# Problem set 3

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June 10, 2025

**Note 1** *Writing solutions like this in the exams might not fetch you full marks. Please clearly mention the name of any theorem or result you use; and also try to be very specific with ε-δ rigourously in each problem. Don't write one step without proper literal justification from the definition, unless prompted otherwise.*

1. Show that $\sum_{n=1}^{\infty} \frac{1-n}{1+2n}$ and $\sum_{n=1}^{\infty} (-1)^n$ diverge. Do $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ and $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$ converge?

*Soln*. $\left|\frac{1-n}{1+2n}\right| = \left|\frac{\frac{1}{n}-1}{\frac{1}{n}+2}\right| \to \left|-\frac{1}{2}\right| \neq 0$, so $\sum_{n=1}^{\infty} \frac{1-n}{1+2n}$ does not converge.

For even $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} (-1)^n = 0$, and for odd $n$, $\sum_{n=1}^{\infty} (-1)^n = 1$. The sequence of partial sums $\left\{\sum_{i=1}^{n} (-1)^i\right\}_{n\in\mathbb{N}}$ oscillates hence does not converge.

The third one is similar to the first one.

$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right) = \log\left(\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\right) = \log\left(\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)\right)$. Consider the sequence of partial sums: $\left\{\log\left(\prod_{i=1}^{n}\left(\frac{i+1}{i}\right)\right)\right\}_{n\in\mathbb{N}} = \left\{\log\left(\frac{n+1}{1}\right)\right\}_{n\in\mathbb{N}} = \{\log(n+1)\}_{n\in\mathbb{N}}$ which is increasing. Thus the series is divergent. A series like this is called a *telescoping series*, because it just brings the initial and the distant end term close together, skipping the intermediate distance like a telescope does.

2(i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$.

*Soln*. Let $a_n = \frac{(-1)^n}{(2n)!}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(2n+1)(2n+2)}\right| \to 0$ as $n \to \infty$. Therefore, the series is convergent.

2(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$.

*Soln*. The series, being absolutely convergent, is also convergent.

2(iii) $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}\right)$.

*Soln.* $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}+\frac{(-1)^{n-1}}{n}\right)=\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{1}{\sqrt{n}}+\frac{1}{n}\right)$. For $n$ odd, the subseries dominates the series $\sum\frac{1}{n}$ which is divergent. For $n$ odd, all the individual entries are positive. So, the series in all is divergent.

2(iv) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{2n-1}\right)$.

*Soln.* $\frac{n}{2n-1}-\frac{n+1}{2n+1}=\frac{1}{(2n+1)(2n-1)}$. So, $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{2n-1}\right)=\sum_{k=1}^{\infty}\frac{1}{(4k-1)(4k-3)}$, taking $n+1=2k$ and seeing two terms of the series at a time, then the next two terms etc. This shows that the series is divergent.