|^2 \left| \frac{\phi_{N,N}}{h} \right|^2,$
- $\displaystyle\int_0^T |\dot{\varphi}_N(t)|^2\,dt = Th^2\lambda_{N,4} \left| \frac{\phi_{N,N}}{h} \right|^2.$

$$\left| \frac{\phi_{N,N}}{h} \right|^2 = 2\lambda_{N,4} \left(1 - \frac{h^2\lambda_{N,2}}{4} \right) \left(1 + \frac{h^2\lambda_{N,2}}{12} \right)^{-1}$$

$$TE_h(0)\left(1 - \frac{h^2\lambda_{N,2}}{4} \right) = \left(\frac{1}{2} + \frac{c_N^2}{2} \left| \frac{2\cos(N\pi h) - 8}{6} \right|^2 + \frac{h^2\lambda_{N,4}}{6} \right) B$$

Blow-up of the discrete observability constant

Behavior of the highest modes

For each $T > 2$, we have that

$$\sup_{\varphi_h \text{ solution of } \star} \frac{E_h(0)}{\underbrace{\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2\,dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt}_{B}} \to \infty \text{ as } h \to 0.$$

Steps

$$\lambda_{N,2} = \frac{4}{h^2} \sin^2\left(\frac{\pi N h}{2}\right)$$

$$TE_h(0)\left(1 - \frac{h^2\lambda_{N,2}}{4}\right) = \underbrace{\left(\frac{1}{2} + \frac{c_N^2}{2}\left|\frac{2\cos(N\pi h) - 8}{6}\right|^2 + \frac{h^2\lambda_{N,4}}{6}\right)}_{\text{bounded}} B$$

$$\left(1 - \frac{h^2\lambda_{N,2}}{4}\right) \to 0, \text{ when } h \to 0$$

Restoring the observability

Proof sketch

- discrete multipliers method
- Ingham's inequality

Proposition

Let $\delta \in (0,1)$. There exist constants $C, T > 0$ such that

$$\frac{1}{12}\int_0^T |\dot{\varphi}_N(t)|^2\, dt + \frac{1}{4}\int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt \geq C\, E_h(0),$$

for any solution φ_h of the problem ★ of the form

$$\varphi_h(t) = \frac{1}{\sqrt{2}} \sum_{|k|=1}^{[\delta N]} \frac{a_k}{\sqrt{\lambda_{|k|,4}}}\, e^{i\,\mathrm{sign}(k)\,\sqrt{\lambda_{|k|,4}}\,t}\, \phi_{|k|}.$$

Ingham's inequality

Let $(\mu_n)_{n\in\mathbb{Z}}$ be strictly increasing with $\mu_{n+1} - \mu_n \geq \gamma > 0$. If $T > 2\pi/\gamma$, then there exist $C_1, C_2 > 0$ such that for any finite sequence $(b_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$,

$$C_1 \sum_n |b_n|^2 \;\leq\; \int_0^T \left| \sum_n b_n e^{i\mu_n t} \right|^2 dt \;\leq\; C_2 \sum_n |b_n|^2.$$

Lemma

Let $\delta \in (0, 1)$. There exist $C_1, C_2 > 0$ such that

$$C_1\left(\sqrt{\lambda_{n+1,2}} - \sqrt{\lambda_{n,2}}\right) \leq \sqrt{\lambda_{n+1,4}} - \sqrt{\lambda_{n,4}} \leq C_2\left(\sqrt{\lambda_{n+1,2}} - \sqrt{\lambda_{n,2}}\right), \quad n = 1, \ldots, N$$

$$2C_1(1-\delta) \leq \sqrt{\lambda_{n+1,4}} - \sqrt{\lambda_{n,4}} \leq C_2\pi, \qquad (1 \leq n \leq \delta N).$$

For every fixed integer $n \geq 1$, there exists $a_n,\ b_n > 0$, depending on h, such that

$$a_n \leq \sqrt{\lambda_{n+1,4}} - \sqrt{\lambda_{n,4}} \leq b_n \text{ satisfying } \lim_{h\to 0} a_n = \pi, \lim_{h\to 0} b_n = \frac{(n+1)\pi}{n}.$$

Moreover, we have that

$$\lim_{h\to 0} \sqrt{\lambda_{N,4}} - \sqrt{\lambda_{N-1,4}} = 0.$$

Restoring the observability

Energy computation

$$E_h(0) = \frac{1}{2} \sum_{|k|=1}^{[\delta N]} |a_k|^2$$

Proposition

Let $\delta \in (0,1)$. There exist constants $C, T > 0$ such that

$$\frac{1}{12} \int_0^T |\dot{\varphi}_N(t)|^2 \, dt + \frac{1}{4} \int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt \geq \mathcal{C}_h \, E_h(0),$$

for any solution φ_h of the problem ★ of the form $\varphi_h(t) = \frac{1}{\sqrt{2}} \sum_{|k|=1}^{[\delta N]} \frac{a_k}{\sqrt{\lambda_{|k|,4}}} e^{i \operatorname{sign}(k) \sqrt{\lambda_{|k|,4}} t} \phi_{|k|}$.

Proof sketch

$$\int_0^T |\dot{\varphi}_N(t)|^2 dt = \int_0^T \left| \sum_{|k|=1}^{\delta N} a_k c_k \sin(Nk\pi h) \, e^{i \operatorname{sign}(k) \sqrt{\lambda_{|k|,4}} \, t} \right|^2 dt,$$

$$\int_0^T \left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 dt = \int_0^T \left| \sum_{|k|=1}^{\delta N} \frac{a_k c_k}{\sqrt{\lambda_{|k|,4}}} \frac{\sin((N-1)k\pi h) - 8 \sin(Nk\pi h)}{6h} e^{i \operatorname{sign}(k) \sqrt{\lambda_{|k|,4}} \, t} \right|^2 dt,$$

$$\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt = \int_0^T \left| \sum_{|k|=1}^{\delta N} \frac{a_k c_k}{\sqrt{\lambda_{|k|,4}}} \operatorname{sign}(k) \, \sin(Nk\pi h) \, e^{i \operatorname{sign}(k) \sqrt{\lambda_{|k|,4}} \, t} \right|^2 dt$$

Restoring the observability

Energy computation

$$E_h(0) = \frac{1}{2} \sum_{|k|=1}^{[\delta N]} |a_k|^2$$

Proposition

Let $\delta \in (0,1)$. There exist constants $C, T > 0$ such that

$$\frac{1}{12} \int_0^T |\dot{\varphi}_N(t)|^2 \, dt + \frac{1}{4} \int_0^T \left(\left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 + \left| \frac{\varphi_N(t)}{h} \right|^2 \right) dt \geq \mathcal{C}_h \, E_h(0),$$

for any solution φ_h of the problem ★ of the form $\varphi_h(t) = \frac{1}{\sqrt{2}} \sum_{|k|=1}^{[\delta N]} \frac{a_k}{\sqrt{\lambda_{|k|,4}}} e^{i \, \text{sign}(k) \sqrt{\lambda_{|k|,4}} t} \phi_{|k|}$.

Proof sketch

$$\int_0^T |\dot{\varphi}_N(t)|^2 dt \geq \mathcal{C} \sum_{|k|=1}^{\delta N} |a_k|^2 k^2 h^2,$$

$$\int_0^T \left| \frac{\varphi_{N-1}(t) - 8\varphi_N(t)}{6h} \right|^2 dt \geq \mathcal{C} \sum_{|k|=1}^{\delta N} |a_k|^2,$$

$$\int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt \geq \mathcal{C} \sum_{|k|=1}^{\delta N} |a_k|^2.$$

Construction of numerical operators with enhanced accuracy order

$$\star \begin{cases} \partial_t^2 \varphi_h(t) + A_h^n \varphi_h(t) = 0, & t \in (0,T), \\ \varphi_h(0) = \varphi_{0,h}, & \dot{\varphi}_h(0) = \varphi_{1,h}, \end{cases}$$

$n = 2m$

High-order operator representation

$$A_h^n = \sum_{j=0}^{(n-2)/2} (-1)^j a_{j+1}^{2,2(j+1)} h^{2j} \left(A_h^2\right)^{j+1}$$

j	0	1	2	3	4	5	6	7	8
$a_{j+1}^{2,2(j+1)}$	1	$-\frac{1}{12}$	$\frac{1}{90}$	$-\frac{1}{560}$	$\frac{1}{3150}$	$-\frac{1}{16632}$	$\frac{1}{84084}$	$-\frac{1}{411840}$	$\frac{1}{1969119}$

Spectral properties

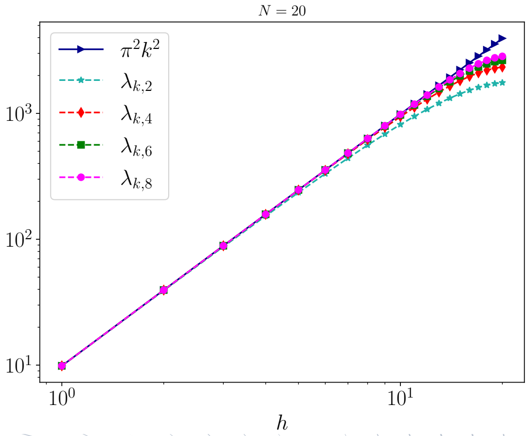
- A_h and A_h^2 have exactly the same eigenvectors.

$$\lambda_{k,n} = \sum_{j=0}^{(n-2)/2} (-1)^j a_{j+1}^{2,2(j+1)} h^{2j} (\lambda_{k,2})^{j+1}$$

- Using **Ingham's inequality**, we can establish observability results similar to Infante & Zuazua (1999).

Visualization of eigenvalues for multiple orders

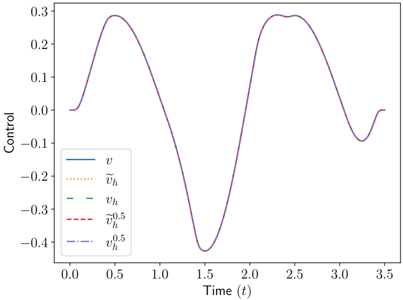
$$\lambda_{k,n} = \sum_{j=0}^{(n-2)/2} (-1)^j a_{j+1}^{2,2(j+1)} h^{2j} (\lambda_{k,2})^{j+1}$$

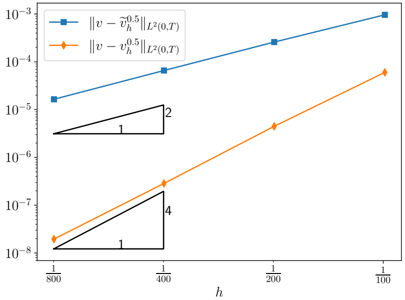


Numerical experiments
Case 1

Initial data to control

$$u_0(x) = \sin(\pi x),$$
$$u_1(x) = 0 \qquad \text{for } x \in (0,1). \qquad \text{(EX1)}$$





Control	$h = \frac{1}{100}$	$h = \frac{1}{200}$	$h = \frac{1}{400}$	$h = \frac{1}{800}$
$\widetilde{v}_h$	86	10	10	10
v_h	62	10	10	10

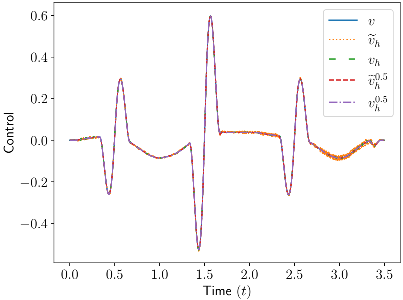
Number of iterations needed for the convergence of the conjugate gradient

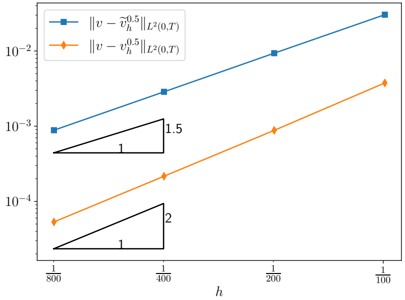
Numerical experiments
Case 2

Initial data to control

$$w_0(x) = 400(3x-1)(3x-2)(2x-1)\left|\left|x-\frac{1}{2}\right|-\frac{1}{6}\right|\mathbb{1}_{\left(\frac{1}{3},\frac{2}{3}\right)}(x), \quad \text{(EX2)}$$

$$w_1(x) = 20x\left(x-\frac{1}{2}\right)\mathbb{1}_{\left(0,\frac{1}{2}\right)}(x), \text{ for every } x \in (0,1).$$





Control	$h=\frac{1}{100}$	$h=\frac{1}{200}$	$h=\frac{1}{400}$	$h=\frac{1}{800}$
$\widetilde{v}_h$	-	-	-	-
v_h	-	-	-	-
$\widetilde{v}_h^{0.5}$	88	44	9	9
$v_h^{0.5}$	-	9	9	9

Number of iterations needed for the convergence of the conjugate gradient

Clamped Euler-Bernoulli beam equation
-work in progress-

$$\begin{cases} \ddot{\varphi}(x,t) + \partial_x^4 \varphi(x,t) = 0, & (x,t) \in (0,1) \times (0,T) \\ \varphi(0,t) = \varphi(1,t) = 0, & t \in (0,T) \\ \partial_x^2 \varphi(0,t) = \partial_x^2 \varphi(1,t) = 0, & t \in (0,T) \\ \varphi(x,0) = \varphi_0(x), \quad \dot{\varphi}(x,0) = \varphi_1(x), & x \in (0,1). \end{cases}$$

Finite-differences semi-discretization of order $n = 2m$

$$\bigstar \begin{cases} \ddot{\varphi}_h(t) + B_h^n \varphi_h(t) = 0, & t \in (0,T), \\ \varphi_h(0) = \varphi_{0,h}, \quad \dot{\varphi}_h(0) = \varphi_{1,h}. \end{cases}$$

$$B_h^n = \sum_{j=0}^{(n-2)/2} (-1)^j a_{j+2}^{4,2(j+1)} h^{2j} \left(A_h^2\right)^{j+2}$$

j	0	1	2	3	4	5	6
$a_{j+2}^{4,2(j+1)}$	1	$-\frac{1}{6}$	$\frac{7}{240}$	$-\frac{41}{7560}$	$\frac{479}{453600}$	$-\frac{59}{277200}$	$\frac{266681}{6054048000}$

Perspectives

- Establish the observability result for the beam equation.
- Extend the analysis to ***2d*** problems.
- Apply similar techniques to finite element discretizations.

Thank you!

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