Coast of a fuzzy set as a crisper subset of the boundary

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Abstract

Calculations such as intersection matrices involving fuzzy boundary face a major drawback in the sense that fuzzy boundary does not clearly demarcate between a set and its complement, or the interior and the exterior. To overcome this, we define coast of a fuzzy set consisting of points having membership $\frac{1}{2}$. Similar subsets have been used previously by Shi and Liu but without specifically mentioning the set or stating its properties. We further explore topological properties of the coast and its alternative definitions, and demonstrate using 9-intersection matrices how better results can be achieved using the coast in place of previously used height, fringe or boundary.

1 Introduction and Motivation

While modeling natural objects, we occasionally come across situations where it is impossible to distinguish two objects from each other. Despite their visible differences, their partition becomes critical due to the natural vagueness of the boundary. Naturally, these objects are not distinguishable using a crisp line; hence, several methods have been employed for the task [?]. The mathematical description of the vague/uncertain object as the fuzzy set has been embraced by the majority of researchers subsequent to the introduction of the fuzzy set by Zadeh [Za]. Rather than having complete inclusion or exclusion the objects are allowed to have partial belongingness specified by membership values.

On the other hand, in several real-life applications of fuzzy sets, existing studies distinguish the fuzzy objects with the help of membership value only. These types of studies are widespread in the literature with their diverse applications, e.g., [Bj, TaKa, TaKaWa]. Ambiguity arises when trying to point out the boundary between two geographical objects that are fuzzy in nature, whereas for crisp objects with exact belongingness there is a sharp demarcation called the boundary. In a geographic information system (GIS), the boundary plays a significant role in determining the interaction of an object with another object. GIS is a database system storage, manipulation, analysis and management of geographical data. The 9-intersection model gives topological relations between two fuzzy objects or regions, but were essentially based on point-set topology [Eg]. Generalizations further determine relationship between uncertain objects, objects in 3 dimensions, spatio-temporal objects, spatial objects with holes, etc [TaKa]. The boundary definition of the objects plays a vital role in the construction of these algebraic models. After the inception of the fuzzy set theory, several authors attempted to theoretically define the fuzzy boundary with the help of fuzzy topology. The existing concepts of the fuzzy boundary are rarely used in real-life applications, except in the fuzzy 9-intersection matrix, which is designed to determine the topological relation between the vague spatial objects in GIS.

Our main objective in this article is to bridge the gap between these two. Hence, we first review the application-related studies that have the practical implementation of the concept of boundary (these studies have not used any definition of the fuzzy boundary but they utilized the membership function to distinguish between the fuzzy sets without introducing any new definition). Here, we study those sets and define them using fuzzy topology, and then we compare them with each other and with the fuzzy boundary. However, the real usability of such a construction lies in its practical implementation where fuzzy boundary fails to solve

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much purpose except for the 9-intersection matrices. Instead, the gap between definition and practical use can be overcome by using those points as a boundary whose membership value is just $\frac{1}{2}$.

Thus the boundary is a key ingredient to study the topological relations between two spatial objects or regions, and their extent of interaction. At first, studies on topological relations between regions focused on non-fuzzy situations. Examination for the fuzzy cases opened a new domain here [TaKa]. Those models, which are described by the topological relations between two regions with a crisp boundary or simply between polygons are concerned mainly with the general cases of spatial objects. The topological relations are not always a matter of region-to-region. Cases have been come across for fuzzy line-to-region with special appearance in GIS problems. An example of fuzzy topological relations between lines and regions is study of the effect of Severe Acute Respiratory Syndrome (SARS) on a community [ShLi]. The steps are:

- to trace path of an infected person (the result is a fuzzy line), and
- to investigate the effect of this person's movements (as a fuzzy line) through the community (as a fuzzy region).

In previous studies on fuzzy relations between uncertain objects, a GIS object is normally partitioned as two parts: an interior certain part with a membership value as one, and an uncertain boundary part with a membership of between zero and one. A value is then assigned to each part. Problems arising from this description are:

- It may be difficult to determine the spatial distribution of the uncertainties for each object in GIS.
- We only have a one-value result on the relations (topological or other) of each intersection part when we compare two fuzzy objects in GIS.
- There is little information on the effect of a fuzzy object on another in this description.

An essential object of interest is a demarcation between a spatial object and its exterior. For a sharper demarcation we usually have to go with a crisper description of the topology which renders the study practically useless for many real-life purposes. On the other hand, a vague demarcation hinders our identfication of objects of study as separate entities. Thus, it is desirable to have a crisp demarcation and it serves our purpose if that does not affect our data regarding fuzziness. This is a problem our paper aspires to solve for good, and also show many feasible applications of our approach for GIS and image enhancement.

The aforesaid problem of demarcating between the interior of a fuzzy spatial object and its exterior has previously been addressed in the definitions of *edge* and *fringe* [TaKaWa], and this method has been in use for increasing image clarity, map demarcation etc. However, the main problem in edge detemination is identifying sets that are *open* in an induced fuzzy topology. Two problems mostly motivated our research to characterize the members of a fuzzy set having membership exactly or 'somewhat' $\frac{1}{2}$, and these are briefly described in the subsequent part of this section. We define the coast of a fuzzy set as the subset containing exactly those points whose membership is $\frac{1}{2}$ and discuss some of its topological properties in the context of Warren's postulates regarding an ideal fuzzy boundary. Finally, we redefine the 9-intersection matrix by replacing the existing fuzzy boundary with the coast and determine the topological relation using the proposed one.

The rest of the paper is arranged as follows:

1. The preliminaries section notes down the ideas that will be needed for our further calculations, which are beyond standard coursework.
2. We then briefly describe two related studies which establishes the usability of our proposed construction.
3. We introduce the coast of a fuzzy set and describe its set-theoretic properties. We further study how much the coast fits in the framework of Warren's hypothesized properties for a fuzzy boundary. The coast is then shown to be a subset of the fuzzy boundary, often proper.
4. In the next section, we show how the coast fits in the problem of demarcating in a crisp manner between the interior of a fuzzy set and its exterior. Further, a separate study of the properties of the coast helps in simplifying the approaches described in [ShLi] and [Li]. We also describe some intricacies of its application in practical mapmaking with easy examples.

5. In the final section we show how the coast can replace the boundary and fringe of a fuzzy region with better results.

6. Finally, concluding reamrks are given followed by future research ideas.

2 Preliminaries

Fuzzy set theory can be seen as a generalization of crisp set theory, where the degree of membership μ of an element x in a set A can take values throughout the continuum between 0 (not belonging) and 1 (belonging). Thus, for a generic element $x \in A$, the membership of x in A is denoted by $\mu_A(x)$. We call $\mu_A : A \to [0,1]$ as the *membership function* of the fuzzy set (A, μ_A). This enables us to define a topology on fuzzy set, whence we can generalize notions from point-set topology to have a wide range applications to realistic problems apart from mathematical depth.

Let X be the universal set.

- A is an *empty* fuzzy set [denoted by 0_X] in X iff $\mu_A(x) = 0 \; \forall \; x \in X$. The whole fuzzy set $A = X$ is denoted by 1_X.
- Two fuzzy sets A and B are *equal*, i.e., $A = B$ iff $\mu_A(x) = \mu_B(x) \; \forall \; x \in X$.
- *Complement* of a fuzzy set A is denoted by A^C and is defined by $\mu_{A^C} = 1 - \mu_A$.
- A fuzzy set A is *contained* in a fuzzy set B or A is a *subset of* B or A is *smaller than or equal to* B iff $\mu_A(x) \leq \mu_B(x) \; \forall_x \in X$.

Definition 2.1. *Let (A, μ_A) and (B, μ_B) be two fuzzy sets.*

1. *Their* union *is a fuzzy set $C = A \cup B$ whose membership function μ_C is given by $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$ $\forall \; x \in X$, or, $\mu_C = \mu_A \vee \mu_B$. Alternatively, The* union *of fuzzy sets A and B is the smallest fuzzy set that covers both A and B.*

2. Intersection *$C = A \cap B$ has membership function*

   $\mu_C = \min\{\mu_A, \mu_B\}$ *or,* $\mu_C = \mu_A \wedge \mu_B$.

   Alternatively, the intersection $A \cap B$ is the largest fuzzy set which is contained in both A and B. Two fuzzy sets A and B are disjoint *if $A \cap B$ is empty.*

Often, we don't mention the membership functions explicitly, and A would mean the fuzzy set (A, μ_A). In that case we denote union and intersection of two fuzzy sets A and B by $A \vee B$ and $A \wedge B$ respectively.

Point-set topological notion of *interior* of a set A, A° refers to the collection of those points x for which $\exists \; G_x^{\text{open}} \ni x \in G_x \subseteq A$. The *closure* of A, denoted by $\overline{A}$ is the intersection of all the closed sets containing A [Mu]. The fact whether two fuzzy sets are separated or connected plays a vital role in determining the extent of interaction between the respective regions in the study of geography. For that, definitions of separatedness and connectedness in crisp topology can be variously extended to fuzzy sets, as in [TaKa].

Definition 2.2. *[Ch] A family τ of fuzzy sets in X is called a* fuzzy topology, *and (X, τ) is called a* fuzzy topological space *or* fts *in short, if*

1. $0_X, 1_X \in \tau$;

2. $\{A_i\}_{i=1}^n \; [n \in \mathbb{N}] \in \tau \Rightarrow \bigwedge_{i=1}^n A_i \in \tau$;

3. $\{A_\lambda\}_{\lambda \in \Lambda} \in \tau \Rightarrow \bigvee_{\lambda \in \Lambda} A_\lambda \in \tau$.

Definition 2.3. *Fuzzy sets A, B in the fts (X, τ) are said to be* separated *iff $\exists \; U, V \in \tau \ni U \supseteq A, V \supseteq B, U \wedge B = V \wedge A = 0_X$.*

Definition 2.4. *Fuzzy sets A, B in the fts (X, τ) are said to be* Q-separated *iff* $\exists\ H, K \ni H^C, K^C \in \tau; H \supseteq A, K \supseteq B, H \wedge B = K \wedge A = 0_X$.

We can then define the notion of topological invariance in the following way.

Definition 2.5. *Let $(X, \tau), (Y, \upsilon)$ be fts. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be* F-continuous *if* $\forall\ A \in \upsilon$, $f^{-1}(A) \in \tau$. *A* fuzzy homeomorphism *is an F-continuous injective map from an fts X onto an fts Y, such that the inverse of the map is F-continous as well. In that case X and Y are said to be* F-homeomorphic *and each is a* fuzzy homeomorph *of the other.*

Definition 2.6. *[TaKa] A fuzzy relation is said to be a* topological relation *if it is preserved under a fuzzy homeomorphism of its embedding fts.*

The boundary for a fuzzy region has many, including the following, alternative definitions.

Definition 2.7. *(B1) [Wa] The* fuzzy boundary *of a fuzzy set A is the infimum of all closed fuzzy sets D in X such that $\mu_D(y) \geq \mu_{\overline{A}}(y)\ \forall\ y \in \left\{ x \in X \mid \mu_{\overline{A}}(x) \wedge \mu_{\overline{A^C}}(x) > 0 \right\}$.*

(B2) [PaYi] The fuzzy boundary *of a fuzzy set A is defined as: $\partial A = \overline{A} \wedge \overline{A^C}$.*

(B2′) [Bj] The membership function of the fuzzy boundary *∂A of A is defined as $\mu_{\partial A}(x, y) = 2 \min \left\{ \mu_A(x, y), 1 - \mu_A(x, y) \right\}$, where the factor 2 normalizes the membership values.*

(B3) [CuTa] The boundary of A is the infimum of all closed fuzzy sets B in X such that $B(x) \geq \overline{A}(x)\ \forall\ x \in X \ni \overline{A}(x) - A^\circ(x) > 0$.

(B4) [Gu] $\partial A = \vee \left\{ x_\beta \mid \beta \leq \overline{A}(x), \beta > A^\circ(x) \right\}$.

(B5) [MaDa] $\partial A = \left(\overline{A}\right)^\circ \wedge \overline{A^\circ}$.

Richard Warren [Wa] postulated some conditions a reasonably good definition for fuzzy boundary (∂A) must satisfy [though all the definitions don't satisfy all the properties; the definitions that satisfy a particular property are written in brackets, and is summarized in Table 1]:

(W1) ∂A is closed. [(B1), (B2), (B3), (B4)]

(W2) The closure is the supremum of the interior and the boundary, i.e., $\overline{A} = A^\circ \vee \partial A$. [(B1), (B2), (B3), (B4)]

(W3) This fuzzy definition of boudary becomes the usual topological boundary when the sets concerned are crisp. [(B1), (B2), (B3), (B4)]

(W4) The boundary operator is an equivalent way of defining a fuzzy topology. [(B1), (B2), (B3), (B4)]

(W5) The boundary of a fuzzy set is identical to the boundary of the complement of the set. [(B2), (B4)]

(W6) If a fuzzy set is closed (or open), then the interior of the boundary is empty. [(B3)]

(W7) If a fuzzy set is both open and closed, then the boundary is empty.

Since a major part of the problems we shall deal with are essentially of a geographical point of view, the definition of a fuzzy region will be useful in the course of this paper. But mostly we shall not go in all those rigorous details and shall deal with a simple structure of a fuzzy region. Needless to say, the details are just as fine with our results with some tedious checking. Before a fuzzy region, the following need to be defined.

Subsets of the boundary ∂A: [TaKa]

- c-boundary, $\partial^c A = \left\{ x_\lambda \in \partial A : \mu_{\partial A}(x) = \mu_{\overline{A}}(x) \right\}$;
- i-boundary, $\partial^i A = \left\{ x_\lambda \in \partial A : \mu_{\partial A}(x) < \mu_{\overline{A}}(x) \right\}$.

Subsets of the closure $\overline{A}$:

- i-closure, $A^{\pm} = \left\{x_\lambda \in \overline{A} \mid \mu_{\partial A}(x) < \mu_{\overline{A}}(x)\right\}$;
- c-closure, $A^{\mp} = \left\{x_\lambda \in \overline{A} \mid \mu_{\partial A}(x) = \mu_{\overline{A}}(x)\right\}$.

Crisp subsets:

- Core, $A^{\oplus} = \left\{x_\lambda \in X \mid \mu_{A^\circ}(x) = 1\right\} \subseteq A^\circ$;
- Outer, $A^{=} = \left\{x_\lambda \in A^e \mid \mu_{A^e}(x) = 1\right\} \subseteq A^e$, where $A^e = \left(A^C\right)^\circ$.

Other subsets:

- b-closure of A in the fts, $A^{\perp} = A^{\pm} - A^{\oplus}$;
- fringe of A, $\ell A = \partial^c A \cup A^{\perp}$.

Definition 2.8. *[TaKa] A* simple fuzzy region *is a non-empty subset of an fts which is a double connected regular closed set whose core is the interior of a simple closed region, the i-closure is a non-empty double connected regular open set, the interior of the b-closure is a non-empty double-connected regular open set, the c-boundary is a non-empty double-connected closed set and the outer is a non-empty double-connected open set.*

Properties	$B1$	$B2$	$B3$	$B4$	$B5$
	$\partial_1 A = \inf \left\{D^{\text{closed}} \subseteq X \ : \mu_D(y) \geq \mu_{\overline{A}}(y)\right\}$	$\partial_2 A = \overline{A} \wedge \overline{A^C}$	$\inf B^{\text{closed}} \subseteq X$ for which $B(x) \geq \overline{A}(x)$ $\forall\, x \in X \ni \overline{A}(x) - A^\circ(x) > 0$	$\partial_4 A = \bigvee \{x_\beta \mid \beta \leq \overline{A}(x), \beta > A^\circ(x)\}$	$\partial_5 A = \left(\overline{A}\right)^\circ \wedge \overline{A^\circ}$
W1 (*Boundary is closed*)	✓	✓	✓	✓	
W2 (*Closure is supremum of interior and boundary*)	✓	✓	✓	✓	✓
W3 (*Becomes usual boundary when sets are crisp*)	✓	✓	✓	✓	
W4 (*Boundary operator equivalent way to define fuzzy topology*)	✓	✓	✓	✓	
W5 ($\partial A = \partial A^C$)		✓		✓	
W6 (*Interior of boundary empty for closed or open sets*)			✓		
W7 (*Boundary of clopen set is empty*)					

Table 1: Table showing Warren's properties against definitions of Fuzzy Boundary

3 Related Study

3.1 Entropy

Entropy of a fuzzy set can be thought of as a measure of fuzziness of the fuzzy set. In Xuechung's [Li] study of entropy and associated calculations with functions defined on fuzzy sets, repeated use is seen of that subset which contains elements of membership $\frac{1}{2}$. While in some cases, just the *support* of that subset is enough for consideration, one comes across situations that motivates the study in particulat of those sets. We denote here by $\mathcal{F}(X)$ the class of all fuzzy sets on X and by $\mathcal{P}(X)$ the class of all crisp sets on X.

Definition 3.1. *[Li]* $\left[\frac{1}{2}\right]_X$ *is defined as that set in* $\mathcal{F}(X)$ *for which* $\mu_{\left[\frac{1}{2}\right]_X}(x) = \frac{1}{2}\ \forall\ x \in X$.

Thus $\left[\frac{1}{2}\right]_X$ is a set which forms, as we shall see later in this paper, the *boundary line* of some fuzzy superset of X. Let $F \subseteq \mathcal{F}(X) \ni$

1. $\mathcal{P}(X) \subset F$,
2. $\left[\frac{1}{2}\right]_X \in F$,
3. $A, B \in F \Rightarrow A \vee B \in F, A^C \in \mathcal{F}$.

Definition 3.2. *[Li] A real function* $e : F \to \mathbb{R}^+$ *is called an entropy on* F *if* e *has the following properties*

1. $e(D) = 0\ \forall\ D \in \mathcal{P}(X)$;
2. $e\left(\left[\frac{1}{2}\right]_X\right) = \max\limits_{A \in F} e(A)$;
3. $\forall\ A, B \in F$, *and either* $\mu_B(x) \geq \mu_A(x) \geq \frac{1}{2}$ *or* $\mu_B(x) \leq \mu_A(x) \leq \frac{1}{2}$, *then* $e(A) \geq e(B)$;
4. $e\left(A^C\right) = e(A)\ \forall\ A \in F$.

Example. [Li] Let $X = \left\{x_1, x_2, \dots, x_n\right\}$. For any $A \in \mathcal{F}(X)$, $\hat{A} \in \mathcal{P}(X)\ \ni$

$$\mu_{\hat{A}}(x) = \begin{cases} 1 & \text{when } \mu_A(x) > \frac{1}{2} \\ 0 & \text{when } \mu_A(x) \leq \frac{1}{2} \end{cases}$$

and $e'(A) = \left(\sum\limits_{i=1}^{n} \left|\mu_A\left(x_i\right) - \mu_{\hat{A}}\left(x_i\right)\right|^{\omega}\right)^{\frac{1}{\omega}}\ \forall\ A \in \mathcal{F}(X)$. Then e' is an entropy on $\mathcal{F}(X)$, where $\omega \geq 1$.

Definition 3.3. *[Li] If an entropy* e *on* F *satisfies* $e\left(\left[\frac{1}{2}\right]_X\right) = 1$, *then* e *is called a* normal entropy *on* F.

Result 3.4. *[Li] Let* e *be an entropy of* F, *then* $\hat{e}$ *given by* $\hat{e}(A) = \dfrac{e(A)}{e\left(\left[\frac{1}{2}\right]_X\right)}\ \forall\ A \in F$ *is a normal entropy on* F.

$\left[\frac{1}{2}\right]_X$ is used in deriving core results regarding entropy.
A certain aspect of another work is henceforth showed in brief, that uses similar sets for computation.

3.2 Distribution of SARS virus among people in a community

In [ShLi], the set $A_{\frac{1}{2}}$ has been defined such that

$$\mu_{A_{\frac{1}{2}}}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) > \frac{1}{2}; \\ 0 & otherwise. \end{cases}$$

On the other hand, to detect the distribution of SARS in a community [ShLi] by tracing the path of an affected person, investigating the effect of that person as well as an infected region on the community, five topologically invariant components:

- $(A \wedge B)_{\mu_A+\mu_B \leq 1}$, denoting memebership in intersection of two such sets, sum of whose individual memberships for a given point is at most 1;
- $\bigwedge (A \wedge B)^{\mu_A \text{ or } \mu_B \leq 0.5}_{\mu_A+\mu_B>1}$, denoting memebership in intersection of all possible such intersection of two such sets, sum of whose individual memberships for a given point is greater than 1 but membership in one of them is less than or equal to $\frac{1}{2}$;
- $\bigwedge \left(A \wedge B^C\right)^{\mu_A \text{ and } \mu_B>0.5}_{\mu_A+\mu_B>1}$, denoting memebership in intersection of all possible such intersection of two such sets, sum of whose individual memberships for a given point is greater than 1 and membership in both is greater than or equal to $\frac{1}{2}$;
- $\bigwedge \left(A^C \wedge B\right)^{\mu_A \text{ and } \mu_B>0.5}_{\mu_A+\mu_B>1}$, denoting memebership in intersection of all possible such intersection of two such sets, sum of whose individual memberships for a given point is greater than 1 and membership in both is greater than or equal to $\frac{1}{2}$;
- $\left\{x \in X \mid \mu_A(x) = \mu_B(x)\right\}$, those points which have equal membership in A and B.

were used, involving the idea of a set exclusively containing those points whose membership is $\frac{1}{2}$. Study of such a set which can be beneficial to the above procedures, has been conducted after stating some preliminary definitions.

4 Definition and properties of Coast of a fuzzy set

4.1 Properties of coast

Let A be a fuzzy set. The *Coast* of A is the set of those elements whose degrees of membership in A are exactly $\frac{1}{2}$. Alternatively, seeing the Coast as a subset of $\overline{A}$ makes more sense because there might be elements in $\overline{A}$ for which the membership function in A is $\frac{1}{2}$. As a third option, we can define the Coast as a subset of the boundary (B2) [PaYi].

Definition 4.1. *1. The first Coast $A^{(1)}_L$ of a fuzzy set A is defined by the membership function*

$$\mu_{A^{(1)}_L}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

2. The second Coast $A^{(2)}_L$ of a fuzzy set A is defined by the membership function

$$\mu_{A^{(2)}_L}(x) = \begin{cases} \mu_{\overline{A}}(x) & \text{if } \mu_{\overline{A}}(x) = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

3. *The third Coast $A_L^{(3)}$ of a fuzzy set A is defined by the membership function*

$$\mu_{A_L^{(3)}}(x) = \begin{cases} \mu_{\overline{A}\cap\overline{A^C}}(x) & \textit{if } \mu_{\overline{A}\cap\overline{A^C}}(x) = \frac{1}{2}; \\ 0 & \textit{otherwise.} \end{cases}.$$

We can also imagine this third definition as a boundary line *that makes a line demarcation out of the fuzzy boundary.*

Note that $A_L^{(1)}$ is the fuzzy set consisting of elements which have same membership function values in A and A^C. We define $L = \text{supp}\left(A_L^{(1)}\right)$ as its crisp version, i.e., $L = \text{supp}\left(A_L^{(1)}\right) = \left\{x \in X \ : \ \mu_A(x) = \frac{1}{2}\right\}$.

Definition 4.2. *The* enclosure *of a Coast is the interior of* $\left\{x \in X \ : \ \mu_A(x) \leq \frac{1}{2}\right\}$, *i.e., In*$(A_L) = \{B \subseteq X \ : \ h(B) \leq \frac{1}{2}\right\}$.

We henceforth study some properties of the coast of a fuzzy set from a purely theoretical viewpoint.

Result 4.3. *If A is closed, then* $A_L^{(1)} = A_L^{(2)} \subset A_L^{(3)}$.

Proof. If A is closed, then $A = \overline{A}$ so exactly those elements which have membership $\frac{1}{2}$ in A shall have membership $\frac{1}{2}$ in $\overline{A}$. This shows that $A_L^{(1)} = A_L^{(2)}$. Let $\mu_A(x) = \frac{1}{2} \Rightarrow \mu_{\overline{A}}(x) = \mu_A(x) = \frac{1}{2}$ and $\mu_{\overline{A^C}}(x) \geq \frac{1}{2} \Rightarrow \mu_{\overline{A}} \wedge \mu_{\overline{A^C}}(x) = \frac{1}{2}$ which proves that $A_L^{(1)} \subset A_L^{(3)}$. ☐

Theorem 4.4. *If A is clopen,* $A_L^{(1)} = A_L^{(3)}$.

Proof. Let $x \in A_L^{(3)} \Rightarrow \mu_{\overline{A}\wedge\overline{A^C}}(x) = \frac{1}{2}$. If $\mu_{\overline{A^C}}(x) = \frac{1}{2}$ then $\mu_{A^C}(x) = \frac{1}{2}$ [$\because A$ is open $\Rightarrow A^C$ is closed $\Rightarrow \mu_{A(x)} = \frac{1}{2}$. If $\mu_{\overline{A}}(x) = \frac{1}{2}$, then $\mu_A(x) = \frac{1}{2}$ [$\because A$ is closed]. $\therefore A_L^{(1)} \supset A_L^{(3)}$. The rest follows from Result **??**. ☐

Theorem 4.5. $A_L^{(2)} < A_L^{(3)}$. *In particular, if A is clopen,* $A_L^{(2)} \subset A_L^{(3)}$.

Proof. Follows from the fact that $\mu_{\overline{A}}(x) = \frac{1}{2} \Rightarrow \mu_A(x) \leq \frac{1}{2} \Rightarrow \mu_{A^C}(x) \geq \frac{1}{2}$. ☐

Theorem 4.6. $A_L^{(1)}$ *is a topological invariant.*

Proof. Let us consider an I-fuzzy homemorphism $f^{\rightarrow} \ : \ \left(I^X, \delta\right) \rightarrow \left(I^Y, \mu\right)$. Then, $\forall\, y \in Y\ \exists$ unique $x_0 \in X \ni f\left(x_0\right) = y$. Thus, $f^{\rightarrow}\left(\mu_A\right)(y) = \vee\left\{\mu_A(x) \ : \ x \in X \ni f(x) = y\right\} = \mu_A\left(x_0\right)$. Now,

$$\begin{aligned} f^{\rightarrow}\left(\mu_{A_L^{(1)}}\right)(y) &= \begin{cases} f^{\rightarrow}\left(\mu_{A_L^{(1)}}\right)(x_0) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_{A_L^{(1)}}\left(x_0\right) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_A\left(x_0\right) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Again,

$$\begin{aligned} \left(f^{\rightarrow}(\mu_A)\right)_L^{(1)}(y) &= \begin{cases} f^{\rightarrow}(\mu_A)(y) & \text{if } f^{\rightarrow}(\mu_A)(y) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu_A\left(x_0\right) & \text{if } \mu_A(x_0) = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

$\therefore f^{\rightarrow}\left(\mu_{A_L}^{(1)}\right)(y) = \left(f^{\rightarrow}(\mu_A)\right)_L^{(1)}(y)\ \forall\, y \in Y$, i.e., $\mu_{A_L^{(1)}}$ is an invariant of homeomorphism. ☐

Theorem 4.7. $A_L^{(1)} \subseteq A_L^{(3)}$.

Proof. Let $x \in A_L^{(1)}$. $\therefore \mu_A(x) = \frac{1}{2} \Rightarrow \mu_{\overline{A}}(x) = \frac{1}{2} = \varepsilon$ for some $\varepsilon \geq 0$. Again, $\mu_{A^C}(x) = 1 - A(x) = \frac{1}{2} \Rightarrow \mu_{\overline{A^C}}(x) = \frac{1}{2} + \varepsilon'$ [$\because \mu_A \leq \mu_{\overline{A}}$]. But then one of A and A^C must be closed. Let $B \in \{A, A^C\}$ is closed. Then $\overline{B}(x) = B(x) = \frac{1}{2}$. Then $\left(\mu_{\overline{A}} \wedge \mu_{\overline{A^C}}\right)(x) = \frac{1}{2} \Rightarrow x \in \overline{A} \cap \overline{A^C}$. $\therefore A_L^{(1)} \subseteq A_L^{(3)}$. $\square$

We end this section by giving a simple example to illustrate the definition in a subset of $\mathbb{R}^2$. Let us suppose A be a fuzzy set in X where

$$\mu_A(x) = \begin{cases} 0 & \text{if } |x| > 3, \\ \varepsilon & \text{if } |x| = 3, \\ ||x| - 3| & \text{if } 2 < |x| < 3, \\ 1 & \text{if } |x| \leq 2. \end{cases}$$

Then A is closed in $\mathbb{R}$ as it is upper semicontinuous from $\mathbb{R}$ to I. Then,

$$\mu_{A^\circ} = \begin{cases} 0 & \text{if } |x| > 3, \\ 0 & \text{if } |x| = 3, \\ ||x| - 3| & \text{if } 2 < |x| < 3, \\ 1 - \varepsilon_1 & \text{if } |x| = 2, \\ 1 & \text{if } |x| < 2. \end{cases}$$

Here, $L = \{\pm 2.5\}$ and $\mu_{A_L^{(1)}} = \left\{\left(2.5, \frac{1}{2}\right), \left(-2.5, \frac{1}{2}\right)\right\}$.

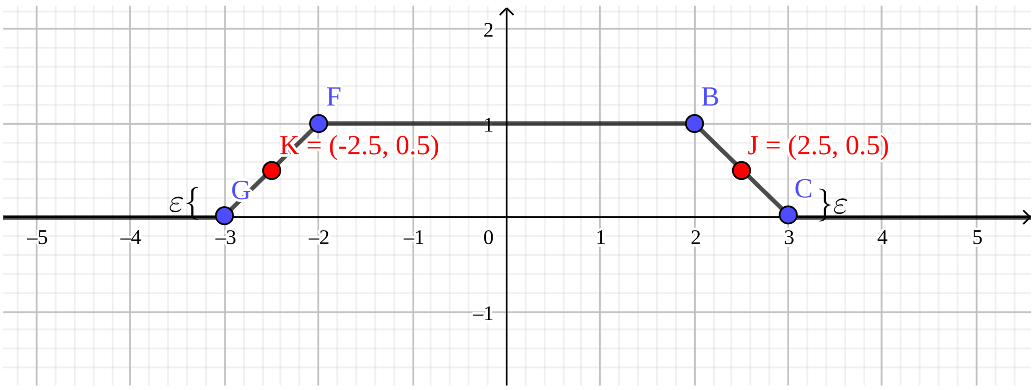


Figure 1: The fuzzy set A, its support $L = \{\pm 2.5\}$ and coast $\mu_{A_L^{(1)}} = \left\{\left(2.5, \frac{1}{2}\right), \left(-2.5, \frac{1}{2}\right)\right\}$

Each of these three formulations of the coast have their own usability depending on whether the membership values vary continuously over the object, whether we want the thinnest possible boundary or a boundary that covers up more 'gray area' between the set and its exterior, whether the considered set is open or closed etc. For the rest of our discussion, we go with $A_L^{(1)}$ as the coast of A. The reasons for such a choice are suited to our objective. These are:

1. It is, in any case, either the thinnest or one of the thinnest boundary lines;
2. Since we shall be dealing with spatial objects where the membership varies continuously, $A_L^{(1)}$ shall never be discrete; moreover, it shall always be a continuous loop or line, and in some cases 2-dimensional;
3. The support of the set will be just the points of the set, and can be thought of with a pre-determined fixed value of membership;
4. Since we don't characterize any set as either closed or open in our discussion, an assumption of clopen sets also makes $A_L^{(1)}$ a set consisting of those points of $A_L^{(3)}$, the crisp boundary, that have membership $\frac{1}{2}$;

5. Comparison with Warren's properties in the next section also shows that $A_L^{(1)}$ is a suitable candidate for our purpose.

6. The formulation is simpler and requires less computation to implement;

One might wish to work with another formulation in a different context, e.g., when the membership function is not continuous over the spatial object and $A_L^{(1)}$ is a set of disconnected or discrete points. In that case, a broader coast might give a continuous demarcation to separate the set from its exterior.

4.2 Results about Coast similar to Warren's properties

It should be repeated again that our definition of the Coast is not intended to serve as a definition of boundary. On the very contrary, is it applicable only in those cases where we find it difficult to deal with more formal definitions of fuzzy boundary but still would like our sets to retain their innate sense of fuzziness. It is only in those cases that we would like a "line" between a set and its complement, a line that marks the bounds of the set. It gives us a somewhat crisp boundary in an otherwise fuzzy environment. But its application lies in an entirely different environment than where fuzzy boundary is applied.

A Coast has nothing to do with these properties, and indeed, satisfying them is not a criteria. We have in this section formulated some properties the coast satistfies. These properties are comparable with Warren's properties and hence might give a clearer idea of the comparison and contrast between fuzzy boundary and Coast.

(W1′) Unlike the conventional boundaries, Coast is not closed.

(W2′) $\overline{A} > A^\circ \vee A_L^{(1)}$, $\overline{A} > A^\circ \vee A_L^{(2)}$ however it is not necessarily the supremum as stated in the condition.

(W5′) The Coast of the set A and its complement A^C are equal if we consider $A_L^{(1)}$ or $A_L^{(3)}$ but not necessarily for $A_L^{(2)}$.

(W6′) If the interior of the Coast in $\mathbb{R}^2$ is empty, then the Coast is a line or a set of scattered points. Else, it is two-dimensional.

Proof.

(W2′) We know, $\overline{A} \supset A^\circ$. Also $\mu_{\overline{A}}(x) \geq \mu_A(x) \Rightarrow \mu_{\overline{A}}(x) \geq \frac{1}{2}$ when $\mu_{A(x)} \geq \frac{1}{2}$. So, $\overline{A} \supset A^\circ \vee A_L^{(1)}$. The other part follows the same way.

(W5′) This holds since $\mu_A(x) = \frac{1}{2} \Rightarrow \mu_{A^C}(x) = \frac{1}{2}$.

For $A_L^{(2)}$ this is true only if $\mu_A(x) = \mu_{\overline{A}}(x)$, i.e., if A is closed.

For $A_L^{(3)}$, we can see that $\mu_{\overline{A}} \wedge \mu_{\overline{A^C}} = \frac{1}{2} \Leftrightarrow \mu_{\overline{(A^C)^C}} \wedge \mu_{\overline{A^C}} = \frac{1}{2}$. So the Coasts for A and A^C are same.

(W6′) It is easy to see that every two-dimensional area has a non-empty interior. So, if the interior is empty, the Coast must be a line or a scattered set of points. Figures 2a, 2b and 2c show that the Coast can indeed be two-dimensional.

The above comparison, however, suggests that in favorable cases, using the coast as a thin boundary or edge or fringe can indeed solve our purpose with better outcomes. This is verified in detail in the penultimate section.

4.3 Coast forms a subset of the boundary

By outward monotonicity, we mean that whenever the central point of the region (with membership 1) is joined with a point on the boundary by a straight line, with distance from the center the membership value monotonically decreases (or remains same). Thus the points with membership $\frac{1}{2}$ should lie together on each such line. Continuity over $\mathbb{R}^2$ implies that those points shall form a connected set.

If the membership value is strictly decreasing on each outward line, then each line can have at most 1 point with membership $\frac{1}{2}$. However, continuity implies that running from 1 to 0, the line shall have memberships each value in between. So, A_L is a one-dimensional loop. Thus,

Theorem 4.8. *In a simple closed region with outwardly monotone and continuous membership function, A_L is a closed loop. If it is strictly decreasing outward then A_L is a 1-dimensional loop.*

Proof. Let us mark a point $a \in A$ that has highest membership function. Let there exists some circular boundary of A, each point of which has membership 0; call it C. If we join an arbitrary point $c \in C$ with a by a straight line, since μ_A is outwardly continuous and monotone, there exist a point c' such that $\mu_A(c') = \frac{1}{2}$, or a line segment $C' \subset C$ each of whose points have membership $\frac{1}{2}$. Since the membership function is continuous, we see that the set of points with membership $\frac{1}{2}$ is path-connected, hence form a closed loop.

If additionally, μ_A is strictly decreasing, there is only one point c' on each outward line from a such that $\mu_A(c') = \frac{1}{2}$. From continuity of μ_A, it follows that the set $\Gamma = \left\{c' \mid \mu_A\left(c'\right) = \frac{1}{2}\right\}$ is connected, hence a loop with zero-width. □

5 Using the Coast for mapmaking [This section might be omitted]

Broadly, any point between absolute membership 1 and 0 is a boundary point for a fuzzy set. However, a more precise categorization of boundary is necessary. For image enhancement we sometimes need a clearer view of the set and its exterior, and Coast solves this purpose to some extent.

The membership of a point in the boundary is at most half. The more we go both ways to the interior or exterior of the set, membership in the boundary decreases. The definition of Coast has been conceived such that it considers as boundary, in a very crisp way, only those points which have the maximum possible membership in other definitions of boundary, making $A_L^{(3)}$ a proper subset of our usual definition of ∂A and other A_L's alternatives in appropriate situations. This gives a thinner boundary, demarcating the set and its exterior more prominently.

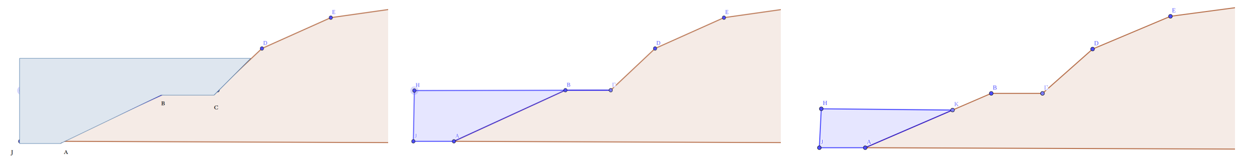
Example 1

Where the exact shoreline varies due to daily high and low tides and lunar cycles, the Coast can be used to draw a precise outline map. We can paint the interior as inland and the exterior blue for the sea. For a gross idea or for a political map this will be more suited. The sand beach is usually the boundary between land and water and being a part of each during each time of the day. Now, if A denotes underwater place, $\mu_A(x) = 0$ implies the place x is inland. Similarly, $\mu_A(x) = 1$ for the sea.

1. One likely option is to mark the exact water-lines during the high-tides and low-tides everyday over a lunar cycle and take their average mid-point on map to mark the place with membership m_A as $\frac{1}{2}$. But this approach has its problems that (i) the slope of the land is not reflected in the map; (ii) land undulations or nature of rocks and soils can affect tide water level, and hence these marked points may not exactly reflect 'the state of being halfway inland'.

2. As a better alternative, we can set $\mu_A(x) = \frac{1}{2}$ is the point x remains dry for exactly 12 hours a day in all, or on an average $\frac{1}{2}$ of the lunar cycle. For political or jurisdictory maps (which indeed is needed very much in river-delta areas, where calculation of the areas of islands is necessary to estimate the amount of land under each police station and fix patrolling duties) or for drawing one-pixel wide coastal line in maps, these points with μ_A as $\frac{1}{2}$ can be used. Then we consider the border as A_L.

Calculation of Coast in the above-posed situation. We take $2n$ number of aerial photographs of the beach in question within the same frame over a lunar cycle, captured at regular time intervals. Superimposing the images, we get $2n$ water-lines stretched almost all across the beach width. We mark black the strip between the n and $(n+1)$th lines as the *Coast*. Of course, this is not exactly the Coast as defined, but the best possible approximation from our pictorial data. No better conclusion from the given assumed data is possible. Each point within this strip remains underwater in exactly half of the pictures. As n increases, we get a finer strip that approaches the exact fuzzy line $\left\{x \mid \mu_A(x) = \frac{1}{2}\right\}$ as $n \to \infty$. It might be noted as well, that a fuzzy line may or may not be a line. If the slope at one point is 0 and its membership is $\frac{1}{2}$, then the fuzzy line is two-dimensional there. Figure 2(b) show such a case where the whole strip BC has $\mu_A = \frac{1}{2}$, i.e., a case when the Coast is two-dimensional, while Figures 2(a) and 2(c) show instances when the Coast is one-dimensional.



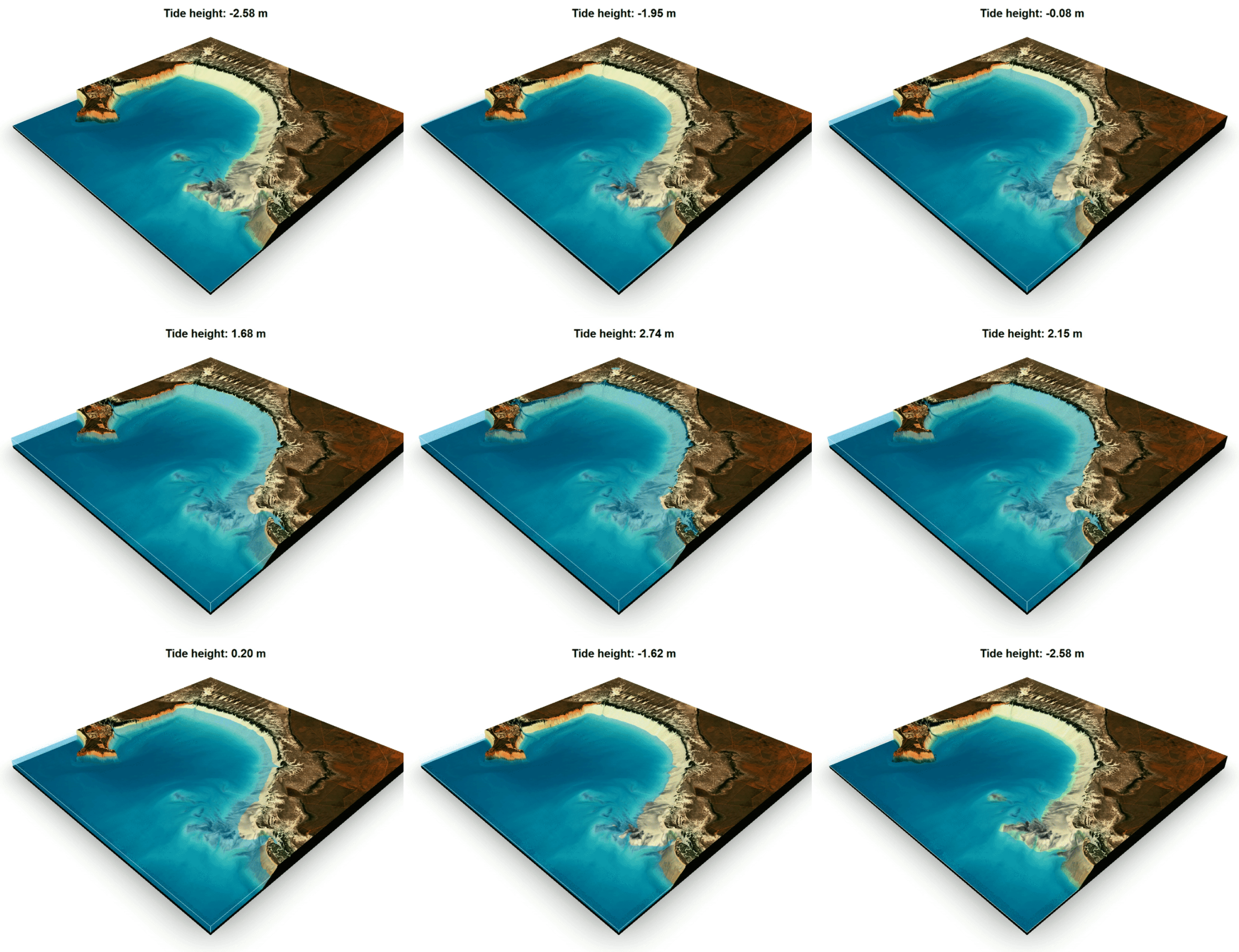
(a) The whole strip BC has $\mu_A = \frac{1}{2}$, when the water level is above BC

(b) The whole strip BC has $\mu_A = \frac{1}{2}$, when the water level is at BC

(c) The whole strip BC has $\mu_A = \frac{1}{2}$, when the water level is below BC

Figure 2: Examples where coast along a sea-beach can be one-dimensional or two-dimensional.

Also, Theorem 4.8 might or might not be applicable. The slope at some point might be infinity, and hence the graph discontinuous. Take for example rocky cliff sea-shores which naturally give us Coasts predefined. The data used has been taken from `https://earthobservatory.nasa.gov/images/145237/mapping-the-land-between-the-tides`. A 3D Landsat model of Roebuck coast of Australia is shown below, which was photographed 116 times in a 24-hour cycle. Below, we show a selection from that, each photograph of time 3 hours apart in a 24-hour cycle.

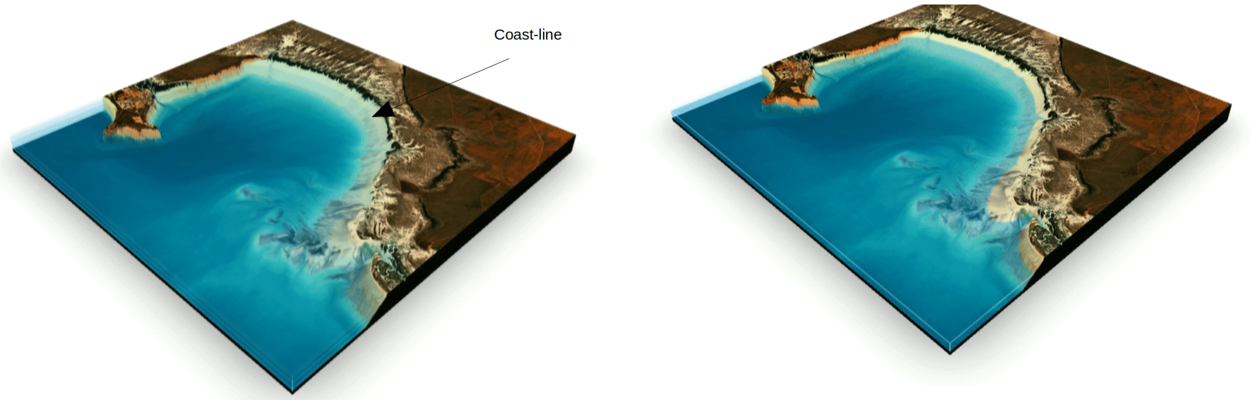


Superimposed, the images give a Coast somewhere through the middle of the beach (Figure 4a). This is the Coast line of minimum width that can be drawn out of from the available data set, and the best approximation given the number of frames. Im Figure 4b the Coast given by roughly the 29th, 30th, 87th and 88th frames is shown, from a 116-frame data of the same shore over a diurnal tide cycle, and it can be checked that it gives the same result.

Method What we essentially do at this point, when the frames itself have an extent of fuzziness when treated as images, is to

1. Draw out a coast line using image enhancement techniques for each still image, and then
2. Superimpose the lines and mark the middlemost one or the area between the two middlemost ones, if we need the finest boundary line available.

This effort can be reduced by selecting the middlemost few frames and then applying image enhancement only for them.



(a) Coast given by superimposing 8 frames a day

(b) Coast given by superimposing 116 frames a day with the middlemost frames

Figure 4: Coast given by superimposition of time-series boundary lines

6 An Example

In order to validate the consistency of the introduced technique, we arbitrarily take ten observations with two simple fuzzy areas. Then, for each of the observations, we obtain the relationships using the proposed method, the methodologies given in the studies in [Bj] and [TaKa]. Here, we have considered the raster-based data. The standard fuzzification method [PaKietal] is followed to convert the images into the fuzzy set.

Table 2 shows previous similar calculation in [Bj], computing the 4-intersection matrix using height of a fuzzy set, and that in [TaKa] using core, fringe and outer of a fuzzy set; and in [JaMa], where the membership of the boundary for each raster element is calculated by $2 * \left(A_{i,j}(x), A^c_{i,j}(x) \right)$ and the membership of the interior as $A_{i,j}(x) - 2 * \left(A_{i,j}(x), A^c_{i,j}(x) \right)$.

In our calculations, we have used the coast instead of fringe or fuzzy boundary, and have replaced the interior by those points which do not have membership $\frac{1}{2}$. The relationship values thus obtained are tabulated in Table 3.

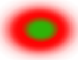
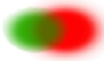
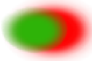
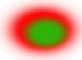
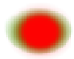
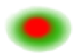
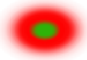
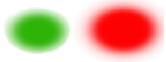
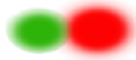
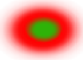
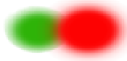
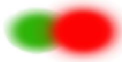
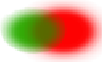
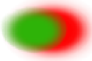
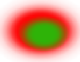
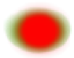
Sl. no	Objects	with edge			Bjørke		Tang & Kainz
		Matrix	Relationship		Matrix	Relationship	
			S_1	S_2			
1.		$\begin{pmatrix}0.18 & 0.36 & 0.67\\ 0.44 & 0.47 & 0.55\\ 0.66 & 0.59 & 1\end{pmatrix}$	I_0–0.85, I_4–0.66, I_7–0.55	I_0–0.81, I_4–0.64, I_0–0.50	$\begin{pmatrix}0.40 & 0.77\\ 0.88 & 0.98\end{pmatrix}$	I_7–0.40, I_6–0.23	$\begin{pmatrix}0 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$
2.		$\begin{pmatrix}0.67 & 0.45 & 0.56\\ 0.03 & 0.04 & 0.47\\ 0.03 & 0.04 & 1\end{pmatrix}$	I_1–0.78, I_5–0.68, I_3–0.67	I_1–0.74, I_3–0.52, I_5–0.52	$\begin{pmatrix}1 & 0.98\\ 0.05 & 0.07\end{pmatrix}$	I_1–0.67, I_5–0.08	$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$
3.		$\begin{pmatrix}0.53 & 0.55 & 0.53\\ 0.54 & 0.50 & 0.53\\ 0.43 & 0.54 & 1\end{pmatrix}$	I_7–0.58, I_1–0.57, I_0–0.57	I_7–0.56, I_1–0.55, I_0–0.55	$\begin{pmatrix}0.97 & 1\\ 0.98 & 0.93\end{pmatrix}$	I_7–0.93, I_1–0.01	$\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$
4.		$\begin{pmatrix}0.60 & 0.31 & 0.42\\ 0.28 & 0.29 & 0.44\\ 0.28 & 0.35 & 1\end{pmatrix}$	I_3–0.65, I_1–0.62, I_5–0.57	I_3–0.62, I_1–0.59, I_5–0.53	$\begin{pmatrix}1 & 1\\ 0.86 & 0.96\end{pmatrix}$	I_7–0.86, I_5–0.14	$\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}$
5.		$\begin{pmatrix}0.66 & 0.35 & 0.32\\ 0.10 & 0.23 & 0.47\\ 0.10 & 0.23 & 1\end{pmatrix}$	I_3–0.70, I_1–0.68, I_5–0.62	I_3–0.65, I_1–0.63, I_5–0.55	$\begin{pmatrix}1 & 0.79\\ 0.31 & 0.77\end{pmatrix}$	I_5–0.69,I_1–0.22	$\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}$
6.		$\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}$	I_0–1, I_4–0.88	I_0–1	$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$	I_0–1	$\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}$
7.		$\begin{pmatrix}0.67 & 0.45 & 0.47\\ 0.05 & 0.08 & 0.47\\ 0.05 & 0.08 & 1\end{pmatrix}$	I_1–0.76, I_3–0.69, I_5–0.66	I_1–0.72, I_3–0.57, I_5–0.54	$\begin{pmatrix}1 & 0.97\\ 0.21 & 0.36\end{pmatrix}$	I_1–0.63, I_7–0.20	$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$
8.		$\begin{pmatrix}0.67 & 0.17 & 0.17\\ 0.25 & 0.40 & 0.42\\ 0.25 & 0.45 & 1\end{pmatrix}$	I_3–0.71, I_2–0.61, I_6–0.58	I_3–0.68, I_2–0.55, I_6–0.52,	$\begin{pmatrix}1 & 0.52\\ 0.70 & 0.98\end{pmatrix}$	I_7–0.52, I_6–0.48	$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}$
10.		$\begin{pmatrix}0.68 & 0.05 & 0.05\\ 0.47 & 0.14 & 0.14\\ 0.50 & 0.45 & 1\end{pmatrix}$	I_2–0.75, I_3–0.68, I_6–0.67	I_2–0.71, I_3–0.60, I_6–0.58	$\begin{pmatrix}1 & 0.17\\ 0.98 & 049\end{pmatrix}$	I_2–0.50, I_6–0.49	$\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$
10.		$\begin{pmatrix}0.68 & 0.45 & 0.60\\ 0.03 & 0.03 & 0.47\\ 0.03 & 0.03 & 1\end{pmatrix}$	I_1–0.75, I_5–0.68, I_3–0.68	I_1–0.75, I_5–0.50, I_3–0.58	$\begin{pmatrix}1 & 1\\ 0.03 & 03\end{pmatrix}$	I_1–0.97, I_7–0.03	$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$

Table 2: Relationship between fuzzy areas using different algebraic methods

Table 3: 9-Intersection matrices using coast instead of fuzzy boundary

Sl no.	Object	Matrix	Relationship
1		$\begin{pmatrix} 0.7864 & 0.9953 & 0.6128 \\ 0 & 0 & 0 \\ 0.5782 & 0.0118 & 1.0000 \end{pmatrix}$	0.4912 0.7560 0.5298 0.6209 0.3801 0.6449 0.4187 0.4427
2		$\begin{pmatrix} 0.8792 & 0.8137 & 0.5754 \\ 0 & 0 & 0 \\ 0.5724 & 0.5569 & 1.0000 \end{pmatrix}$	0.5569 0.6821 0.6244 0.5956 0.4458 0.5710 0.5133 0.4886
3		$\begin{pmatrix} 0.9206 & 0.8118 & 0.5465 \\ 0 & 0 & 0 \\ 0.5617 & 0.5647 & 1.0000 \end{pmatrix}$	0.5490 0.6836 0.6321 0.6040 0.4378 0.5725 0.5210 0.4895
4		$\begin{pmatrix} 0.9132 & 1.0000 & 0.4525 \\ 0 & 0 & 0 \\ 0.4827 & 0.6392 & 1.0000 \end{pmatrix}$	0.5179 0.6938 0.6203 0.5932 0.4068 0.5827 0.5092 0.4986
5		$\begin{pmatrix} 0.9373 & 0.2706 & 0.3289 \\ 0 & 0 & 0 \\ 0.3662 & 0.7231 & 1.0000 \end{pmatrix}$	0.5789 0.6053 0.7141 0.6943 0.4678 0.4942 0.6030 0.4029
6		$\begin{pmatrix} 0.9626 & 0.3129 & 0.2991 \\ 0 & 0 & 0 \\ 0.4050 & 0.3435 & 1.0000 \end{pmatrix}$	0.5302 0.6473 0.6777 0.7336 0.4191 0.5362 0.5666 0.3692
7		$\begin{pmatrix} 0.8279 & 0 & 0.1523 \\ 0 & 0 & 0 \\ 0.6653 & 0 & 1.0000 \end{pmatrix}$	0.5544 0.5905 0.7045 0.7789 0.4433 0.4794 0.5934 0.2939
8		$\begin{pmatrix} 0.8481 & 0 & 0.1641 \\ 0 & 0 & 0 \\ 0.6570 & 0 & 1.0000 \end{pmatrix}$	0.5526 0.5950 0.7045 0.7808 0.4415 0.4839 0.5934 0.2966
9		$\begin{pmatrix} 0.8148 & 0 & 0.1058 \\ 0 & 0 & 0 \\ 0.6536 & 0 & 1.0000 \end{pmatrix}$	0.5494 0.5852 0.7070 0.7839 0.4383 0.4741 0.5958 0.2860
10		$\begin{pmatrix} 0.7151 & 0 & 0.4037 \\ 0 & 0 & 0 \\ 0.6724 & 0 & 1.0000 \end{pmatrix}$	0.5957 0.6052 0.6649 0.7377 0.4846 0.4940 0.5538 0.3101

BjØrke's method has the highest membership corresponding to 'overlap' relation. Similarly, in observation 1 we get the highest membership corresponding to the 'disjoint' relation followed by the 'meet' and 'overlap' relation, whereas Bjørke's method again has the highest membership corresponding to the 'overlap' relation. Through manual visualization, one may observe that the obtained result in our cases matches with the visualized result. If we compare with the Tang and Kainz method, we see that it only provides the intersection matrix; we compute those matrices for each of the cases and find that all the generated matrices belong to the set of valid forty-four matrices [TaKa].

7 Conclusion

From the use of related concepts in [Li, Example 2.2] and [ShLi] we have essentially formulated here 3 definitions for the coast of a fuzzy set. There might be situations in which each shall prove useful; broadly speaking, the coast consists of those points in the boundary whose membership either in the set or its closure or in the boundary is $\frac{1}{2}$. We have further shown how these three coastlines interact among themselves, as well as with the standard definitions of fuzzy boundary.

Our application of the coast to 9-intersection matrices can further be extended to other tools used to study fuzzy regions. The present paper only considers the relations between two area objects; further study is needed to investigate the area–line and line–line objects relations. Exploring other possible theoretical properties of the coast and its interaction with other subsets of a region might give interesting results, which in turn has possible applications in GIS and image-processing. Language models can be explained with fuzzy topology, and only basic works have been done in this approach. Coast of a fuzzy topology shall probably find use in error detection, and demarcating a correct command from an absurd command, in language models aiding deep learning. Our ideas about this are in nascent stage, but given the already-explored applicability of fuzzy topology in machine learning, these ideas might not be far-fetched.

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