EXAMEN CALCUL DIFERENTIAL SI INTEGRAL
SERIA 13

OFICIU: **1 punct**

SUBIECTUL 1. *(2 puncte)*

Sa se studieze natura seriei $\sum\limits_{n=0}^{+\infty} \frac{1.5.9\cdots(4n+1)}{2.6.10\cdots(4n+2)} \cdot \frac{1}{n+1}$.

SUBIECTUL 2. *(2 puncte)*

Sa se determine punctele de extrem local ale functiei $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$, $f(x, y) = y \ln\left(x^2 + y^2\right)$ $\forall (x, y) \in \mathbb{R} \times (0, +\infty)$.

SUBIECTUL 3. *(2 puncte)*

Sa se demonstreze inegalitatea $\ln(1 + x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ $\forall x \in (0, +\infty)$.

SUBIECTUL 4. *(3 puncte)*

a) Sa se calculeze $\iint\limits_D \sqrt{x+y}\,dxdy$, unde $D = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x + y \le 1, y \ge -1\right\}$.

b) Se considera o functie derivabila $f : (0, +\infty) \to \mathbb{R}$ cu proprietatea că $\exists \lim\limits_{x\to\infty} (3f(x) + xf'(x)) = l \in \mathbb{R}$. Sa se demonstreze ca $\exists \lim\limits_{x\to\infty} f(x) = \frac{l}{3}$.

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EXAMEN LA CDI

Subiectul I

$$\sum_{n=0}^{+\infty} \frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4n+1)}{2\cdot 6\cdot 10\cdot \ldots \cdot (4n+2)} \cdot \frac{1}{n+1}$$

Fie $x_n = \frac{1\cdot 5\cdot 9\cdot \ldots \cdot (4n+1)}{2\cdot 6\cdot 10\cdot \ldots \cdot (4n+2)} \cdot \frac{1}{n+1}$, $x_n > 0 \ \forall n \in \mathbb{N}$

$$\frac{x_n}{x_{n+1}} = \frac{\cancel{1\cdot 5\cdot 9\cdots(4n+1)}}{\cancel{2\cdot 6\cdot 10\cdots(4n+2)}} \cdot \frac{1}{n+1} \cdot \frac{\cancel{2\cdot 6\cdot 10\cdots(4n+2)}(4n+6)}{\cancel{1\cdot 5\cdot 9\cdots(4n+1)}(4n+5)} \cdot (n+2) =$$

$$= \frac{(4n+6)(n+2)}{(n+1)(4n+5)} = \frac{4n^2+8n+6n+12}{4n^2+5n+4n+5} = \frac{4n^2+14n+12}{4n^2+9n+5}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+14n+12}{4n^2+9n+5} - \frac{4n^2+9n+5}{1}\right) = \lim_{n\to\infty} n\left(\frac{4n^2+14n+12-4n^2-9n-5}{4n^2+9n+5}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{5n+7}{4n^2+9n+5}\right) = \lim_{n\to\infty} \frac{5n^2+7n}{4n^2+9n+5} = \frac{5}{4} > 1 \Rightarrow \sum_{n=0}^{+\infty} x_n \text{ e convergentă}$$

(Criteriul lui Raabe-Duhamel)

Subiectul II

$f : \mathbb{R}\times(0,\infty) \to \mathbb{R}$, $f(x,y) = y\ln(x^2+y^2)\ \forall (x,y)\in\mathbb{R}\times(0,\infty)$

$D = \mathbb{R}\times(0,\infty)$ mulțime deschisă

f funcție continuă pe $\mathbb{R}\times(0,\infty)$ (combinație de funcții elementare)

$D_f = \{x \in D \mid f \text{ nu e continuă în } x\} = \emptyset$

$$\frac{\partial f}{\partial x}(x,y) = (y\ln(x^2+y^2))'_x = y\frac{2x}{x^2+y^2} = \frac{2xy}{x^2+y^2}\ \forall (x,y)\in\mathbb{R}\times(0,\infty)$$

$$\frac{\partial f}{\partial y}(x,y) = (y\ln(x^2+y^2))'_y = \ln(x^2+y^2) + y\frac{2y}{x^2+y^2} = \ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}\ \forall (x,y)\in\mathbb{R}\times(0,\infty)$$

$\left.\begin{array}{l}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ funcții continue pe} \\ \text{mulțime deschisă}\end{array}\right\} \Rightarrow f$ funcție de clasă $C^1 \Rightarrow f$ diferențiabilă pe $\mathbb{R}\times(0,\infty)$

$D_1 = \{x\in D \mid f \text{ nu e diferențiabilă în } x\} = \emptyset$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0 \end{cases} \Rightarrow \begin{cases} \frac{2xy}{x^2+y^2}=0 \\ \ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0 \end{cases} \Rightarrow \left.\begin{cases} \left.\begin{array}{l} 2xy=0 \\ y\in(0,\infty) \end{array}\right\} \Rightarrow x=0 \\ \ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0 \end{cases}\right\} \Rightarrow$$

$$\Rightarrow \ln(0+y^2)+\frac{2y^2}{0+y^2}=0 \Rightarrow \ln(y^2)+\cancel{\frac{2y^2}{y^2}}=0 \Rightarrow \ln(y^2)=-2 \Rightarrow y^2=e^{-2} \Rightarrow$$

$$\Rightarrow |y|=\frac{1}{\sqrt{e^2}}\ (y\in(0,\infty)) \Rightarrow y=\frac{1}{e} \Rightarrow \left(0,\frac{1}{e}\right)\in \mathbb{R}\times(0,\infty) \Rightarrow C=\left\{\left(0,\frac{1}{e}\right)\right\}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)=\left(\frac{2xy}{x^2+y^2}\right)'_x=\frac{2y(x^2+y^2)-2xy(2x)}{(x^2+y^2)^2}=\frac{2y^3-2x^2y}{(x^2+y^2)^2}\ \forall(x,y)\in \mathbb{R}\times(0,\infty)$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=\left(\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}\right)'_x=\frac{2x}{x^2+y^2}+2y^2\left(-\frac{2x}{(x^2+y^2)^2}\right)=\frac{2x^3-2xy^2}{(x^2+y^2)^2}\ \forall(x,y)\in \mathbb{R}\times(0,\infty)$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)=\left(\frac{2xy}{x^2+y^2}\right)'_y=\frac{2x(x^2+y^2)-2xy(x^2+2y)}{(x^2+y^2)^2}=\frac{2x^3-2x^3y-2xy^2}{(x^2+y^2)^2}\ \forall(x,y)\in \mathbb{R}\times(0,\infty)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)=\left(\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}\right)'_y=\frac{2y}{x^2+y^2}+\frac{4y(x^2+y^2)-2y^2(x^2+2y)}{(x^2+y^2)^2}=\frac{6yx^2+2y^3-2y^2}{(x^2+y^2)^2}$$
$\forall (x,y)\in \mathbb{R}\times(0,\infty)$

$\left.\begin{array}{l} \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y\partial x}, \frac{\partial^2 f}{\partial y^2} \text{ funcții continue pe } \mathbb{R}\times(0,\infty) \\ \mathbb{R}\times(0,\infty) \text{ mulțime deschisă} \end{array}\right\} \Rightarrow f$ e funcție de clasă $C^2 \Rightarrow$

$\Rightarrow f$ e diferențiabilă de două ori pe $\mathbb{R}\times(0,\infty)$

$\Delta_2=\{x\in D \mid f$ nu este diferențiabilă de două ori în $x\}=\emptyset$

$$Hf\left(0,\frac{1}{e}\right)=\begin{pmatrix} \frac{\partial^2 f}{\partial x\partial x}(0,\frac{1}{e}) & \frac{\partial^2 f}{\partial x\partial y}(0,\frac{1}{e}) \\ \frac{\partial^2 f}{\partial y\partial x}(0,\frac{1}{e}) & \frac{\partial^2 f}{\partial y\partial y}(0,\frac{1}{e}) \end{pmatrix}=\begin{pmatrix} \frac{\frac{2}{\cancel{e^3}}}{\frac{1}{e^4}} & 0 \\ 0 & \frac{\frac{2}{\cancel{e^3}}}{\frac{1}{e^4}} \end{pmatrix}=\begin{pmatrix} 2e & 0 \\ 0 & 2e \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1=2e>0 \\ \Delta_2=4e^2>0 \end{array}\right\} \xRightarrow{\text{cazul a)}} \left(0,\frac{1}{e}\right) \text{ punct de minim local} \quad (1)$$

$\Delta_3=\{x\in D \mid$ criteriul se pronunță în $x\}=\{(0,\frac{1}{e})\}$

$\Delta_4=\{x\in D \mid$ criteriul nu se pronunță în $x\}=\emptyset$

$$E\subseteq \underbrace{D_f}_{\emptyset}\cup \underbrace{\Delta_1}_{\emptyset}\cup \underbrace{\Delta_2}_{\emptyset}\cup \Delta_3\cup \underbrace{\Delta_4}_{\emptyset} \quad (2)$$

Din (1) și (2) $\Rightarrow E=\{(0,\frac{1}{e})\}$

Subiectul III

$\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad \forall x \in (0,\infty)$

Alegem funcția $f:(0,\infty)\to\mathbb{R}$, $f(x)=\ln(1+x)$ $\forall x\in(0,\infty)$

f derivabilă de 4 ori

$f'(x)=\frac{1}{1+x} \quad \forall x\in(0,\infty)$

$f''(x)=-\frac{1}{(1+x)^2} \quad \forall x\in(0,\infty)$

$f'''(x)=\left(-\frac{1}{(1+x)^2}\right)'=\frac{2}{(x+1)^3} \quad \forall x\in(0,\infty)$

$f^{(IV)}(x)=\left(\frac{2}{(x+1)^3}\right)'=\frac{-6}{(x+1)^4} \quad \forall x\in(0,\infty)$

Fie $x_0=0$. Aplicăm formula lui Taylor cu restul sub forma lui Lagrange:

$\forall x\in(0,\infty)$ cu $x\neq 0$ există un $c\in(0,\infty)$ situat între x și 0 astfel încât $f(x)=T_{f,3,0}+\frac{f^{(4)}(c)}{4!}x^4$

$T_{f,3,0}=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3=0+\frac{x}{1}-\frac{x^2}{2}+\frac{\cancel{2}x^3}{\cancel{6}_3}=$

$=x-\frac{x^2}{2}+\frac{x^3}{3}$

Rezultă că $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\frac{f^{(4)}(c)}{4!}x^4$

$$\left.\begin{array}{l}\left.\begin{array}{l} 0<x \Rightarrow c\in(0,x) \\ f^{(4)}(c)=-\frac{6}{(c+1)^4}\end{array}\right\} \Rightarrow f^{(4)}(c)<0 \Rightarrow \frac{f^{(4)}(c)}{4!}x^4<0\end{array}\right\} \Rightarrow$$

$\Rightarrow \ln(1+x) < x-\frac{x^2}{2}+\frac{x^3}{3} \quad \forall x\in(0,\infty)$

Subiectul IV

a) $\iint_\Delta \sqrt{x+y}\,dx\,dy \quad \Delta = \{(x,y) \in \mathbb{R} \mid 0 \le x+y \le 1,\ y \ge -1\}$

$\Delta: \begin{cases} 0 \le x+y \le 1 \\ y \ge -1 \end{cases}$

$\left.\begin{array}{l} x+y=0 \Rightarrow y=-x \\ x+y \ge 0 \end{array}\right\} \Rightarrow$ punctele din plan deasupra dreptei de ecuație $y=-x$ sau pe dreapta $y=-x$

$\left.\begin{array}{l} x+y=1 \Rightarrow y=1-x \Rightarrow y=-x+1 \\ x+y \le 1 \end{array}\right\} \Rightarrow$ punctele din plan de sub dreapta $y=-x+1$ sau pe dreapta $y=-x+1$

$y \ge -1 \Rightarrow$ porțiunea din plan cu $y \ge -1$

$\left.\begin{array}{l} 0 \le x+y \le 1 \mid -y \Rightarrow -y \le x \le 1-y \Rightarrow -y \le 1-y \Rightarrow y \in \mathbb{R} \\ y \ge -1 \mid \cdot(-1) \Rightarrow -y \le 1 \end{array}\right\} \Rightarrow y \in [-1, \infty)$

Fie $g, h : \mathbb{R} \to \mathbb{R}$, $g(y) = -y$, $h(y) = 1-y$

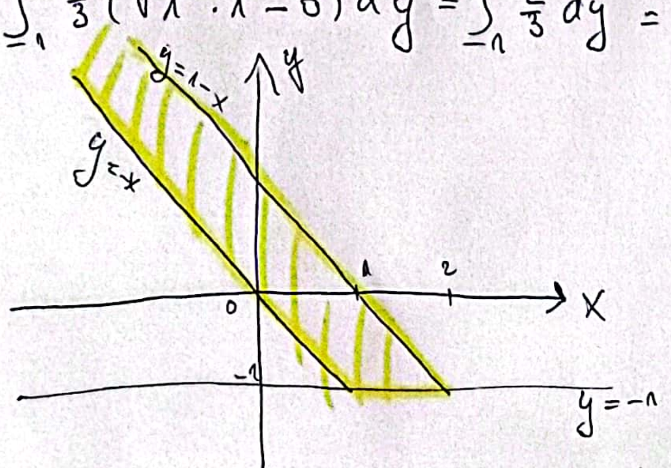
$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [-1, y),\ -y \le x \le 1-y\}$

A mulțime simplă în raport cu axa Ox. Integrarea se face în ordinea $dxdy$.

$$\iint_\Delta \sqrt{x+y}\,dx\,dy = \int_{-1}^{\infty}\left(\int_{-y}^{1-y} \sqrt{x+y}\,dx\right)dy = \int_{-1}^{\infty} \frac{2}{3}\sqrt{x+y}\,(x+y)\Big|_{-y}^{1-y} dx =$$

$$= \int_{-1}^{\infty} \frac{2}{3}\left[\sqrt{1-y+y}\,(1-y+y) - \sqrt{-y+y}\,(-y+y)\right]dy =$$

$$= \int_{-1}^{\infty} \frac{2}{3}(\sqrt{1}\cdot 1 - 0)\,dy = \int_{-1}^{\infty} \frac{2}{3}\,dy = \frac{2}{3}\,y\Big|_{-1}^{\infty} = +\infty \text{ (integrala divergentă)}$$



h) $f:(0,\infty)\to\mathbb{R}$ derivabilă

$\exists \lim\limits_{x\to\infty}(3f(x)+xf'(x))=l\in\mathbb{R}$

$\exists \lim\limits_{x\to\infty} f(x)=\frac{l}{3}$

Se studiază teorema lui L'Hospital pentru rezolvarea acestui exercițiu. Se aleg funcțiile $g,h:(0,\infty)\to\mathbb{R}$ definite prin $g(x)=x^3f(x)\ \forall x\in(0,\infty)$ și $h(x)=\frac{x^3}{3}\ \forall x\in(0,\infty)$
(g,h, derivabile)

$$\frac{g'(x)}{h'(x)}=\frac{(x^3f(x))'}{\left(\frac{x^3}{3}\right)'}=\frac{3x^2f(x)+x^3f'(x)}{\frac{\cancel{3}x^2}{\cancel{3}}}=\frac{\cancel{x^2}(3f(x)+xf'(x))}{\cancel{x^2}}$$

$\forall x\in(0,\infty)$

$$\left.\begin{array}{l} g,h \text{ derivabile pe } (0,\infty)\\ h'(x)\neq 0\ \forall x\in(0,\infty)\\ \exists \lim\limits_{x\to\infty}\frac{g'(x)}{h'(x)}=\lim\limits_{x\to\infty}(3f(x)+xf'(x))=l\\ \lim\limits_{x\to\infty}h(x)=\lim\limits_{x\to\infty}\frac{x^3}{3}=+\infty \end{array}\right\}\overset{L'H}{\Rightarrow}\exists\lim\limits_{x\to\infty}\frac{g(x)}{h(x)}=l\Rightarrow$$

$$\Rightarrow\exists\lim\limits_{x\to\infty}\frac{x^3f(x)}{\frac{x^3}{3}}=l\in\mathbb{R}\Rightarrow\exists\lim\limits_{x\to\infty}3\frac{\cancel{x^3}f(x)}{\cancel{x^3}}=l\Rightarrow$$

$$\Rightarrow\exists\lim\limits_{x\to\infty}3f(x)=l\Rightarrow\exists\, 3\lim\limits_{x\to\infty}f(x)=l\Rightarrow\exists\lim\limits_{x\to\infty}f(x)=\frac{l}{3}$$