Aspects of the Davenport Constant for Finite Abelian Groups

Anamitro Biswas
anamitroappu@gmail.com

[joint work with Dr. Eshita Mazumdar]

18th International Conference of IMBIC
Mathematical Sciences for Advancement of Science and Technology (MSAST) 2024
Dec 21st-23rd

A sequence, also called a *multiset*, is a member of the free abelian group $\mathcal{F}(G)$ generated by G. We shall denote by

- *juxtaposition* when elements form a **sequence**, and by
- *addition* the **group operation**.

Zero-sum problems in additive number theory:

1. Conditions which ensure that given sequences have non-empty zero-sum subsequences with prescribed properties.
2. Structure of extremal sequences which have no zero-sum subsequences.

Baayen, Erdős and Davenport posed the problem to determine

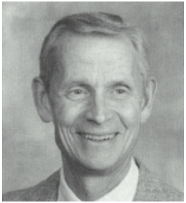
$$D(G) = \min\{|S| \; : \quad S \in \mathcal{F}(G) \text{ has a non-trivial zero subsum}\}$$

called the *Davenport constant* for group G.

$D(G) \leq |G|$ for any group G.

Proof. Let $S = x_1 x_2 \dots x_n \in \mathcal{F}(G)$ where $|G| = n$. Consider

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ &\vdots \\ s_n &= x_1 + x_2 + \cdots + x_n \end{aligned}$$



If all the s_i's are distinct, we must have $0 \in \{s_1, \dots, s_n\}$ since G contains only n elements. Else, we have by the Pigeon-Hole Principle, $\exists\ i \neq j \ni s_i = s_j$. Then, $x_{i+1} \dots x_j$ is a zero-sum subsequence. So, $D(G) \leq |G|$. $\square$

Particularly for $G = C_n = \langle 1 \rangle$ (the cyclic group of order n) we can construct the sequence $S = \underbrace{11\dots1}_{n-1}$ such that $0 \notin [S]$ (the set of subsums of sequence S including $\sigma(S)$). Thus $D(C_n) \geq n \Rightarrow D(C_n) = n$.

Theorem (Olson, 1961)

$$D\left(\bigoplus_{i=1}^{d} C_{p^{e_i}}\right) = 1 + \sum_{i=1}^{d} \left(p^{e_i} - 1\right)$$

Olson furthered conjectured that for any finite abelian group

Conjecture (Olson, 1961)

$D\left(\bigoplus_{i=1}^{d} C_{n_i}\right) = 1 + \sum_{i=1}^{d} \left(n_i - 1\right) = D^*\left(G\right)$ *where* $n_i \mid n_{i+1}$ *for* $i \in \{1, \ldots, d-1\}$.

Too good to be true, this conjecture isn't in fact, since, for example, we have

Geroldinger and Schneider, 1992

For odd $m, n \ni 3 \leq m|n$,

$$D\left(C_m \oplus C_n^2 \oplus C_{2n}\right) > D^*\left(C_m \oplus C_n^2 \oplus C_{2n}\right).$$

However, we have $D(C_m \oplus C_n) = D^*(C_m \oplus C_n)$ where $m \mid n$ and all known counterexamples are of rank ≥ 4. So, apart from characterization of the groups for which this conjecture holds, it is also an open problem whether the conjecture holds for groups of rank 3.

Theorem (Bhowmik and Schlage-Puchta)

$D(C_3 \oplus C_3 \oplus C_{3d}) = D^*(C_3 \oplus C_3 \oplus C_{3d})\ \forall\ d \in \mathbb{N}$.

Conjecture

Fixed $5 \leq p \in \mathbb{P}$, $G \cong C_p^3 \oplus C_2$. $S = (x_1, y_1) \ldots (x_{4p-2}, y_{4p-2}) \in \mathcal{F}(G)\ \ni$

$$\begin{aligned} y_1 = \cdots = y_r &= 1 \\ y_{r+1} = \cdots = y_{4p-2} &= 0 \end{aligned}$$

for even $r \in [2p+2, 4p-6]$. Then $0 \in [S]$.

Theorem (Sheikh, 2017)

$D(C_p \oplus C_p \oplus C_{2p}) = D^*(C_p \oplus C_p \oplus C_{2p})$ for all those primes p for which the previous Conjecture holds.

Theorem (Sheikh, 2017)

$D(C_5 \oplus C_5 \oplus C_{5d}) \leq D^*(C_5 \oplus C_5 \oplus C_{5d}) + 4\ \forall\ d \in \mathbb{N}$.

With the help of computer, Sheikh further confirmed, $D\left(C_5 \oplus C_5 \oplus C_{10}\right) = D^*\left(C_5 \oplus C_5 \oplus C_{10}\right) = 18$. Generalizing Sheikh's approach,

Conjecture

Fix $p, q\ (p \neq q), p \in \mathbb{P}$; define $G := C_p^d \oplus C_q$.
Let $m = p(q+2) - 2$.
Let $S = (x_1, y_1) \ldots (x_m, y_m) \in \mathcal{F}\left(C_p^d \oplus C_q\right)$. Suppose

$$y_{\sum_{i=1}^{t} r_i + 1} = \cdots = y_{\sum_{i=1}^{t+1} r_i} = t + 1 \ \ (t \in [0, q-1])$$

where $r = \sum_{i=1}^{q-1} r_i$. If

1. $r \in [pq+1, p(q+2) - 2]$ and
2. $\sum_{i=1}^{q-1} i r_i \equiv 0 \pmod{q}$,

then $0 \in [S]$.

Theorem (me and Eshita Ma'am)

Let p be a prime such that Conjecture 2 holds. Then,
$D\left(C_p^d \oplus C_q\right) = D^*\left(C_p^d \oplus C_q\right)$.

(Bhowmik and Schlage-Puchta, 2007) For $G \cong C_3 \oplus C_3 \oplus C_{3q},\ q \in \mathbb{N}$, $D(G) = D^*(G)$. So Conjecture 2 is true for $p = 3, d = 3$ at least.
For the group $C_5 \oplus C_5 \oplus C_{15}$, the conjecture can be further reduced to the assumption for only the cases where $10 \leq r_2 - \left\lfloor \frac{r_2 - r_1}{3} \right\rfloor$, by constructing subsequences of this form:

$$\begin{aligned} S' &= \left(\prod_{i=1}^{\alpha} (x_{r+i}, y_{r+i})\right) \left(\prod_{i=1}^{r_1} \sigma\left((x_i, y_i)(x_{r_1+i}, y_{r_1+i})\right)\right) \\ & \left(\prod_{i=\frac{r_1+\ell}{3}}^{\left\lfloor \frac{r_2}{3} \right\rfloor} \sigma\left((x_{r_1+i}, y_{r_1+i})(x_{r_1+i+1}, y_{r_1+i+1})(x_{r_1+i+2}, y_{r_1+i+2})\right)\right) \end{aligned}$$

where $\ell = 3\left\lceil \frac{r_1}{3} \right\rceil$, $\alpha = s - r = 23 - r$ and $\beta = \min\left(r_1, 23 - \alpha - r_1\right) = r_1$.

the r-wise Davenport constant

There can be a number of possible ways to generalize the idea of Davenport constant which just requires *one* sequence that adds to *zero*.

Definition (Girard and Schmid)

For $r \in \mathbb{N}$, the r-wise Davenport constant of group G,

$$D_r(G) = \min\{k \in \mathbb{N} \mid S \in \mathcal{F}(G) \ \& \ |S| \geq k \Rightarrow S \text{ has } r \text{ disjoint zero-sum subsequences}\}.$$

It is clear that $D_r(G) \leq D_{r+1}(G)$.

Theorem (Girard and Schmid, 2019)

- *If $n, r \in \mathbb{N}$, $D_r(C_n) = rn$.*
- *Let $G \cong C_m \oplus C_n$ where $m|n$. Then, $D_r(G) = rn + m - 1$.*

Let p be an odd prime and $G \cong C_{p^{e_1}} \oplus C_{p^{e_2}} \oplus \cdots \oplus C_{p^{e_d}} \ni e_i \leq e_{i+1}$ for $i \in [1, d-1]$. $S = (\underbrace{1, 0, \ldots, 0}_{d})^{rp^{e_d}-1}(0, 1, \ldots, 0)^{p^{e_{d-1}}-1} \ldots$

$(0, 0, \ldots, 0, 1)^{p^{e_1}-1}$ does not have r disjoint zero-sum subsequences.

Therefore, $D_r(G) \geq rp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - d + 1$.

$\eta_r(G) = \min\{k \in \mathbb{N} \mid \forall S \in \mathcal{F}(G) \ni |S| \geq k, (0_{\text{small}})^r \in [S]\}$. By 0_{small} we mean a small zero subsum, i.e., a subsum of length $\leq \exp(G)$.

Theorem (Fan, Gao, Wang, Zhong (2013))

$m \in \mathbb{N}$; let H be a finite abelian group $\ni$

- $exp(H)|m$
- $m \geq D(H)$
- $D(C_m \oplus C_m \oplus H) = 2m + D(H) - 2$

Then $\eta_1(C_m \oplus H) \leq 2m + D(H) - 2$.

If $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$, then $\eta_1(G) \leq D(G) + \exp(G)$. Let $1 \leq r \in \mathbb{Z}$.

If $\eta_1(G) \leq D(G) + \exp(G)$, then $\eta_r(G) \leq D(G) + r \exp(G)$. For such a group, $D_r(G) \leq \eta_r(G) = D(G) + (r-1)\exp(G)$.

Theorem (Geroldinger and Halter-Koch, 2006)

Let p be an odd prime and $G \cong C_{p^{e_1}} \oplus C_{p^{e_2}} \oplus \cdots \oplus C_{p^{e_d}} \ni e_i \leq e_{i+1}$ for $i \in [1, d-1]$. Then, $D_r(G) = rp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - d + 1$.

Theorem (Delorme, Ordaz and Quiroz (2001))

$p \in \mathbb{P}, n \geq 2$ & $(m, p^n) = 1$. Then, for $G \cong C_p \oplus C_p \oplus C_{p^n m}$, $D(G) = D^*(G)$.

Delorme, Ordaz and Quiroz (2001)

Let $H \lhd G$ and $r \in \mathbb{N}$, then $D_r(G) \leq D_{D_r(H)}(G/H)$ and $D(G) \geq D(H) + D(G/H) - 1$.

Theorem (Me and Eshita Ma'am)

$G \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_d} \;\; (n_1, n_2, \ldots, n_d \in \mathbb{N})$
i.e., $G = C_{\prod_{j=1}^{\ell} p_j^{e_1^{(j)}}} \oplus C_{\prod_{j=1}^{\ell} p_j^{e_2^{(j)}}} \oplus \cdots \oplus C_{\prod_{j=1}^{\ell} p_j^{e_d^{(j)}}}$ *where*

- *w.l.o.g.* $e_i^{(j)} \in \mathbb{Z} \ni 0 \leq e_i^{(j)} \leq e_{i+1}^{(j)} \; \forall \; 1 \leq j \leq \ell$
- *but all* $e_i^{(j)}$*'s are not zero for each* $j \in \{1, \ldots, \ell\}$ $(1 \leq i \leq d)$
- $p_1, p_2, \ldots, p_\ell$ *primes* $\ni p_j^{e_d^{(j)}} \geq 1 + \sum_{i=1}^{k-1} \left(p_j^{e_i^{(j)}} - 1 \right) \; \forall \; j = 1, 2, \ldots, \ell.$

Let $\varphi(p_j) = \sum_{i=1}^{d-1} p_j^{e_i^{(j)}} - d + 1$ *for* $j = 1, \ldots, \ell.$

$$\begin{aligned} \textit{Then, } r \prod_{j=1}^{\ell} p_j^{e_d^{(j)}} \;\; + \;\; \sum_{i=1}^{d-1} \left(\prod_{j=1}^{\ell} p_j^{e_i^{(j)}} - 1 \right) &\leq D_r(G) \\ &\leq \;\; r \prod_{j=1}^{\ell} p_j^{e_d^{(j)}} + \sum_{m=1}^{\ell-1} \left(\left(\prod_{j=m+1}^{\ell} p_j^{e_d^{(j)}} \right) \varphi(p_m) \right) + \varphi(p_\ell). \end{aligned}$$

Corollary (a generalization of Delorme et al)

For $G = C_{p^{e_1}} \oplus C_{p^{e_2}} \oplus \cdots \oplus C_{p^{e_{d-1}}} \oplus C_{mp^{e_d}}$ *with* $e_i \leq e_{i+1}$ *such that* $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$,

$$D_r(G) = rmp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - d + 1.$$

Result

The error becomes negligible, i.e., $\frac{\text{upper bound}}{\text{lower bound}} \to 1$ if

- either p_j's $[j \in \{1, \ldots, \ell\}]$ are large;
- $e_i^{(j)}$ $(j = 1, \ldots, \ell; i = 1, \ldots, d)$'s are higher natural numbers;
- r increases.

$$\text{error} := \frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}}$$

[error has been throughout multiplied by 100 for easy visualization]

Group		$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$C_{2.3.5} \oplus C_{2^2.3^2.5^2} \oplus C_{2^3.3^3.5^3}$	UB	41778	68778	95778	122778	149778
	LB	27928	54928	81928	108928	135928
	diff	13850				
	err	49.59181	25.21483	16.90509	12.71482	10.18922
$C_{3.5.7} \oplus C_{3^2.5^2.7^2} \oplus C_{3^3.5^3.7^3}$	UB	1596033	2753658	3911283	5068908	6226533
	LB	1168753	2326378	3484003	4.641628	5799253
	diff	427280				
	err	36.55862	18.36675	12.26405	9.205391	7.367845
$C_{5.7.11} \oplus C_{5^2.7^2.11^2} \oplus C_{5^3.7^3.11^3}$	UB	69921553	126988178	184054803	241121428	298188053
	LB	57215233	114281858	171348483	228415108	285481733
	diff	12706320				
	err	22.207932	11.118405	7.415484	5.562819	4.450835

Group		$r=1$	$r=5$
	UB	3292613286703417	16039460139018988
$C_{31.47.101} \oplus C_{31^2.47^2.101^2} \oplus C_{31^3.47^3.101^3}$	LB	3186733368408697	15933580220724268
	diff	105879918294720	
	err	3.3225220	0.6645080

$C_{31^2.47^3.101^3} \oplus C_{31^8.47^9.101^5} \oplus C_{31^{17}.47^{21}.101^7}$

$r=1$

Upper bound=31437892770703911707659464196047220569991824639379613485534754890454138880
Lower bound= 314378927707027215691704348813743973043171844658271204789424845095104937984
Difference= 118904470778768425226721896032537391818273330798491684160143 36

Error= 0.00000000000037822023

$r=5$

Upper bound= 157189463853514772875934262174821754444102869672022430991882915072358062489 6
Lower bound=157189463853513587759126621169493542247059768064603070867175000262642027724 8

Error= 0.00000000000000756440

References

G. Bhowmik and J. Schlage-Puchta, Davenport's constant for groups of the form $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$. In Additive combinatorics, volume 43 of CRM Proc. Lecture Notes, 307-326. *Amer. Math. Soc.,* Providence, RI, (2007).

A. Biswas and E. Mazumdar, *Davenport constant for finite abelian groups with higher rank*, arXiv:2402.09999 (2023).

A. Geroldinger and F. Halter-Koch, Non-Unique Factorizations: Algebraic, Combinatorial and Analytic Theory, *Chapman & Hall/CRC* (2006).

A. Geroldinger, R. Schneider, On Davenport's constant, *J. Combin. Theory Ser. A* 61, 147—152, (1992).

B. Girard and W. A. Schmid, Direct zero-sum problems for certain groups of rank three,*J. Number Theory,* 197, 297—316, (2019).

J. E. Olson, A Combinatorial Problem on Finite Abelian Groups, I, *J. Number Theory,* 1, 8–10, (1969).

K. Rogers, A Combinatorial problem in Abelian groups, *Proc. Cambridge Phil. Soc.* 59, 559-562, (1963).

A. Sheikh, The Davenport constant of finite abelian groups, *Thesis, University of London,* (2017).

Thank you :)
A handout for this talk can be found at: https://anamitro.github.io/files/anamitro_msast24.pdf