Homework: Probabilistic Graphical Models.

8.3. $a, b, c \in \{0,1\}$

Show: ① $p(a,b) \neq p(a)\,p(b)$; marginally dependent

② $p(a,b|c) = p(a|c)\,p(b|c)$ for $c=0$ & $c=1$

① $p(a,b) \overset{?}{=} p(a)\,p(b)$

- we need to find $p(a,b)$, $p(a)$ and $p(b)$

① $p(a,b) = p(a,b,c=0) + p(a,b,c=1)$

② $p(a) = p(a,b=0) + p(a,b=1)$

③ $p(b) = p(a=0,b) + p(a=1,b)$

- Since we want to prove that $p(a,b) \neq p(a)\,p(b)$, we just need to find a counterexample that proves it.

Let's try $a=0$ & $b=0$

- $a=0$, $b=0$:

① $p(a,b) = 0.192 + 0.144 = 0.336$

② $p(a) = \underbrace{p(a=0,b=0)}_{0.336} + \underbrace{p(a=0,b=1)}_{0.048 + 0.216} = 0.6$

③ $p(b) = \underbrace{p(a=0,b=0)}_{0.336} + \underbrace{p(a=1,b=0)}_{0.192 + 0.064} = 0.592$

$\Rightarrow$ Then, $p(a,b) \overset{?}{=} p(a)\,p(b)$.

$0.336 \overset{?}{=} 0.6 \cdot 0.592$

$0.336 \neq 0.3552$ $\Longrightarrow$ a & b are marginally dependent

② $p(a, b|c) \overset{?}{=} p(a|c)\, p(b|c)$

- We need:

① $p(a,b|c) = \dfrac{p(a,b,c)}{p(c)}$

② $p(a|c) = \dfrac{p(a,c)}{p(c)}$

③ $p(b|c) = \dfrac{p(b,c)}{p(c)}$

- Also, $p(c)$ is needed:

$p(c=0) = \sum p(a,b,c=0) = 0.192 + 0.048 + 0.192 + 0.048$

$= 0'48$

$p(c=1) = 1 - p(c=0) = 0'52$

- $p(a,c) = p(a, b=0, c) + p(a, b=1, c)$
- $p(b,c) = p(a=0, b, c) + p(a=1, b, c)$

a	b	c	p(a,b,c)	p(a,c)	p(b,c)	p(c)	p(a|c)	p(b|c)
0	0	0	0.192	0.24	0.384	0.48	0.5	0.8
0	0	1	0.144	0.36	0.208	0.52	0.692	0.4
0	1	0	0.048	0.24	0.096	0.48	0.5	0.2
0	1	1	0.216	0.36	0.312	0.52	0.692	0.6
1	0	0	0.192	0.24	0.384	0.48	0.5	0.8
1	0	1	0.064	0.16	0.208	0.52	0.308	0.4
1	1	0	0.048	0.24	0.096	0.48	0.5	0.2
1	1	1	0.096	0.16	0.312	0.52	0.308	0.6

a	b	c	p(a,b|c)	p(a|c)p(b|c)
0	0	0	0.192/0.48 = 0.4	0.5 · 0.8 = 0.4
0	0	1	0.277	0.277
0	1	0	0.1	0.1
0	1	1	0.42	0.42
1	0	0	0.4	0.4
1	0	1	0.123	0.123
1	1	0	0.1	0.1
1	1	1	0.185	0.185

$p(a,b|c) = p(a|c)\, p(b|c)$ for all values of a, b, c.

8.4. Evaluate the distributions: $p(a)$, $p(b|c)$ and $p(c|a)$

Show that: $p(a,b,c) = p(a)\,p(c|a)\,p(b|c)$ & draw the graph.

- From the previous exercise, we already have the values for $p(b|c)$. As has been done for $p(b|c)$, we can compute $p(a)$ & $p(c|a)$

- $p(c|a) = \dfrac{p(c,a)}{p(a)}$ (we already know $p(c,a)$)

- $p(a) = p(a, b=0) + p(a, b=1)$

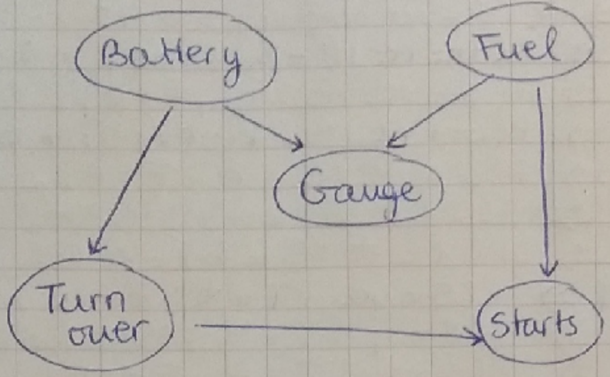
a	b	c	$p(b\|c)$	$p(a)$	$p(c,a)$	$p(c\|a)$	$p(a,b,c)$	$p(a)\,p(c\|a)\,p(b\|c)$
0	0	0	0.8	0.6	0.24	0.4	0.192	0.192
0	0	1	0.4	0.6	0.36	0.6	0.144	0.144
0	1	0	0.2	0.6	0.24	0.4	0.048	0.048
0	1	1	0.6	0.6	0.36	0.6	0.216	0.216
1	0	0	0.8	0.4	0.24	0.6	0.192	0.192
1	0	1	0.4	0.4	0.16	0.4	0.064	0.064
1	1	0	0.2	0.4	0.24	0.6	0.048	0.048
1	1	1	0.6	0.4	0.16	0.4	0.096	0.096

they are equivalent!

$p(a)p(c|a)p(b|c) \Longrightarrow$ $a \longrightarrow c \longrightarrow b$

Frost

② Bayesian network ⇒ Probability of a car engine starting.



- B ⇒ can be good or bad condition
- F ⇒ empty or not empty
- G ⇒ empty or not empty
- T ⇒ False or True
- S ⇒ yes or no

a) Factorization $p(B,F,G,T,S)$

$$p(B,F,G,T,S) = p(B)\,p(F)\,p(G\mid B,F)\,p(T\mid B)\,p(S\mid T,F)$$

b) Are this satisfied?

i. $T \perp\!\!\!\perp F \mid \emptyset$?

- Path through B: not blocked
- Path through G: blocked
- Path through S: blocked

$\Rightarrow T \perp\!\!\!\perp F$

ii. $T \perp\!\!\!\perp F \mid \neg S$?

- Path through S: not blocked
- Path through B: not blocked
- Path through G: blocked

$\Rightarrow T \not\!\perp\!\!\!\perp F \mid \neg S$

iii. $B \perp\!\!\!\perp F \mid S$?

- Path through G: blocked
- Path through T: not blocked
- Path through S: not blocked

$\Rightarrow B \not\!\perp\!\!\!\perp F \mid S$

c) $P(F = \text{empty} \mid S = \text{no}) = ?$

$P(B=\text{bad}) = 0.02 \Rightarrow P(B=\text{good}) = 0.98$
$P(F=\text{empty}) = 0.05 \rightarrow P(F=\text{not empty}) = 0.95$
$P(G=e \mid B=g, F=ne) = 0.04 \Rightarrow P(G=ne \mid B=g, F=ne) = 0.96$
$P(G=e \mid B=g, F=e) = 0.97 \Rightarrow P(G=ne \mid B=g, F=e) = 0.03$
$P(G=e \mid B=b, F=ne) = 0.1 \Rightarrow P(G=ne \mid B=b, F=ne) = 0.9$
$P(G=e \mid B=b, F=e) = 0.99 \Rightarrow P(G=ne \mid B=b, F=e) = 0.01$
$P(T=f \mid B=g) = 0.03 \Rightarrow P(T=t \mid B=g) = 0.97$
$P(T=f \mid B=b) = 0.98 \Rightarrow P(T=t \mid B=b) = 0.02$
$P(S=no \mid T=\text{true}, F=ne) = 0.01 \Rightarrow P(S=\text{yes} \mid T=t, F=ne) = 0.99$
$P(S=no \mid T=t, F=e) = 0.92 \Rightarrow P(S=y \mid T=t, F=e) = 0.08$
$P(S=no \mid T=f, F=ne) = 1 \Rightarrow P(S=y \mid T=f, F=ne) = 0$
$P(S=no \mid T=f, F=e) = 0.99 \Rightarrow P(S=y \mid T=f, F=e) = 0.01$

$$P(F=\text{empty} \mid S=no) = \frac{P(F=\text{empty}, S=no)}{P(S=no)}$$

$$P(F=\text{empty}, S=no) = \sum_B \sum_G \sum_T P(B, F=e, G, T, S=no) =$$

$$P(B=g)\,P(F=e)\,P(G=e \mid B=g, F=e)\,P(T=t \mid B=g)\,P(S=no \mid F=e, T=t)$$
$$+ P(B=g)\,P(F=e)\,P(G=e \mid B=g, F=e)\,P(T=f \mid B=g)\,P(S=no \mid F=e, T=f)$$
$$+ P(B=g)\,P(F=e)\,P(G=ne \mid B=g, F=e)\,P(T=t \mid B=g)\,P(S=no \mid F=e, T=t)$$
$$+ P(B=g)\,P(F=e)\,P(G=ne \mid B=g, F=e)\,P(T=f \mid B=g)\,P(S=no \mid F=e, T=f)$$
$$+ \sum_G \sum_T P(B=\text{bad}, F=e, G, T, S=no) =$$

$$P(F=e)\Big[P(B=\text{good})\Big(P(S=no \mid F=e, T=t)\,P(T=t \mid B=g)\cdot \underbrace{\big(P(G=e \mid B=g, F=e) + P(G=ne \mid B=g, F=e)\big)}_{=1}$$
$$+ P(S=no \mid F=e, T=f)\,P(T=f \mid B=g)\Big)$$
$$+ P(B=\text{bad})\Big(P(S=no \mid F=e, T=t)\,P(T=t \mid B=b) + P(S=no \mid F=e, T=f)\,P(T=f \mid B=b)\Big)\Big]$$

$$= 0.05\Big[0.98(0.92 \cdot 0.97 + 0.99 \cdot 0.03) + 0.02(0.92 \cdot 0.02 + 0.99 \cdot 0.98)\Big]$$

$$= 0.0462$$

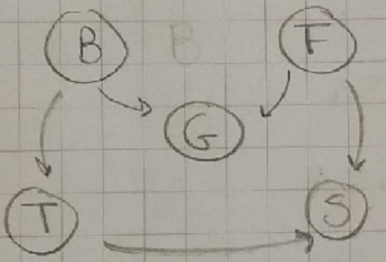
- We now have $P(F=e, S=no)$, we now need to know $P(S=no) = P(F=e, S=no) + P(F=ne, S=no)$
- We calculate $P(F=ne, S=no)$ the same way than before $\Rightarrow P(F=ne, S=no) = 0.0556$

$\Rightarrow P(S=no) = 0.0462 + 0.0556 = 0.102$

Therefore, $P(F=e \mid S=no) = \dfrac{P(F=e, S=no)}{P(S=no)}$

$= \dfrac{0.0462}{0.102} = \boxed{0.454}$

d) Transform into an equivalent factor graph.



$\Rightarrow$

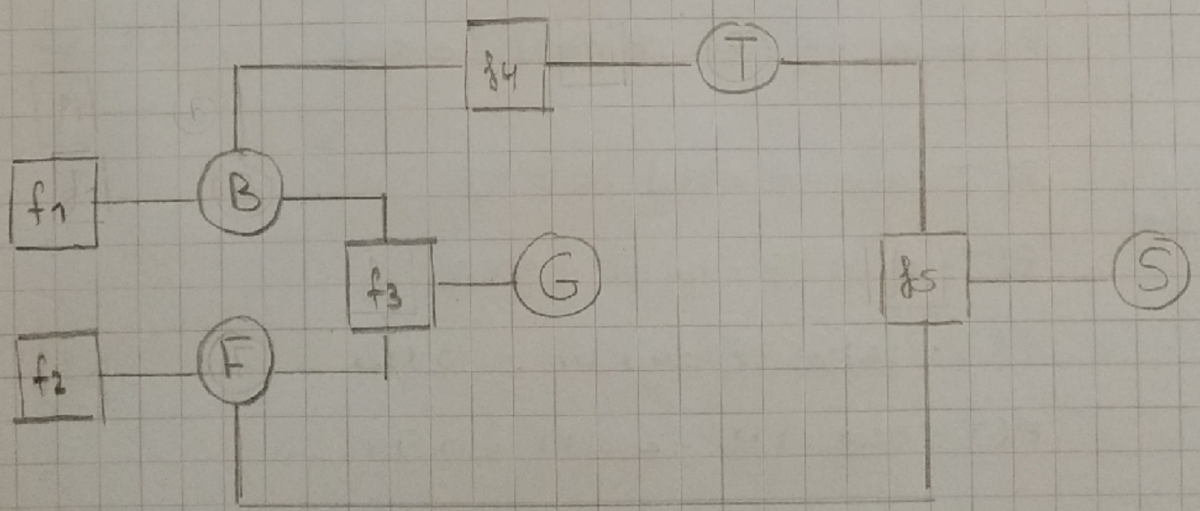
$\boxed{f_1}$ Battery

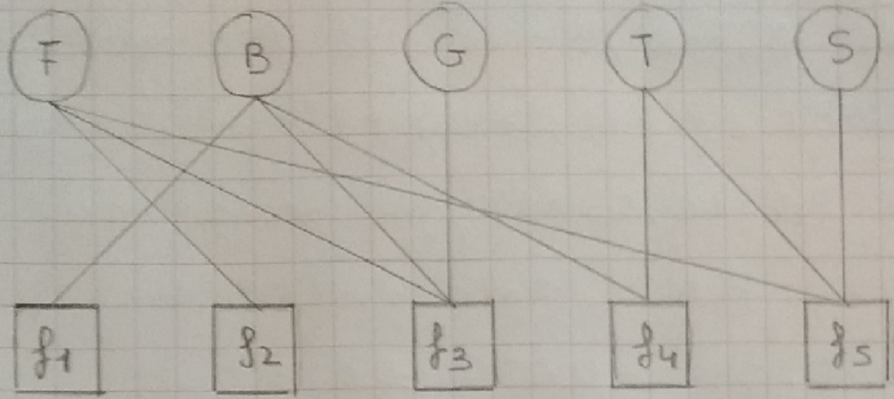
$\boxed{f_2}$ Fuel

$\boxed{f_3}$ Gauge

$\boxed{f_4}$ Turn over

$\boxed{f_5}$ Starts





- Involved factors and their values.

f_1: Battery : $P(B=good) = 0.98$; $P(B=bad) = 0.02$

f_2: Fuel $P(F=empty) = 0.05$; $P(F=notempty) = 0.95$

f_3: Fuel Battery Gauge

$P(G=e \mid B=g, F=e) = 0.97$

$P(G=ne \mid B=g, F=e) = 0.03$

$P(G=e \mid B=b, F=e) = 0.99$

$P(G=ne \mid B=b, F=e) = 0.01$

$P(G=e \mid B=g, F=ne) = 0.04$

$P(G=ne \mid B=g, F=ne) = 0.96$

$P(G=e \mid B=b, F=ne) = 0.1$

$P(G=ne \mid B=b, F=ne) = 0.9$

f_4: Battery Turn-over

$P(T=false \mid B=bad) = 0.98$

$P(T=true \mid B=bad) = 0.02$

$P(T=false \mid B=good) = 0.03$

$P(T=true \mid B=good) = 0.97$

f_5: Fuel

$P(S=yes \mid T=t, F=e) = 0.08$

$P(S=no \mid T=t, F=e) = 0.92$

$P(S=no \mid T=f, F=e) = 0.99$

$P(S=yes \mid T=f, F=e) = 0.01$

$P(S=no \mid T=t, F=ne) = 0.01$

$P(S=yes \mid T=t, F=ne) = 0.99$

$P(S=no \mid T=f, F=ne) = 1$

$P(S=yes \mid T=f, F=ne) = 0$

e) Belief propagation messages from B, F to G.

1. $\mu_{f_1 \to B}(B) = f_1(B)$

$\mu_{f_4 \to B}(B) = \text{ones}$

$\mu_{B \to f_3}(B) = \mu_{f_1 \to B}(B)\, \mu_{f_4 \to B}(B) = f_1(B)$

2. $\mu_{f_2 \to F}(F) = f_2(F)$

$\mu_{f_5 \to F}(F) = \text{ones}$

$\mu_{F \to f_3}(F) = \mu_{f_2 \to F}(F)\, \mu_{f_5 \to F}(F) = f_2(F)$

3. Finally

$\mu_{f_3 \to G}(G) = f_3(B,F,G)\, \mu_{B \to f_3}(B)\, \mu_{F \to f_3}(F) =$

$= P(G \mid B,F)\, f_1(B)\, f_2(F)$

$= P(G \mid B,F)\, P(B)\, P(F)$