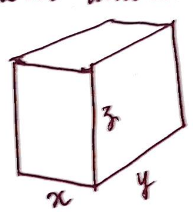
LAGRANGE MUTIPLIER

<u>Lagrange Multiplier problem</u>: Maximise (or minimise) $w = f(x,y,z)$ constrained by $g(x,y,z) = c$.

<u>Lagrange multiplier solution</u>: Local maxima (or minima) must occur at a <u>critical point</u>. This is a point where $\vec{\nabla} f = \lambda \vec{\nabla} g$, and $g(x,y,z) = c$.

1. Making a box out of minimum amount of materials.

A box is made of cardboard with double thick sides, a triple thick bottom, single thick front and ~~no~~ back and no top. Its volume is fixed at 3. What dimensions use the least amount of cardboard.



Area of one side $= yz$. Area of 2 double thick sides $= 4yz$.

Area of front and back (single thick) $= 2xz$.

Area of bottom $= 3xy$.

Total cardboard used,

$$w = f(x,y,z) = 4yz + 2xz + 3xy$$

The fixed volume acts as a constraint. $V = xyz = 3$.

<u>Critical points</u> $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2z + 3y, 4z + 3x, 4y + 2x \rangle$

$\vec{\nabla} V = \langle yz, xz, xy \rangle$

The Lagrange multiplier equations are then

$$\vec{\nabla} f = \lambda \vec{\nabla} V \text{ and } V = 3.$$

$$\Leftrightarrow \langle 2z + 3y,\ 4z + 3x,\ 4y + 2x \rangle = \lambda \langle yz, xz, xy \rangle,\ xyz = 3$$

$$\frac{2z+3y}{yz} = \lambda = \frac{4z+3x}{xz} = \frac{4y+2x}{xy} = xyz \Rightarrow \frac{2}{y} + \frac{3}{z} = \frac{4}{x} + \frac{3}{z} = \frac{4}{x} + \frac{2}{y}$$

$$\Rightarrow \frac{2}{y} = \frac{4}{x} \Rightarrow x = 2y \text{ and } \frac{3}{z} = \frac{2}{y} \Rightarrow z = \frac{3}{2} y$$

Now, $xyz = 3 \Rightarrow 3y^3 = 3 \Rightarrow y = 1$

Ans: $x = 2,\ y = 1,\ z = \frac{3}{2},\ w = 18$.

2. (checking the boundary) A rectangle in the plane is placed on the first quadrant so that one corner O is at the origin and the two sides adjacent to O are on the axes. The corner P opposite O is on the curve $x + 2y = 1$. Using Lagrange's multipliers find for which P the rectangle has maximum area. Say how you know this point gives the maximum.

Soln. $g(x,y) = x + 2y = 1$ = the constraint and $f(x,y) = xy$ = the area.

The gradients are: $\vec{\nabla} g = \hat{i} + 2\hat{j}$, $\vec{\nabla} f = y\hat{i} + x\hat{j}$.

Lagrange's multipliers: $y = \lambda,\ x = 2\lambda,\ x + 2y = 1$

$\Rightarrow \lambda = \frac{1}{4}$ etc.

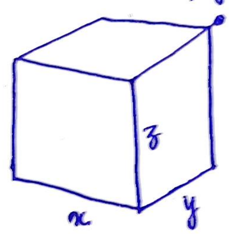
We know this is maximum because maximum occurs either at a critical point or on the boundary. In this case, the boundary points are on the axes at $(1,0)$ and $(0,\frac{1}{2})$ which gives a rectangle with area 0.

3. [A different (and might be slightly more difficult) version of problem ①]

In a open-top wooden drawer, the two sides and the back cost Rs. 200/sq. ft, the bottom Rs. 100/sq ft and the front Rs. 400/sq. ft. Using Lagranges multipliers find the dimensions of the drawer with the largest capacity that can be made for Rs. 7200.

[I told the problem in class with each amount multiplied by 2.]

Soln. The box has dimensions x, y and z.



Area of each side $= yz$; the area of the front (and the back) $= \cancel{yz}\ xz$; the area of the bottom $= xy$. Thus the cost of wood is

$$C(x,y,z) = 200(2y + xz) \overset{+100xy + 400xz}{\cancel{+100xy + 400xz}} = (4yz + 6xz + xy)\,100$$

And, $100(4yz + 6xz + xy) = 7200 \Rightarrow 4yz + 6xz + xy = 72.$

This is our constraint. We are trying to maximise the volume

$$f = xyz.$$

The Lagrange multiplier equations are then

$$\vec{\nabla} f = \lambda \vec{\nabla} g \;;\quad g = 72$$

$\Leftrightarrow \langle yz, xz, xy \rangle = \lambda \langle 6z+y, 4z+x, 4y+6x \rangle$, $4yz + 6xz + xy = 72$

We solve the critical points by isolating $\frac{1}{\lambda}$.

$$\frac{1}{\lambda} = \frac{6}{y} + \frac{1}{z} = \frac{4}{x} + \frac{1}{z} = \frac{4}{x} + \frac{6}{y}$$

Comparing the third and fourth terms gives $\frac{1}{z} = \frac{6}{y} \Rightarrow y = 6z$.

Likewise the second and fourth terms gives $x = 4z$.

Substituting this in the constraint gives $72 z^2 = 72 \Rightarrow z = 1$. Thus,

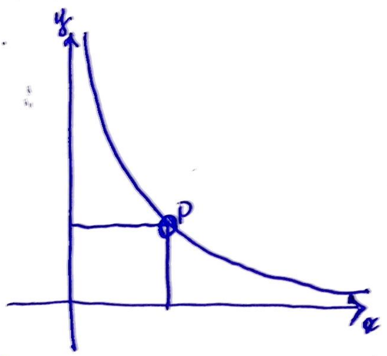
$$z = 1, \quad x = 4, \quad y = 6.$$

4. (boundary at ∞) — compare with 3

A rectangle in the plane is placed in the first quadrant so that one corner O is at the origin and the two sides adjacent to O are on the axes. The corner P opposite O is on the curve $xy = 1$. Using Lagrange's multiplier, find out for which point P the rectangle has minimum perimeter. Say how you know this point gives the minimum.

Soln. Let $g(x, y) = xy = 1$ = the constraint and $f(x, y) = 2x + 2y$ = the perimeter.

Gradients: $\vec{\nabla} g = y\hat{i} + x\hat{j}$, $\vec{\nabla} f = 2\hat{i} + 2\hat{j}$.



Lagrange multipliers: $2 = \lambda y$, $2 = \lambda x$, $xy = 1$.

This gives: $P = (1, 1)$.

We know that this is the minimum because minimum occurs either at a critical point or on the boundary. In this case the boundary points are infinitely far out on the axes which gives a rectangle with perimeter $= \infty$.

5. Find the maximum and minimum values of $f(x,y) = x^2+x+2y^2$ on the unit circle.

Soln. $f(x,y) = x^2+x+2y^2$

$g(x,y) = x^2+y^2+1$

$\vec{\nabla} f = (f_x, f_y) = (2x+1, 4y)$

$\vec{\nabla} g = (g_x, g_y) = (2x, 2y)$

$\rightarrow \vec{\nabla} f = \lambda \vec{\nabla} g$

$\Rightarrow 2x+1 = \lambda 2x$

$4y = \lambda 2y$

and $x^2+y^2=1$

$4y = \lambda 2y \Rightarrow \lambda = 2$ or $y = 0$.

$\boxed{\lambda = 2} \Rightarrow 2x+1 = \lambda . 2x = 4x \Rightarrow x = \frac{1}{2} \Rightarrow y = \pm\sqrt{1-x^2} = \pm\frac{\sqrt{3}}{2}$.

$\boxed{y=0} \Rightarrow x = \pm 1$.

Thus the critical points are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), (1,0), (-1,0)$.

$f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$ (max)

$f(1,0) = 2$ (neither max nor min)

$f(-1,0) = 0$ (min).

6. Find the max and min values of $f(x,y) = x^2 - xy + y^2$ on the quarter circle $x^2+y^2 = 1$; $x, y \geq 0$.

Soln. The constraint function is ~~to~~ $g(x,y) = x^2+y^2 = 1$.

The max and min values of $f(x,y)$ will occur where $\vec{\nabla} f = \vec{\nabla} g$ or at the endpoints of the quarter circle.

$$\vec{\nabla} f = (2x-y, -x+2y), \quad \vec{\nabla} g = (2x, 2y).$$

Setting $\vec{\nabla} f = \vec{\nabla} g$, we get $2x - y = \lambda \cdot 2x$ and $-x+2y = \lambda \cdot 2y$

$$\lambda = \frac{2x-y}{2x} = \frac{-x+2y}{2y}$$

$$\Rightarrow 2xy - y^2 = -x^2 + 2xy$$

$$\Rightarrow x^2 = y^2$$

Because we are constrained to $x^2+y^2=1$ $(x,y \geq 0)$, $x = y = \frac{1}{\sqrt{2}}$.

Thus the extrema of $f(x,y)$ will be at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(1,0)$ or $(0,1)$.

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$, minimum value of f on this quarter circle.

$f(1,0) = f(0,1) = 1$, maximum values of f on this quarter circle.