A crisper alternative to the fuzzy boundary

Anamitro Biswas
joint work with
Subhankar Jana and Juthika Mahanta

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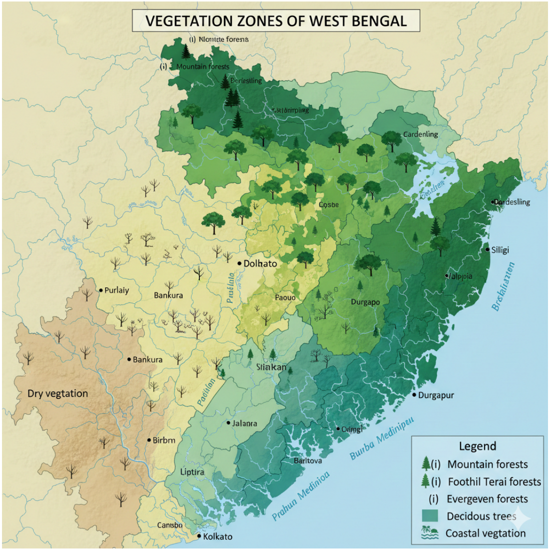


A is a *spatial region* in a connected topological space X if $\phi \neq A^{\text{connected}} \subset X$ such that $A = \overline{A^\circ}$.

If A and B are sets, spatial relations between A and B $(\partial A \wedge \partial B,\ A^\circ \wedge B^\circ,\ \partial A \wedge B^\circ,\ A^\circ \wedge \partial B)$.

[Egenhofer-Franzosa] Feasible tuples: $(\varnothing, \varnothing, \varnothing, \varnothing)$, $(\odot, \varnothing, \varnothing, \varnothing)$, $(\odot, \odot, \varnothing, \varnothing)$, $(\varnothing, \odot, \odot, \varnothing)$, $(\odot, \odot, \odot, \varnothing)$, $(\varnothing, \odot, \varnothing, \odot)$, $(\odot, \odot, \varnothing, \odot)$, $(\varnothing, \odot, \odot, \odot)$, $(\odot, \odot, \odot, \odot)$ [$\odot$ stands for non-empty set]

Fuzzy set

[Zadeh] $\mu : \Omega \to [0, 1]$

$$A = \{(x, \mu_A(x)) \mid x \in \Omega\}$$

crisp set: $\mu_A(x) \in \{0, 1\}$

1. A *empty* iff $\mu_A(x) = 0 \; \forall \; x \in \Omega$.
2. $A = B$ iff $\mu_A(x) = \mu_B(x) \; \forall \; x \in \Omega$.
3. *Complement* of a fuzzy set A is denoted by A^C and is defined by

$$\begin{aligned} \mu_{A^C}(x) &= 1 - \mu_A(x) \; \forall \; x \in \Omega \\ \text{or, } \mu_{A^C} &= 1 - \mu_A. \end{aligned}$$

4. A fuzzy set A is *contained* in a fuzzy set B or A is a *subset of* B or A is *smaller than or equal to* B iff

$$\mu_A \leq \mu_B.$$

$$\mu_{A\cup B} = \max\{\mu_A, \mu_B\} \text{ or, } \mu_C = \mu_A \vee \mu_B$$
$$\mu_{A\cap B} = \min\{\mu_A, \mu_B\} \text{ or, } \mu_C = \mu_A \wedge \mu_B$$

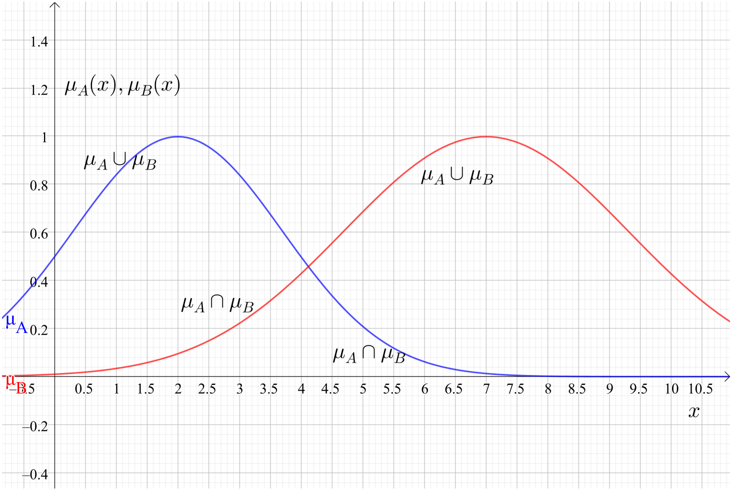


Figure: ▲ Union and intersection of fuzzy sets in $\mathbb{R}^1$

Fuzzy topology I

[Chang] Let τ be a family of fuzzy sets in X such that

1. $0_X, 1_X \in \tau$;
2. $A, B \in \tau \Rightarrow A \wedge B \in \tau$;
3. $A_i \in \tau \ \forall \ i \in I \Rightarrow \vee_{i \in I} A_i \in \tau$.

τ *fuzzy topology* for X; (X, τ) *fuzzy topological space* or *fts* for short. Properties that remain invariant under fuzzy homeomorphism, are called *topological properties*.

Fuzzy boundary

1. (Warren) Infimum of all closed fuzzy sets D in X with the property $\mu_D(y) \geq \mu_{\overline{A}}(y) \ \forall \ y \in \left\{x \in X \mid \mu_{\overline{A}}(x) \wedge \mu_{\overline{A^C}}(x) > 0\right\}$.
2. (Pu, Liu) $\partial A = \overline{A} \cap \overline{A^C}$
   (Bjϕrke) $\mu_{\partial A}(x, y) = 2 \min \{\mu_A(x, y), 1 - \mu_A(x, y)\}$, factor 2 normalizes the membership values.

Fuzzy topology II

3. (Cuchillo-Ibáñez, Tarrés) Infimum of all closed fuzzy sets B in X such that $B(x) \geq \overline{A}(x) \ \forall \ x \in X \ni \overline{A}(x) - A^{\circ}(x) > 0$.
4. (Guang-qing, Chong-you) $\partial A = \vee \left\{ x_{\beta} \mid \beta \leq \overline{A}(x), \beta > A^{\circ}(x) \right\}$.
5. (Mahanta, Das) $\partial A = \left(\overline{A}\right)^{\circ} \wedge \overline{A^{\circ}}$.

Subsets of the boundary ∂A:

- c-boundary, $\partial^c A = \{x_\lambda \in \partial A : \mu_{\partial A}(x) = \mu_{\overline{A}}(x)\}$;
- i-boundary, $\partial^i A = \{x_\lambda \in \partial A : \mu_{\partial A}(x) < \mu_{\overline{A}}(x)\}$.

Subsets of the closure $\overline{A}$:

- i-closure, $A^{\pm} = \left\{x_\lambda \in \overline{A} \mid \mu_{\partial A}(x) < \mu_{\overline{A}}(x)\right\}$;
- c-closure, $A^{\mp} = \left\{x_\lambda \in \overline{A} \mid \mu_{\partial A}(x) = \mu_{\overline{A}}(x)\right\}$.

Crisp subsets:

- Core, $A^{\oplus} = \{x_\lambda \in X \mid \mu_{A^\circ}(x) = 1\} \subseteq A^\circ$;
- Outer, $A^{=} = \{x_\lambda \in A^e \mid \mu_{A^e}(x) = 1\} \subseteq A^e$, where $A^e = \left(A^C\right)^\circ$.

More invariants:

- b-closure of A in the fts, $A^\perp = A^\pm - A^\oplus$;
- fringe of A, $\ell A = \partial^c A \vee A^\perp$.

The following are topological properties: (i) c-boundary, (ii) i-boundary, (iii) c-closure, (iv) i-closure, (v) b-closure, (vi) core, (vii) outer.

A simple fuzzy region is a subset of an fts that is

- non-empty double connected regular closed set
- whose core is the interior of a simple closed region,
- the i-closure is a non-empty double connected regular open set,
- the interior of the b-closure is a non-empty double-connected regular open set,
- the c-boundary is a non-empty double-connected closed set
- the outer is a non-empty double-connected open set.

Intersection matrices

[Tang-Kainz]
9-*intersection matrix*

$$I_9 = \begin{pmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge \ell B & A^{\oplus} \wedge B^{=} \\ \ell A \wedge B^{\oplus} & \ell A \wedge \ell B & \ell A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge \ell B & A^{=} \wedge B^{=} \end{pmatrix},$$

4×4 *intersection matrix*

$$I_{4\times 4} = \begin{pmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge B^{\perp} & A^{\oplus} \cap \partial^c B & A^{\oplus} \wedge B^{=} \\ A^{\perp} \wedge B^{\oplus} & A^{\perp} \wedge B^{\perp} & A^{\perp} \cap \partial^c B & A^{\perp} \wedge B^{=} \\ \partial^c A \wedge B^{\oplus} & \partial^c A \wedge B^{\perp} & \partial^c A \cap \partial^c B & \partial^c A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge B^{\perp} & A^{=} \cap \partial^c B & A^{=} \wedge B^{=} \end{pmatrix}.$$

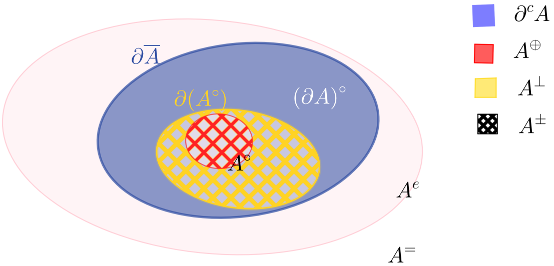


Figure: A pictorial example of a simple region

For a simple fuzzy region,
9-intersection matrix: $2^9 = 512 \to 44$
4×4 intersection matrix: $2^{16} = 65536 \to 152$

Coast

For $\alpha \in I = [0,1]$, we define the α-*interior* of a set A as

$$A_\alpha = \begin{cases} A & \text{if } A > \alpha, \\ 0 & \text{if } A \leq \alpha; \end{cases}$$

and its α-*closure* as

$$A^\alpha = \begin{cases} 1 & \text{if } A \geq \alpha, \\ A & \text{if } A < \alpha; \end{cases}$$

and α-*coast* as

$$A_{L,\alpha} = \begin{cases} A & \text{if } A = \alpha, \\ 0 & \text{if } A \neq \alpha; \end{cases}$$

[Shi and Liu] we know that the interior and the closure thus defined generate together a unique topology.

Entropy

$\mathcal{F}(X)$ be the class of all fuzzy sets on X, $\mathcal{P}(X)$ be the class of all crisp sets on X. [Liu Xuechung]
$\left[\frac{1}{2}\right]_X \in \mathcal{F}(X) \ni \mu_{\left[\frac{1}{2}\right]_X}(x) = \frac{1}{2} \;\forall\; x \in X$.
Define $\mathcal{F} \subseteq \mathcal{F}(X) \ni$

1. $\mathcal{P}(X) \subset \mathcal{F}$,
2. $\left[\frac{1}{2}\right]_X \in \mathcal{F}$,
3. $A, B \in \mathcal{F} \Rightarrow A \vee B \in \mathcal{F}, A^C \in \mathcal{F}$.

Definition. $e : \mathcal{F} \to \mathbb{R}^+$ *entropy* on $\mathcal{F}$ if

1. $e(D) = 0 \;\forall\; D \in \mathcal{P}(x)$;
2. $e\left(\left[\frac{1}{2}\right]_X\right) = \max_{A \in \mathcal{F}} e(A)$;
3. $\forall\; A, B \in \mathcal{F}$, and either $\mu_B(x) \geq \mu_A(x) \geq \frac{1}{2}$ or $\mu_B(x) \leq \mu_A(x) \leq \frac{1}{2}$, then $e(A) \geq e(B)$;
4. $e\left(A^C\right) = e(A) \;\forall\; A \in \mathcal{F}$.

Example. Let $X = \{x_1, x_2, \ldots, x_n\}$. For any $A \in \mathcal{F}(X)$,

$\hat{A} \in \mathcal{P}(X) \ni \mu_{\hat{A}}(x) = \begin{cases} 1 & \text{when } \mu_A(x) > \frac{1}{2} \\ 0 & \text{when } \mu_A(x) \leq \frac{1}{2} \end{cases}$,

$e'(A) = \left(\sum_{i=1}^{n} \left| \mu_A(x_i) - \mu_{\hat{A}}(x_i) \right|^{\omega} \right)^{\frac{1}{\omega}}$ $\forall A \in \mathcal{F}(X)$. Then e' is an entropy on $\mathcal{F}(X)$, where $\omega \geq 1$.

If an entropy e on $\mathcal{F}$ satisfies $e\left(\left[\frac{1}{2}\right]_X\right) = 1$, then e is called a *normal entropy* on $\mathcal{F}$.

Let e be an entropy of $\mathcal{F}$, then $\hat{e}$ given by $\hat{e}(A) = \dfrac{e(A)}{e\left(\left[\frac{1}{2}\right]_X\right)}$ $\forall A \in \mathcal{F}$ is a normal entropy on $\mathcal{F}$.

$$\left[\frac{1}{2}\right]_X$$

Distribution of SARS among people in a community

[Shi and Liu] $\mu_{A_{1/2}}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) > \frac{1}{2}; \\ 0 & \textit{otherwise}. \end{cases}$

1. 7-tupled topological relation two fuzzy objects, used in GIS to quantify the effect of one fuzzy entity on another.
2. (SARS) Tracing the path of an affected person, investigating the effect of that person as well as an infected region on the community.

5 topologically invariant components:

1. $(A \wedge B)_{\mu_A+\mu_B \leq 1}$,
2. $\bigcap (A \wedge B)^{\mu_A \text{ or } \mu_B \leq 0.5}_{\mu_A+\mu_B>1}$
3. $\bigcap \left(A \wedge B^C\right)^{\mu_A \text{ and } \mu_B>0.5}_{\mu_A+\mu_B>1}$,
4. $\bigcap \left(A^C \wedge B\right)^{\mu_A \text{ and } \mu_B>0.5}_{\mu_A+\mu_B>1}$,
5. $\{x \in X | \mu_A(x) = \mu_B(x)\}$

Properties of coast with $\alpha = \frac{1}{2}$

Theorem

$A_L = A_{L,\frac{1}{2}}$ is a topological invariant.

If $A, B \subseteq X$ be fuzzy sets, then $A_L \wedge B_L = (A \wedge B)_L$, $A_L \vee B_L = (A \wedge B)_L$.
If $A, B \subseteq X$ be fuzzy sets, then $A_L \wedge B_L = (A \wedge B)_L$, $A_L \vee B_L = (A \wedge B)_L$.

1. Unlike the conventional boundaries, Coast is not closed.
2. $\overline{A} > A^\circ \vee A_L$, however it is not necessarily the supremum as stated in the condition.
3. Coast of a set is identical to the coast of its complement.
4. If the interior of the Coast in $\mathbb{R}^2$ is empty, then the Coast is a line or a set of scattered points. Else, it is two-dimensional.

In a simple closed region with outwardly monotone and continuous membership function, A_L is a closed loop. If it is strictly decreasing outward then A_L is a 1-dimensional loop.

Warren's properties

(W1) ∂A is closed. [(B1), (B2), (B3), (B4)]

(W2) The closure is the supremum of the interior and the boundary, i.e., $\overline{A} = A^{\circ} \vee \partial A$. [(B1), (B2), (B3), (B4)]

(W3) This fuzzy definition of boudary becomes the usual topological boundary when the sets concerned are crisp. [(B1), (B2), (B3), (B4)]

(W4) The boundary operator is an equivalent way of defining a fuzzy topology. [(B1), (B2), (B3), (B4)]

(W5) The boundary of a fuzzy set is identical to the boundary of the complement of the set. [(B2), (B4)]

(W6) If a fuzzy set is closed (or open), then the interior of the boundary is empty. [(B3)]

(W7) If a fuzzy set is both open and closed, then the boundary is empty.

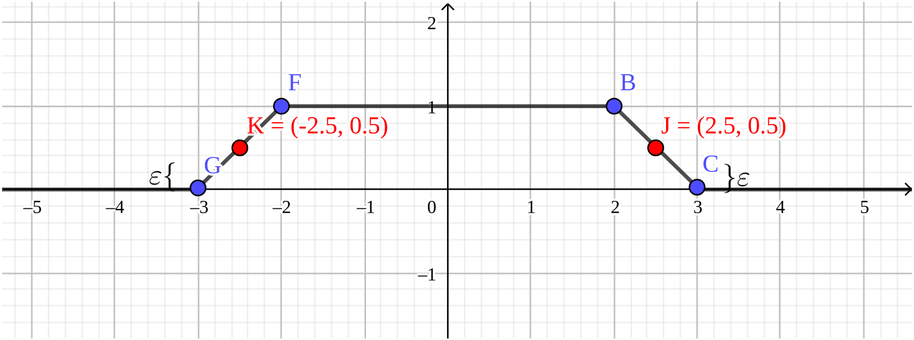
A be a fuzzy set in X where

$$\mu_A = \begin{cases} 0 & \text{if } |x| > 3, \\ \varepsilon & \text{if } |x| = 3, \\ ||x| - 3| & \text{if } 2 < |x| < 3, \\ 1 & \text{if } |x| \leq 2. \end{cases}$$

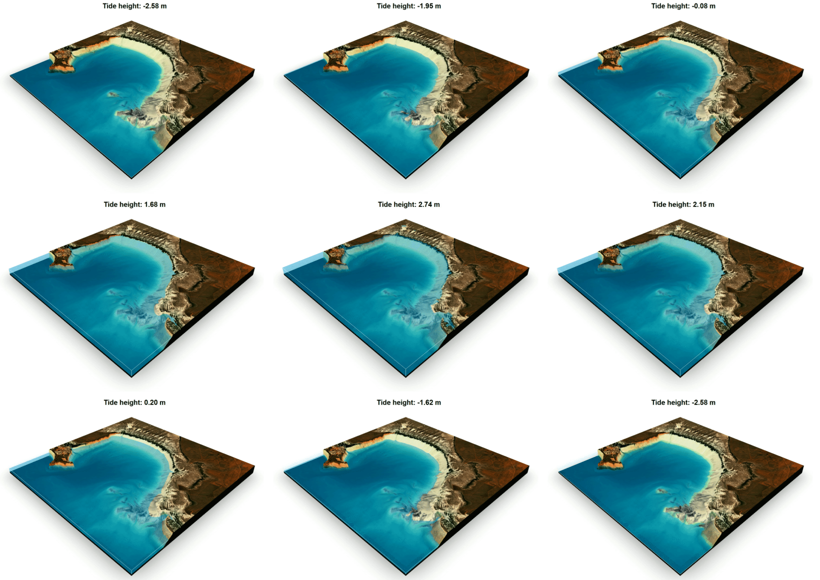
Then A is closed in $\mathbb{R}$ as it is upper semicontinuous from $\mathbb{R}$ to I. Then,

$$\mu_{A^\circ} = \begin{cases} 0 & \text{if } |x| > 3, \\ 0 & \text{if } |x| = 3, \\ ||x| - 3| & \text{if } 2 < |x| < 3, \\ 1 - \varepsilon_1 & \text{if } |x| = 2, \\ 1 & \text{if } |x| < 2. \end{cases}$$

Here, $L = \{\pm 2.5\}$ and $\mu_{A_L} = \left\{ \left(2.5, \frac{1}{2}\right), \left(-2.5, \frac{1}{2}\right) \right\}$.


F
B
K = (-2.5, 0.5)
J = (2.5, 0.5)
G
C
ε
ε
2
1
0
–1
–5
–4
–3
–2
–1
1
2
3
4
5

Roebuck Coast, Australia



Coast given by superimposing 116 frames a day

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Thank you :-)