Practical 1 : Limits and Continuity

1 $$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}\right]$$

→ $$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}}\right]$$

$$= \lim_{x \to a} \left[\frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})}{3a + x - 4x}\right]$$

$$= \lim_{x \to a} \left[\frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})}{3a - 3x}\right]$$

$$= \lim_{x \to a} \left[\frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{3a+x} + 2\sqrt{x})}{3(a-x)}\right]$$

$$= \lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{3(a-x)} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}}\right] \times \sqrt{3a+x} + 2\sqrt{x}$$

$$= \lim_{x \to a} \left[\frac{(a + 2x - 3x)\sqrt{3a+x} + 2\sqrt{x}}{3(a-x)(\sqrt{a+2x} + \sqrt{3x})}\right]$$

$$= \lim_{x \to a} \left[\frac{\cancel{a-x}}{3\cancel{(a-x)}} \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{a+2x} + \sqrt{3x}}\right]$$

$$= \lim_{x \to a} \frac{1}{3} \left[\frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{a+2x} + \sqrt{3x}}\right]$$

$$= \frac{1}{3} \left[\frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}\right]$$

$$= \frac{1}{3}\left[\frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3}a}\right]$$

$$= \frac{1}{3}\left[\frac{2\sqrt{a} + 2\sqrt{a}}{\sqrt{3}a + \sqrt{3}a}\right]$$

$$= \frac{1}{3}\left[\frac{\cancel{(2)}\, 2\sqrt{a}}{\cancel{(2)}\sqrt{3}a}\right]$$

$$= \frac{1}{3} \times \frac{2\sqrt{a}}{\sqrt{3}a}$$

$$= \frac{2\cancel{\sqrt{a}}}{3\sqrt{3}\cancel{a}} = \frac{2}{3\sqrt{3}}$$

2) $$\lim_{y\to 0}\left[\frac{\sqrt{9+x} - \sqrt{a}}{y\sqrt{a+y}}\right]$$

$$= \lim_{y\to 0} \frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+x} + \sqrt{a}}{\sqrt{a+x} + \sqrt{a}}$$

$$= \lim_{y\to 0} \frac{(\cancel{a}+y-\cancel{a})}{(y\sqrt{a+y})(\sqrt{a+y} + \sqrt{a})}$$

$$= \lim_{y\to 0} \frac{\cancel{y}}{\cancel{y}\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})}$$

$$= \lim_{y\to 0} \frac{1}{\sqrt{a}\,(\sqrt{a} + \sqrt{a})}$$

$$= \lim_{y \to 0} \frac{1}{\sqrt{a}\,(2\sqrt{a})}$$

$$= \frac{1}{2a}$$

3) $$\lim_{x \to \frac{\pi}{6}} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$$

$$x - \frac{\pi}{6} = h \qquad x = h + \frac{\pi}{6} \qquad h \to 0$$

$$= \lim_{h \to 0} \frac{\cos(h + \pi/6) - \sqrt{3}\,(h + \pi/6)}{\pi - 6(h + \pi/6)}$$

$$= \lim_{h \to 0} \frac{(\cos h \cdot \cos \pi/6 - \sin h \cdot \sin \pi/6) - \sqrt{3}\,(\sin h \ \cos \pi/6 + \cos h \sin \pi/6)}{-6h}$$

$$= \lim_{h \to 0} \frac{\left[\left(\cos h \ \frac{\sqrt{3}}{2}\right) - \left(\sin h \frac{1}{2}\right)\right] - \sqrt{3}\left(\sin h \frac{\sqrt{3}}{2} + \cos h \frac{1}{2}\right)}{-6h}$$

$$= \lim_{h \to 0} \frac{\left(\cos \frac{\sqrt{3}}{2} h - \sin \frac{h}{2}\right) - \left(\sin \frac{3h}{2} + \cos \frac{\sqrt{3}}{2} h\right)}{-6h}$$

$$= \lim_{h \to 0} \frac{\frac{-\sin 4h}{2}}{-6h}$$

$$= \lim_{h \to 0} \frac{\sin 2h}{6h}$$

$$= \frac{1}{3} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{3}$$

4) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}}$$

$$= \lim_{x \to \infty} \frac{(x^2+5-x^2+3)(\sqrt{x^2+3} + \sqrt{x^2+1})}{(x^2+3-x^2-1)(\sqrt{x^2+5} + \sqrt{x^2-3})}$$

$$= \lim_{x \to \infty} \frac{8(\sqrt{x^2+3} + \sqrt{x^2+1})}{2(\sqrt{x^2+5} + \sqrt{x^2-3})}$$

$$= \lim_{x \to \infty} \frac{4(\sqrt{x^2+3}) + (\sqrt{x^2+1})}{\sqrt{x^2+5} + \sqrt{x^2-3}}$$

$$= \lim_{x \to \infty} \frac{4\sqrt{x^2(1+3/x^2)} + \sqrt{x^2(1+1/x^2)}}{\sqrt{x^2(1+5/x^2)} + \sqrt{x^2(1-3/x^2)}}$$

$$= 4 \lim_{x \to \infty} \frac{\sqrt{x^2(1+3/x^2)} + \sqrt{x^2(1+1/x^2)}}{\sqrt{x^2(1+5/x^2)} + \sqrt{x^2(1-3/x^2)}}$$

$= 4$

5) Examine the continuity of the following function

(i) $f(x) = \left.\begin{cases} \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}, & 0 < x \leq x/2 \\ \dfrac{\cos x}{\pi - 2x}, & \pi/2 < x < \pi \end{cases}\right\}$ at $x = \pi/2$

(ii) $f(x) = \left.\begin{cases} \dfrac{x^2-9}{x-3}, & 0 < x < 3 \\ x+3, & 3 \leq x < 6 \\ \dfrac{x^2-9}{x+3}, & 6 \leq x \leq 9 \end{cases}\right\}$ at $x = 3$ and $x = 6$

→ (i) $\because f(\pi/2) = \frac{\sin(x/2)}{\sqrt{1-\cos 2(\pi/2)}}$

1) $f(\pi/2) = 0$

$\therefore f$ at $x = \pi/2$ define

2) $\lim_{x \to \pi/2^+} f(x) = \lim_{x \to \pi/2^+} \frac{\cos x}{\pi - 2x}$

put $x - \pi/2 = h$

$\therefore x = \pi/2 + h$

as $x \to \pi/2 \quad h \to 0+$

$= \lim_{h \to 0+} \frac{\cos(\pi/2 + h)}{x - 2(\pi/2 + h)} = \lim_{h \to 0+} \frac{-\sin h}{x - \pi - 2h}$

$= \lim_{h \to 0+} \frac{-\sin h}{-2h} = \frac{1}{2}$

$= \lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$

$= \lim_{x \to \pi/2^-} \frac{2\sin x \cos x}{\sqrt{2\sin^2 x}} = \lim_{x \to \pi/2^-} \frac{2\sin x \cos x}{\sqrt{2}\sin x}$

$= \frac{2}{\sqrt{2}} \lim_{x \to \pi/2} \cos x = 0$

$\therefore$ LHL $\neq$ RHL

$\therefore f$ is not continuous at $x = \pi/2$

iii) $f(x) = \dfrac{x^2 - a}{x-3} \quad 0 < x < 3$

$= x + 3 \quad 3 \le x \le 6$

$= \dfrac{x^2 - a}{x+3} \quad 6 \le x \le 9$

at $x = 3$ and $x = 6$

at $\dfrac{x = 3}{f(3)} = \dfrac{x^2 - a}{x-3} = 0$

f at $x = 3$ define

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+}$

$f(3) = x + 3 = 3 + 3 = 6$

f is define at $x = 3$

2) $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (x+3) = 6$

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \dfrac{x^2 - 9}{x - 3} = \dfrac{(x-3)(x+3)}{(x-3)}$

$= LHL = RHL$

f is continuous at $x = 3$

for $x = 6$

$f(6) = \dfrac{x^2 - 9}{x + 3} = \dfrac{36 - 9}{6 + 3} = \dfrac{27}{9}$

$\lim\limits_{x \to 6^+} = \dfrac{x^2 - 9}{x + 3}$

$$\lim_{x \to 6^+} \frac{(x-3)(x+3)}{(x+3)} = \lim_{x \to 6^+} (x-3) = 6-3 = 3.$$

$$\lim_{x \to 6^-} x+3 = 3+6 = 9$$

d

LHL = RHL

function is not continuous.

6) Find value of k so that the function $f(x)$ is continuous at the indicated point.

(i) $f(x) = \left.\begin{cases} \dfrac{1-\cos 4x}{x^2}, & x<0 \\ = k, & x=0 \end{cases}\right\}$ at $x=0$

(ii) $f(x) = \left.\begin{array}{l} (\sec^2 x)^{\cot^2 x} \\ = k \end{array}\right\}$ at $x=0$

(iii) $f(x) = \left.\begin{cases} \dfrac{\sqrt{3}-\tan x}{\pi - 3x}, & x \neq \pi/3 \\ = k, & x = \pi/3 \end{cases}\right\}$ at $x = \pi/3$

→ i) f is continuous at $x=0$

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = k$$

$$= 2\lim_{x \to 0} \frac{2\sin^2 x}{x^2} = k \quad = 2\lim_{x \to 0} \left(\frac{\sin^2 2x}{x}\right) = k$$

$2(2)^2 = k$

$k = 8$

ii) $f(x) = \begin{cases} (\sec^2 x)^{\cot^2 x} & x \neq 0 \\ k & x = 0 \end{cases}$ at $x = 0$.

$$\frac{\sqrt{3} - 3\tan h - \sqrt{3} - \tan h}{\left(\frac{1 - \sqrt{3}\tan h}{-3h}\right)} = \frac{\frac{-4\tan h}{(-3h)(1-\sqrt{3}\tan h)}}{-3h}$$

$$= \frac{4}{3}\left(\frac{1}{1-0}\right)$$

$$= \frac{4}{3}$$

iii)

$x - \pi/3 = h$

$x = h + \pi/3 \quad h \to 0$

$f(\pi/3 + h) = \sqrt{3}$

$\therefore \lim_{x \to \pi/3} f(x) = f(\pi/3)$

$$\lim_{x \to \pi/3} \frac{\sqrt{3} - \tan x}{\pi - 3x} = k$$

$$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

$$\lim_{h \to 0} \frac{\sqrt{3}\left(\frac{\tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}\right)}{\pi - \pi - 3h}$$

$$\lim_{h\to 0} \frac{\sqrt{3}(1-\tan\pi/3 \cdot \tan h) - \tan(\pi/3 - \tan h)}{(-3h)(1-\tan\pi/3 \cdot \tan h)}$$

$$\lim_{h\to 0} \frac{\sqrt{3}(1-\sqrt{3}\tan h) - \sqrt{3} - \tan h}{-3h(1-\sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{\sqrt{3} - 3\tan h - \sqrt{3} - \tan h}{-3h(1-\sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3h(1-\sqrt{3}\tan h)}$$

$$= -\frac{4}{3} \lim_{h\to 0} \left(\frac{\tan h}{h}\right)\left(\frac{1}{1-\sqrt{3}\tan h}\right)$$

$$= \frac{4}{3} \frac{1}{1-\sqrt{3}/0}$$

$$= \frac{4}{3}$$

7) discuss the continuity of the following functions which of these function have removable discontinuity ? Redefine function so as to remove the discontinuity.

(i) $f(x) = \begin{cases} \dfrac{1-\cos 3x}{x \tan x}, & x \neq 0 \\ 9 & x = 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} \dfrac{(e^{3x}-1)\sin x^\circ}{x^2}, & x \neq 0 \\ \pi/60, & x = 0 \end{cases}$ at $x = 0$

i) → $f(x) = \dfrac{1-\cos 3x}{x \tan x}$

$= \dfrac{2\sin^2 \frac{3}{2}x}{x \tan x}$

$= \dfrac{2\sin^2 3x/2 \times x^2}{x^2} \Big/ x \cdot \dfrac{\tan x}{x^2} \times x^2$

$= 2 \lim_{x \to 0} (3/2)^2/1$

$= 2 \times \dfrac{9}{4} = \dfrac{9}{2}$

$\lim_{x \to 0} f(x) = \dfrac{9}{2} \quad 9 = f(0)$

∴ f is not continuous at $x = 0$.

redefine function:

$f(x) = \dfrac{1-\cos 3x}{x \tan x} \quad x \neq 0$

$9/2 \quad x = 0$.

Now $\lim_{x \to 0} f(x) = f(0)$

f has removable discontinuity at $x=0$

ii) at $x=0$

$$\lim_{x\to 0} (e^{3x}-1)\sin\left(\frac{\pi x}{180}\right) / x^2$$

$$3\lim_{x\to 0} \frac{e^{3x}-1}{3x} \lim_{x\to 0} \frac{\sin \pi x/180}{x}$$

$$3\log e \frac{\pi}{180} = \frac{\pi}{60} = f(0)$$

f is continuous at $x=0$.

8) If $f(x) = \left(\frac{ex^2 - \cos x}{x^2}\right)$ for $x \neq 0$ is continuous at $x=0$ find $f(0)$

$$f(x) = \frac{ex^2 - \cos x}{x^2} \quad x=0$$

$\therefore f$ is continuous at $x=0$.

$$\lim_{x\to 0} f(x) = f(0)$$

$$= \frac{ex^2 - \cos x}{x^2} = f(0)$$

$$\frac{ex^2 - \cos x - 1 + 1}{x^2}$$

$$(e^{x^2}-1) + (1-\cos x) / x^2$$

$$\frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{2\sin^2 x/2}{}$$

$$= \log e + 2 \left(\frac{\sin x/2}{x}\right)^2 \frac{x^2}{2}$$

multiply with 2 on numerator and denominator

$$= 1 + 2 \times \frac{1}{9} = \frac{3}{2} = f(0)$$

9) If $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$ for $x \neq \pi/2$ is continuous at $x = \pi/2$ find $f(\pi/2)$

$$f(x) = \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \quad x \neq \pi/2$$

$f(0)$ is continuous at $x = \pi/2$

$$= \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$$

$$= \frac{2 - 1 + \sin x}{\cos^2 x(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{1 + \sin x}{1 - \sin^2 x(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{1}{(1-\sin x)(\sqrt{2} + \sqrt{1+\sin x})}$$

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2 \times 2\sqrt{2}} = \frac{1}{4\sqrt{2}}$$

$$f(\pi/2) = \frac{1}{4\sqrt{2}}$$

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6)ii) $f(x) = (\sec^2 x)^{\cot^2 x}$ $\quad x \neq 0$ } at $x = 0$
$= R$ $\quad x = 0$ }

$f(x) = (\sec^2 x)^{\cot^2 x}$

using
$\sec^2 x - \tan^2 x = \sec^2 x = 1$
$\therefore \sec^2 x = 1 + \tan^2 x$

and
$\cot^2 x = \frac{1}{\tan^2 x}$

$\lim_{x \to 0} (\sec^2 x)^{\cot^2 x}$

$\lim_{x \to 0} (1 + \tan^2 x)^{\frac{1}{\tan^2 x}}$

we know that,

$\lim_{x \to 0} (1 + px)^{\frac{1}{px}} = e$

$\therefore = e$
$\therefore R = e$

Practical : 02

Topic : Derivative.

I Show that the following function defined from $\mathbb{R}$ G $\mathbb{R}$ differentiable

1) $\cot x$ 2) $\operatorname{cosec} x$ 3) $\sec x$

II If $f(x) = 4x + 1 \quad x \leqslant 2$
$= x^2 + 5 \quad x > 0$ at $x = 2$, then find f is differentiable or not.

III If $f(x) = 4x + 7 \quad x < 3$
$= x^2 + 3x \quad x \geqslant 3$ at $x = 3$ then find f is differentiable or not

IV If $f(x) = 8x - 5 \quad x \leqslant 2$
$= 3x^2 - 4x + 7 \quad x < 2$ at $x = 2$ then find f is differentiable or not

Solution :

I) 1) $\cot x$

$f(x) = \cot x$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a}$$

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$$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x-a) \tan x \tan a}$$

put $x - a = h$ $\quad x = a+h \quad$ as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a) \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h) \tan a}$$

formula $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$$\tan A - \tan B = \tan(A-B)(1 + \tan A \cdot \tan B)$$

$$= \lim_{h \to 0} \frac{\tan(a-a-h) - (1 + \tan a + \tan(a+h))}{h \times \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} -\frac{\tan h}{h} \times \frac{1 + \tan a \tan(a+h)}{\tan(a+h) \tan a}$$

$$= -1 \times \frac{1 + \tan^2 a}{\tan^2 a}$$

$$= -\frac{\sec^2 a}{\tan^2 a} = -\frac{1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$$Df(a) = -\cos^2 a$$

$\therefore f$ is differentiable $\forall a \in R$.

2) $\operatorname{cosec} x$

$$f(x) = \operatorname{cosec} x$$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\sin x - 1/\sin a}{x - a}$$

$x \to a \quad (x-a)\sin a \, \sin x$

put $x - a = h \quad x = a+h \quad$ as $x \to a \quad h \to 0$.

$$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$$

$$Df(h) = \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$$

formula

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \cdot \sin(a+h)}$$

$$= \lim_{h \to 0} \frac{-\sin h/2}{h/2} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \, \sin(a+h)}$$

$$= \frac{-\frac{1}{2} \times 2\cos\left(\frac{2a}{2}\right)}{\sin(a+0)}$$

$$= \frac{-\cos a}{\sin^2 a} = -\cot a \, \operatorname{cosec} a$$

3) $\sec x$

$f(x) = \sec x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\sec(x) - \sec(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a}$$

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$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a)\cos a \ \cos x}$$

put $x - a = h$

$x = a + h$

as $x \to a$ $\quad h \to 0$

$$Df(h) - \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cdot \cos(a+h)}$$

Formula: $-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right)\sin\left(\frac{a-a-h}{2}\right)}{\cos a \quad \cos(a+h) \quad x - \frac{h}{2}}$$

$$= \frac{-1}{2} \quad x - 2 \quad \frac{\sin a}{\cos a \times \cos a}$$

$= \tan a \ \sec a$

II LHD :-

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4 \times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+8-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{(x-2)} = 4$$

$Df(2^-) = 4$

RHD:-

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)(x-2)}{(x-2)}$$

$$= 2+2 = 4$$

$Df(2+) = 4$

RHD = LHD

f is differentiable at $x=2$

III RHD:-

$$Df(3+) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(3^2+3-3+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-19}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$$

$$\lim_{x \to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$\lim_{x \to 3^+} \frac{x(x+6) - 3(x+6)}{x-3}$$

$$\lim_{x \to 3^+} \frac{(x+6)(x-3)}{(x-3)} = 3+6 = 9$$

$Df(3^+) = 9$

LHD = $Df(3^-)$

$$= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4(x-3)}{(x-3)}$$

$Df(3^-) = 4$

RHD = LHD.

$\therefore$ f is not differentiable at $x = 3$.

IV

$f(2) = 8 \times 2 - 5 = 16 - 5 = 11$

RHD: $Df(2+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)(x-2)}{(x-2)}$$

$$= 3 \times 2 + 2 = 8$$

$$Df(2^+) = 8$$

LHD:- $Df(2^-) = \lim_{x \to 2^-} \dfrac{f(x) - f(2)}{x - 2}$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8(x-2)}{(x-2)}$$

$$Df(2^-) = 8$$

LHD = RHD

f is differentiable at $x = 3$

Practical : 03

Topic: Application of Derivative

I Find the intervals in which function is increasing or decreasing.

I $f(x) = x^3 - 5x - 11$
II $f(x) = x^2 - 4x$
III $f(x) = 2x^3 + x^2 - 20x + 4$
IV $f(x) = x^3 - 27x + 5$
V $f(x) = 69 - 24x - 9x^2 + 2x^3$

II Find the intervals in which function is concave upwards

I $y = 3x^2 - 2x^3$
II $y = x^4 - 6x^3 + 12x^2 + 5x + 7$
III $y = x^3 - 27x + 5$
IV $y = 69 - 24x - 9x^2 + 2x^3$
V $y = 2x^3 + x^2 - 20x + 4$

Solution:

I

1) $f(x) = x^3 - 5x - 11$

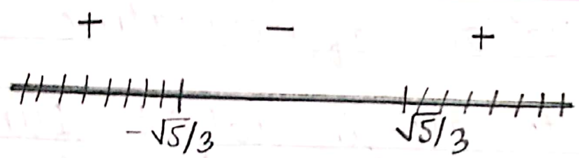
$\therefore f'(x) = 3x^2 - 5$

$\therefore f$ is increasing iff $f'(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$(x - \sqrt{5/3})(x + \sqrt{5/3}) > 0$



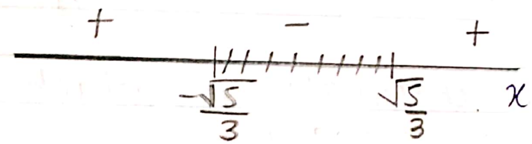
$x \in (-\infty, -\sqrt{5}/3) \cup (\sqrt{5}/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore 3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5}/3)(x + \sqrt{5}/3) < 0$



$x \in (-\sqrt{5}/3, \sqrt{5}/3)$

$f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$\because f(x)$ is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$x \in (2, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$x \in (-\infty, 2)$

3) $f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\because f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

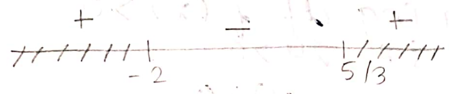
$\therefore 2(3x^2 + x - 10) > 0$

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

$\therefore 3x(x+2) - 5(x+2) > 0$

$\therefore (x+2)(3x-5) > 0$



and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

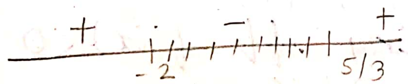
$\therefore 2(3x^2 + x - 10) < 0$

$\therefore 3x^2 + x - 10 < 0$

$\therefore 3x^2 + 6x - 5x - 10 < 0$

$\therefore 3x(x+2) - 5(x+2) < 0$

$\therefore (x+2)(3x-5) < 0$



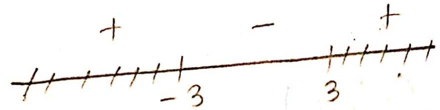
4) $f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$\because f$ is increasing iff $f'(x) > 0$

$\therefore 3(x^2 - 9) > 0$

$\therefore (x-3)(x+3) > 0$



and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$\therefore\ 3(x^2-9)<0$

$\therefore\ (x-3)(x+3)<0$

5) $f(x) = 2x^3 - 9x^2 - 24x + 69$

$f'(x) = 6x^2 - 18x - 24$

$\because$ f is increasing iff $f'(x)>0$

$\therefore\ 6x^2-18x-24>0$

$\therefore\ 6(x^2-3x-4)>0$

$\therefore\ x^2-4x+x-4>0$

$\therefore\ x(x-4)+1(x-4)>0$

$\therefore\ (x+1)(x-4)>0$

$\therefore\ x \in (-\infty,-1)\cup(4,\infty)$

and f is decreasing iff $f'(x)<0$

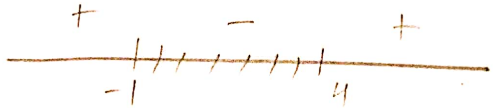
$\therefore\ 6x^2-18x-24<0$

$\therefore\ 6(x^2-3x-4)<0$

$\therefore\ x^2-4x+x-4<0$

$\therefore\ x(x-4)+1(x-4)<0$

$\therefore\ (x-4)(x+1)<0$



$\therefore\ x \in (-1,4)$

II (i)

(ii

$y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

f is concave upward if $f''(x) > 0$

$\therefore (6 - 12x) > 0$

$\therefore 12\ (6/12 - x) > 0$

$x - 1/2 > 0$

$x > 1/2$

$\because f''(x) > 0$

$\therefore x \in (1/2, \infty)$

$y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upward if $f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 2x - x + 2 > 0$

$\therefore x(x-2) - 1(x-2) > 0$

$\therefore (x-2)(x-1) > 0$

iii

$y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff

$f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x \in (0, \infty)$

iv

$y = 69 - 24x - 9x^2 + 2x^3$

$f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upward iff $f''(x) > 0$

$\therefore 12x - 18 > 0$

$\therefore 12(x - 18/12) > 0$

$\therefore x - 3/2 > 0$

$\therefore x > \frac{3}{2}$

$\therefore x \in (3/2, \infty)$

v) $y = 2x^3 + x^2 - 20x + 4$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$\therefore f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 12(x + 2/12) > 0$

$\therefore x + 1/6 > 0$

$\therefore x < -1/6$

$\because f''(x) \not> 0$

$\therefore$ There exist ~~no~~ interval $(-1/6, \infty)$

Practical: 04

Topic: Application of derivative and Newtons method.

I) Find maximum and minimum value of following.

i) $f(x) = x^2 + \frac{16}{x^2}$

ii) $f(x) = 3 - 5x^3 + 3x^5$

iii) $f(x) = x^3 - 3x^2 + 1 \quad [-1/2, 4]$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1 \quad [-2, 3]$

II) Find the root of the following equation by newtons (Take 4 iteration only) correct upto 4 decimal.

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

I) i) $f(x) = x^2 + \dfrac{16}{x^2}$

$f'(x) = 2x - \dfrac{32}{x^3}$

Now, consider, $f'(x) = 0$

$\therefore 2x - \dfrac{32}{x^3} = 0$

$2x = \dfrac{32}{x^3}$

$x^4 = \dfrac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \dfrac{96}{x^4}$

$f''(2) = 2 + \dfrac{96}{2^4}$

$= 2 + \dfrac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore f(2) = 2^2 + \dfrac{16}{2^2}$

$= 4 + \dfrac{16}{4}$

$= 4 + 4$

$= 8$

$\therefore f''(-2) = 2 + 96/(-2)^4$

$= 2 + 96/16 = 2 + 6 = 8 > 0$

$\therefore$ f has minimum value at $x=-2$

$\therefore$ function reaches minimum value at $x=2$ and $x=-2$

ii) $f(x) = 3-5x^3+3x^5$

$\therefore f'(x) = -15x^2+15x^4$

consider, $f'(x)=0$

$\therefore 15x^2 + 15x^4 = 0$

$\therefore 15x^4 = 15x^2$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$f(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x=1$

$\therefore f''(-1) = -30(-1)+60(-1)^3$

$= 30-60$

$= -30 < 0$

$\therefore$ f has maximum value at $x=-1$

$\therefore f(-1) = 3-5(-1)^3 + 3(-1)^5$

$= 3+5-3$

$= 5$

$\therefore$ f has maximum value 5 at $x=-1$ and has the minimum value 1 at $x=1$

iii)

iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

Consider, $f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$\therefore 3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

$\therefore f''(x) = 6x - 6$

$\therefore f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore$ f has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1 = 1$

$\therefore f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore$ f has maximum value 1 at $x = 0$ and f has minimum value -3 at $x = 2$

$f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$

Consider, $f'(x) = 0$

$\cancel{\therefore 6(x^2 - x - 2) =}$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$\therefore x^2 - x - 2 = 0$

$\therefore x^2 + x - 2x - 2 = 0$

$\therefore x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore f$ has minimum value at $x = 2$

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$\therefore f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(1) + 1$

$= -2 - 3 - 12 + 1$

$= 8$

II (i)

$\therefore$ f has maximum value 8 at $x=-1$ and f has minimum value -19 at $x=2$

II (i)

$f(x) = x^3 - 3x^2 - 55x + 9.5$ $\quad x_0 = 0 \rightarrow$ given.

$f'(x) = 3x^2 - 6x - 55$

By Newton's method

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$\therefore x_1 = x_0 - f(x_0) / f'(x_0)$

$\therefore x_1 = 0 + 9.5/55$

$\therefore x_1 = \underline{0.1727}$

$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= \underline{-0.0829}$

$\therefore f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467$

$\therefore x_2 = x_1 - f(x_1) / f'(x_1)$

$= 0.1727 - 0.0829 / 55.9467$

$= 0.1712$

$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= \underline{0.0011}$

$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55$

$= -55.9393$

$\therefore x_3 = x_2 - f(x_2) / f'(x_2)$

$= 0.1712 + 0.0011 / 55.9393$

$= \underline{0.1712}$

$\therefore$ The root of equation is $\underline{0.1712}$

(ii) $f(x) = 3 - 5x^3 + 3x^5$

$\therefore f'(x) = -15x^2 + 15x^4$

consider , $f'(x) = 0$

$\therefore 15x^2 + 15x^4 = 0$

$\therefore 15x^4 = 15x^2$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$f(1) = -30 + 60$

$= 30 > 0$

$\therefore f$ has minimum value at $x = 1$

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$\therefore f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$

$\therefore f$ has maximum value at $x = -1$

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ $[1,2]$

$f(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(7)^2 - 10(1) + 17$

$= -1.8 - 10 + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17 = +2.2$

Let $x_0 = 2$ be initial approximation

By newtons method

$x_{n+1} = x_n - f(x_n) / f'(x_n)$

$x_1 = x - f(x_2) / f'(x_2)$

$= 2 - 2.2/5.2$

$= 2 - 0.4230 = \underline{1.577}$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= \underline{0.6755}$

$f'(x) = 3(1.577)^2 - 3.6(1.577) - 10$

$= 7.4608 - 5.6772 - 10$

$= \underline{-8.2164}$

$\therefore x_2 = x_1 - f(x_1) / f'(x_1)$

$= 1.577 + 0.6755 / 8.2164$

$= 1.577 + 0.0822$

$= \underline{1.6592}$

$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$

$= 4.5677 - 4.9553 - 16.592 + 17$

$= \underline{0.0204}$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= 8.2588 - 5.97312 - 10$

$= \underline{\underline{-7.7143}}$

$x_3 = x_2 - f(x_2) / f'(x_2)$

$= 1.6592 + 0.0204 / 7.7143$

$= 1.6592 + 0.0026$

$= \underline{\underline{1.6618}}$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.5892 - 4.9708 - 16.618 + 17$

$= 0.004$

$f(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$

$= 8.2847 - 5.9824 - 10$

$= \underline{\underline{-7.6977}}$

$$x_4 = x_3 - f(x_3)/f'(x_3)$$

$$= 1.8618 + \frac{0.0004}{7.6977}$$

$$= \underline{1.6618}$$

Practical : 05

Topic: Integration

I) Solve the following integrations

1 $\dfrac{dx}{\sqrt{x^2+2x-3}}$

2 $\int (4e^{3x}+1)\, dx$

3 $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

4 $\int \dfrac{x^3+3x+4}{\sqrt{x}}\, dx$

5 $\int t^{7} \sin(2t^4)\, dt$

6 $\int \sqrt{x}\,(x^2-1)\, dx$

7 $\int \dfrac{1}{x^3} \sin\left(\dfrac{1}{x^2}\right) dx$

8 $\int \dfrac{\cos x}{\sqrt[3]{\sin^2 x}}\, dx$

9 $\int e^{\cos^2 x} \sin 2x\, dx$

10 $\int \left(\dfrac{x^2-2x}{x^3-3x^2+1}\right) dx$

1) $\int \frac{1}{x^2+2x-3} \, dx$

$= \int \frac{1}{\sqrt{x^2+2x-3}} \, dx$

$= \int \frac{1}{\sqrt{x^2+2x+1-4}}$

\# $a^2+2ab+b^2 = (a+b)^2$

$= \int \frac{1}{\sqrt{(x+1)^2-4}} \, dx$

Substitute,

put $x-1 = t$

$dx = \frac{1}{t} \times dt$ where $t=1$ $t = x+1$

$\int \frac{1}{\sqrt{t^2-4}} \, dt$

using,

\# $\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \ln(|x+\sqrt{x^2-a^2}|)$

$= \ln(|t+\sqrt{t^2-4}|)$
$t = x+1$

$= \ln (|x+1| + \sqrt{(x+1)^2 - 4|})$

$= \ln (|x+1| + \sqrt{x^2+2x-3|})$

$= \ln (|x+1| + \sqrt{x^2+2x-3}|) + c$

2) $\int (4e^{3x} + 1)\, dx$

$= \int 4e^{3x} dx + \int 1\, dx$

$= 4 \int e^{3x} dx + \int 1\, dx$ # $\int e^{dx} dx = \frac{1}{2} x e^{ax}$

$= \frac{4e^{3x}}{3} + x$

$= \frac{4e^{3x}}{3} + x + c$

3) $\int 2x^2 - 3\sin(x) + 5\sqrt{x}\, dx$

$= \int 2x^2 - 3\sin(x) + 5x^{1/2} dx$ # $\sqrt[n]{a^m} = a^{m/n}$

$= \int 2x^2 dx - \int 3\sin(x)\, dx + \int 5x^{1/2} dx$

$= \frac{2x^3}{3} + 3\cos x + \frac{10x\sqrt{x}}{3} + c$

$= \frac{2x^3 + 10x\sqrt{x}}{3} + 3\cos x + c$

4) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$= \int \frac{x^3+3x+4}{x^{1/2}}\,dx$

split the denominator

$= \int \frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\,dx$

$= \int x^{5/2} + 3x^{1/2} + \frac{4}{x^{1/2}}\,dx$

$= \int x^{5/2}\,dx + \int 3x^{1/2}\,dx + \int \frac{4}{x^{1/2}}\,dx$

$= \frac{x^{5/2+1}}{5/2+1}$

$= \frac{2x^3\sqrt{x}}{7} + 2x\sqrt{x} + 8\sqrt{x} + C$

5) . $t^7 \times \sin(2t^4)\,dt$

put $u = 2t^4$

$du = 2\times 4t^3$

$= \int t^7 \times \sin(2t^4) \times \frac{1}{2\times 4t^3}\,du$

$= \int t^4 \sin(2t^4) \times \frac{1}{2\times 4}\,du$

$= \int t^4 \sin(2t^4) \times \frac{1}{8}\,du = \frac{t^4 \times \sin(2t^4)}{8}\,du$

Substitute t^4 with $\frac{u}{2}$

$$= \int \frac{u/2 \times \sin(u)}{8} \, du$$

$$= \int \frac{u \times \sin(u)}{2} / 8 \quad du$$

$$= \int \frac{u \times \sin(u)}{16} \quad du$$

$$= \frac{1}{16} \int u \times \sin(u) \, du$$

$\int u\,dv = uv - \int v\,du$

where $u = u$

$dv = \sin(u) \times du$

$du = 1\,du \quad v = -\cos(u)$

$$= \frac{1}{16} (u) \times (-\cos(u)) - \int -\cos(u)\,du$$

$$= \frac{1}{16} \times (u \times (-\cos u)) + \int \cos(u)\,du$$

$\int \cos x \, dx = \sin(x)$

$$= \frac{1}{16} \times (u \times (-\cos(u)) + \sin(u))$$

Return the substitution $\quad u = 2t^4$

$= \frac{1}{16} \times \left(2t^4 \times (-\cos(2t^4)) + \sin(2t^4)\right)$

$= -\frac{t^4 \times \cos(2t^4)}{8} + \frac{\sin(2t^4)}{16} + C$

6) $\int \sqrt{x}\,(x^2 - 1)\,dx$

$= \int \sqrt{x}\; x^2 - \sqrt{x}\; dx$

$= \int x^{1/2} \times x^2 - x^{1/2}\, dx$

$= \int x^{5/2} - x^{1/2}\, dx$

$= \int x^{5/2}\, dx - \int x^{1/2}\, dx$

$= I_1 = \frac{x^{5/2+1}}{} = \frac{x^{7/2}}{} = \frac{2x^{7/2}}{} = \frac{2\sqrt{x^7}}{}$

8) $\int \frac{\cos x}{\sqrt[3]{\sin(x)^2}} dx$

$= \int \frac{\cos x}{\sin x^{2/3}} dx$

Put $t = \sin(x)$

$t = \cos x$

$= \int \frac{\cos(x)}{\sin(x)^{3/2}} \times \frac{1}{\cos(x)\,dt}$

$= \frac{1}{\sin x^{3/2}} dt$

$= \frac{1}{t^{2/3}} dt$

$I = \int \frac{1}{t^{2/3}} dt = \frac{-1}{(2/3 - 1)t^{2/3-1}} = \frac{-1}{(2/3 - 1)\, t^{2/3-1}}$

$= \frac{-1}{-1/3\, t^{2/3-1}} = \frac{1}{1/3\, t^{-1/3}} = \frac{t^{1/3}}{1/3} = 3t^{1/3}$

$= 3\sqrt[3]{t}$

Return substitution $t = \sin(x)$.

$= 3\sqrt[3]{\sin(x)} + C$

10)

$$\int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 3 \times 2x} \, dt$$

put $x^3 - 3x^2 + 1 = dt$

$$I = \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 3 \times 2x} \, dt$$

$$= \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3x^2 - 6x} \, dt$$

$$= \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3(x^2 - 2x)} \, dt$$

$$= \int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \times \frac{1}{3(x^2 - 2x)} \, dt$$

$$= \int \frac{1}{x^3 - 3x^2 + 1} \times \frac{1}{3} \, dt$$

$$= \int \frac{1}{3(x^3 - 3x^2 + 1)} \, dt = \int \frac{1}{3t} \, dt$$

$$= \frac{1}{3} \int \frac{1}{t} \, dt \qquad \int \frac{1}{x} \, dx = \ln|x|$$

$$= \frac{1}{3} \times \ln|t| + c$$

$$= \frac{1}{3} \times \ln(|x^3 - 3x^2 + 1|) + c$$

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07/02/2022

Practical : 6

Topic : Application of Integration and Numerical Integration

Q1 Find the length of the following curve:

1 $x = t\sin t$, $y = 1 - \cos t$ $t \in [0, 2\pi]$

2 $y = \sqrt{4 - x^2}$ $x \in [-2, 2]$

3 $y = x^{3/2}$ in $[0, 4]$

4 $x = 3\sin t$, $y = 3\cos t$ $t \in [0, 2\pi]$

5 $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$

Q2 Using simpsons rule solve the following:

1 $\int_0^2 e^{x^2}\, dx$ with $n = 4$

2 $\int_0^4 x^2\, dx$ with $n = 4$

3 $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ with $n = 6$

52.

Q1) i) $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = t - \sin t$ $\quad \therefore \frac{dx}{dt} = 1 - \cos t$

$y = 1 - \cos t$ $\quad \therefore \frac{dy}{dt} = 0 - (-\sin t) = \sin t$

$L = \int_0^{2\pi} \sqrt{(1 - \cos t + \sin t)^2 + (\sin t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} \sqrt{2(1 - \cos t)}$

$= \int_0^{2\pi} \sqrt{2 \times 2 \sin^2 t/2} = \sqrt{4 \sin^2 t/2}$

$= \int^{2\pi} 2 \left| \sin \frac{t}{2} \right| dt$ $\quad \therefore \sin^2 \frac{t}{2} = \frac{1 - \cos t}{2}$

$= \left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi}$

$= (-4\cos\pi) - (-4\cos 0)$

$= 4+4$

$= \underline{\underline{8}}$

ii) $y = \sqrt{4-x^2}$ $\quad x \in [-2,2]$

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$y = \sqrt{4-x^2}$ $\quad \therefore \frac{dy}{dx} = 2\int_0^2 1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2 dx$

$= 2\int_0^2 \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$

$= 4\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$

$= 4\left(\sin^{-1}(x/2)\right)_0^2$

$= 2\pi$

iii) $y = x^{3/2}$. in $[0,4]$

$f(x) = \frac{3}{2} x^{1/2}$

$[f'(x)]^2 = \frac{9}{4} x$

$L = \int_0^b \sqrt{1 + [f'(x)]^2}\, dx$

$$= \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx$$

$$= \int_0^4 \sqrt{\frac{4+9x}{4}}\, dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{1/2+1}}{\frac{1}{2}+1}\right]_0^4 dx$$

$$= \frac{1}{27}\left[(4+9x)^{3/2}\right]_0^4$$

$$= \frac{1}{27}\left[(4+0)^{3/2}-(4+36)^{3/2}\right]$$

$$= \frac{1}{27}\left[(4)^{3/2}-(40)^{3/2}\right]$$

iv) $x = 3\sin t \qquad y = 3\cos t$

$$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t$$

$$L = \int_0^{2\pi} \sqrt{(3\cos t)^2+(-3\sin)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(1)}\quad dt$$

$$= \int_0^{2\pi} 3\, dt \quad = 3\int_0^{2\pi} dt$$

$$= 3[x]_0^{2\pi}$$

$$= 3(2\pi - 0)$$

$$= 6\pi$$

v) $x = \frac{1}{6}y^3 + \frac{1}{2y} \qquad y = [1, 2]$

$$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} = \frac{y^4 - 1}{2y^2}$$

$$= \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y)^2}}\, dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$$

$$= \frac{1}{2}\int_1^2 y^2\,dy + \frac{1}{2}\int_1^2 y^{-2}\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right] = \frac{17}{12}$$

Q2) i) $\int_0^2 e^{x^2}\,dx$ with $n=4$

$a = 0 \quad b = 2 \quad n = 4$

$h = \frac{2-0}{4} = \frac{1}{2} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.2840	2.7182	9.4877	54.5981
	y_0	y_1	y_2	y_3	y_4

By simpsons rule,

$$\int_0^2 e^{x^2}\,dx = \frac{0.5}{3}\left[(1 + 54.5981) + 4(1.28400 + 9.4877) + 2(2.7182 + 54.5981)\right]$$

$$= \frac{0.5}{3}\left[55.5981 + 43.0868 + 114.6326\right]$$

$$= 1.1779$$

ii) $\int_0^4 x^2 \, dx$

$l = \frac{4-0}{4} = 1$

x	0	1	2	3	4		
y	0	1	4	9	16		

$\int_0^4 x^2 \, dx = \frac{1}{3}[16+4(10)+8]$

$= \frac{64}{3}$

$\int_0^4 x^2 \, dx = \underline{21.533}$

iii) $\int_0^{\pi/3} \sqrt{\sin x} \, dx \quad n = 6$

$l = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$6\pi/18$	$8\pi/18$
y	0	0.4166	0.58	0.70	0.80087	0.8727	0.9904

$\int_0^{\pi/3} \sqrt{\sin x} = {}'' x_3 \quad \pi/54 \times 12.1163$

$= \int_0^{\pi/3} \sqrt{\sin x} \, dx$

$= 0.7049$

Practical - 07

Topic: differential equation.

Q1) Solve the following :-

i) $x \frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$P(x) = \frac{1}{x}$ $\quad Q(x) = \frac{e^x}{x}$

$I.F = e^{\int 1/(x)\, dx}$

$= e^{\int 1/x\, dx}$

$= e^{\ln x}$

$I.F = x$

$y(I.F) = \int Q(x)\,(I.F)\, dx + c$

$= \int \frac{e^x}{x} \cdot x \cdot dx + c$

$= \iint e^x dx + 1$

$xy = e^x + c$

2)

3

2) $e^x \frac{dy}{dx} + 2e^x y = 1$.

$$\frac{dy}{dx} + \frac{2e^x}{e^x} y = \frac{1}{e^x}$$

$$\frac{dy}{dx} + 2y = \frac{1}{e^x}$$

$$P(x) = 2 \qquad Q = e^{-x}$$

$$IF = e^{\int 2dx}$$
$$= e^{2x}$$

$$y(IF) = \int Q(IF)\, dx + c$$
$$= \int \frac{1}{e^x}(e^x \cdot e^x)\, dx + c$$
$$= \int \frac{(e^{2x})}{e^x}\, dx + c$$

$$= \int e^x\, dx + c$$

$$\therefore y \cdot e^{2x} = e^x + c$$

3) $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$$\frac{x}{2} \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$$

$$\frac{dy}{dx} + \frac{2}{x} \times y = \frac{\cos x}{x^2}$$

$P = \frac{2}{x}$ $\quad Q = \cos x / x^2$

$IF = e^{\int P dx} = e^{\int 2/x \, dx} = e^{2 \log x} = e^{\log x^2}$

$= x^2$

4) $y(IF) = \int Q(IF)\, dx + C$

$\therefore y(x^2) = \int \frac{\cos x}{x^2} \cdot x^2 + C = \sin x + C$

$$x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$$

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$$

($\div$ by x on both sides)

$P(x) = \frac{3}{x}$ $\quad Q(x) = \frac{\sin x}{x^3}$

$= e^{\int P(x) dx}$

$= e^{\int 3/x \, dx}$

$= e^{3/x} \; x$

$= e^{\ln x^3}$

$I.F = x^3$

5) $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\therefore \frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$ dividing both sides by e^{2x}

$\therefore P = 2 \quad Q = 2x/e^{2x}$

$IF = e^{\int P dx} = e^{\int 2x dx / c} - e^{2x}$

$\therefore IF = e^{2x}$

$y(IF) = \int Q(IF)\,dx + c = \int 2x/e^{2x} \cdot e^{2x}\,dx + c$

$\therefore ye^{2x} = x + c$

6) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$

$\sec^2 x \tan y\,dx = -\sec^2 y \tan x\,dy$

$\frac{\sec^2 x\,dx}{\tan x} = \frac{-\sec^2 y\,dy}{\tan y}$

$\int \frac{\sec^2 x\,dx}{\tan x} = -\int \frac{\sec^2 y}{\tan y}\,dx$

$\therefore \log(\tan x) = -\log|\tan y| + C$

$\therefore \log|\tan x - \tan y| = c$

$\therefore \tan x \cdot \tan y = e^c.$

7) $\frac{dy}{dx} = \sin^2(x-y+1)$

Put $x-y+1 = v$

differentiating on both sides.

$$x-y+1 = v$$

$$1 - dy/dx = dv/dx$$

$$1 - \frac{dv}{dx} = dy/dx$$

$$1 - dv/dx = \sin^2 v$$

$$\frac{dv}{dx} = 1 - \sin^2 v$$

$$\frac{dv}{dx} = \cos^2 v$$

$$\frac{dv}{\cos^2 v} = dx$$

$$\int \sec^2 v\, dx = \int dx$$

$$\tan v = x + c$$

$$\therefore \tan(x+y-1) = x+c$$

8) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Put $2x+3y = v$

$$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$$

$$dy/dx = 1/3\,(dv/dx - 2)$$

$$1/3\,(dv/dx - 2) = 1/3\,\frac{(v-1)}{(v+2)}$$

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$$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$$

$$= \frac{3v+2}{v+2}$$

$$= \frac{3(v+1)}{v+2}$$

$$\int \frac{(v+2)}{(v+1)} dv = 3dx$$

$$\int \frac{v+1}{v} dx + \int \frac{1}{v+1} dv = 3x$$

$$\therefore v + \log|x| = 3x + c$$

$$\therefore 2x + 3y + \log|2x+3y+1| = 3x + c$$

$$\therefore 3y = x - \log|2x+3y+1| + c.$$

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Practical - 08

Topic : Euler's method

1) $\frac{dy}{dx} = y + e^{x} - 2$ $\quad y(0) = 2$, $h = 0.5$ find $y(2)$

2) $\frac{dy}{dx} = 1 + y^2$ $\quad y(0) = 0$, $h = 0.2$ find $y(1)$

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $\quad y(0) = 1$, $h = 0.2$ find $y(1)$

4) $\frac{dy}{dx} = 3x^2 + 1$ $\quad y(1) = 2$ find $y(2)$ for $h = 0.5$, $h = 0.25$

5) $\frac{dy}{dx} = \sqrt{xy} + 2$ $\quad y(1) = 1$ find $y(1.2)$ with $h = 0.2$

Answers:-

1) $\frac{dy}{dx} = y + e^{x-2}$

$y_0 = 2$, $x_0 = 0$, $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.487	3.57435
2	1	3.5743	4.2925	5.3615

$y_{n+1} = y_n + h f(x_n + y_n)$

n
3
4
2)

3	1.5	5.3615	7.8431	
4	2	9.2831		9.283064

$\therefore$ By euler's formula,

$y(2) = 9.2831$

2) $\frac{dy}{dx} = 1 + y^2$

$f(x,y) = 1 + y^2$, $y_0 = 0$, $x_0 = 0$, $h = 0.2$

Using euler's iteration formula.

$y_{n+1} = y_n + h\, f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8630	1.2942
5	1	1.2442		

$\therefore$ By euler's formula

$y(1) = 1.2942$

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $\quad y(0) = 1 \quad x_0 = 0 \quad h = 0.2$

using Euler's iteration formula,

$y_{n+1} = y_n + h f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	0
1	0.2	0		
2	0.4			
3	0.6			
4	0.8			
5	1			

4) $\frac{dy}{dx} = 3x^2 + 1 \quad y_0 = 2, \quad x_0 = 1 \quad h = 0$

for $h = 0.5$

Using eulers iteration formula,

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	$y(n+1)$
0	1	2	4	4
1	1.5	4	49	28.5
2	2	28.6		

By euler's formula,
$y(2) = 28.6$

for $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4219
2	1.5	4.4219	7.75	6.3594
3	1.75	6.3594	10.1815	8.9048
4	2	8.9048		

$\therefore$ By eulers formula

$y(2) = 8.9048$

5) $\frac{dy}{dx} = \sqrt{xy} + 2 \qquad y_0 = 1, \; x_0 = 1, \; h = 0.2$

Using euler's iteration formula,

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	1.6
1	1.2	1.6		

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$\therefore$ By euler's formula,

$y(1.2) = 1.6$

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Practical - 09

Topic: Limits and Partial Order Derivatives.

Q1 Evaluate the following limits:

1) $\lim_{(x,y)\to(-4,-1)} \dfrac{x^3-3y+y^2-1}{xy+5}$

at $(-4,-1)$ denominator $\neq 0$

$\therefore$ by applying limit

$= \dfrac{(-4)^3-3(-1)+(-1)^2-1}{-4(-1)+5}$

$= \dfrac{-64+3-1}{4+5}$

$= \dfrac{-61}{9}$

2) $\lim_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$

at $(2,0)$, denominator $\neq 0$

$\therefore$ by applying limit,

$= \dfrac{(0+1)((2)^2+0-4(2))}{2+0}$

$= \dfrac{1(4+0-8)}{2}$

$= \dfrac{-4}{2} = -2$

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3) $\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3-x^2yz}$

at $(1,1,1)$, denominator $= 0$

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$$

$$= \lim_{(x,y,z)\to(1,1,1)} \frac{(x-yz)(x+yz)}{x^2\,(x-yz)}$$

$$= \lim_{(x,y,z)\to(1,1,1)} \frac{x+yz}{x^2}$$

on applying limit

$$= \frac{1+1(1)}{(1)^2} = 2$$

Q2).

1) $f(x,y) = xye^{x^2+y^2}$

$\therefore f_x = \dfrac{\partial}{\partial x}(f(x,y))$

$= \dfrac{\partial}{\partial x}(xye^{x^2+y^2})$

$= ye^{x^2+y^2}(2x)$

$\therefore f_x = 2xye^{x^2+y^2}$

$f_y = \dfrac{\partial}{\partial y}(f(x,y))$

$= \dfrac{\partial}{\partial y}(xye^{x^2+y^2})$

$= xe^{x^2+y^2}(2y)$

2) $f(x,y) = e^{x}\cos y$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(e^{x}\cos y)$

$\therefore f_x = e^{x}\cos y$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(e^{x}\cos y)$

$\therefore f_y = -e^{x}\sin y$

3) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(x^3y^2 - 3x^2y + y^3 + 1)$

$\therefore f_x = 3x^2y^2 - 6xy$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(x^3y^2 - 3x^2y + y^3 + 1)$

$\therefore f_y = 2x^3y - 3x^2 + 3y^2$

Q3)

1) $f(x,y) = \frac{2x}{1+y^2}$

$f_x = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$

$$= \frac{1+y^2 \frac{\partial}{dx}(2x) - 2x\frac{d}{dx}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{2+2y^2-0}{(1+y^2)^2}$$

$$= \frac{2(1+y^2)}{(1+y^2)(1+y)^2}$$

$$= \frac{2}{1+y^2}$$

at $(0,0)$ $= \frac{2}{1+0} = 2$

$$f(y) = \frac{d}{dy}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{d}{dx}(2x) - 2x\frac{d}{dx}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

at $(0,0)$, $= \frac{-4(0)(0)}{(1+0)^2} = 0$

Q4)

1) $f(x,y) = \frac{y^2-xy}{x^2}$

$$fx = \frac{x^2 \frac{d}{dx}(y^2-xy) - (y^2-xy)\frac{d}{dx}(x^2}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$$

$$= \frac{-x^2y - 2x(y^2 - xy)}{x^4}$$

$$fy = \frac{2y - x}{x^2}$$

$$fxx = \frac{d}{dx}\left(\frac{-x^2y - 2x(y^2 - xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{d}{dx}(-x^2y - 2xy^2 + 2x^2y)\right) - (-x^2y - 2xy + 2x^2y)\frac{d}{dx}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2y - 2xy + 2x^2y)}{x^6}$$

$$fyy = \frac{d}{dy}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{2 - 0}{x^2} = \frac{2}{x^2}$$

$$fxy = \frac{d}{dy}\left(\frac{-x^2y - 2xy^2 + 2x^2y}{x^4}\right) \text{ — ②}$$

$$= \frac{\cancel{-x^2} - 4xy + 2x^2}{x^4} \text{ — ③}$$

$$fyx = \frac{d}{dx}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{x^2\frac{d}{dx}(2y - x) - (2y - x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

fy =

fyy

fx

fy

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} \longrightarrow (4)$$

from 3 and 4
$f_{xy} - f_{yx}$

2) $f(x,y) = x^3 - 3x^2y^2 - \log(x^2+1)$

$$f_x = \frac{d}{dx}(x^3 + 3x^2y^2 - \log(x^2+1))$$

$$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$$

$$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1 \frac{d(2x)}{dx} - 2x \frac{d(x^2+1)}{dx}}{(x^2+1)^2} \right)$$

$$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2} \right) \longrightarrow (1)$$

$$f_y = \frac{d}{dy}(x^3 + 3x^2y^2 - \log(x^2+1))$$

$$= 0 + 6x^2y - 0$$

$$= 6x^2y$$

$$f_{yy} = \frac{d}{dy}(6x^2y)$$

$$= 6x^2$$

$$f_{xy} = \frac{d}{dy}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$$

$$= 0 + 12xy - 0$$

$$= 12xy \longrightarrow (3)$$

$$f_{yx} = \frac{d}{dx}(6x^2y)$$

$$= 12xy$$

From (3) and (4) $\longrightarrow$ (4)

$\therefore f_{xy} = f_{yx}$

3) $f(x,y) = \sin(xy) + e^{x+y}$

$fx = y\cos xy + e^{x+y}(1)$

$= y\cos xy + e^{x+y}$

$fy = x(\cos xy + e^{x+y}(1)$

$= x\cos xy + e^{x+y}$

$\therefore fxx = \dfrac{d}{dx}(y\cos(xy) + e^{x+y})$

$= -y\sin(xy)(y) + e^{x+y}$

$= -y^2\sin(xy) + e^{x+y}$ —— 1

$fyy = \dfrac{d}{dx}(x\cos x(xy) + e^{x+y})$

$= -x\sin(xy)(x) + e^{x+y}$ —— 2

$= -x^2\sin xy + e^{x+y}$

$fxy = \dfrac{d}{dy}(y\cos(xy) + e^{x+y})$

$= -y^2\sin(xy) + \cos(xy) + e^{x+y}$ —— 3.

$fyx = \dfrac{d}{dx}(x\cos(xy) + e^{x+y})$

$= -x^2\sin(xy) + \cos(xy) + e^{x+y}$ —— 4.

$\therefore$ from 3 and 4

$fxy \neq fyx$

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Q5) (i)

$f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$ $\qquad f_y = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{x}{\sqrt{x^2+y^2}}$ $\qquad = \frac{y}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$ $\qquad f_y$ at $(1,1) = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{2}{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

ii $f(x,y) = 1-x+y\sin x$ at $(\pi/2, 0)$

$f(\pi/2, 0) = 1-\frac{\pi}{2}+0 = 1-\frac{\pi}{2}$

$fx = 0-1+y\cos x$ $\qquad$ $fy = 0-0+\sin x$

fx at $(\pi/2, 0 = -1+0$
$= -1$

fy at $(\pi/2, 0) = \sin\frac{\pi}{3} = 1$

$L(x,y) = f(a,b)+fx(a,b)(x-a)+fy(a,b)(y-b)$
$= 1-\frac{\pi}{2}+(-1)(x-\frac{\pi}{2})+1(y-0)$
$= 1-\cancel{\frac{\pi}{2}}-x+\cancel{\frac{\pi}{2}}+y$
$= 1-x+y$

iii) $f(x,y) = \log x+\log y$ at $(1,1)$
$f(1,1) = \log(1)+\log(1) = 0$

$fx = \frac{1}{x}+0$ $\qquad$ $fy = 0+\frac{1}{y}$

fx at $(1,1) = 1$ $\qquad$ fy at $(1,1) = 1$

$\therefore L(x,y) = f(a,b)+fx(a,b)(x-a)+fy(a,b)(y-b)$
$= 0+1(x-1)+1(y-1)$
$= x-1+y-1$
$= x+y-2$

Q1)
i)

Practical - 10

Topic: Directional derivative, gradient vector, maxima, minima, tangent and normal vector.

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Q1)

i) $f(x,y) = x+2y-3$ $a = (1,-1)$ $u = 3i-j$

→ Here,

$u = 3i-j$ is not a unit vector

$\bar{u} = 3\bar{i} - \bar{j}$

$|u| = \sqrt{10}$

∴ unit vector along u is $\frac{\bar{u}}{|u|} = \frac{1}{\sqrt{10}}(3i-j)$

$= \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

Now,

$f(a+hu) = f\left((1,-1)+h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= 1+2(-1)-3+\left(\frac{3h}{\sqrt{10}}, \frac{-h}{\sqrt{10}}\right)+\left(1+\frac{3h}{\sqrt{10}}, -1-\frac{h}{\sqrt{10}}\right)$

$= 4+\frac{3h}{\sqrt{10}} - 4 \quad 1+\frac{3}{\sqrt{10}} + 2\left(-1-\frac{h}{\sqrt{10}}\right) - 3$

$= 1-2-3+\frac{3h}{\sqrt{10}} - \frac{2h}{\sqrt{10}}$

$= -4+\frac{h}{\sqrt{10}}$

$\therefore D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$

$= \lim_{h\to 0} \frac{-4+\frac{h}{\sqrt{10}}-(-4)}{h}$

$= \lim_{\substack{h \to 0}} \frac{h}{h\sqrt{10}}$

$= \frac{1}{\sqrt{10}}$

ii) $f(x,y) = y^2 - 4x + 1 \qquad a = (3,4) \qquad u = i + 5j$

Here,

$u = i + 5j$ is not a unit vector

$\bar{u} = \bar{i} + 5\bar{j}$

$|\bar{u}| = \sqrt{26}$

$\therefore$ unit vector along u is $\frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{26}}(\bar{i} + 5\bar{j})$

$= \frac{1}{\sqrt{26}}(1, 5)$

$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

Now,

$f(a+hu) = f\left[(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right]$

$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$

$= \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$

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$$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5 - 5}{h}$$

$$= \lim_{h\to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$= \frac{25(0)}{26} + \frac{36}{\sqrt{26}}$$

$$= \frac{36}{\sqrt{26}}$$

ii $f(x,y) = 2x + 3y$

$a = (1, 2)$, $u = 3i + aj$

Here,

$u = 3i + 4j$ is not a unit vector

$\bar{u} = 3\bar{i} + 4\bar{j}$

$|\bar{u}| = \sqrt{25} = 5$

$\therefore$ unit vector along $u = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{5}(3\bar{i} + 4\bar{j})$

$= \frac{1}{5}(3,4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

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Now,

$f(a+hu) = f((1,2) + h(\frac{3}{5}, \frac{4}{5}))$

$= f(1 + \frac{3h}{5}, 2 + \frac{4h}{5})$

$= 2(1 + \frac{3h}{5}) + 3(2 + \frac{4h}{5})$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= 8 + \frac{18h}{5}$

$\therefore D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(u)}{h}$

$= \lim_{h \to 0} \frac{8 + \frac{18h}{5} - 8}{h}$

$= \lim_{h \to 0} \frac{18h}{5h}$

$= \frac{18}{5}$

Q2

i $f(x,y) = x^y + y^x$ $a = (1,1)$

$f_x = y(x^{y-1}) + y^x \log y$

$f_y = x(y^{x-1}) + x^y \log x$

$\nabla f(x,y) = f_x, f_y$

$= (y x^{x-1} + y^2 \log y, \; x y^{x-1} + x^y \log x)$

$\nabla f(x,y)$ at $(1,1)$

$= (1(1)^0 + 1^1 \log(1), \; 1(1)^{(1-1)} + 1^1 \log 1)$

$= (1,1).$

ii) $f(x,y) = (\tan^{-1} x) \cdot y^2$

$a = (1,-1)$

$f_x = y^2 \left(\frac{1}{1+x^2}\right) = \frac{y^2}{1+x^2}$

$f_y = 2y \tan^{-1} x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\frac{y^2}{1+x^2}, \; 2y \tan^{-1} x\right)$

$\nabla f(x,y)$ at $(1,-1)$

$= \left(\frac{(-1)^2}{1+(1)^2}, \; 2(-1) \tan^{-1}(1)\right)$

$= \left(\frac{1}{2}, \; \frac{-2\pi}{4}\right)$

$= \left(\frac{1}{2}, \frac{\pi}{2}\right)$

iii) $f(x,y,z) = xyz - e^{x+y+z}$ $\quad a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

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$= (yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z})$

$\nabla f(x,y,z)$ at $(1,-1,0)$

$= (-1(0) - e^{1-1+0}, 1(0) - e^{1-1+0}, 1(-1) - e^{1-1+0})$

$= (0-1, 0-1, -1-1)$

$= (-1, -1, 2)$

Q3)

i) $x^2 \cos y + e^{xy} = 2$

at $(1,0)$

$f(x,y) = x^2 \cos y + e^{xy} - 2$

$f_x = 2x \cos y + y e^{xy}$

$f_y = -x^2 \sin y + x e^{xy}$

$(x_0, y_0) = (1,0)$

f_x at $(1,0) = 2(1) \cos 0 + 0$

$= 2$

f_y at $(1,0) = -(1)^2 \sin(0) + 1(e)^{1(0)}$

$= 1$

$f_x(x - x_0) + f_y(y - y_0) = 0$

$\therefore 2(x-1) + 1(y-0) = 0$

$\therefore 2x - 2 + y = 0$

$\therefore 2x + y - 2 = 0 \rightarrow$ eqn of tangent.

NO
fo

ii) x

f
f

f

for

Now,
for equation of normal;

$$bx + ay + d = 0$$
$$x + 2y + d = 0$$
$$(1) + 2(0) + d = 0$$
$$1 + d = 0 \qquad \text{at } (1,0)$$
$$d = -1$$
$$x + 2y - 1 = 0 \rightarrow \text{equation of normal}$$

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$
at $(2,-2)$

$$f(x,y) = x^2 + y^2 - 2x + 3y + 2$$
$$f_x = 2x + 0 - 2 + 0 + 0 \qquad f_x \text{ at } (2,-2) = 2(2) - 2$$
$$= 2x - 2 \qquad = 2$$
$$f_y = 0 + 2y - 0 + 3 + 0$$
$$= 2y + 3 \qquad f_y \text{ at } (2,-2) = 2(-2) + 3$$
$$= -1$$

∴ equation of tangent

$$f_x(x - x_0) + f_y(y - y_0) = 0$$
$$2(x - 2) + (-1)(y + 2) = 0$$
$$2x - 4 - y - 2 = 0$$
$$2x - y - 6 = 0 \rightarrow \text{eq}^n \text{ of tangent}$$

for equation of normal at $(2,-2)$

$$bx + ay + d = 0$$
$$-x + 2y + d = 0$$
$$-(2) + 2(-2) + d = 0$$
$$-24 - 4 + d = 0$$
$$d = +6$$

$\therefore -x+2y+6=0 \rightarrow$ equ of normal

Q4)

i) $x^2-2yz+3y+xz=7$ at $(2,1,0)$

$f(x,y,z) = x^2-2yz+3y+xz-7$

$f_x = 2x-0+0+z-0$

$= 2x+z$

$\therefore f_x$ at $(2,1,0) = 2(2)+0$

$= 4$

$f_y = -2z+3+0-0$

$= -2z+3$

f_y at $(2,1,0) = -2(0)+3$

$= 3$

$f_z = 0-2y+0+x-0$

$= -2y+x$

f_z at $(2,1,0) = -2(1)+2$

$= 0$

Equation of tangent;

$f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$

$4(x-2)+3(y-1)+0(z-0)=0$

$4x-8+3y-3=0$

$\therefore 4x+3y-11=0$

Equation of normal,

$$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$$

Equation of normal,

$$\frac{x-x_0}{fx} = \frac{y-y_0}{fy} = \frac{z-z_0}{fz}$$

$$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z-0}{0}$$

) $3xyz - x - y + z = 4$

at $(1, -1, 2)$

$f(x,y,z) = 3xyz - x - y + z + 4$

$fx = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

fx at $(1, -1, 2) = 3(-1)(2) - 1$

$= -7$

$fy = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1$

fy at $(1, -1, 2) = 3(1)(2) - 1$

$= 5$

$fz = 3xy - 0 - 0 + 1 - 0$

$= 3xy + 1$

fz at $(1, -1, 2) = 3(1)(-1) + 1$

$= -2$

Equation of tangent

$fx(x - x_0) + fy(y - y_0) + fz(z - z_0) = 0$

$\therefore -7(x-1) + 5(y+1) + (-2)(z-2) = 0$

$\therefore -7x + 7 + 5y + 5 - 2z + 4 = 0$

$\therefore -7x + 5y - 2z + 16 = 0$

Equation of normal;

$$\frac{x-x_0}{fx} = \frac{y-y_0}{fy} = \frac{z-z_0}{fz}$$

$$\frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$

Q5)

i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$\therefore f_x = 6x + 0 - 3y + 6 - 0$

$= 6x - 3y + 6$ —— ①

$f_y = 2y - 3x + 0 - 4$

$= 2y - 3x - 4$ —— ②

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + z) = 0$

$2x - y + z = 0$

$2x - y = -z$ —— ③

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4$ —— ④

multiplying ③ by 2 and subtracting ④ from ③.

$$\begin{array}{r} 4x - 2y = -4 \\ -\ \ 2y - 3x = 4 \\ \hline 7x = 0 \\ x = 0 \end{array}$$

Substituting value of x in 3

$2(0) - y = -2$

$-y = -2$

$y = 2$

$\therefore$ critical points are $(0,2)$

Now,

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

$rt - s^2 = 12 - 9$

$= 3 > 0$

here, $r > 0$ and $rt - s^2 > 0$

$\therefore$ f has minimum at $(0,2)$

$\therefore f(0,2) = 3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$= 0 + 4 - 0 + 0 - 8$

$= -4$

$f(x,y) = 2x^4 + 3x^2y - y^2$

$f_x = 8x^3 + 6xy - 0$

$= 8x^3 + 6xy$

$f_y = 0 + 3x^2 - 2y$

$= 3x^2 - 2y$

Now,

$f_x = 0$

$8x^3 + 6xy = 0$

$2x(4x^2 + 6xy) = 0$

$4x^2 + 6xy = 0$ — ①

$f_y = 0$

$3x^2 - 2y = 0$ — ②

Multiply eqn ① by 3 and ② by 4 and subtracting ② from ①:

$12x^2 + 18y = 0$

$-12x^2 - 8y = 0$

$24y = 0$

$\boxed{y = 0}$ — ③

Substituting ③ in ②, we get

$3x^2 - 2(0) = 0$

$3x^2 = 0$

$x^2 = 0$

$\boxed{x = 0}$ — 4

∴ Critical points are $(0, 0)$

Now,

$r = f_{xx} = 24x^2 + 6y$

$t = f_{yy} = -2$

$s = f_{xy} = 6x$

$rt - s^2 = (24x^2 + 6y)(-2) - (6x)^2$

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$= -48x^2 - 12y - 36x^2$

$= -84x^2 - 12y$

at $(0,0)$

$r = 24(0)^2 + 6(0)$

$= 0$

$s = 6(0) = 0$

$rt - s^2 = -84(0)^2 - 12(0) = 0$

$r = 0$ and $rt - s^2 = 0$

$\therefore$ nothing can be said

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