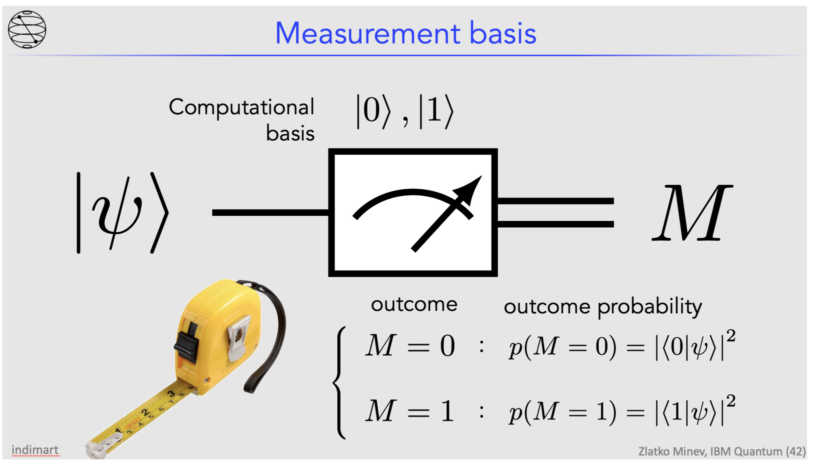
QGSS22 Intro to projective noise

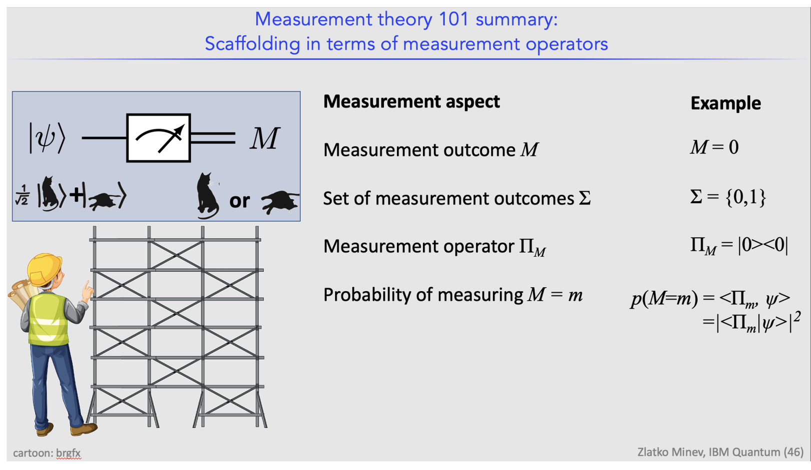
Wednesday, June 1, 2022 10:38 AM

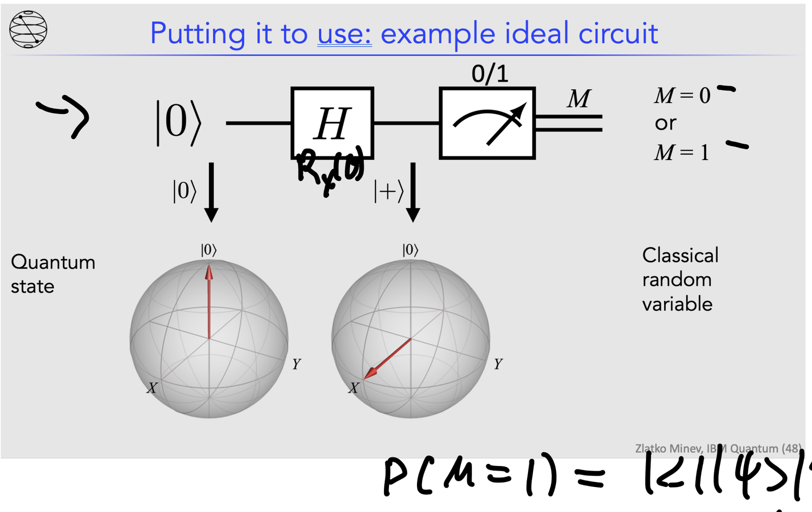
Introduction to quantum noise
Measurement theory & projection noise

Qiskit Global Summer School on Quantum Simulations

Zlatko K. Minev (www.zlatko-minev.com)





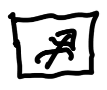


$\Sigma = \{0,1\}$

$\hat{\Pi}_0 = |0\rangle\langle 0|$

$\hat{\Pi}_1 = |1\rangle\langle 1|$

$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$



$|\psi\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|\psi\rangle^\dagger = \begin{pmatrix}1\\0\end{pmatrix}^+$

$= (1 \;\; 0)$

$$\begin{aligned} p(M=1) &= |\langle 1|\psi\rangle|^2 \\ &= \langle\psi|\hat{\Pi}_1|\psi\rangle \\ &= \tfrac{1}{2}(1\;1)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \tfrac{1}{2}(1\;1)\begin{pmatrix}0\\1\end{pmatrix} \\ &= \tfrac{1}{2}\cdot 1 \\ &= \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} p(M=0) &= |\langle 0|\psi\rangle|^2 \\ &= \hat{\Pi}_0|\psi\rangle \\ &= \tfrac{1}{2}(1\;1)\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \tfrac{1}{2}(1\;1)\cdot\begin{pmatrix}1\\0\end{pmatrix} \\ &= \tfrac{1}{2}\cdot 1 \\ &= \tfrac{1}{2} \end{aligned}$$

$p(M=1) + p(M=0) = 1 \Rightarrow p(M=1) = 1 - p(M=0)$

$M: \begin{cases} 1 & p = 1/2 \\ 0 & p = 1/2 \end{cases}$

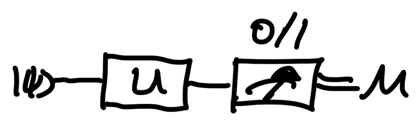
Statistics: Mean

Random variable M drawn from distribution

$\mathbb{E}[M] = \sum_{m\in\Sigma} m\, p(M=m)$

classical expectation value

$= 0\cdot p(M=0) + 1\, p(M=1) = 0\tfrac{1}{2} + 1\tfrac{1}{2} = \tfrac{1}{2}$



more general case

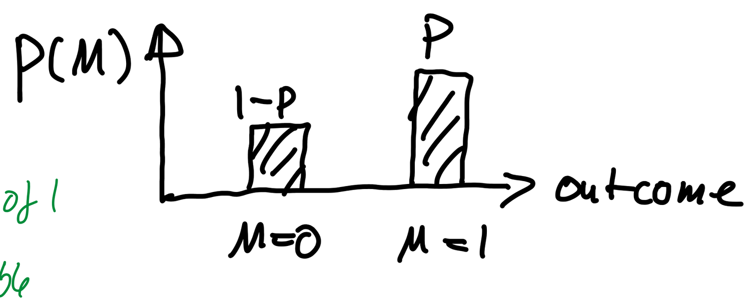
Prob to find $M=1$: $p(M=1) = p$ $\quad p \in [0,1]$

every possible qubit case

Bernoulli Distribution

$p(M) \sim B(p)$ ← prob of 1

↑ classical rand variable probability



$$\begin{aligned} \mathbb{E}[M] &= \sum_{m\in\Sigma} m\, p(M=m) \\ &= 0\, p(M=0) + 1\, p(M=1) \\ &= 0(1-p) + 1\cdot p \\ &= p \end{aligned}$$

classical

$$\begin{aligned} \mathbb{E}[M] &= \sum_{m\in\Sigma} m\, p(M=m) \\ &= \sum_{m\in\Sigma} m\,\langle\hat{\Pi}_m\rangle \\ &= \sum_{m\in\Sigma} m\,\langle\psi|\hat{\Pi}_m|\psi\rangle \\ &= \langle\psi|\boxed{\sum_{m\in\Sigma} m\,\hat{\Pi}_m}|\psi\rangle \\ &= \langle\psi|\hat{M}|\psi\rangle \\ &= \underbrace{\langle\hat{M}\rangle} \end{aligned}$$

Quantum

$\langle\hat{\Pi}_m\rangle = \langle\psi|\hat{\Pi}_m|\psi\rangle$

$= \langle\psi|m\rangle\langle m|\psi\rangle$

linear

$\langle a\hat{B}\rangle = a\langle\hat{B}\rangle$

$\langle\hat{A} + \hat{B}\rangle = \langle\hat{A}\rangle + \langle\hat{B}\rangle$

define

$\hat{M} := \sum_{m\in\Sigma} m\,\hat{\Pi}_m$

defn

Observable

Different observables

For example $\hat{M} = 0|0\rangle\langle 0| + 1|1\rangle\langle 1| = \frac{1}{2}(\hat{I} + \hat{Z}) = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

$\hat{M} = (+1)|0\rangle\langle 0| - 1|1\rangle\langle 1| = \hat{Z} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

For $\hat{M} = \hat{Z}$ $\quad \Sigma = \{-1, +1\}$ $\quad$ Pauli Observables

$\hat{M} = \hat{X}$ $\quad \mathcal{Q} = \{+, -\}$

$\hat{\Pi}_+ = |+\rangle\langle +|$

$\hat{\Pi}_- = |-\rangle\langle -|$

$$\hat{X} = \sum_{m\in\mathcal{Q}} m\,|m\rangle\langle m|$$
$$= +1|+\rangle\langle +| - |-\rangle\langle -|$$

$\hat{X}|+\rangle = +1|x\rangle$

$\hat{x}|-\rangle = -1|x\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

Statistic: Variance

$$\mathbb{V}[M] = \mathbb{E}[M^2] - \mathbb{E}[M]^2$$
$$= \underline{\langle \hat{M}^2\rangle} - \langle \hat{M}\rangle^2$$

$\langle \hat{M}\rangle = \langle \psi|\hat{M}|\psi\rangle$

$\langle \hat{M}^2\rangle = \langle \psi|\hat{M}\cdot\hat{M}\cdot|\psi\rangle$

Return to $\mathcal{Q} = \{0, 1\}$

$\hat{\Pi}_0 = |0\rangle\langle 0|$

$\hat{\Pi}_1 = |1\rangle\langle 1|$

$p(M=1) = p$

$$\mathbb{E}[M^2] = \sum_{m\in\mathcal{Q}} m^2\, p(M=m)$$
$$= \sum m^2 \langle |m\rangle\langle m|\rangle$$
$$= \left\langle \sum m^2 |m\rangle\langle m| \right\rangle$$
$$= \langle \hat{M}^2\rangle$$
$$= \langle |1\rangle\langle 1|\rangle$$
$$= \langle \hat{\Pi}_1\rangle$$
$$= p = \mathbb{E}[M]$$

$p(M=m) = \langle |m\rangle\langle m|\rangle$

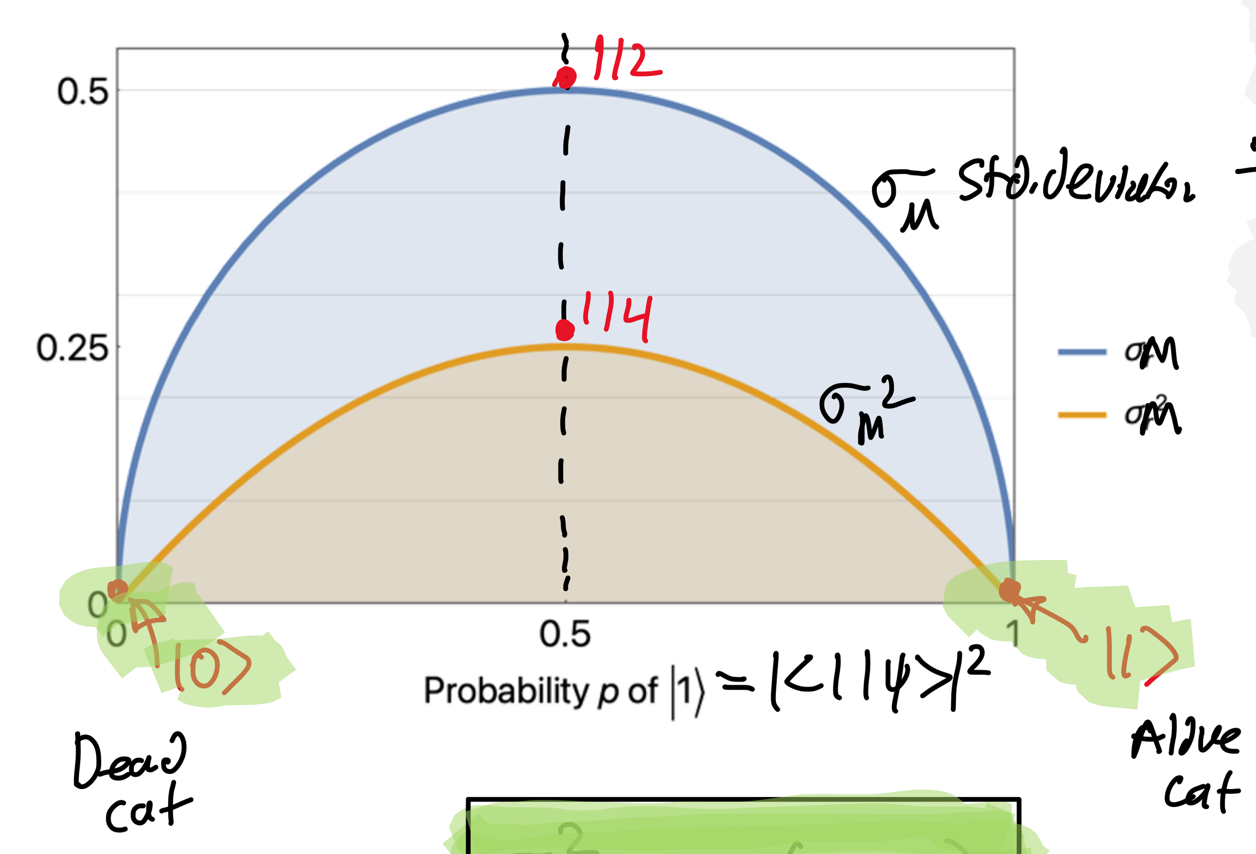
$\hat{M} = 0|0\rangle\langle 0| + 1|1\rangle\langle 1|$

$\hat{M}^2 = |1\rangle\langle 1| = M$

$$\mathbb{V}[M] = p - p^2$$
$$= p(1-p)$$
$$= \sigma_M^2$$

0 iff $p=0$ or $p=1$

Variance of the random classical variable vs. probability to obtain 1



$$\boxed{\sigma_M^2 = p(1-p)}$$

Quantum system prepared by Quantum algo

M

Random coin toss
Governed by Bernoulli distribution
Parameters determined by quantum algorithm

Projection noise and sampling error

Let's turn to the example of finite number of shots we execute for our experiment.
Perform N experiment, each giving us a single shot result 0 or 1

shot 1: 10) 10) shot 2: 10) shot 3: 10) shot 4:

M_1

M_2

M_3

M_4

$\vdots$

M_N Total # of shots will be N

For 3 samples, there are 2^3 possible outcome sequences $(N=3)$

M_1	M_2	M_3
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

2^N possible combination

Sample mean?

$$S := \frac{1}{N}\sum_{n=1}^{N} M_n$$

shots

random variable

index of the shot #

indep. identically distributed (i.i.d)

$M_n \sim M$

examp $S = \frac{1}{3}(1+0+1) = 2/3$

$$\mathbb{E}[S] = \mathbb{E}[\frac{1}{N}\sum_{n=1}^{N} M_n]$$

$$= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[M]$$

$\mathbb{E}[M_n] = \mathbb{E}[M_{n'}] = \mathbb{E}[M]$

$$= \frac{1}{\cancel{N}}\cancel{N}\,\mathbb{E}[M]$$

$$= \mathbb{E}[M]$$

$$\mathbb{E}[S] = \mathbb{E}[M] = \langle \hat{M} \rangle$$

$= p$ for qubit $\hat{M} = |1\rangle\langle 1|$

$$\mathbb{V}[S] = \frac{\mathbb{V}[M]}{N} = \frac{p(1-p)}{N}$$

$$\sigma_S = \sqrt{\frac{p(1-p)}{N}}$$

shots

error bar (std dev) on sample mean estimator for our expectation value.

QGSS22 Measurement error

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Introduction to quantum noise
Measurement error

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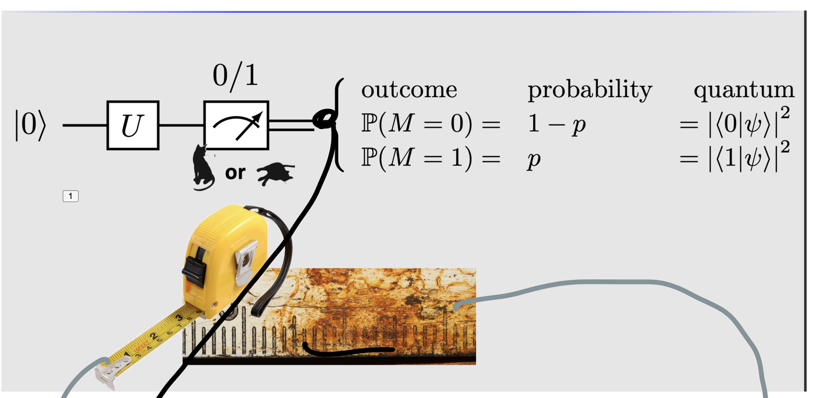
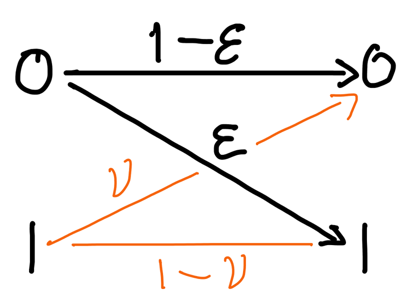


M classical random variable (ideal) (sender)



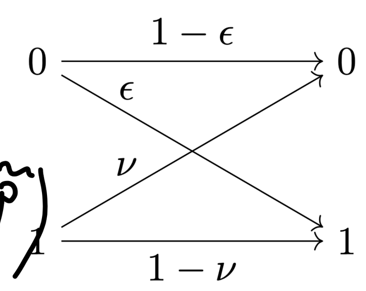
$\tilde M$ noisy measurement outcome

$p(\tilde M=1 \mid M=0) = \varepsilon$

$p(\tilde M=0 \mid M=1) = 1-\varepsilon$

$p(\tilde M=0 \mid M=1) = \nu$

$p(\tilde M=1 \mid M=1) = 1-\nu$



$$P_M = \begin{pmatrix} p(M=0) \\ p(M=1) \end{pmatrix} = \begin{pmatrix} 1-p \\ p \end{pmatrix}$$

$$P_{\tilde M} = \begin{pmatrix} p(\tilde M=0) \\ p(\tilde M=1) \end{pmatrix} = \begin{pmatrix} 1-\tilde p \\ \tilde p \end{pmatrix}$$

$\tilde p$ is noisy probability

$$\begin{cases} p(\tilde M=0) = \sum_{m\in S} (\tilde M=0 \mid M=m) \\ \qquad = p(\tilde M=0 \mid M=0)\, p(M=0) + p(\tilde M=0 \mid M=1)\, p(M=1) \\ p(\tilde M=1) = p(\tilde M=1 \mid M=0)\, p(M=0) + p(\tilde M=1 \mid M=1)\, p(M=1) \end{cases}$$

$$\tilde P_M = \begin{pmatrix} p(\tilde M=0) \\ p(\tilde M=1) \end{pmatrix} = A P_M = A \begin{pmatrix} p(M=0) \\ p(M=1) \end{pmatrix}$$

$$A = \begin{array}{c c} & \begin{array}{cc} M=0 & M=1 \end{array} \\ \begin{array}{c} \tilde M=0 \\ \tilde M=1 \end{array} & \begin{pmatrix} p(\tilde M=0 \mid M=0) & p(\tilde M=0 \mid M=1) \\ p(\tilde M=1 \mid M=0) & p(\tilde M=1 \mid M=1) \end{pmatrix} \end{array}$$

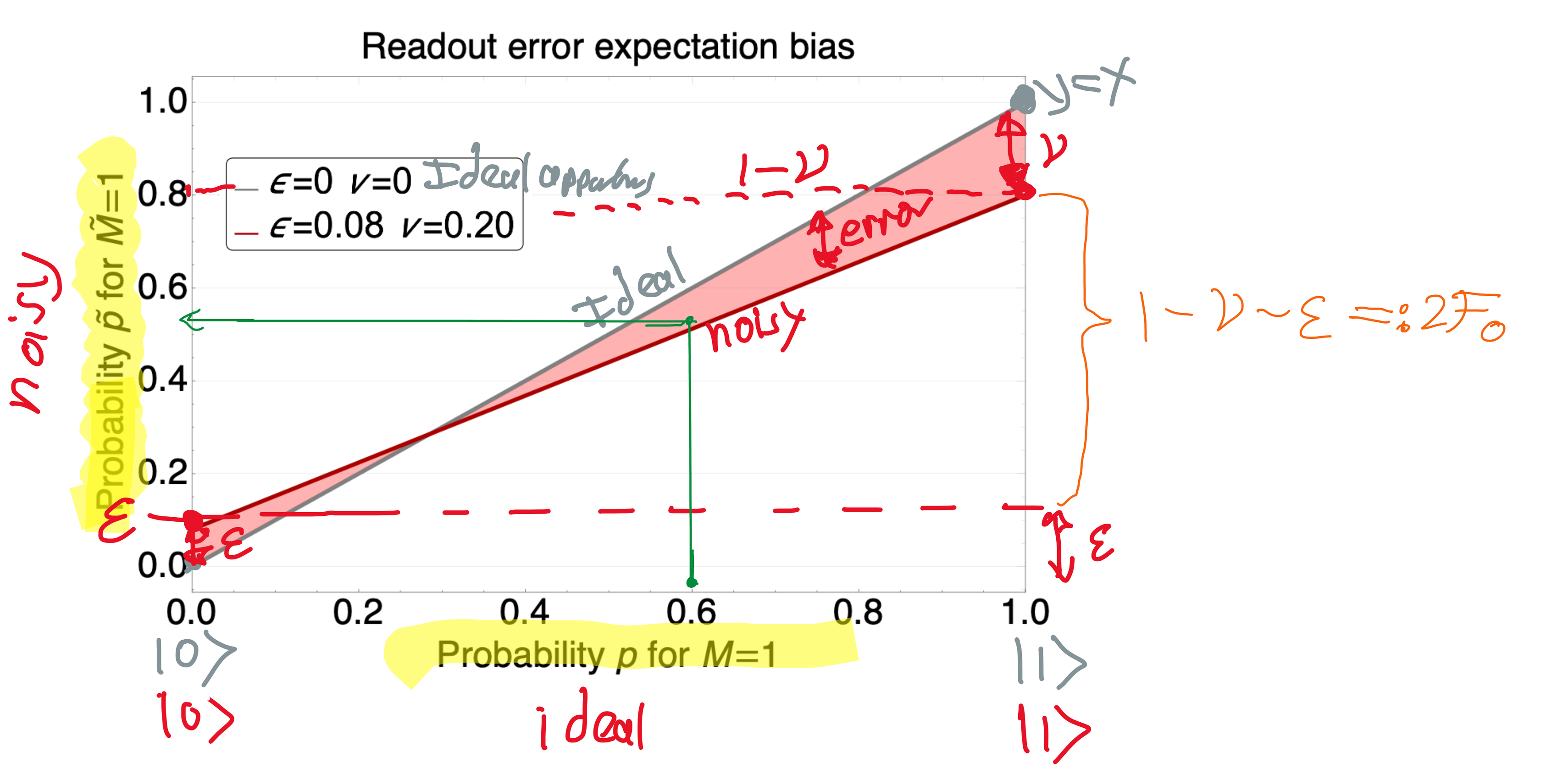
$$= \begin{pmatrix} \underbrace{\begin{pmatrix} 1-\varepsilon \\ \varepsilon \end{pmatrix}}_{\Sigma = 1} & \underbrace{\begin{pmatrix} \nu \\ 1-\nu \end{pmatrix}}_{\Sigma = 1} \end{pmatrix}$$

$\left\{ \begin{array}{l} \text{~~Ideal case~~ for no error} \quad \nu = \varepsilon = 0 \\ \Rightarrow \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \end{array} \right.$

Stochastic matrix $\qquad \sum_n A_{mn} = 1$ for any n

$$\tilde p = p(\tilde M=1) = p(\tilde M=1 \mid M=0)\, p(M=0) + p(\tilde M=1 \mid M=1)\, p(M=1)$$
$$= \varepsilon(1-p) + (1-\nu)p$$
$$\tilde p = p - \underbrace{(\nu+\varepsilon)}p + \underbrace{\varepsilon}_{\text{offset}}$$

Visualizing readout error



QGSS22 Incoherent noise

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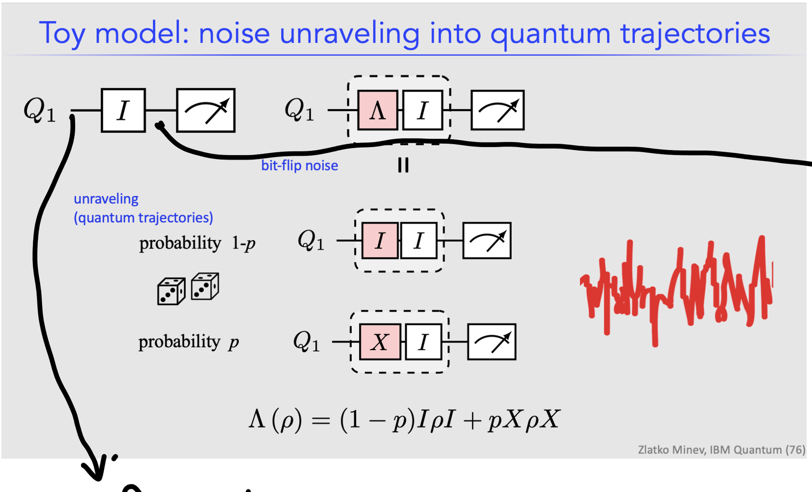
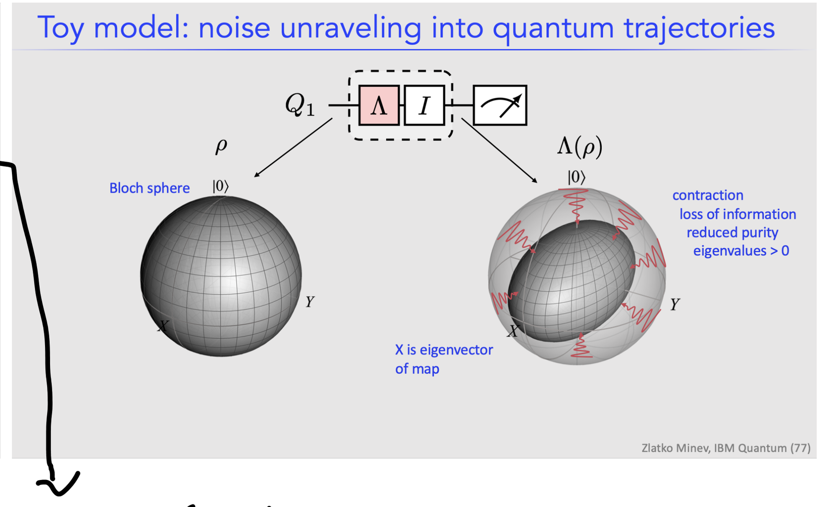


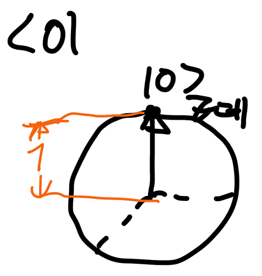


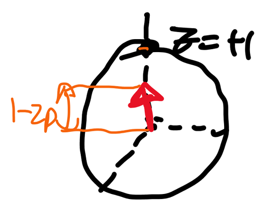
$\rho_0 = |0\rangle\langle 0|$

$\langle X \rangle = 0$

$\langle Y \rangle = 0$

$\langle Z \rangle = 1$

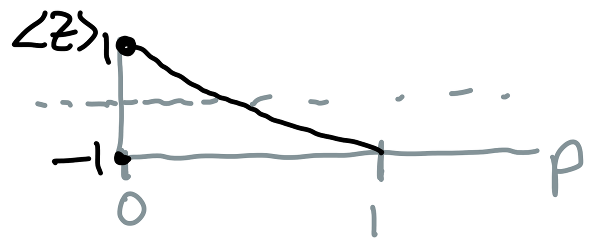




$$\rho = (1-p)I|0\rangle\langle 0|I + pX\rho X$$
$$= (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$$
$$= \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix} = (\cdot\cdot)I + (\cdot\cdot)Z$$

$$\rho_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{I+Z}{2}$$
$$\langle X \rangle = \mathrm{Tr}(X\rho_0)$$
$$= \tfrac{1}{2}\mathrm{Tr}(X(I+Z))$$
$$= \tfrac{1}{2}\mathrm{Tr}(X + X\cdot Z)$$
$$\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$$
$$\mathrm{Tr}(I) = 2$$
$$\langle X \rangle = \tfrac{1}{2}\left[\cancel{\mathrm{Tr}(X)}^{0} + \cancel{\mathrm{Tr}(X\cdot Z)}^{0}\right]$$
$$X\cdot Z \simeq Y$$
$$= 0$$
$$\langle Y \rangle = 0$$
$$\langle Z \rangle = \mathrm{Tr}(Z\rho_0)$$
$$= \tfrac{1}{2}\mathrm{Tr}(Z + Z\cdot Z)$$
$$= \tfrac{1}{2}\left[\cancel{\mathrm{Tr}(Z)}^{0} + \cancel{\mathrm{Tr}(I)}^{2}\right]$$
$$= 2/2 = 1$$

$$\mathrm{Tr}(\rho) = 1$$
$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\begin{pmatrix} (1-p)^2 & \\ & p^2 \end{pmatrix} = (1-p)^2 + p^2$$
$$= 1 - 2p + p^2 + p^2$$
$$= 1 - 2p + 2p^2$$
$$= 1 + 2p(p-1)$$
$$= \begin{cases} 1 & \text{if } p = 0 \\ 1 & \text{if } p = 1 \end{cases}$$
$$\langle X \rangle = ? = 0$$
$$\langle Y \rangle = ? = 0$$
$$\langle Z \rangle = \mathrm{Tr}\left(Z\cdot\begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}\right)$$
$$= \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}\right)$$
$$= \mathrm{Tr}\begin{pmatrix} 1-p & \\ & -p \end{pmatrix}$$
$$= 1 - 2p$$



Coherent error: ZZ [bonus]

Friday, July 2, 2021 5:14 PM

Introduction to quantum noise

Coherent errors

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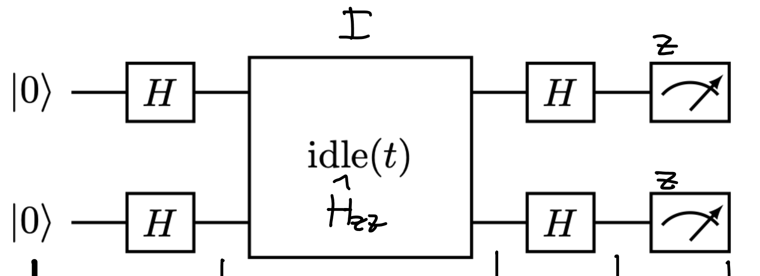

$|0\rangle|0\rangle$ $\quad |X\rangle|X\rangle$ $\quad |+\rangle|+\rangle$ $\quad |X\rangle|X\rangle$ $\quad |0\rangle|0\rangle$

$$\langle ZI\rangle = +1$$
$$\langle IZ\rangle = +1$$
$$\langle ZZ\rangle = \langle 0|\langle 0| Z Z |0\rangle|0\rangle$$
$$= \langle 0|Z|0\rangle \langle 0|Z|0\rangle$$
$$= (+1)\ (+1)$$
$$= +1$$

Hadamard gate

$$H = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ |1\rangle & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}$$

$$\begin{cases} H|0\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \\ H|1\rangle = |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \end{cases}$$

$$X|+X\rangle = +1|+X\rangle$$
$$X|-X\rangle = -1|-X\rangle$$
$$|+X\rangle :=: |+\rangle$$
$$|-X\rangle :=: |-\rangle$$

$$\begin{cases} Z|0\rangle = +1|0\rangle \\ Z|1\rangle = -1|1\rangle \end{cases}$$

NOISY $\qquad$ ZZ Interaction

$$\hat{H} = \frac{1}{2}\hbar\omega \hat{Z}\hat{Z}$$
$$\hat{U}(t) = \exp\left(-i\hbar^{-1}\hat{H}t\right)$$
$$= \exp\left(-i\frac{\omega t}{2}\hat{Z}\hat{Z}\right)$$
$$= \cos\left(\frac{\omega t}{2}\right)\hat{I} - i\sin\left(\frac{\omega t}{2}\right)\hat{Z}\hat{Z}$$
$$= \hat{R}_{ZZ}(\theta = \omega t)$$

$$\hat{R}_X(\theta) = \exp\left(-i\frac{\theta}{2}X\right) \qquad X^2 = I$$
$$= \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)X$$
$$(\hat{Z}\hat{Z})^2 = Z^2Z^2 = I_2 \otimes I_2 = I_4$$

$$|0\rangle|0\rangle \xrightarrow{HH} |+\rangle|+\rangle \xrightarrow{\text{idle}} R_{zz}(\theta)|+\rangle|+\rangle = \cos\frac{\theta}{2}|+\rangle|+\rangle - i\sin\frac{\theta}{2}|-\rangle|-\rangle \xrightarrow{HH} \underline{\cos\theta/2\,|0\rangle|0\rangle - i\sin\theta/2\,|1\rangle|1\rangle}$$

$$R_{zz}(\theta)|+\rangle|+\rangle = \cos\frac{\theta}{2}|+\rangle|+\rangle - i\sin\frac{\theta}{2}(Z|+\rangle)_1 \otimes (Z|+\rangle)_2$$
$$Z|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle$$

$$Z|+\rangle = |-\rangle$$
$$Z|-\rangle = |+\rangle$$

$$\langle ZI\rangle = \cos\omega t$$
$$\langle IZ\rangle = \cos\omega t$$
$$\langle ZZ\rangle = 1$$

State prep [bonus]

Saturday, July 3, 2021 2:00 PM

Introduction to quantum noise

State preparation noise

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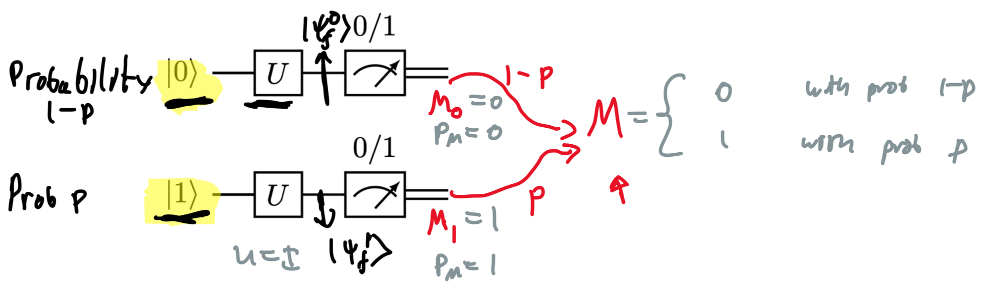


<u>Density operator</u>

$$\rho = (1-p)\,\underbrace{|0\rangle\langle 0|}_{|0\rangle} + p\,\underbrace{|1\rangle\langle 1|}_{|1\rangle}$$

$$= P_{\text{no error}}\, \rho_{\text{ideal}} + P_{\text{error}}\, \rho_{\text{error}}$$

$$\rho_f = P_{\text{no error}}\, \rho^{\text{final}}_{\text{ideal}} + P_{\text{error}}\, \rho^{\text{final}}_{\text{error}}$$

$$= (1-p)\, |\psi_f^0\rangle\langle\psi_f^0| + p\, |\psi_f^1\rangle\langle\psi_f^1|$$

$$= (1-p)\, U|0\rangle\langle 0|U^\dagger + p\, U|1\rangle\langle 1|U^\dagger$$

$$= U\left[(1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|\right]U^\dagger$$

$$= U \rho_{\text{initial}} U^\dagger$$

$$(U|\psi\rangle)^\dagger = (|\psi_f\rangle)^\dagger$$

$$\langle\psi_f| = \langle\psi|U^\dagger$$

$$\rho_U = U \rho U^\dagger$$

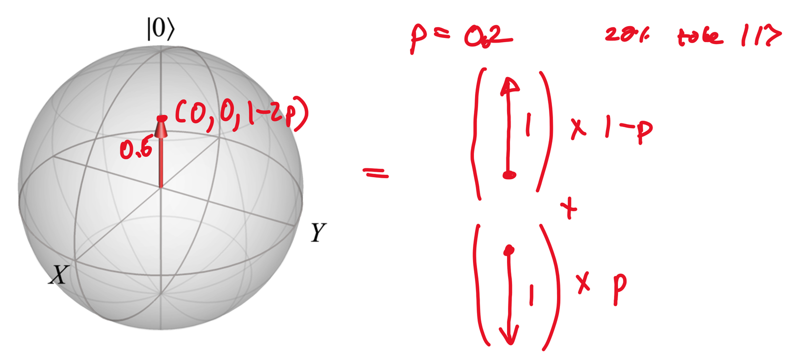
$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

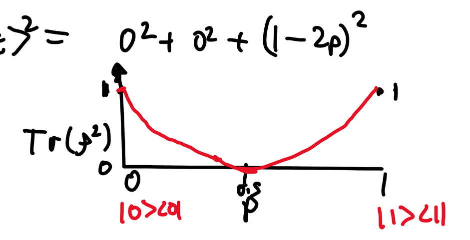
$$\rho = \begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & 1-p & 0 \\ |1\rangle & 0 & p \end{array} = \frac{1}{2}\left(\hat{I} + (1-2p)\hat{Z}\right)$$

$$\begin{cases} \langle X\rangle = \mathrm{Tr}[X\rho] = 0 \\ \langle Y\rangle = \mathrm{Tr}[Y\rho] = 0 \\ \langle Z\rangle = \mathrm{Tr}[Z\rho] = 1-2p \end{cases}$$

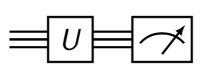
$$\mathrm{Tr}[P_\alpha P_\beta] = 2\delta_{\alpha\beta} \quad \text{for } \alpha,\beta \in \{X, Y, Z\}$$



$$\mathrm{Tr}(\rho^2) = \langle X\rangle^2 + \langle Y\rangle^2 + \langle Z\rangle^2 = 0^2 + 0^2 + (1-2p)^2$$



Scaling to larger number of qubits

$[(1-p)\,|0\rangle\langle 0| + p\,|1\rangle\langle 1|]^{\otimes n}$ 

$$\rho_0 = \left[(1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|\right]^{\otimes n}$$

$$= (1-p)^n |000\ldots0\rangle\langle 0\ldots0| + \ldots\ |0100\ldots\rangle\langle\ldots1\ldots| \ldots$$

$$P(M = 00000\ldots0) = \begin{cases} 1 & \text{if } p = 0 \quad (\text{ideal case}) \\ (1-p)^n \simeq 1 - np + O(p^2) & \text{for } p > 0 \end{cases}$$

$$\langle Z_k\rangle = \langle IIII\overset{k^{th}}{Z}III\ldots\rangle = 1-2p \xrightarrow{\text{much faster}}$$

$$\langle ZZZ\ldots\ldots Z\rangle = \mathrm{Tr}[Z^{\otimes n}\rho] = (1-2p)^n$$

$$\rho_0 = \prod_{k=1}^{n}\otimes \frac{1}{2}\left(\hat{I}_k + (1-2p)\hat{Z}_k\right)$$

$$= \frac{1}{2^n}(1 + \ldots Z)\otimes(1 + \ldots Z)\otimes \ldots$$

$$= \frac{1}{2^n}\left[(1-2p)^n Z^{\otimes n} + \ldots\right]$$

$$\mathrm{Tr}[\rho_0^2] = \prod_{k=1}^{n} \mathrm{Tr}(\rho_{0,k}^2) = (1-2p)^{2n}$$