r-wise Davenport constant for finite abelian groups

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[based on joint work with Dr. Eshita Mazumdar]

COmbinatorial Number Theory And Connected Topics-II
(CONTACT-II)
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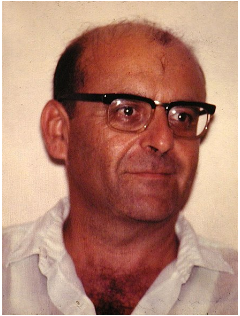
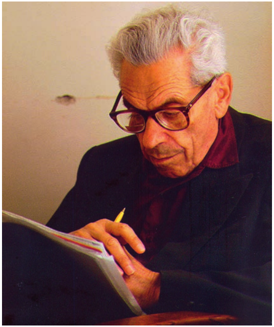
Introduction

- $(G, +, 0)$ finite abelian group.
- G-sequence of length k: $S = (x_1, \ldots, x_k)$ with $x_i \in G$ for each i.
- *zero-sum* G-sequence: $S = (x_1, \ldots, x_k) \ni \sum_i x_i = 0$.
- (Erdős, Ginzburg and Ziv, 1961) Every $(2n-1)$-length sequence from C_n shall have a zero-sum subsequence of length n.





This started the study of zero-sum problems in Additive group theory.

1. conditions which ensure that given sequences have non-empty zero-sum subsequences with prescribed properties.
2. structure of extremal sequences which have no zero-sum subsequences.

Davenport constant

- Baayen, Erdős and Davenport posed the problem to determine

$$D(G) = \min\{|S| \ : \ S \in \mathcal{F}(G) \text{ has a non-trivial zero subsum}\}$$

Davenport constant for group G.

- (K. Rogers, 1963) The Davenport constant is important invariant of the ideal class group of the ring of integers of an algebraic number field.

$D(G) \leq |G|$

$S = (x_1, x_2, \ldots, x_n) \in \mathcal{F}(G)$ where $|G| = n$.

$$
\begin{aligned}
s_1 &= x_1 \\
s_2 &= x_1 + x_2 \\
&\vdots \\
s_n &= x_1 + x_2 + \cdots + x_n
\end{aligned}
$$

- **Case 1** All s_i's are distinct, so $0 \in \{s_1, \ldots, s_n\}$.
- **Case 2** By Pigeon-Hole Principle, $\exists\ i \neq j \ni s_i = s_j$. Then, $(x_{i+1}, \ldots, x_j)$ is a zero-sum subsequence.

So, $D(G) \leq |G|$.

- $G = C_n = \langle 1 \rangle$. Then $D(C_n) \leq n$.
  Again, $\underbrace{11\ldots1}_{n-1}$ does not have any non-trivial zero-sum subsequence.
  So $D(C_n) = n$.
- (Olson, 1969) $G \cong C_{n_1} \times C_{n_2}$, then $D(G) = n_1 + n_2 - 1$.

- (Olson, 1968) For a p-group $G \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_d}}$

$$D(G) = 1 + \sum_{i=1}^{d} (p^{e_i} - 1) \,.$$

Conjecture (Olson)

For any $G \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d} \;\ni n_i | n_{i+1}$,

$$D(G) = 1 + \sum_{i=1}^{d} (n_i - 1) = D^*(G).$$

Is this conjecture true?

This conjecture is false in general. For infinitely many groups of rank 4 this conjecture does not hold.

Geroldinger and Schneider, 1992

For odd $m, n \ni 3 \leq m|n$,

$$D\left(C_m \oplus C_n^2 \oplus C_{2n}\right) > D^*\left(C_m \oplus C_n^2 \oplus C_{2n}\right)$$

Yet it remains to be seen

1. for which groups Olson's conjecture holds
2. whether true for all groups of rank 3.

Number of possible generalizations of Davenport constant.

Definition (Girard and Schmid, 2019)

For a finite abelian group G and $r \in \mathbb{N}$, r-wise Davenport Constant, denoted by $D_r(G)$, is defined to be the least positive integer k such that every sequence of length at least k has r disjoint zero-sum subsequences.

- $D_r(G) \leq D_{r+1}(G)$.
- $D_r(G) = D(G)$ for $r = 1$.

known results

- (Girard and Schmid, 2019) If $n, r \in \mathbb{N}$,

$$D_r(C_n) = rn.$$

- (Girard and Schmid, 2019) Let $G \cong C_m \times C_n$ where $m|n$. Then,

$$D_r(G) = rn + m - 1.$$

- **Question:** What about higher ranks?

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- Let p be an odd prime.
- $G \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_d}}$ such that $e_i \leq e_{i+1}$.
- Max lower bound is given by
  $\underbrace{(1, 0, \ldots, 0)}_{d}^{rp^{e_d}-1}(0, 1, \ldots, 0)^{p^{e_{d-1}}-1} \ldots (0, 0, \ldots, 0, 1)^{p^{e_1}-1}$.

$$\therefore D_r(G) \geq rp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - k + 1.$$

Theorem

If $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$, *we have*

$$D_r(G) = rp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - k + 1.$$

Further results that follow

For $G = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_{d-1}}} \times C_{mp^{e_d}}$ with $e_i \leq e_{i+1}$ such that $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$, we have,

$$D_r(G) = rmp^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - d + 1.$$

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Theorem

Let $G \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_{d-2}}} \times C_{mp^{e_{d-1}}} \times C_{np^{e_d}}$ with $e_i \leq e_{i+1}$ and $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$.

If $m \mid n$ then,

$$\begin{aligned} & rnp^{e_d} + (m-1)p^{e_{d-1}} + \sum_{i=1}^{d-1} p^{e_i} - d + 1 \\ \leq \quad & D_r(G) \\ \leq \quad & rnp^{e_d} + (m-1)p^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - d + 1. \end{aligned}$$

Theorem

Let $G \cong C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_{d-2}}} \times C_{mp^{e_{d-1}}} \times C_{np^{e_d}}$ with $e_i \leq e_{i+1}$ and $p^{e_d} \geq 1 + \sum_{i=1}^{d-1} (p^{e_i} - 1)$.

If $n|m$ then,

$$\begin{aligned}
\max \quad & \Big\{ rnp^{e_d} + (m-1)p^{e_{d-1}} + \sum_{i=1}^{d-1} p^{e_i} - d + 1, \\
& np^{e_d} + (rm-1)p^{e_{d-1}} + \sum_{i=1}^{d-1} p^{e_i} - d + 1 \Big\} \\
\leq \quad & D_r(G) \\
\leq \quad & rnp^{e_d} + (m-1)p^{e_d} + \sum_{i=1}^{d-1} p^{e_i} - d + 1.
\end{aligned}$$

Theorem

$G \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_d}$ $(n_1, n_2, \ldots, n_d \in \mathbb{N})$

i.e., $G = C_{\prod_{j=1}^{\ell} p_j^{e_1^{(j)}}} \times C_{\prod_{j=1}^{\ell} p_j^{e_2^{(j)}}} \times \cdots \times C_{\prod_{j=1}^{\ell} p_j^{e_d^{(j)}}}$

1. *where w.l.o.g.* $e_i^{(j)} \in \mathbb{Z} \ni 0 \leq e_i^{(j)} \leq e_{i+1}^{(j)}$ $\forall\ 1 \leq j \leq \ell$
2. *but all* $e_i^{(j)}$*'s are not zero for each* $j \in \{1, \ldots, \ell\}$ $(1 \leq i \leq d)$
3. $p_1, p_2, \ldots, p_\ell$ *primes* $\ni p_j^{e_d^{(j)}} \geq 1 + \sum_{i=1}^{k-1} \left(p_j^{e_i^{(j)}} - 1 \right)$ $\forall\ j = 1, 2, \ldots, \ell.$

Theorem (Continued)

- *Let* $\varphi(p_j) = \sum_{i=1}^{d-1} p_j^{e_i^{(j)}} - d + 1$ *for* $j = 1, \ldots, \ell$.

Then,
$$r\prod_{j=1}^{\ell} p_j^{e_d^{(j)}} + \sum_{i=1}^{d-1}\left(\prod_{j=1}^{\ell} p_j^{e_i^{(j)}} - 1\right) \leq D_r(G)$$
$$\leq r\prod_{j=1}^{\ell} p_j^{e_d^{(j)}} + \sum_{m=1}^{\ell-1}\left(\left(\prod_{j=m+1}^{\ell} p_j^{e_d^{(j)}}\right)\varphi(p_m)\right) + \varphi(p_\ell).$$

Some comments on the previous theorem

- For $d = 1$, these two bounds coincide.

$$
\begin{aligned}
\bullet\ \text{difference} &= \text{upper bound} - \text{lower bound} \\
&= \sum_{m=1}^{\ell-1} \left(\left(\prod_{j=m+1}^{\ell} p_j^{e_d^{(j)}} \right) \varphi(p_m) \right) + \varphi(p_\ell) - \sum_{i=1}^{d-1} \left(\prod_{j=1}^{\ell} p_j^{e_i^{(j)}} - 1 \right)
\end{aligned}
$$

INDEPENDENT OF r

Define, $\text{error} = \dfrac{\text{upper bound} - \text{lower bound}}{\text{lower bound}}$.

We observe that

1. increasing r
2. with larger primes $p_1, \ldots, p_\ell$
3. higher powers e_i^j $(j = 1, \ldots, \ell; i = 1, \ldots, d)$

$$\text{error} \to 0$$

$$\text{i.e., } \frac{\text{upper bound}}{\text{lower bound}} \to 1$$

.

[error has been throughout multiplied by 100 for easy visualization]

Group		$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$C_{2.3.5} \oplus C_{2^2.3^2.5^2} \oplus C_{2^3.3^3.5^3}$	UB	41778	68778	95778	122778	149778
	LB	27928	54928	81928	108928	135928
	diff	13850				
	err	49.59181	25.21483	16.90509	12.71482	10.18922
$C_{3.5.7} \oplus C_{3^2.5^2.7^2} \oplus C_{3^3.5^3.7^3}$	UB	1596033	2753658	3911283	5068908	6226533
	LB	1168753	2326378	3484003	4.641628	5799253
	diff	427280				
	err	36.55862	18.36675	12.26405	9.205391	7.367845
$C_{5.7.11} \oplus C_{5^2.7^2.11^2} \oplus C_{5^3.7^3.11^3}$	UB	69921553	126988178	184054803	241121428	298188053
	LB	57215233	114281858	171348483	228415108	285481733
	diff	12706320				
	err	22.207932	11.118405	7.415484	5.562819	4.450835

Group		$r=1$	$r=5$
$C_{31.47.101} \oplus C_{31^2.47^2.101^2} \oplus C_{31^3.47^3.101^3}$	UB	3292613286703417	16039460139018988
	LB	3186733368408697	15933580220724268
	diff	105879918294720	
	err	3.3225220	0.6645080

$C_{31^2.47^3.101^3} \oplus C_{31^8.47^9.101^5} \oplus C_{31^{17}.47^{21}.101^7}$

$r = 1$

Upper bound=314378927707039117076594641960472205699918246393796134855347548904541388800

Lower bound= 314378927707027215691704348813743973043171844658271204789424845095104937984

Difference= 11890447077876842522672189603253739181827333079849168416014336

Error= 0.00000000000037822023

$r = 5$

Upper bound= 1571894638535147728759342621748217544441028696720224309918829150723580624896

Lower bound=1571894638535135877591266211694935422470597680646030708671750002626420277248

Error= 0.00000000000000756440

- We are able to conclude about $D_r(G)$ for certain class of G with $\exp(G) = p^{e_d}$, where $e_d > 1$

Question

What if $\exp(G) = p$? (i.e., group of the form C_p^d for $d \geq 3$.)

Conjecture (1)

For prime p, and positive integer r, d, $D_r((C_p)^d) = (r + d - 1)p - (d - 1)$.

- For $d = 2$, the conjecture is satisfied $\forall$ r.

If the Conjecture is true, we have

Theorem

Let p, q be distinct primes and $G \cong C_p^{d-1} \times C_{pq}$ of rank $d \geq 3$. If the previous Conjecture holds for prime p, then $D(G) = D^*(G)$.

$D^*(G) = (r + d - 1)p - (d - 1)$ in this case.

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$G = C_p \times C_p \times C_{pq}$ with primes $p \neq q$

We shall see that $D(G) = D^*(G)$ holds for a certain class of primes p, q that satisfy the following condition. (Whether all primes do, again, we don't know! $p = 3$ does, at least.)

Conjecture (2)

Fix p, q $(p \neq q)$; *define* $G := C_p^3 \times C_q$. *Let* $m = p(q+2) - 2$. *Let* $x_1 \dots x_m$ *be a sequence over* C_p^3 *and* $y_1 \dots y_m$ *over* C_q. *Suppose*

$$y_{\sum_{i=1}^{t} r_i + 1} = \cdots = y_{\sum_{i=1}^{t+1} r_i} = t + 1 \ \ (t \in [0, q-1])$$

where $r = \sum_{i=1}^{q-1} r_i$. *If* $r \in [pq+1, p(q+2)-2]$ *and* $\sum_{i=1}^{q-1} i r_i \equiv 0 \ (mod\ q)$, *then the sequence* $S := (x_1, y_1) \dots (x_m, y_m)$ *over* G *has a nontrivial zero-sum subsequence.*

Theorem

Let p be a prime such that Conjecture 2 holds. Then, for group $G = C_p^3 \times C_q$,

$$D(G) = D^*(G).$$

- (Bhowmik and Schlage-Puchta, 2007) For $G \cong C_3 \times C_3 \times C_{3d},\ d \in \mathbb{N}$, $D(G) = D^*(G)$. So Conjecture 2 is true for $p = 3$ atleast.
- One can observe that Conjecture 1 is much stronger than Conjecture 2, because

$$D_r\left(C_p^d\right) = (r+d-1)p-(d-1) \implies D\left(C_p^{d-1} \times C_{pr}\right) = (r+d-1)p-(d-1).$$

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Thank you :)