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Temă seminarul 1

- $A^{2022} = ?$ $\quad A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$

$$A^2 = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3} & \cos\frac{\pi}{3}\sin\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{3} \\ -\sin\frac{\pi}{3}\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\cos\frac{\pi}{3} & -\sin^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{2\pi}{3} & 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} \\ -2\sin\frac{\pi}{3}\cos\frac{\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{2\pi}{3}\cos\frac{\pi}{3} - \sin\frac{2\pi}{3}\sin\frac{\pi}{3} & \cos\frac{2\pi}{3}\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}\cos\frac{\pi}{3} \\ -\sin\frac{2\pi}{3}\cos\frac{\pi}{3} - \cos\frac{2\pi}{3}\sin\frac{\pi}{3} & -\sin\frac{2\pi}{3}\sin\frac{\pi}{3} + \cos\frac{2\pi}{3}\cos\frac{\pi}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\left(\frac{2\pi}{3}+\frac{\pi}{3}\right) & \sin\left(\frac{2\pi}{3}+\frac{\pi}{3}\right) \\ -\sin\left(\frac{2\pi}{3}+\frac{\pi}{3}\right) & \cos\left(\frac{2\pi}{3}+\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{3\pi}{3} & \sin\frac{3\pi}{3} \\ -\sin\frac{3\pi}{3} & \cos\frac{3\pi}{3} \end{pmatrix}$$

Demonstrăm prin inducție că $A^n = \begin{pmatrix} \cos\frac{n\pi}{3} & \sin\frac{n\pi}{3} \\ -\sin\frac{n\pi}{3} & \cos\frac{n\pi}{3} \end{pmatrix}$

Presupunem că $A^k = \begin{pmatrix} \cos\frac{k\pi}{3} & \sin\frac{k\pi}{3} \\ -\sin\frac{k\pi}{3} & \cos\frac{k\pi}{3} \end{pmatrix}$ și demonstrăm că

$$A^{k+1} = \begin{pmatrix} \cos\frac{(k+1)\pi}{3} & \sin\frac{(k+1)\pi}{3} \\ -\sin\frac{(k+1)\pi}{3} & \cos\frac{(k+1)\pi}{3} \end{pmatrix}.$$

$$A^{k+1} = \begin{pmatrix} \cos\frac{k\pi}{3} & \sin\frac{k\pi}{3} \\ -\sin\frac{k\pi}{3} & \cos\frac{k\pi}{3} \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{k\pi}{3}\cos\frac{\pi}{3} - \sin\frac{k\pi}{3}\sin\frac{\pi}{3} & \cos\frac{k\pi}{3}\sin\frac{\pi}{3} + \sin\frac{k\pi}{3}\cos\frac{\pi}{3} \\ -\sin\frac{k\pi}{3}\cos\frac{\pi}{3} - \cos\frac{k\pi}{3}\sin\frac{\pi}{3} & -\sin\frac{k\pi}{3}\sin\frac{\pi}{3} + \cos\frac{k\pi}{3}\cos\frac{\pi}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\left(\frac{k\pi}{3}+\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}+\frac{k\pi}{3}\right) \\ -\sin\left(\frac{k\pi}{3}+\frac{\pi}{3}\right) & \cos\left(\frac{k\pi}{3}+\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{(k+1)\pi}{3} & \sin\frac{(k+1)\pi}{3} \\ -\sin\frac{(k+1)\pi}{3} & \cos\frac{(k+1)\pi}{3} \end{pmatrix}$$

Conform principiului inducției matematice $\Rightarrow$

$$\Rightarrow A^n = \begin{pmatrix} \cos\frac{n\pi}{3} & \sin\frac{n\pi}{3} \\ -\sin\frac{n\pi}{3} & \cos\frac{n\pi}{3} \end{pmatrix}, n \geq 1$$

1) $$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} \xlongequal{L_1'=L_1+L_2+L_3} \begin{vmatrix} a+2b & a+2b & a+2b \\ b & a & b \\ b & b & a \end{vmatrix} = (a+2b)\begin{vmatrix} 1 & 1 & 1 \\ b & a & b \\ b & b & a \end{vmatrix} =$$

$$\xlongequal[C_3'=C_3-C_1]{C_2'=C_2-C_1} (a+2b)\begin{vmatrix} 1 & 0 & 0 \\ b & a-b & 0 \\ b & 0 & a-b \end{vmatrix} = (a+2b)(-1)^2\cdot 1\cdot \begin{vmatrix} a-b & 0 \\ 0 & a-b \end{vmatrix} = (a+2b)(a-b)^2$$

2) $$\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} \xlongequal{L_1'=L_1+L_2+L_3} \begin{vmatrix} (a+b)^2 & (a+b)^2 & (a+b)^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} =$$

$$= (a+b)^2 \begin{vmatrix} 1 & 1 & 1 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} \xlongequal[C_2'=C_2-C_1]{C_3'=C_3-C_1} (a+b)^2 \begin{vmatrix} 1 & 0 & 0 \\ b^2 & a^2-b^2 & 2ab-b^2 \\ 2ab & b^2-2ab & a^2-2ab \end{vmatrix} =$$

$$= (a+b)^2 \begin{vmatrix} a^2-b^2 & 2ab-b^2 \\ b^2-2ab & a^2-2ab \end{vmatrix} = (a+b)^2\left[(a^2-b^2)(a^2-2ab)-(b^2-2ab)(2ab-b^2)\right] =$$

$$= (a+b)^2(4a^2b^2-2ab^3-2a^3b+a^2b^2-a^2b^2+a^4+b^4-a^2b^2) =$$

$$= (a+b)^2(3a^2b^2-2ab^3-2a^3b+a^4+b^4) = (a+b)^2\left[2(ab)^2+(ab)^2+a^4\right.$$

$$\left.+b^4-2ab^3-2a^3b\right] = (a+b)^2\left[(a^2+b^2)^2+(ab)^2-2a^3b-2ab^3\right] = (a+b)^2\left[(a^2+b^2)^2+(ab)^2-\right.$$

$$\left.-2ab(a^2+b^2)\right] = (a+b)^2(a^2+b^2-ab)^2 = (a^3+b^3)^2$$

3) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \xlongequal{L_1'=L_1+L_2+L_3} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} =$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \xlongequal[C_3'=C_3-C_1]{C_2'=C_2-C_1} (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} =$$

$$= (a+b+c)(-a-b-c)^2 = (a+b+c)^3$$

Temă seminarul 2

• Dezvoltare după coloana 4 pentru determinantul următor:

$$\Delta = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} = 3(-1)^{1+4} \begin{vmatrix} 1 & 1 & 3 \\ 2 & 5 & 1 \\ -1 & -2 & 2 \end{vmatrix} + 4(-1)^{2+4} \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 1 \\ -1 & -2 & 2 \end{vmatrix} +$$

$$+ (-1)(-1)^{3+4} \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ -1 & -2 & 2 \end{vmatrix} + 4(-1)^{4+4} \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 5 & 1 \end{vmatrix} =$$

$$= (-3)(10-12-1+15+2-4)+4(10-8-1+10+2-4)+(2-4-3+2+6-2)+4(1+10+6-4-15-1) =$$

$$= (-3)\cdot 10 + 4\cdot 9 + 1 + 4(-3) = -30+36+1-12 = -5$$

• Fie $\det A_n(a,x) = \begin{vmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \dots & x \end{vmatrix}$. Să se calculeze $A_3^{-1}(1,2)$.

$$A_3(1,2) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad A_3^t(1,2) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A_3^*(1,2) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\det A_3(1,2) = 8+1+1-2-2-2 = 4$$

$$A_3^{-1}(1,2) = \frac{1}{\det A_3(1,2)} A_3^*(1,2) = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Temă seminarul 4

• Verificați dacă următoarele mulțimi se constituie în subspații vectoriale ale lui $\mathcal{M}_n(\mathbb{R})$:

a) $AS = \{ B = (b_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R}) \mid b_{ij} = -b_{ij} \} \subset \mathcal{M}_n(\mathbb{R})$

Fie $A, C \in AS$ și $\alpha, \beta \in \mathbb{R}$

$A \in AS \Rightarrow A = (a_{ij})_{i,j=\overline{1,n}},\ a_{ij} = -a_{ji},\ \forall i,j = \overline{1,n}$

$C \in AS \Rightarrow C = (c_{ij})_{i,j=\overline{1,n}},\ c_{ij} = -c_{ji},\ \forall i,j = \overline{1,n}$

$\alpha A + \beta C = (\alpha a_{ij} + \beta c_{ij})_{i,j=\overline{1,n}}$ (1)

$\alpha A+\beta C_{ij} = \alpha(-a_{ji})+\beta(-c_{ji}) = -(\alpha a_{ji}+\beta c_{ji}),\ \forall i,j=\overline{1,n}$ (2)

Din (1) și (2) $\Rightarrow \alpha A+\beta C \in AS \Rightarrow AS \subset M_n(\mathbb{R})$ subspațiu vectorial

b) $\Delta = \{B=(b_{ij})_{i,j=\overline{1,n}} \in M_n(\mathbb{R}) \mid b_{ij}=0,\ \forall 1\leq i\neq j\leq n\}$

Fie $A, B \in \Delta$, $\alpha,\beta \in \mathbb{R}$

$$\left.\begin{array}{l} A\in\Delta \Rightarrow A=(a_{ij})_{i,j=\overline{1,n}},\ a_{ij}=0,\ \forall 1\leq i\neq j\leq n \\ B\in\Delta \Rightarrow B=(b_{ij})_{i,j=\overline{1,n}},\ b_{ij}=0,\ \forall 1\leq i\neq j\leq n \end{array}\right\} \Rightarrow \alpha a_{ij}+\beta b_{ij}=0,\ \forall 1\leq i\neq j\leq n$$

$\Rightarrow \alpha A+\beta B = (\alpha a_{ij}+\beta b_{ij})_{i,j=\overline{1,n}},\ \alpha a_{ij}+\beta b_{ij}=0,\ \forall 1\leq i\neq j\leq n \Rightarrow$

$\Rightarrow \alpha A+\beta B\in\Delta \Rightarrow \Delta \subset M_n(\mathbb{R})$ subspațiu vectorial

c) $sl_n(\mathbb{R}) = \{X\in M_n(\mathbb{R}) \mid Tr(X)=0\}$

Fie $A,B\in sl_n(\mathbb{R})$ și $\alpha,\beta\in\mathbb{R}$

$A\in sl_n(\mathbb{R}) \Rightarrow Tr(A)=0$

$B\in sl_n(\mathbb{R}) \Rightarrow Tr(B)=0$

$Tr(\alpha A+\beta B) = \alpha Tr(A)+\beta Tr(B) = \alpha\cdot 0+\beta\cdot 0 = 0 \Rightarrow \alpha A+\beta B\in sl_n(\mathbb{R}) \Rightarrow$

$\Rightarrow sl_n(\mathbb{R}) \subset M_n(\mathbb{R})$ subspațiu vectorial

<u>Temă seminarul 5</u>

- Să se determine baze pentru H_1, H_2, $H_1\cap H_2$ și să se verifice teorema Grassmann.

$H_1 = \{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=0\}$

$H_2 = \{(x,y,z)\in\mathbb{R}^3 \mid x-y+z=0\}$

$H_1 \ni (x,y,z)$ unde $x+y+z=0$, $x,y,z\in\mathbb{R}$, deci $y=-x-z$

$= \{(-y-z,y,z) \mid y,z\in\mathbb{R}\} = (-y,y,0)+(-z,0,z) =$

$= y\underbrace{(-1,1,0)}_{v_1} + z\underbrace{(-1,0,1)}_{v_2} = yv_1+zv_2 \Rightarrow S_1=\{v_1,v_2\}\subset H_1$

sistem de generatori

$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Delta_1 = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = 0+1 = 1\neq 0 \Rightarrow$ rang $A_1 = 2 \Rightarrow S_1$ e

sistem de vectori liniar independent $\Rightarrow S_1 \subset H_1$ este bază
$=\{v_1,v_2\}$

$\dim H_1 = 2$ (plan vectorial)

$H_2 \ni (x,y,z),\ x,y,z \in \mathbb{R}$ a.î. $x-y+z=0 \Leftrightarrow y = x+z$

$(x, x+z, z) = (x,x,0)+(0,z,z) = x(1,1,0)+z(0,1,1) = xw_1 + zw_2$

$\Rightarrow S_2 = \{w_1, w_2\} \subset H_2$ sistem de generatori pentru H_2 (1)

$A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix},\ \Delta_2 = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1-0 = 1 \neq 0 \Rightarrow$ rang $A_2 = 2 \Rightarrow$

$\Rightarrow S_2$ e liniar independent (2)

Din (1) și (2) $\Rightarrow S_2 \subset H_2$ e bază pentru H_2
$= \{v_1, v_2\}$

$\dim H_2 = 2$ (plan vectorial)

$H_1 \cap H_2 = \{(x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x+y+z=0 \\ x-y+z=0 \end{cases}\}$

$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix},\ \Delta_3 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0 \Rightarrow x, y$ nec principale)

z nec. secundare
$= \alpha \in \mathbb{R}$

$\begin{cases} x+y=-\alpha \\ x-y=-\alpha \end{cases} \Rightarrow \begin{cases} 2x=-2\alpha \\ x+y=-\alpha \end{cases} \Rightarrow \begin{cases} x=-\alpha \\ -\alpha+y=-\alpha \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x=-\alpha \\ y=0 \end{cases} \Rightarrow H_1 \cap H_2 = \{(x,y,z) \mid x=-z, y=0\} =$

$= \{(-\alpha, 0, \alpha) \mid \alpha \in \mathbb{R}\} = \{\alpha(-1,0,1) \mid \alpha \in \mathbb{R}\} = \langle(-1,0,1)\rangle$

$S_3 = \{(-1,0,1)\} \subset H_1 \cap H_2$ bază $\Rightarrow \dim H_1 \cap H_2 = 1$ (dr vect)

$\dim(H_1+H_2) = \dim H_1 + \dim H_2 + \dim H_1 \cap H_2 = 2+2-1 = 3$

$\left.\begin{array}{l} \dim H_1 + H_2 = 3 \\ H_1 + H_2 \subseteq \mathbb{R}^3 \\ \text{ssp vect} \end{array}\right\} \Rightarrow H_1 + H_2 = \mathbb{R}^3$

Temă seminarul 6

- Să se aducă la forma eșalon-redusă următoarele matrice:

$A_1 = \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 3 & 1 & 4 \end{pmatrix} \in \mathcal{M}_{(3,4)}(\mathbb{R})$

$A_1 \underset{-2L_1+L_3}{\overset{-L_1+L_2}{\sim}} \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -5 & -4 \end{pmatrix} \overset{-L_3+L_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & \boxed{1} & 6 & 3 \\ 0 & -1 & -5 & -4 \end{pmatrix} \underset{L_2+L_3}{\overset{-2L_2+L_1}{\sim}} \begin{pmatrix} 1 & 0 & -9 & -2 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & \boxed{1} & -1 \end{pmatrix}$

$$\xrightarrow[3L_3+L_1]{-6L_3+L_2} \begin{pmatrix} \boxed{1} & 0 & 0 & -11 \\ 0 & \boxed{1} & 0 & 9 \\ 0 & 0 & \boxed{1} & -1 \end{pmatrix} = E$$

$$A_2 = \begin{pmatrix} \boxed{1} & 3 & 1 & 2 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & -2 & 1 \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$$

$$A_2 \xrightarrow[-3L_1+L_3]{-2L_1+L_2} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -5 & 1 & -5 \\ 0 & -7 & -5 & -5 \end{pmatrix} \xrightarrow{-\frac{1}{5}L_2} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & \boxed{1} & -\frac{1}{5} & 1 \\ 0 & -7 & -5 & -5 \end{pmatrix} \xrightarrow[7L_2+L_3]{-3L_2+L_1} \begin{pmatrix} 1 & 0 & \frac{8}{5} & -1 \\ 0 & 1 & -\frac{1}{5} & 1 \\ 0 & 0 & -\frac{32}{5} & 2 \end{pmatrix}$$

$$\xrightarrow{-\frac{5}{32}L_3} \begin{pmatrix} 1 & 0 & \frac{8}{5} & -1 \\ 0 & 1 & -\frac{1}{5} & 1 \\ 0 & 0 & \boxed{1} & -\frac{10}{32} \end{pmatrix} \xrightarrow[-\frac{8}{5}L_3+L_1]{\frac{1}{5}L_3+L_2} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{15}{16} \\ 0 & 0 & 1 & -\frac{5}{16} \end{pmatrix} = E$$

• $E = E_{12}$, $F = E_{21}$, $H = E_{11} - E_{22}$ din $\mathcal{M}_2(\mathbb{R})$. Calculați valorile aplicațiilor f_E, f_F, f_H în fiecare din aceste elemente, unde $f_A : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$, $f_A(X) = AX - XA$, $A \in \mathcal{M}_n(\mathbb{R})$.

$f_E : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$, $f_E(X) = E_{12}X - XE_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$

$f_F : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$, $f_F(X) = E_{21}X - XE_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} - \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} -b & 0 \\ a-d & b \end{pmatrix}$

$f_H : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$, $f_H(X) = HX - XH = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} -$

$- \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} - \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}$

Temă seminarul 7

1. $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ -1 & 3 & 2 & -2 \\ 4 & 3 & -2 & -4 \end{pmatrix} \xrightarrow{L_2' = L_2 + L_1} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 4 & 4 & 0 \\ 4 & 3 & -2 & -4 \end{pmatrix} \xrightarrow{L_3' = L_3 - 4L_1}$

$\xrightarrow{L_3' = L_3 - 4L_1} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 4 & 4 & 0 \\ 0 & -1 & -10 & -12 \end{pmatrix} \xrightarrow{L_2' = \frac{L_2}{4}} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -10 & -12 \end{pmatrix} \xrightarrow[L_3' = L_3 + L_2]{L_1' = L_1 - L_2}$

$\xrightarrow[L_3'=L_3+L_2]{L_1'=L_1-L_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -9 & -12 \end{pmatrix} \xrightarrow{L_3'=L_3/(-9)} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4/3 \end{pmatrix} \xrightarrow[L_2'=L_2-L_3]{L_1'=L_1-L_3}$

$\xrightarrow[L_2'=L_1-L_2]{L_1'=L_1-L_3} \begin{pmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$

2. Demonstrați că aplicația considerată este morfism. Să se găsească matricea morfismului considerând pt $\mathbb{R}^n$, $\forall n$, baza canonică.
Să se găsească $Ker f$, $\dim(Ker(f))$, $Im(f)$ și $\dim(Im(f))$

$f:\mathbb{R}^3 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-z \end{pmatrix}$

Există morfism injectiv între $\mathbb{R}^3$ și $\mathbb{R}^2$?

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \in \mathcal{M}_{(2,3)}(\mathbb{R})$

$rg\, A = 2$, $\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0-1 = -1 \neq 0$

$\begin{cases} x+y+z = x' \\ x - z = y' \end{cases}$ $(x', y') = (x+y+z, x-z) = (x, x) + (y, 0) + (z, -z)$
$= x(1,1) + y(1,0) + z(1,-1)$

$Im f = \{(x', y') \in \mathbb{R}^2 \mid \exists (x, y, z) \in \mathbb{R}^3$ a.î. $f((x,y,z)) = (x', y')\}$

Luăm $S_1 = \{v_1 = (1,1), v_2 = (1,0), v_3 = (1,-1)\}$ sist. de gen.

Dar $\dim Im f = 2$ pt că rang $A = 2$ și $\dim \mathbb{R}^2 = 2$

$Ker f = \{(x, y, z) \in \mathbb{R}^3 \mid f((x,y,z)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

$\begin{cases} x+y+z = 0 \\ x - z = 0 \end{cases} \iff \begin{cases} y = -2z \\ x = z \end{cases}$ $z = \alpha \Rightarrow \begin{cases} x = \alpha \\ y = -2\alpha \\ z = \alpha \end{cases}$

$\Rightarrow Ker f = \{(\alpha, -2\alpha, \alpha) \mid \alpha \in \mathbb{R}\} = \{\alpha(1,-2,1) \mid \alpha \in \mathbb{R}\}$

$S_2 = \{w_1 = (1,-2,1)\}$ sist de gen și bază pt $Ker f$

$\Rightarrow \dim Ker f = 1$

$\dim Ker f + \dim Im f = \dim \mathbb{R}^3 \iff 1 + 2 = 3$ (A)

f e morfism $\iff \forall v_1, v_2 \in V$ și $\forall \alpha, \beta \in \mathbb{R}$ avem $f(\alpha v_1 + \beta v_2) =$
$= \alpha f(v_1) + \beta f(v_2)$

$f(v) = Av$, unde $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ și $v \in \mathbb{R}^3$

$$f(\alpha v_1 + \beta v_2) = A \cdot (\alpha v_1 + \beta v_2) = A \alpha v_1 + A \beta v_2 =$$

$$= \alpha A v_1 + \beta A v_2 = \alpha f(v_1) + \beta f(v_2) \Rightarrow f \text{ e morfism de spații vectoriale}$$

Temă seminarul 8

1. $f: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f(x,y,z) = (x+2y,\ 2y,\ -2y+z)$

a) f aplicație liniară

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad AX = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y \\ 2y \\ -2y+z \end{pmatrix} = f(X)$$

$f(X) = AX$

f aplicație $\Leftrightarrow \forall X_1, X_2 \in \mathbb{R}^3$; $\alpha_1, \alpha_2 \in \mathbb{R}$; $f(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 f(X_1) + \alpha_2 f(X_2)$

$f(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1) + A(\alpha_2 X_2) =$
$= (A\alpha_1) X_1 + (A\alpha_2) X_2 = (\alpha_1 A) X_1 + (\alpha_2 A) X_2 = \alpha_1 (A X_1) + \alpha_2 (A X_2) =$
$= \alpha_1 f(X_1) + \alpha_2 f(X_2) \Rightarrow f$ aplicație liniară

f) $A_f^n = ?$

$C \mid \Delta = C^{-1} A_f C \mid \overset{C^{-1}}{\Rightarrow} C \Delta C^{-1} = A_f$

$A_f^2 = C\Delta C^{-1} C \Delta C^{-1} = C\Delta^2 C^{-1}$

$A_f^3 = C\Delta^2 C^{-1} C \Delta C^{-1} = C\Delta^3 C^{-1}$

$\vdots$

$A_f^n = C\Delta^n C^{-1}$

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$\det C = -1$

$$\Delta^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

$a_{11} = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$

$a_{12} = (-1)\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} = 0$

$a_{13} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$a_{21} = (-1)\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} = -2$

$a_{22} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2$

$a_{23} = (-1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$

$a_{31} = \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0$

$a_{32} = (-1)\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1$

$a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$$\Rightarrow C^{-1} = \frac{1}{\det C}\begin{pmatrix} -1 & -2 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$CD^n = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2^{n+1} \\ 0 & 0 & 2^n \\ 0 & 1 & -2^{n+1} \end{pmatrix}$$

$$CD^nC^{-1} = \begin{pmatrix} 1 & 0 & 2^{n+1} \\ 0 & 0 & 2^n \\ 0 & 1 & -2^{n+1} \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2+2^{n+1} & 0 \\ 0 & 2^n & 0 \\ 0 & 2-2^{n+1} & 1 \end{pmatrix}$$

$$Af^n = \begin{pmatrix} 1 & 2(1+2^n) & 0 \\ 0 & 2^n & 0 \\ 0 & 2(1-2^n) & 1 \end{pmatrix}$$

Temă seminarul 9

1. $f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(x,y,z) = (2x-y+2z, -x+2y-z, x+y+z) \ \forall (x,y,z) \in \mathbb{R}^3$

a) f aplicație liniară

$f(\alpha v_1 + \beta v_2) = f(\alpha v_1) + f(\beta v_2)$

$$A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad AX = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$f(X) = AX$

f aplicație liniară $\Leftrightarrow \forall X_1, X_2 \in \mathbb{R}^3, \forall \alpha_1, \alpha_2 \in \mathbb{R}, f(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 f(X_1) + \alpha_2 f(X_2)$

$f(\alpha_1 X_1 + \alpha_2 X_2) = A \cdot (\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1) + A(\alpha_2 X_2) = (A\alpha_1)X_1 + (A\alpha_2)X_2 = (\alpha_1 A)X_1 + (\alpha_2 A)X_2 = \alpha_1 (AX_1) + \alpha_2 (AX_2) = \alpha_1 f(X_1) + \alpha_2 f(X_2) \Rightarrow f$ aplicație liniară

2. 1) $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = (-x+3y-z, -3x+5y-z, -3x+3y+z)$

a) f aplicație liniară

b) mat asoc lui f

c) det valorile proprii și subspațiile proprii corespunzătoare valorilor proprii

d) stabiliți dc Δ e endomorfism diagonalizabil

e) dc $\exists$, scrieți mat diag C și mat diag Δ

f) verificare

g) A_f^n, $n \in \mathbb{N}^*$

a) $A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad AX = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+3y-z & -x+3y-z & -x+3y-z \\ -3x+5y-z & -3x+5y-z & -3x+5y-z \\ -3x+3y+z & -3x+3y+z & -3x+3y \end{pmatrix}$

$f(X) = AX$

f aplicație liniară $\Leftrightarrow \forall X_1, X_2 \in \mathbb{R}^3,\ \alpha_1, \alpha_2 \in \mathbb{R}$, $f(\alpha_1 X_1 + \alpha_2 X_2) =$

$= \alpha_1 f(X_1) + \alpha_2 f(X_2)$

$f(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1) + A(\alpha_2 X_2) =$

$= (A\alpha_1) X_1 + (A\alpha_2) X_2 = (\alpha_1 A) X_1 + (\alpha_2 A) X_2 = \alpha_1 (A X_1) + \alpha_2 (A X_2)$

$= \alpha_1 f(X_1) + \alpha_2 f(X_2) \Rightarrow f$ aplicație liniară

b) $A_f = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$

c) valorile proprii

$P(\lambda) = 0 \Leftrightarrow \det(A_f - \lambda I_3) = 0$

$A_f - \lambda I_3 = \begin{pmatrix} -1-\lambda & 3 & -1 \\ -3 & 5-\lambda & -1 \\ -3 & 3 & 1-\lambda \end{pmatrix}$

$\det(A_f - \lambda I_3) = -(1+\lambda)(1-\lambda)(5-\lambda) + 9 + 9 - 15 + \cancel{3\lambda} - 3 - \cancel{3\lambda} + 9 - 9\lambda =$

$= -(1-\lambda^2)(5-\lambda)+9-9\lambda = -5+\lambda+5\lambda^2-\lambda^3+9-9\lambda =$

$= -\lambda^3+5\lambda^2-8\lambda+4 = 0 \Leftrightarrow$

	-1	5	-8	4
1	-1	4	-4	0

$(\lambda-1)(-\lambda^2+4\lambda-4)=0$

$-(\lambda-1)(\lambda-2)^2=0$

$\lambda_1 = 1$

$\lambda_2 = 2$

$V_{\lambda_1=1} = \{ v \in \mathbb{R}^3 \mid A_f v = \lambda_1 v \}$

$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (\underbrace{A_f - \lambda_1}_{=1} i_3) v = O_{(3,1)}$

$A_f - i_3 = \begin{pmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix}$

$\det(A_f - i_3) = 9+9-12-6=0$

$$\begin{pmatrix} -2 & 3 & -1 \\ -3 & 4 & -1 \\ -3 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Delta_1 = \begin{vmatrix} -2 & 3 \\ -3 & 4 \end{vmatrix} = -8-9 = -17 \neq 0 \Rightarrow \text{rang}(A_f - i_3) = 2$

x, y nec principale

$z = \alpha$, $\alpha \in \mathbb{R}$ nec secundare

$$\begin{cases} -2x+3y = \alpha \mid \cdot 3 \\ -3x+4y = \alpha \mid \cdot(-2) \\ z = \alpha \end{cases}, \alpha \in \mathbb{R} \Rightarrow \begin{cases} -6x+9y = 3\alpha \\ +6x-8y = -2\alpha \quad (+) \\ z = \alpha \end{cases}$$

$y = \alpha$

$2x = 2\alpha \Rightarrow$

$x = \alpha$

$\Rightarrow V_{\lambda_1=1} = \{(\alpha,\alpha,\alpha) \mid \alpha \in \mathbb{R}\} = \{\alpha(1,1,1) \mid \alpha \in \mathbb{R}\}$