Tutorial Sheet: functions

1. (a) $\lim_{x \to 0} [x]$

$[x] = 0$ for $x < 1$.

So $\lim_{x \to 0} [x] = \lim_{1 > x \to 0} [x] = 0$.

(b) $\left.\begin{array}{l} sgn(x) = 1 \text{ if } x > 0 \\ sgn(x) = -1 \text{ if } x < 0 \end{array}\right\}$ limit does not exist since

$\lim_{x \to 0+} sgn(x) \neq \lim_{x \to 0-} sgn(x)$.

(c) $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Within every interval $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$, however small, $\sin \frac{1}{x}$ takes all the values in $[-1, 1]$.

(d) $\lim_{x \to 0} \underset{\substack{\downarrow \\ 0}}{\sqrt{x}} \; \underbrace{\sin \frac{1}{x}}_{>-1,\ <1} = 0$

(e) $\lim_{x \to 0} x \cos \frac{1}{x}$.

$-1 < \cos \frac{1}{x} < 1$, $\quad x \to 0 \quad \Rightarrow x \cos \frac{1}{x} \xrightarrow{x \to 0} 0$.

2. $\lim_{x \to 0} \frac{x - |x|}{x}$

$x > 0$, $\frac{x - |x|}{x} = \frac{x - x}{x} = 0$

$x < 0$, $\frac{x - |x|}{x} = \frac{x + x}{x} = 2.$

$|x| = -x$

$\therefore \lim_{x \to 0+} \frac{x - |x|}{x} \neq \lim_{x \to 0-} \frac{x - |x|}{x}.$

3. $\lim_{x \to \infty} x^{1 + \sin x}$

~~if $x > M \in \mathbb{R}_+$~~ → We want to see if $x^{1+\sin x}$ exceeds M

Fix $M \in \mathbb{R}_+$. Choose the next $x \in \mathbb{R}_+$ such that $\sin x = 1$. Since $\sin x$ is periodic throughout $\mathbb{R}$, we can always find such x.

Then $x^{1 + \sin x} = x^{1+1} \geq M^{1+1} = M^2 > M$.

So, $x^{1+\sin x}$ is unbounded. So, $\lim_{x \to \infty} x^{1+\sin x}$ does not exist.

3. Since both rational nos. and irrational nos. are dense in $\mathbb{R}$, for any $p \in \mathbb{R}$, $\exists$ a sequence $\{q_n\}_{n \in \mathbb{N}}$ of rational nos. and a sequence $\{r_n\}_{n \in \mathbb{N}}$ of irrational nos. that converges to p.

$\therefore$ By sequential criterion,

$$f(q_n) \xrightarrow{n \to \infty} f(p)$$

$$f(r_n) \xrightarrow{n \to \infty} f(p).$$

Without loss of generality, let's say $p \in \mathbb{Q}$.

Then $\forall\, \varepsilon > 0\ \exists\, N_1 \in \mathbb{N}$ such that $|p - r_m| < \varepsilon\ \forall\, m > N_1$.

Again, $\forall\, \varepsilon > 0\ \exists\, N_2 \in \mathbb{N}$ such that $\underbrace{|f(p) - f(r_m)|}_{= |p - 1 + r_m|} < \varepsilon\ \forall\, m > N_2$.

Take $N = \max(N_1, N_2)$.

Then, $|p - r_m| < \varepsilon\ \forall\, m > N$

$|p - 1 + r_m| < \varepsilon\ \forall\, m > N$.

$\cancel{|(p - r_m) - (p - 1 + r_m)| < |p - r_m| + |p - 1 + r_m| < 2\varepsilon}$

$\cancel{\Rightarrow |1 - 2r_m| < 2\varepsilon}$ ~~for arbitrary~~ $\cancel{\varepsilon > 0}$

$\cancel{\therefore 1 - 2r}$

$$|(p - r_m) + (p - 1 + r_m)| < |p - r_m| + |p - 1 + r_m| < 2\varepsilon$$

$\Rightarrow |2p - 1| < 2\varepsilon$ for arbitrary $\varepsilon > 0$. $\Rightarrow 2p - 1 = 0 \Rightarrow p = \frac{1}{2}$.