Review: Quantum Harmonic Oscillator

Shruti Puri

1 Introduction

Several physical systems are characterized by a Hamiltonian of the form

$$\hat{H} = a^2\hat{P}^2 + b^2\hat{Q}^2 \tag{1}$$

where, $[\hat{Q}, \hat{P}] = i\hbar$. This is the Hamiltonian describing, for example, a particle in a harmonic oscillator potential in which case the hermitian operators $\hat{P}, \hat{Q}$ are associated with the position and momentum of the particle. This Hamiltonian also describes electromagnetic field in an optical or microwave cavity or even vibrational modes of trapped atoms or ions. The Hamiltonian in Eq. (1) is called the harmonic oscillator (HO) Hamiltonian. In order to find the eigenvalues of this Hamiltonian it becomes useful to define a new set of operators,

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\frac{b}{a}}\hat{Q} + i\sqrt{\frac{a}{b}}\hat{P}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{\frac{b}{a}}\hat{Q} - i\sqrt{\frac{a}{b}}\hat{P}\right) \tag{2}$$

so that,

$$\hat{P} = i\sqrt{\frac{\hbar b}{2a}}(\hat{a}^\dagger - \hat{a}), \quad \hat{Q} = \sqrt{\frac{\hbar a}{2b}}(\hat{a}^\dagger + \hat{a}) \tag{3}$$

From. Eq. (2) we find

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{4}$$

Using Eq. (3) and Eq. (4) in Eq. (1) we get,

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{5}$$

where, $\omega = 2ab$. $\hat{a}, \hat{a}^\dagger$ are the famous bosonic annihilation and creation operators respectively. The relation in Eq. (4) is also called bosonic commutation relation.

2 Fock states: Eigenstates of the HO

Next, we want to determine the eigenvalues and eigenstates of the HO. Let $|\lambda\rangle$ be the eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue λ, then $|\lambda\rangle$ will also be the eigenstate of the HO with eigenvalue $\hbar\omega(\lambda + 1/2)$. According to our definition,

$$\hat{a}^\dagger\hat{a}|\lambda\rangle = \lambda|\lambda\rangle \tag{6}$$

Firstly, note that since $\hat{a}^\dagger\hat{a}$ is a positive operator, we have $\lambda \geq 0$. Next, multiply both sides of the above equation by $\hat{a}$,

$$\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle = \lambda\hat{a}|\lambda\rangle \tag{7}$$

Using the commutation relation in Eq. (4) the above equation reduces to,

$$(\hat{a}^\dagger\hat{a} + 1)\hat{a}|\lambda\rangle = \lambda\hat{a}|\lambda\rangle \Rightarrow \hat{a}^\dagger\hat{a}\hat{a} = (\lambda - 1)\hat{a}|\lambda\rangle \tag{8}$$

Hence, we see that if $|\lambda\rangle$ is an eigenvector of $\hat{a}^\dagger\hat{a}$ with eigenvalue λ then up to a c-number factor, $\hat{a}|\lambda\rangle$ is also the eigenvector of $\hat{a}^\dagger\hat{a}$ with eigenvalue $\lambda - 1$. That is,

$$\hat{a}|\lambda\rangle = x|\lambda - 1\rangle, \tag{9}$$

$$\langle\lambda|\hat{a}^\dagger = x^*\langle\lambda - 1| \tag{10}$$

By combining these two equations we have,

$$\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = |x|^2 \tag{11}$$

and since according to the definition $\hat{a}^\dagger\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$, the above equation implies $|x| = \sqrt{\lambda}$. Thus, if we ignore an unimportant phase we have

$$\hat{a}|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle \tag{12}$$

The above equation implies that the application of $\hat{a}$ to an eigenvector of $\hat{a}^\dagger\hat{a}$ (or equivalently an eigenvector of the HO) gives another eigenvector with an eigenvalue reduced by 1. This is why $\hat{a}$ is called the annihilation operator. Similarly we can show that,

$$\hat{a}^\dagger|\lambda\rangle = \sqrt{\lambda+1}|\lambda+1\rangle \tag{13}$$

Since the application of $\hat{a}^\dagger$ transforms an eigenvector of $\hat{a}^\dagger\hat{a}$ (or equivalently an eigenvector of the HO) to another eigenvector with eigenvalue increased by 1, $\hat{a}^\dagger$ is called the creation operator.

Exercise: Prove Eq. (13)

It is now easy to see that,

$$\hat{a}^n = \sqrt{\lambda}\sqrt{\lambda-1}\sqrt{\lambda-2}....\sqrt{\lambda-n+1}|\lambda-n\rangle, \tag{14}$$

$$\hat{a}^{\dagger n} = \sqrt{\lambda+1}\sqrt{\lambda+2}\sqrt{\lambda-2}....\sqrt{\lambda+n}|\lambda+n\rangle \tag{15}$$

That is, applying $\hat{a}(\hat{a}^\dagger)$ n times to an an eigenstate of $\hat{a}^\dagger\hat{a}$ gives another eigenstate with a smaller (larger) eigenvalue by n. So if the smallest eigenvalue is λ_0, then the eigenvalues in the ascending order are $\lambda_0, \lambda_0+1, \lambda_0+2, ...$ What is λ_0? Since $\hat{a}|\lambda_0\rangle = \sqrt{\lambda_0}|\lambda_0-1\rangle$ and hence $|\lambda_0-1\rangle$ should be another eigenstate with a lower eigenvalue. But this is not physically possible since λ_0 is the lowest eigenvalue, unless $\lambda_0 = 0$. Therefore, the lowest eigenvalue of $\hat{a}^\dagger\hat{a}$ is 0 and thus all the eigenvalues are integers: $0, 1, 2, ...$ It is possible to represent the eigenstates of $\hat{a}^\dagger\hat{a}$ as a ladder as shown in Fig.1. Action of $\hat{a}^\dagger$ moves us up the ladder, while the action of $\hat{a}$ moves us down the ladder. From now on we will represent the eigenstates as $|0\rangle, |1\rangle, ..., |n\rangle, ...$ and hence,

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{16}$$

Equation (16) is useful to remember for this course. The eigenstates of the HO are also $|0\rangle, |1\rangle, |2\rangle, ...|n\rangle, ...$ and the eigenvalues are $\hbar\omega(n + 1/2)$. Clearly, the minimum energy of the HO is $\hbar\omega/2$ and is also called as the zero-point energy (ZPE). In this course, for convenience, we work in the units where $\hbar = 1$. We will also neglect explicitly writing the ZPE since it only causes a constant shift in the energies (that being said, the ZPE leads to many interesting effects but we will not be dealing with these in this course).

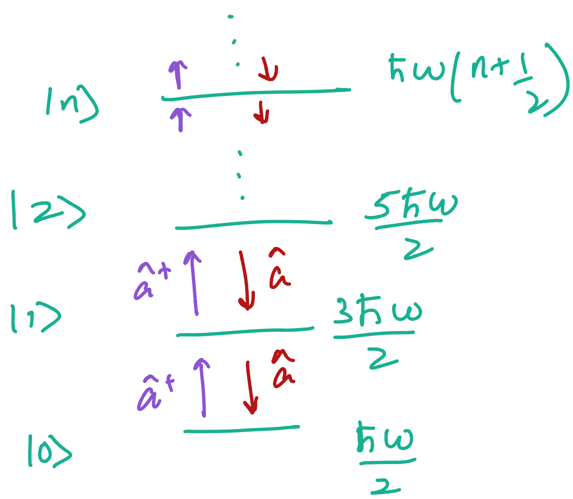

Fig 1 Eigenstates and engenvalues of HO. Action of $\hat{a}$ transforms the n^{th} eigenstate to the $(n-1)^{th}$ eigenstate. Action of $\hat{a}^\dagger$ transforms the n^{th} eigenstate to the $(n+1)^{th}$ eigenstate.

The eigenstates of $\hat{a}^\dagger\hat{a}$, that is $\{|n\rangle\}$, are called **Fock states**. When describing electromagnetic field in optical or microwave cavities, $\{|n\rangle\}$ are also referred to as the **photon number**

Fock states. The lowest energy state, $|0\rangle$, is called the **vacuum** or **0-photon Fock state**. In general $|n\rangle$ is the $\boldsymbol{n}$**-photon Fock state**. The operator $\hat{a}^\dagger \hat{a}$ is called the **photon number operator** and is sometimes denoted as $\hat{\boldsymbol{n}}$. Note that, because $\hat{n}$ is hermitian, the eigenstates also satisfy:

$$\langle n|m\rangle = \delta_{nm}, \quad \text{(orthogonality condition)} \tag{17}$$

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = I, \quad \text{(completeness condition)} \tag{18}$$

3 Quadrature Operators

It is useful to define operators,

$$\hat{p} = i\frac{\hat{a}^\dagger - \hat{a}}{\sqrt{2}}, \quad \hat{q} = \frac{\hat{a}^\dagger + \hat{a}}{\sqrt{2}} \tag{19}$$

These are called quadrature operators and $[\hat{q}, \hat{p}] = i$. In particular, $\hat{p}$ is the momentum quadrature and $\hat{q}$ is the position quadrature. As can be seen from Eq. (3), they are related to $\hat{P}$ and $\hat{Q}$ by scaling factors. These are useful because in realistic experimental systems, $\hat{p}, \hat{q}$ can be measured to get information about the state of an electromagnetic field in a cavity.

4 Normal Ordering

Given a function of $\hat{a}, \hat{a}^\dagger$: $f(\hat{a}, \hat{a}^\dagger)$, we can always write

$$f(\hat{a}, \hat{a}^\dagger) = \sum_{m,n} x_{m,n} \hat{a}^{\dagger m} \hat{a}^n \tag{20}$$

In this form, all the $\hat{a}^\dagger$ operators are on the left of $\hat{a}$ operator and the function is said to be normally ordered. Using commutation relation in Eq. (4), we can always re-write the function as,

$$f(\hat{a}, \hat{a}^\dagger) = \sum_{m,n} y_{m,n} \hat{a}^{m} \hat{a}^{\dagger n} \tag{21}$$

In this form, all the $\hat{a}^\dagger$ operators are on the right of $\hat{a}$ and the function is said to be anti-normally ordered.

5 Examples of Operator Expectation Values

Exercise: If $|n\rangle$ is a Fock state, show that $\langle n|\hat{q}|n\rangle = \langle n|\hat{p}|n\rangle = 0$.

Exercise: If $|n\rangle$ is a Fock state, show that $\langle n|\hat{q}^2|n\rangle = \langle n|\hat{p}^2|n\rangle = n + 1/2$. Therefore, the variance in measurement of $\hat{q}, \hat{p}$ when the HO is in a Fock state is $\Delta\hat{q}^2 = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2, \Delta\hat{p}^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 = n + 1/2$. Clearly, even when the HO is in its ground state or vacuum, that is $n = 0$, the variances are non-zero, $\Delta\hat{q}^2 = \Delta\hat{p}^2 = 1/2$.

Exercise: Consider the superposition of 0- and 4-photon Fock states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |4\rangle) \tag{22}$$

what is average photon number $\langle\hat{a}^\dagger\hat{a}\rangle$?

6 Coherent States

Coherent states are the eigenstates of the annihilation operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \langle\alpha|\alpha\rangle = 1, \quad |\alpha\rangle \rightarrow \text{coherent state} \tag{23}$$

The above equation implies $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$. While the eigenvalues of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ are restricted to be integers 0,1,2,..., the eigenvalues α of $\hat{a}$ can be any complex number. Unlike $\hat{a}$, $\hat{a}^\dagger$ does not have an eigenvector.

A Representation in Fock state basis

Since the Fock states form a complete basis we can always write

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{24}$$

In order to determine what $\{C_n\}$ are we apply $\hat{a}$ to both sides of the above equation and make use of Eq. (23) and Eq. (16),

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n \hat{a}|n\rangle \quad \Rightarrow \quad \alpha|\alpha\rangle = \sum_{n=0}^{\infty} C_n \sqrt{n}|n-1\rangle \tag{25}$$

Equating coefficient of $|n-1\rangle$ on both sides (which is possible because the Fock states are orthogonal) we find,

$$C_n \sqrt{n} = \alpha C_{n-1} \tag{26}$$

$$\Rightarrow C_n = \frac{\alpha}{\sqrt{n}} C_{n-1} = \frac{\alpha}{\sqrt{n}} \frac{\alpha}{\sqrt{n-1}} C_{n-2} = ... = \frac{\alpha^n}{\sqrt{n!}} C_0 \tag{27}$$

Hence,

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{28}$$

Now, in order to determine C_0 we can make use of the normalization condition

$$\langle\alpha|\alpha\rangle = 1 = |C_0|^2 \left(\sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \langle n| \right) \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right) \tag{29}$$

$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^2}{n!} = |C_0|^2 e^{|\alpha|^2} \tag{30}$$

Hence, we can take $C_0 = e^{-|\alpha|^2/2}$ so that,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{31}$$

B Some more properties of coherent states

(a) Two coherent states $|\alpha\rangle$, $|\beta\rangle$ are not orthogonal. In fact,

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \beta^*\alpha\right] \tag{32}$$

(b) Like Fock states, the coherent states also form a complete basis. The completeness relation is,

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1 \tag{33}$$

where the integral is carried out over the complex plane, $d^2\alpha = d\mathrm{Re}[\alpha]d\mathrm{Im}[\alpha]$.

C Operator expectation values and exercises

I. Suppose a HO, such as EM field in a cavity, is in a coherent state $|\alpha\rangle$. Then,

Exercise (a): Show that the expectation values of the quadrature operators are,

$$\langle\hat{q}\rangle = \frac{\alpha^* + \alpha}{\sqrt{2}} = \sqrt{2}\mathrm{Re}[\alpha], \quad \langle\hat{p}\rangle = i\frac{\alpha^* - \alpha}{\sqrt{2}} = \sqrt{2}\mathrm{Im}[\alpha] \tag{34}$$

Exercise (b): Show that the expectation values of the number operator, or the average photon number of the EM field, is $\langle\hat{a}^\dagger\hat{a}\rangle = |\alpha|^2$.

Exercise (c): Show that the variance in the measurement of $\hat{a}^\dagger\hat{a}$ is $|\alpha|$. Thus for a coherent state we find that $\Delta(\hat{a}^\dagger\hat{a}) = \sqrt{\langle\hat{a}^\dagger\hat{a}\rangle}$ which is characteristic of a Poisson process.

Exercise (d): Show that the photon number distribution in the coherent state or the probability that there are n photons in the coherent state is $P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$

II. Suppose the HO such as EM field in a cavity is in a superposition of coherent states

$$|C_+\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle), \quad |C_-\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle - |-\alpha\rangle) \tag{35}$$

These are Schrödinger's famous photonic **cat states**.

Exercise (a): Check that $|C_+\rangle$ and $|C_-\rangle$ are orthogonal.

Exercise (b): What is the photon number distribution of these states? Are there any special values of n for which $P_n = 0$? (Hint: you should see that for $|C_+\rangle$, $P_n = 0$ for odd n, while $|C_-\rangle$, $P_n = 0$ for even n).

Exercise (c): The photon number parity operator is defined as $\hat{\Pi} = \exp(i\pi\hat{a}^\dagger\hat{a})$. Check that the cat states $|C_\pm\rangle$ are eigenstates of $\hat{\Pi}$. What are the eigenvalues?

7 The Displacement Operator

The displacement operator is defined as,

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{36}$$

In fact, it is easy to show that the coherent state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where $|0\rangle$ is the vacuum state. This is why coherent states are sometimes called displaced vacuum states. A useful identity is $\hat{D}(\alpha)\hat{D}(\beta) = e^{\mathrm{Im}(\alpha\beta^*)}\hat{D}(\alpha+\beta)$. Using this identity,

Exercise: Show that for α, β such that $\alpha\beta^* - \beta\alpha^* = i\pi$ the operators $\hat{D}(\alpha)$ and $\hat{D}(\beta)$ anti-commute, while the operators $\hat{D}(2\alpha)$ and $\hat{D}(2\beta)$ commute.