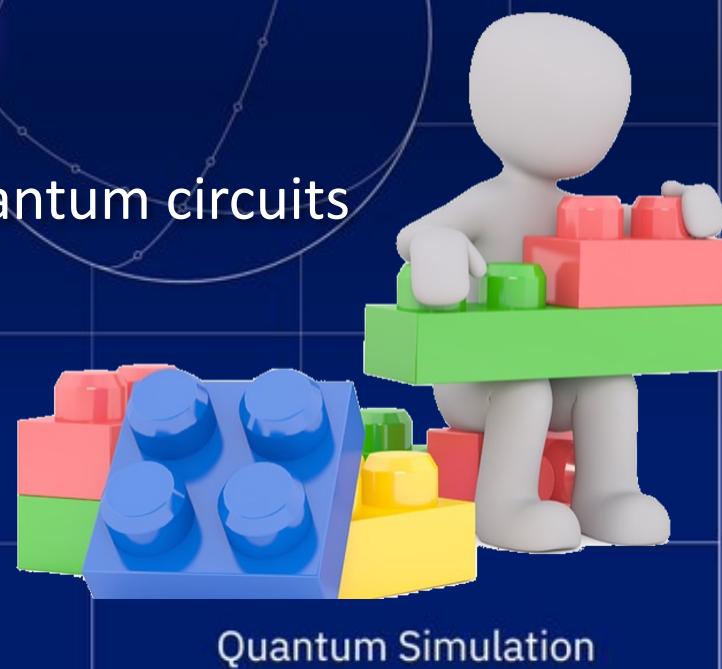
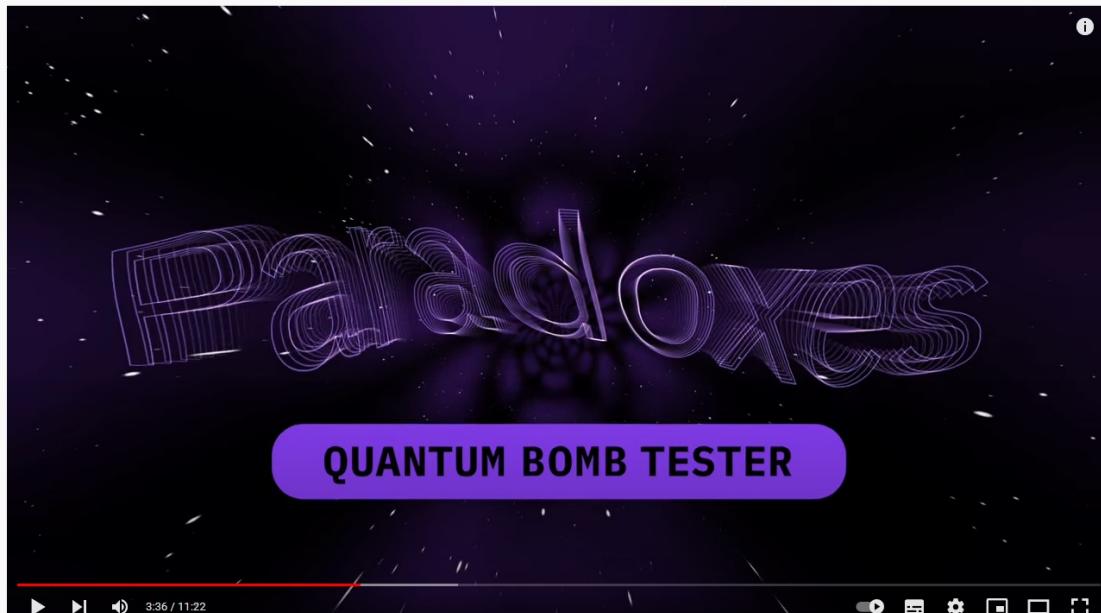


Lecture 2

Building blocks of quantum: from linear algebra to quantum circuits



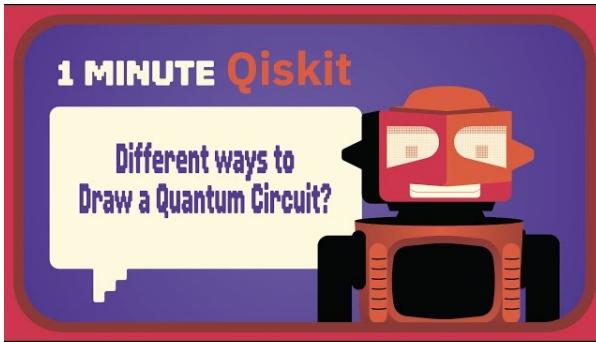
Hello!



Quantum Minesweeper: How to See a Bomb Without Looking

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Building blocks of quantum

- Linear algebra
 - Vector space, basis, bra-ket notation, inner product, dagger, operators, outer product
 - Single qubits
 - Bloch sphere, superposition, NOT, matrix representation, linearity, rotations, Y, Z, H, P, S, T, I, U, unitarity, measurement, bases, global and relative phases
 - Multiple qubits
 - Multiple qubits, Bell states, multi-qubit gates, C-NOT, pure and mixed states, density matrices, universality, quantum teleportation



Vector space

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

$$1. \quad v_1 + v_2 = v_3$$

$$2. \quad \alpha v_1 = v_2$$

linear combination -

$$\alpha \begin{pmatrix} a \\ b \end{pmatrix} + \beta \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \text{real} & \text{real} \\ \alpha a + \beta c & \alpha b + \beta d \end{pmatrix}$$

real
complex complex

Dimensions

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

complex.

$$\downarrow \quad \downarrow$$

$$\alpha v_1 + \beta v_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

2-dimensional

Subspace:

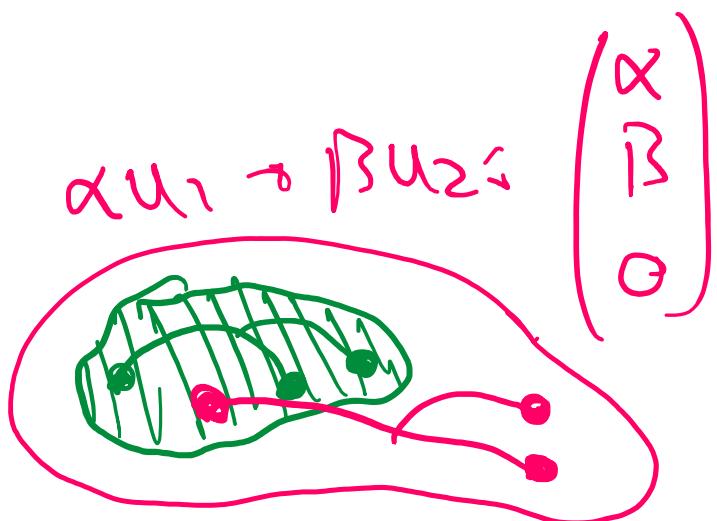
$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \alpha u_1 + \beta u_2 + \gamma u_3 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

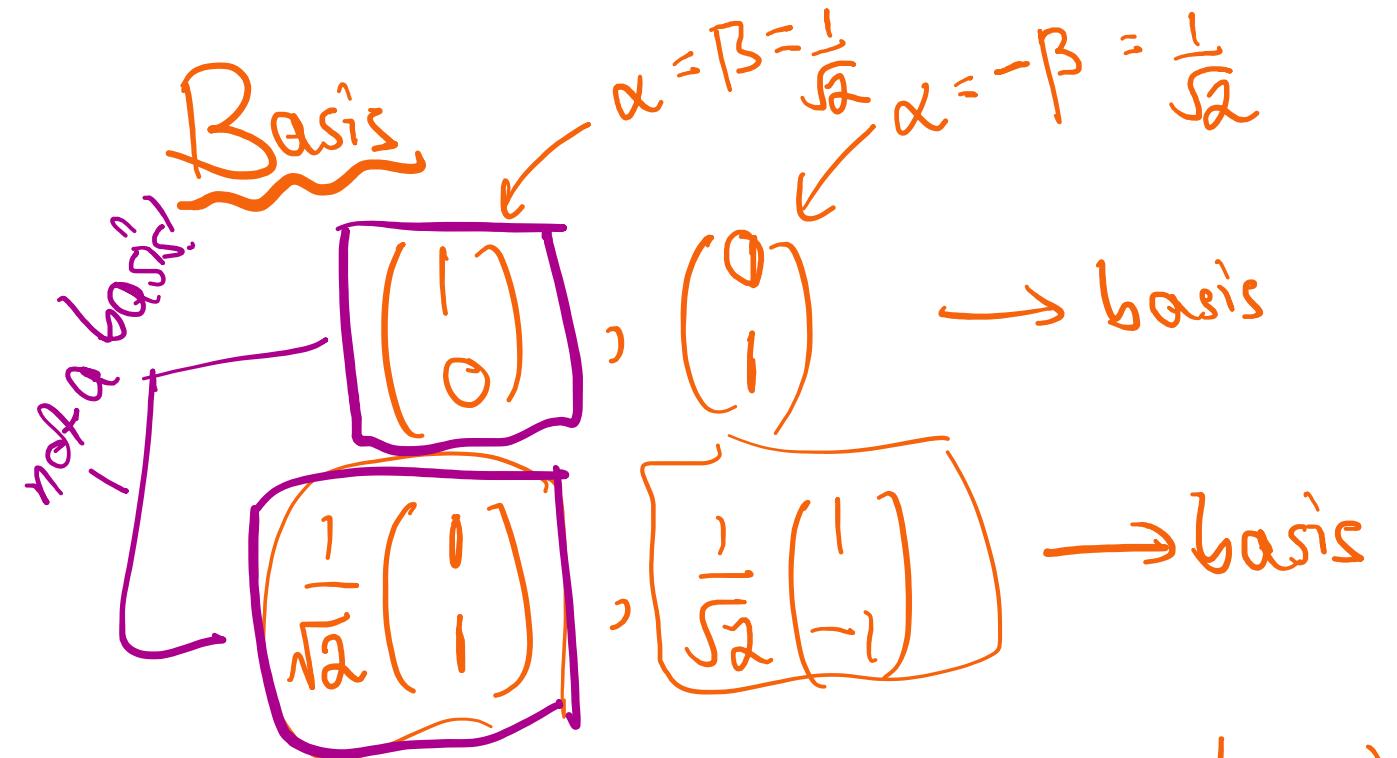
3-dimensional.

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



2-dimensional



$$\alpha \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha + \frac{\beta}{\sqrt{2}} \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times$$

Inner product = ① \rightarrow orthogonal.

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}^*{}^T = \begin{pmatrix} a \\ b \end{pmatrix}^+ = \begin{pmatrix} a^* & b^* \end{pmatrix}$$

$$(a^* \ b^*) \begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$(a_1^* \dots a_n^*) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1^*b_1 + a_2^*b_2 + \dots + a_n^*b_n.$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{S2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Bra-ket notation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = |a\rangle$$

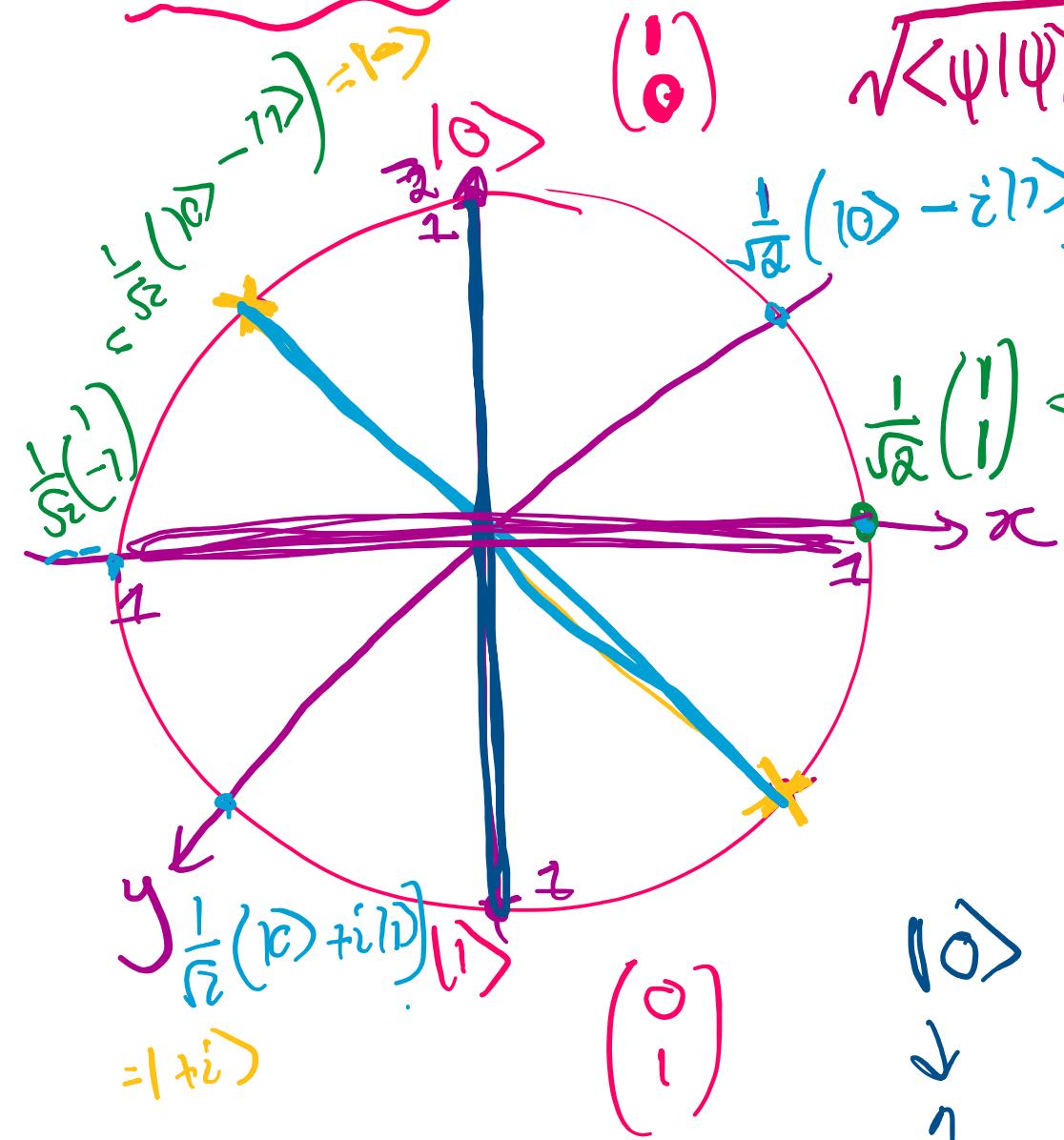
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^+ = (1 \ 0) = \langle 0 | \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}^+ = (0 \ 1) = \langle 1 | \quad (a_1^* \ a_2^* \ \dots \ a_n^*) = \langle a |$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$(a_1^* \ \dots \ a_n^*) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \langle a | b \rangle$$

$\langle a | b \rangle$

Bloch sphere



$$\sqrt{\langle \psi | \psi \rangle} \rightarrow \text{length}$$

$$\alpha|10\rangle + \beta|11\rangle$$

$$\langle 010 \rangle = (1 0)(\begin{pmatrix} 1 \\ 0 \end{pmatrix})^* = 1$$

$$\langle 111 \rangle = 1$$

$$\langle \psi | \psi \rangle = 1$$

$$= \left(\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) \right) \approx |+\rangle$$

$$\langle +1+ \rangle = \frac{1}{2}(\langle 01 \rangle + \langle 11 \rangle) = \frac{1}{2}(1 + 1) = \frac{2}{2} = 1$$

$$|10\rangle, |01\rangle, |+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$

$$\frac{1}{2}, \frac{1}{2}$$

Measurement

$$\underline{\alpha}|0\rangle + \underline{\beta}|1\rangle$$

$$\text{prob. } |0\rangle : |\alpha|^2$$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\langle + | (\alpha|0\rangle + \beta|1\rangle)$$

$$\begin{aligned} &= \alpha \langle + | 0 \rangle + \beta \langle + | 1 \rangle \\ &= \alpha \frac{1}{\sqrt{2}} + \beta \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\alpha + \beta) \end{aligned}$$

$$\text{prob. } |1\rangle : |\beta|^2$$

$$\alpha|0\rangle + \beta|1\rangle$$

$$= \alpha \left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \right) + \beta \left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \right)$$

$$= \frac{1}{\sqrt{2}}(\alpha + \beta)|+\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|-\rangle$$

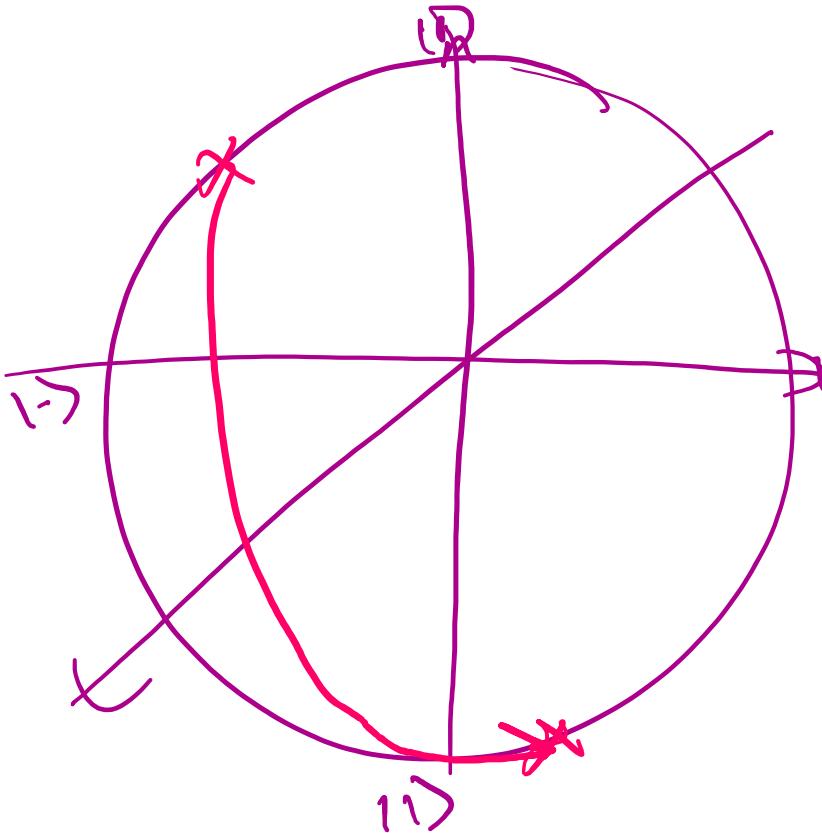
$$\text{prob. } |+\rangle :$$

$$\frac{1}{2}|\alpha + \beta|^2$$

$$\text{prob. } |-\rangle : \frac{1}{2}|\alpha - \beta|^2$$

$$|\langle a | b \rangle|^2$$

Quantum gates



$$|\psi\rangle \rightarrow |\psi'\rangle$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a' \\ b' \end{pmatrix}$$

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

$$M \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a' \\ b' \end{pmatrix}$$

Linear Operators

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} M_1 a + M_2 b \\ M_3 a + M_4 b \end{pmatrix}$$

$$M|\psi\rangle = |\psi'\rangle$$

$$M\left(\frac{|\psi\rangle + |\phi\rangle}{\sqrt{2}}\right) = \underbrace{M|\psi\rangle}_{\sqrt{2}} + \underbrace{M|\phi\rangle}_{\sqrt{2}}$$

*** Unitarity ***

$$U^\dagger U = \mathbb{1}$$

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos \frac{\theta}{2} & e^{i\lambda} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & e^{i(\phi - \lambda)} \cos \frac{\theta}{2} \end{pmatrix}$$

check!

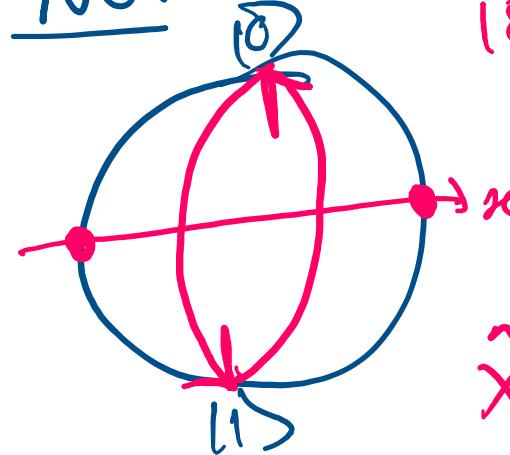
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\langle \psi' | \psi' \rangle = \underbrace{\langle \psi | U^\dagger U | \psi \rangle}_{U^\dagger U = \mathbb{1}} = \langle \psi | \mathbb{1} | \psi \rangle = \langle \psi | \psi \rangle = 1$$

$$U^\dagger U = \mathbb{1}$$

Single qubit gates

NOT



180° about \hat{x}

$$\hat{X}|0\rangle = |1\rangle$$

$$\hat{X}|1\rangle = |0\rangle$$

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{matrix} \text{eigenvectors} \\ + \end{matrix}$$

$$Y: 180^\circ \ y \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \pm i$$

$$Z: 180^\circ \ z \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pm 1$$

$$e^{i\phi}$$

$$U|\psi\rangle = a|1\rangle$$

$$(U|\psi\rangle)^* = \langle \psi|U^\dagger = \langle \psi|a^*$$

$$\begin{aligned} \hat{X}|+\rangle &= \frac{1}{\sqrt{2}} X(|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} (|D\rangle + |D\rangle) \\ \hat{X}|-\rangle &= \frac{X}{\sqrt{2}} (|0\rangle - |1\rangle) = |-\rangle \quad + \\ &= -\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \end{aligned}$$

relative phase.

global phase:

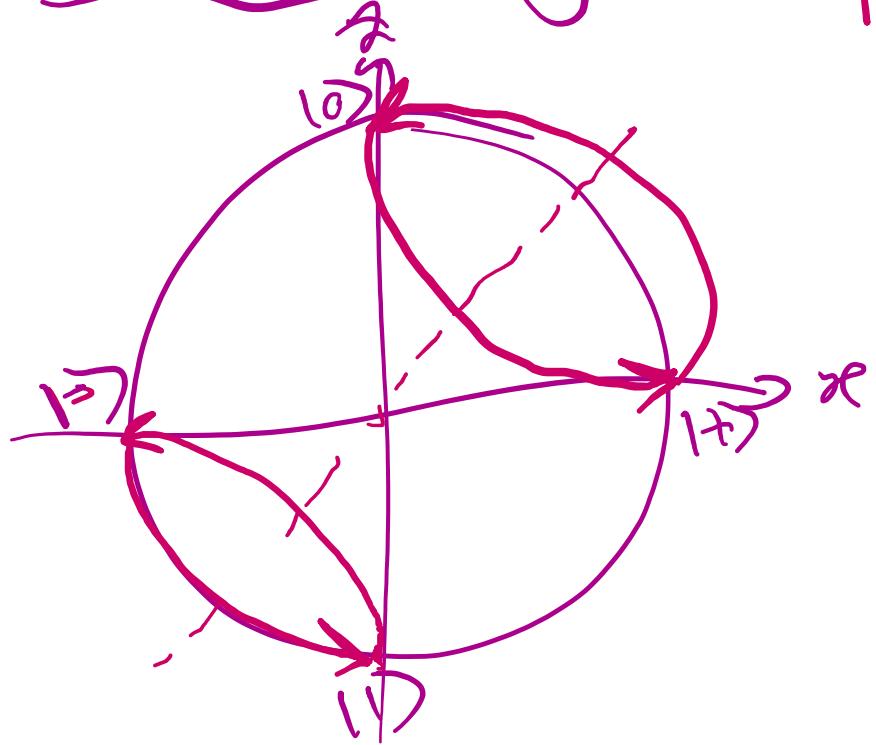
→ doesn't affect measurement

$$\hat{X}\left(\underbrace{\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)}_{|D\rangle}\right) = \frac{1}{\sqrt{2}}\left(\underbrace{|+\rangle}_{|1\rangle} - \underbrace{|-\rangle}_{|1\rangle}\right)$$

$$\langle \psi | \underbrace{U^\dagger U}_{\mathbb{1}} |\psi\rangle = a a^* \langle \psi | \psi \rangle = |a|^2 = 1$$

$$a = e^{i\phi}$$

Hadamard gate



180° π -z axis

$$\hat{H} |0\rangle = |+\rangle$$

$$\hat{H} |+\rangle = |0\rangle$$

$$\hat{H} |-\rangle = |1\rangle$$

$$\hat{H} |1\rangle = |-\rangle$$

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

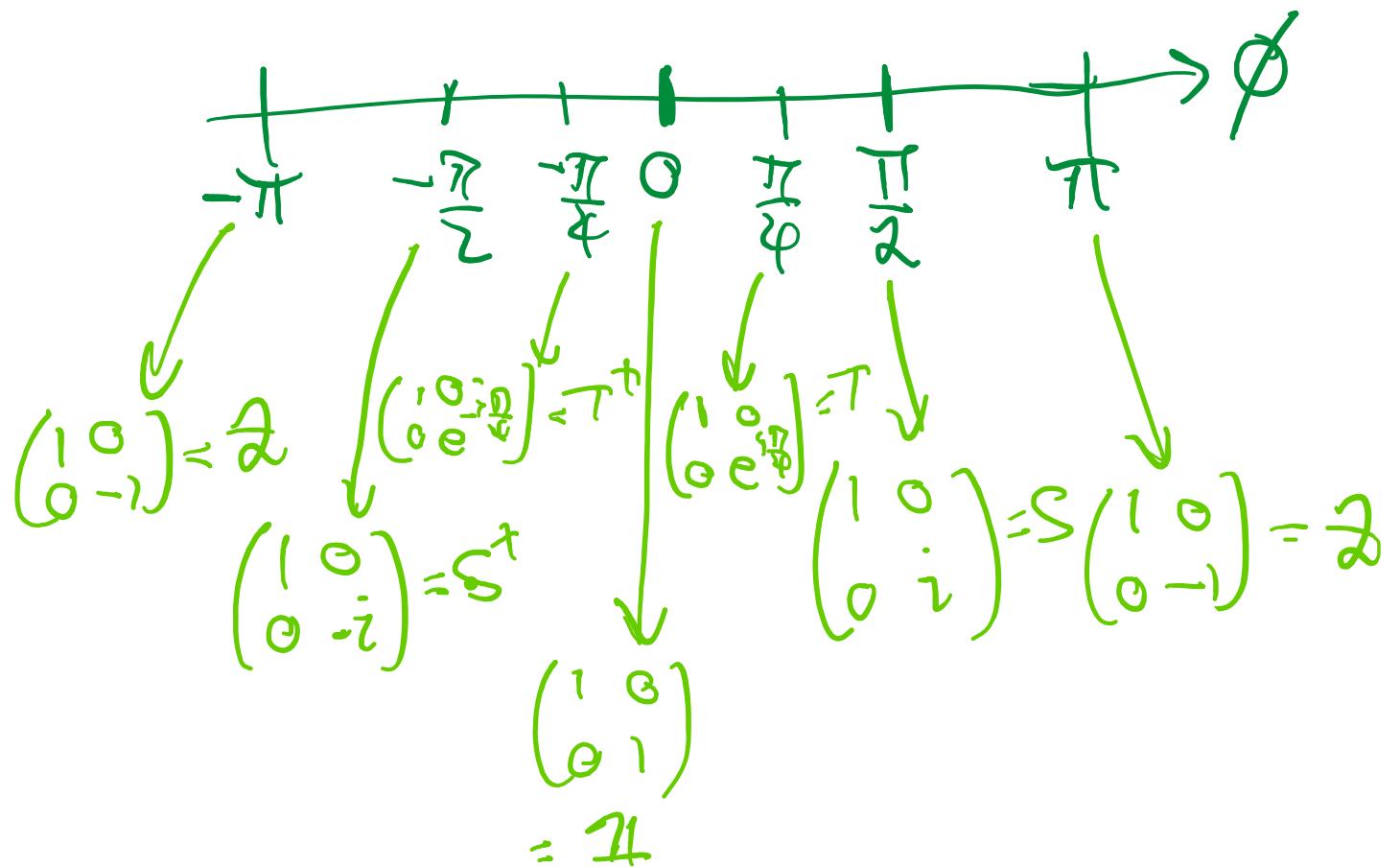
$$H = H^{-1}$$

$$HH|\Psi\rangle = H|1\Psi\rangle = |0\rangle$$

Phase gate

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

Phase gate family tree!



Bra-ket for gates

Outer product:

$$\left\{ \begin{array}{l} |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ |0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ |1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right.$$

$$|a\rangle\langle b| = |a\rangle X^b|$$

$$\begin{aligned} X &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0| \\ X|0\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle \\ &= \cancel{|0\rangle\langle 1|} + |1\rangle\cancel{\langle 0|} \\ &= |1\rangle \end{aligned}$$

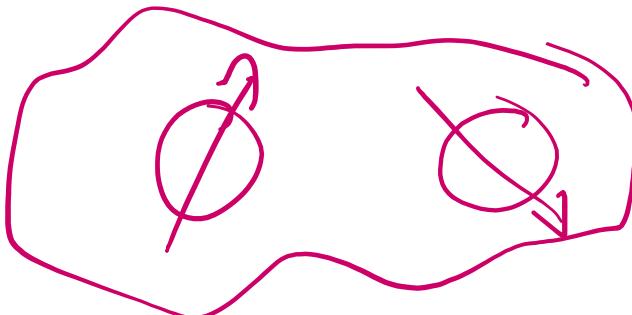
$$\alpha|0\rangle + \beta|0\rangle + \gamma|1\rangle + \delta|1\rangle$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\alpha' |+X+| + \beta' |+X-| + \gamma' |-X+| + \delta' |-X-|$$

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

Multiple qubits!



$$(\hat{X} \otimes \hat{Z})(|01\rangle) = |11\rangle$$

$$(\hat{X} \otimes \hat{Z})|0+\rangle = |1-\rangle$$

$$|a\rangle, |b\rangle$$

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Tensor product! \otimes

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$|a\rangle \otimes |b\rangle = |ab\rangle$$

$$(A \otimes B)(|a\rangle \otimes |b\rangle) = A|a\rangle \otimes B|b\rangle$$

$$\underbrace{(A_1 \ A_2)}_{\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}} \underbrace{(B_1 \ B_2)}_{\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}} = \begin{pmatrix} A_1 \begin{pmatrix} B_1 & B_2 \\ -B_3 & B_4 \end{pmatrix} \\ A_3 \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

2-qubit gates

C-NET gate

controlled - NOT

CNOT , CX

$$C \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C \times 10^3 < 10^3$$

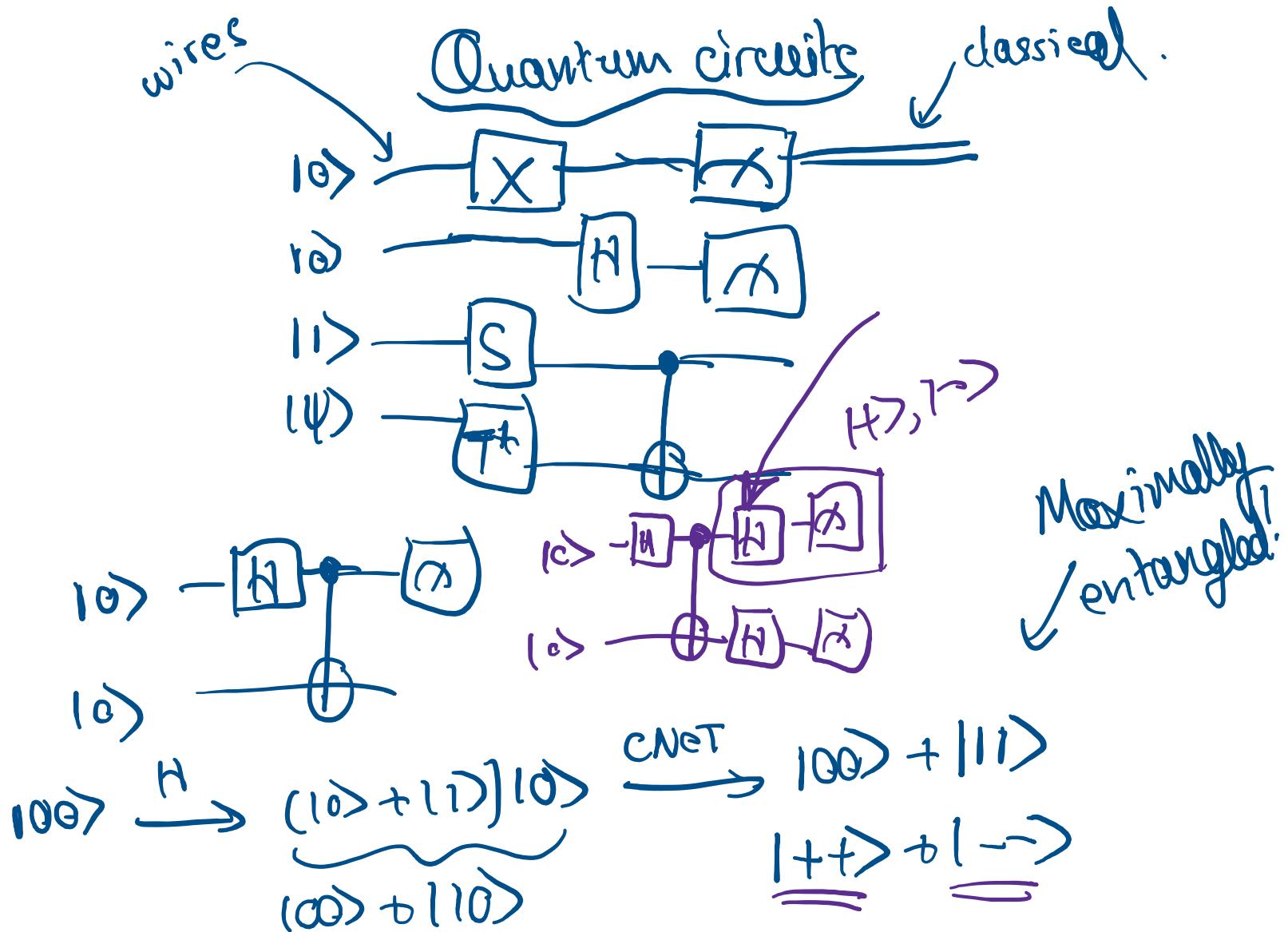
$$cx(1c) = 111$$

$$C \times 111\rangle = 110\rangle$$

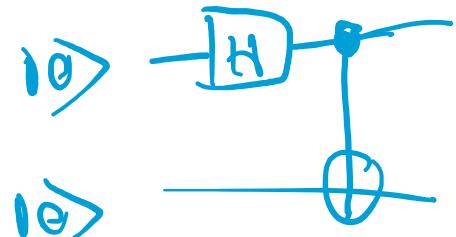
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$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$$|\Psi\rangle = \alpha |100\rangle + \beta \underbrace{|010\rangle}_{\text{I}} + \gamma |110\rangle + \delta |111\rangle$$



Bell states



$$|\phi^+\rangle = |00\rangle + |11\rangle$$

$|0\rangle$ -

$$|\phi^-\rangle = |00\rangle - |11\rangle$$

$|1\rangle$ -

$|1\rangle$ -

$$|\phi^+\rangle = |01\rangle + |10\rangle$$

$|0\rangle$ -

$|1\rangle$ -

$$|\phi^-\rangle = |01\rangle - |10\rangle$$

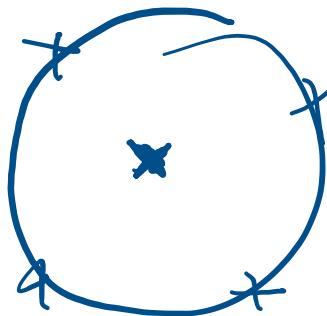
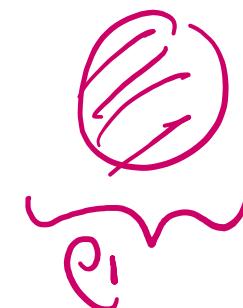
$|1\rangle$ -



$$\langle \phi_i | \phi_j \rangle = 0$$

state vector
Pure and mixed states

$$|\Psi\rangle = |00\rangle + |11\rangle$$



$$\rho = |\Psi\rangle\langle\Psi| = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|$$

density matrix

$$\sum_j e_{1j} e_{12} |\Psi\rangle$$

$$e_1 = \text{tr}_2 e_{12} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$e_2 = \text{tr}_1 e_{12} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

mixed.

$$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

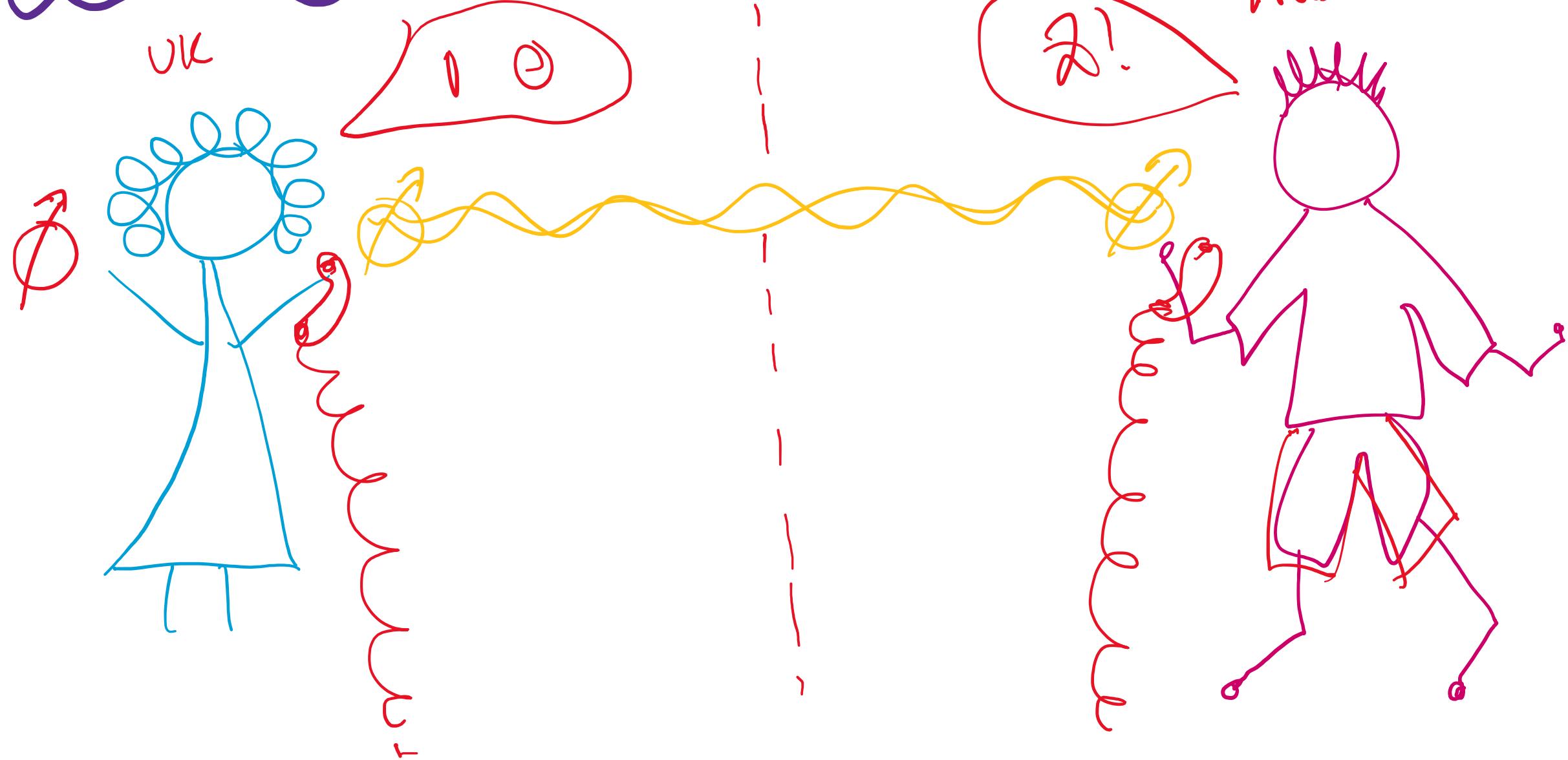
$$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

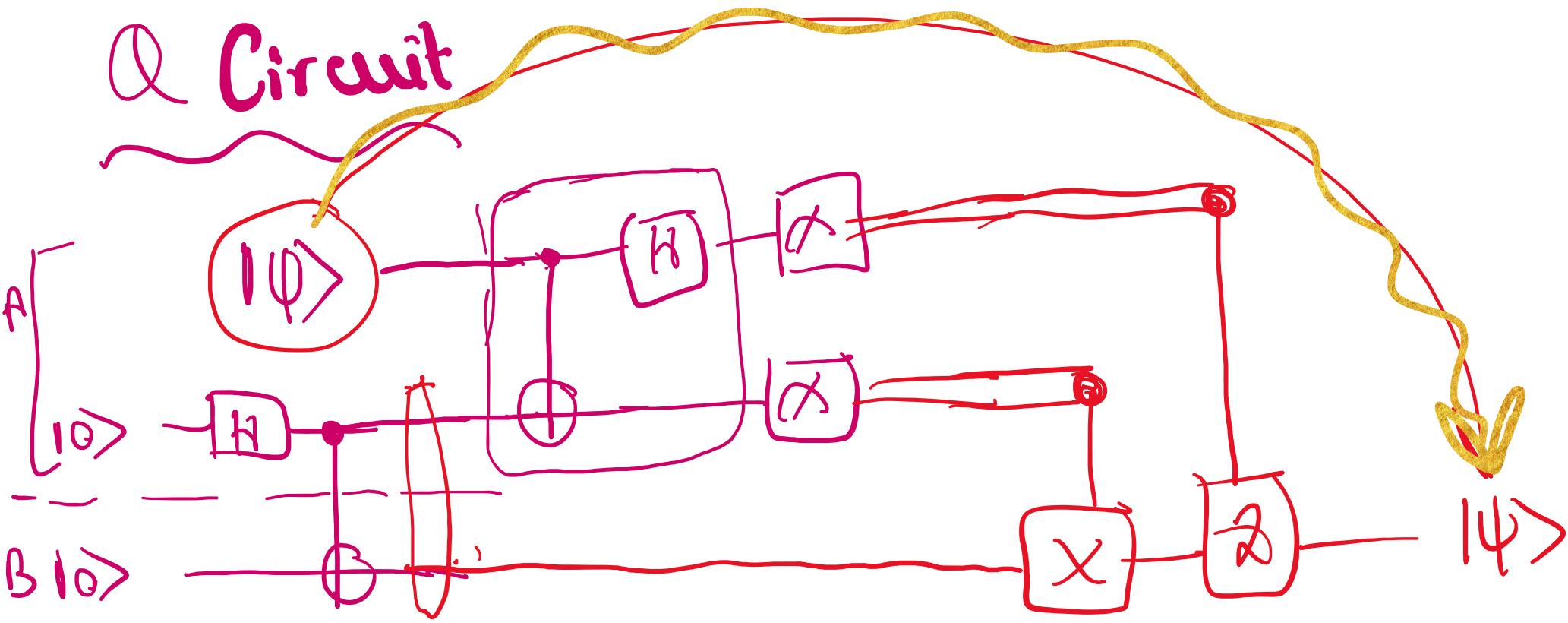
$$(|0\rangle + |1\rangle)$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} & \downarrow \begin{cases} 1 \\ 0 \end{cases} \\ & |0\rangle\langle 0| \otimes |0\rangle\langle 0| \xrightarrow{0} \\ & + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \xrightarrow{0} \\ & + |1\rangle\langle 0| \otimes |1\rangle\langle 0| \xrightarrow{0} \\ & + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \xrightarrow{1} \end{aligned}$$

Quantum teleoperation





$$|\psi\rangle = \alpha|10\rangle + \beta|11\rangle$$

$$|\psi\rangle|10\rangle \rightarrow (\alpha|10\rangle + \beta|11\rangle)(|100\rangle + |111\rangle) = \alpha|1000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|111\rangle$$

$$\text{CNOT} \rightarrow \alpha|1000\rangle + \alpha|1011\rangle + \beta|1110\rangle + \beta|1101\rangle$$

$$\begin{aligned}
 &\xrightarrow{\text{H}} \alpha\cancel{|1000\rangle} + \alpha|100\rangle + \alpha|101\rangle + \alpha|111\rangle + \beta|110\rangle - \beta|111\rangle + \beta\cancel{|1001\rangle} - \beta|101\rangle \\
 &= |100\rangle(\alpha|10\rangle + \beta|11\rangle) + |110\rangle(\alpha|10\rangle - \beta|11\rangle) + |101\rangle(\alpha|11\rangle + \beta|10\rangle) + |111\rangle(\alpha|11\rangle - \beta|10\rangle)
 \end{aligned}$$