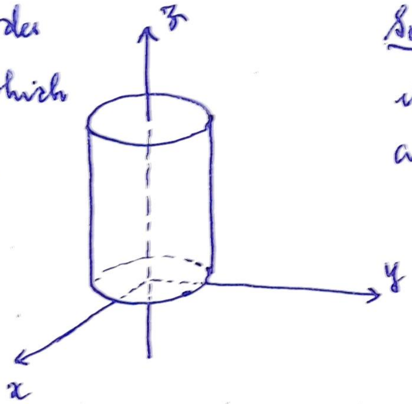
TRIPLE INTEGRALS

1. Find the mass of a cylinder centered on the z-axis which has height h, radius a and density $\delta = x^2+y^2$.



Soln. To find the mass we integrate the product of density and volume:

$$\text{mass} = \iiint_D \delta \, dV$$

$$= \iiint_D r^2 \, dV.$$

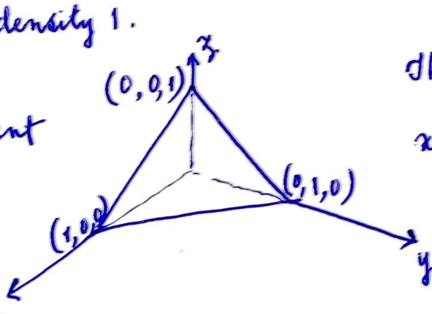
Naturally, we will use cylindrical coordinates in this problem. The limits on z run from 0 to h. The x and y coordinates lie in a disk of radius a, so $0 \le r < a$ and $0 < \theta \le 2\pi$.

~~Inner integral~~ $$\text{Mass} = \iiint_D r^2 \, dV = \int_0^{2\pi} \int_0^a \int_0^h r^2 \, dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^a \int_0^h r^3 \, dz \, dr \, d\theta.$$

Ans: $\dfrac{\pi h a^4}{2}$.

2. Find the moment of inertia of the tetrahedron shown, about the z-axis. Assume the tetrahedron has density 1.



The tetrahedron bounded by $x+y+z=1$ and the coordinate planes.

To compute the moment of inertia, we integrate distance squared from the

z-axis times mass: $\iiint_D (x^2+y^2) \cdot 1 \, dV$

Using cylindrical coordinates about the axis of rotation would give us an "easy" integrand (r) and complicated limits. The integrand x^2+y^2 is not particularly intimidating, so we instead use rectangular coordinates. Integrating first with respect to y or x is preferrable; $(x^2+y^2)(1-x-y)$ is a somewhat more intimidating integrand.

To find our limits of integration, we let y go from 0 to the slanted plane $x+y+z=1$. The x and z coordinates are in R, the <u>projection</u> of D to the xz-plane which is bounded by the x and z-axes and the line $x+z=1$.

$$\text{Moment of inertia} = \int_0^1 \int_0^{1-z} \int_0^{1-x-z} (x^2+y^2) \, dy\, dx\, dz = \frac{1}{30}.$$

3. Find the moment of inertia about the z-axis of a solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=1$. Assume the solid has uniform density 1.

<u>Soln.</u> We use the formula $I = \iiint \rho r^2 dV$ with density $\rho = 1$. Converting to polar coordinates, the equation of the paraboloid becomes $z=r^2$ and we get limits of integration $0 \leq r \leq \sqrt{z}$.

$$I = \iiint_{\text{solid}} \rho r^2 \, dV$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} r^2 r \, dr\, d\theta\, dz = \int_0^1 \int_0^{2\pi} \frac{z^2}{4} \, d\theta\, dz = \frac{\pi}{6}.$$