

Solutions to Problem Set-1

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1. Prove that $\sqrt{5}$ is an irrational number.

Soln. Suppose, on the contrary, that $\sqrt{5}$ is a rational number, and can be written as $\sqrt{5} = \frac{p}{q}$ such that $p, q \in \mathbb{Z}; \gcd(p, q) = 1$. Then $p = q\sqrt{5}$, i.e.,

$$p^2 = 5q^2. \quad (1)$$

Right-hand side is divisible by 5, so 5 divides p^2 , i.e., 5 divides p . Hence, 25 divides p^2 , the left-hand side of (1). So 25 must divide $5q^2 \Rightarrow 5$ divides q^2 , which means 5 divides q . Then $\gcd(p, q) \geq 5$, a contradiction.

So our initial assumption was wrong and $\sqrt{5}$ cannot be expressed as a rational number.

2. Prove that \mathbb{N} is not bounded above.

Soln. Suppose there exists $x \in \mathbb{R}$ such that $x > n \forall n \in \mathbb{N}$, i.e., x is an upper bound of \mathbb{N} . By Least Upper Bound Property, there exists some $M \in \mathbb{R}$ such that $M = \sup \mathbb{N}$. Then,

$$\begin{aligned} n &\leq M \forall n \in \mathbb{N} \\ \Rightarrow n - 1 &\leq M - 1 \forall n \in \mathbb{N} \\ \Rightarrow (n + 1) - 1 &\leq M - 1 \forall n \in \mathbb{N} \text{ (recall Peano's axioms)} \\ \Rightarrow n &\leq M - 1 \forall n \in \mathbb{N}. \end{aligned}$$

Therefore $M - 1$ is also an upper bound of \mathbb{N} but M is supposed to be the least upper bound, a contradiction.

3. Let $A \subseteq \mathbb{R}$ be bounded. Prove that $\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$.

Soln. The point of this exercise is for you to understand why it is not required to include the existence of infimum in the axiom of completeness.

I shall show you how $-\sup(-A) = \inf A$. A is bounded, which means that there exists $x \in \mathbb{R} \ni x \leq a \Rightarrow -x \geq -a \Rightarrow x$ is an upper bound for $-A = \{-a \mid a \in A\}$. Here, x has been an arbitrary lower bound for A . By the axiom of completeness as we know it, $\exists y \in \mathbb{R} \ni y = \sup(-A)$, i.e., $-a \leq y \leq -x \forall a \in A \Rightarrow x \leq -y \leq a$. Here, $-y$ is a lower bound for A , and for any arbitrary $x \leq a$, greater than x , i.e., the greater lower bound or infimum of A . Therefore, $-\sup(-A) = \inf A$.

You should be able to show the other part by replacing A by $-A$.

4. Find the superma and infima of the following sets:

(a) $\{x \in \mathbb{R} \mid 3x^2 + 8x - 3 < 0\}$.

Soln. $3x^2 + 8x - 3 < 0 \Rightarrow 3x^2 + 9x - x - 3 < 0 \Rightarrow (3x-1)(x-3) < 0$. Either $3x-1 < 0, x-3 > 0 \Rightarrow x < \frac{1}{3}, x > 3$ which is not possible, or $3x-1 > 0, x-3 < 0 \Rightarrow x > \frac{1}{3}, x < 3$. So $x \in \left(\frac{1}{3}, 3\right)$. Thus, $\inf x = \frac{1}{3}, \sup x = 3$.

(b) $\left\{ \frac{1}{n} + \frac{1}{m} \mid n, m \in \mathbb{N} \right\}$.

Soln. Check with $n = 1, m \in \mathbb{N}$: $\frac{1}{1} + \frac{1}{1} = 2, \frac{1}{1} + \frac{1}{2} = 1\frac{1}{2}, \frac{1}{1} + \frac{1}{3} = 1\frac{1}{3} \dots$ (decreasing)

Check with $n = 2, m \in \mathbb{N}$: $\frac{1}{2} + \frac{1}{1} = 1\frac{1}{2}, \frac{1}{2} + \frac{1}{2} = 1, \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \dots$ (decreasing)

and so on.

You can see that the supremum has been attained at $n = 1, m = 1$. But what about the infimum? Take any positive real number < 1 , say ε . By Archimedean property, we can find $n \in \mathbb{N}$ such that $n > \frac{2}{\varepsilon} \in \mathbb{R}$ and take $m = n$. We see that $\frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So there exist natural numbers n, m for which $\frac{1}{n} + \frac{1}{m}$ is less than any arbitrary positive real number, but is never negative. What is the infimum of positive real numbers?

(c) $\left\{ n \in \mathbb{N} \mid \frac{n-1}{n+1} \cos \frac{2n\pi}{3} \right\}$.

Soln.

$$\cos \frac{2.1\pi}{3} = 2 \cos^2 \frac{\pi}{3} - 1 = 2 \cdot \frac{1}{4} - 1 = -\frac{1}{2},$$

$$\cos \frac{2.2\pi}{3} = 2 \cos^2 \frac{2\pi}{3} - 1 = 2 \cdot \frac{1}{4} - 1 = -\frac{1}{2},$$

$$\cos \frac{2.3\pi}{3} = \cos \frac{2\pi}{3} \cos \frac{\pi}{3} - \sin \frac{2\pi}{3} \sin \frac{\pi}{3} = \left(-\frac{1}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}\right) - \left(2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) \cdot \frac{\sqrt{3}}{2} = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2},$$

$$\cos \frac{2.4\pi}{3} = \cos \left(2\pi + \frac{2.1\pi}{3}\right) \text{ etc.}$$

Since $\cos \frac{2n\pi}{3}$ is a negative constant, the infimum and supremum of $\left\{ n \in \mathbb{N} \mid \frac{n-1}{n+1} \cos \frac{2n\pi}{3} \right\}$ are attained respectively at the supremum and infimum of $\left\{ n \in \mathbb{N} \mid \frac{n-1}{n+1} \right\}$.

For $n = 1, \frac{n-1}{n+1} = 0$.

For $n = 2, \frac{n-1}{n+1} = \frac{1}{3}$.

For $n = 3, \frac{n-1}{n+1} = \frac{2}{4}$ etc.

Try showing that the supremum will be 1.

So $\inf \left\{ n \in \mathbb{N} \mid \frac{n-1}{n+1} \cos \frac{2n\pi}{3} \right\} = -\frac{1}{2}$, and $\sup \left\{ n \in \mathbb{N} \mid \frac{n-1}{n+1} \cos \frac{2n\pi}{3} \right\}$ is clearly 0.

(d) $\left\{ n \in \mathbb{N} \mid \frac{(n+1)^2}{2^n} \right\}$.

Soln. Let $a_n = \frac{(n+1)^2}{2^n}$. Then $\frac{a_n}{a_{n+1}} = \frac{2(n^2 + 2n + 1)}{n^2 + 4n + 4} = \frac{2(n^2 + 4n + 4)}{n^2 + 4n + 4} - \frac{2(2n + 3)}{n^2 + 4n + 4} = 2 - \frac{4n + 6}{n^2 + 4n + 4} >$

1. So we have $a_{n+1} < a_n$ for all $n \geq 3$. $a_1 = 2, a_2 = \frac{9}{4}, a_3 = 2, a_4 = \frac{25}{16}$ etc. The highest is attained at

$n = 2$. The infimum is 0 since 2^n values increase with a much higher rate with increasing n , than $(n + 1)^2$. We have seen that $a_{n+1} < \frac{2}{3}a_n$, so for any given $\varepsilon > 0$ we can find $n \in \mathbb{N}$ such that $a_n < \varepsilon$.

5. Prove that $[0, 1]$ is uncountable.

Soln. Consider numbers written like this $\overline{0.x_1x_2x_3\dots}$ where the entries after the decimal point are either 0 or 1. It is clear that their collection is a proper subset of $[0, 1]$. Check out Rudin's book, Theorem 2.14 (I have explained this in class), which shows that the collection of such numbers is uncountable. $[0, 1]$, being a superset, is certainly so.

6. Find the sets of all interior points for the following sets: $\mathbb{Q}, (0, 2]$.

Soln. For \mathbb{Q} , Let's say $q \in \mathbb{Q}$. Consider any $\varepsilon > 0$ and the open interval $(q - \varepsilon, q + \varepsilon)$. Since irrational numbers are dense in \mathbb{R} , there exists some $r \notin \mathbb{Q}$ such that $q - \varepsilon < r < q$, i.e., $r \in (q - \varepsilon, q + \varepsilon)$. So q has no interior point.

For $(0, 2]$, If $2 \neq x \in (0, 2]$, then there exists some $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x) \subset (0, 2]$. So, $(0, 2)$ is the interior.

7. Let S be a non-empty bounded set. Consider the set $T = \{|x - y| : x, y \in S\}$. Prove that T is bounded above. Find the supremum of T .

Soln. S is a bounded set. So there exists $M \in \mathbb{R}_{\geq 0}$ such that for any $x \in S$, $|x| < M$. By triangle inequality, $|x - y| \leq |x| + |y| < 2M$. So, T is bounded.

$$\sup T = |\sup S - \inf S|.$$