Temperature Oscillations in a Metal: Probing Aspects of Fourier Analysis

Anamta Asif
Roll no: 2025-10-0098
LUMS School of Science and Engineering

Thursday, November, 16, 2023

1 Abstract

In this experiment, we observed the wave-like behavior and properties of thermal oscillations in a copper rod. It was done using Fourier analysis, which helped in computing the harmonic components and the damping coefficient of the thermal "waves" and provided useful insights about their propagation and diffusivity properties.

2 Introduction

Thermal oscillations come about when the metallic rod is heated at one end by a heater giving out a heating pulse periodically. Diffusive "waves" are generated which dissipate through the length of the rod, and it is damped over time because of the heat losses. Using Fourier analysis, we get helpful information about the diffusivity of thermal oscillations by analyzing the harmonics, more specifically the odd harmonics, spectral power density, and the damping coefficient of these temperature oscillations [1].

These oscillations are the solution to the one-dimensional heat equation, which, on the face of it, looks strikingly similar to the wave equation. However, terming them as "waves" has its drawbacks, as there are some significant differences between them and conventional waves that are the solutions to the actual wave equation. First of all, thermal oscillations are dissipative, non-reflective, and non-refractive in nature. They have no wavefront through which they travel. More mathematically, they involve only a first-order time derivative instead of a second-order one. Understanding these differences will aid in comprehending the respective properties they produce and their respective suitable applications.

3 Theoretical background

The heat equation for one-dimensional heat flow along the metallic rod that relates both the spatial and temporal component of temperature $T(x,t)$ is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{D}\frac{\partial T}{\partial t}, \tag{1}$$

where D is the thermal diffusivity constant. It is defined as

$$D \equiv \frac{\kappa}{\sigma\rho}$$

where κ is the thermal conductivity, σ is the heat capacity per unit mass and ρ is the mass density of the metallic rod.

When the heat flows through the rod, there are energy losses to the environment over a period of time as shown in Equation (1). As energy can neither be created nor destroyed only conserved, we can see that the amount of total heat flow out of the surface of the rod is equal to the rate of heat energy being lost inside the volume of the rod [2]. This loss of thermal energy is related to the thermal diffusivity D, as it is a quantity responsible for the dissipation of thermal oscillations, while at the same time contributing to its damping (as we shall see later on).

Solving Equation (1) gives us the solution for the heat equation, which is temperature $T(x,t)$ for $x \geq 0$,

$$T(x,t) = \sum_{\omega} \mathcal{F}(\omega) \exp\left(-i\sqrt{\frac{\omega}{2D}}x \right) \exp\left[i\left(\sqrt{\frac{\omega}{2D}}x - \omega t \right) \right]. \tag{2}$$

When applying the boundary conditions of $x = 0$, which is the point where the heat source is connected to one end of the rod, Equation (2) reduces to

$$T(0,t) = \sum_{\omega} \mathcal{F}(\omega) \exp\left(-i\omega t\right). \tag{3}$$

A periodic square pulse is applied to a heating source of amplitude A and time period T_p s, which implies that it oscillates from O to A after every $(T_p/2)$ s. For such waves, the average amplitude is $A/2$. Moreover, to express such oscillations in a mathematical manner, we use the Fourier series of a square wave:

$$\begin{aligned} f(t) &= \frac{A}{2} - \frac{A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \frac{\sin(5\omega_0 t)}{5} + ... \right), \\ &= \frac{A}{2} - \frac{A}{\pi}\sum_{n=0}^{\infty}\left(\frac{\sin((2n+1)\,\omega_0 t)}{2n+1} \right), \end{aligned} \tag{4}$$

where $\omega_0 = 2\pi/T_p$ is the fundamental frequency of the wave.

$\mathcal{F}(\omega)$ and $T(0,t)$ from Equation (3) are linked with each other through the Fourier transform via Parseval's Theorem. As Fourier transforms are unitary, $\mathcal{F}(\omega)$ allows only certain frequencies ω's for $T(0,t)$ at this boundary condition of $x = 0$. This is clearly shown in Equation (4), where only odd harmonics are present, whereas no even harmonics exist. Therefore, $T(0,t) = f(t)$ from Equation (4), making Equation (3) and Equation (4) equivalent.

Equation (4) can be written in the form of complex exponentials and its terms be compared with that of Equation (3). We can identify and relate the odd harmonics with the fundamental frequency ω_0.

$$\begin{aligned}
\mathcal{F}(0) &= \frac{A}{2}, \\
\mathcal{F}(2\omega_{2n+1}) &= \frac{-iA}{2\pi(2n+1)}, \\
\mathcal{F}(2\omega_{2n-1}) &= \frac{iA}{2\pi(2n+1)} = \mathcal{F}^*(2\omega_{2n+1}),
\end{aligned}$$

where $\omega_{2n+1} = (2n+1)\omega_0$ and $\omega_{2n-1} = (2n-1)\omega_0$ to simplify our expressions. Hence, after some algebraic shuffling, our final solution $T(x,t)$ to the heat equation, this time over any arbitrary x and time t, is

$$T(x,t) = \frac{A}{2} + \frac{A}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \exp\left(-\sqrt{\frac{\omega_{2n+1}}{2D}} \right) \sin\left(\sqrt{\frac{\omega_{2n+1}}{2D}} x - \omega_{2n+1} t \right), \quad (5)$$

which are our equation for the temperature oscillations.

Furthermore, to analyze damping in our system, we define,

$$\delta = \sqrt{\frac{2D}{\omega_n}},$$

as our damping length. It is the length covered till the original amplitude of oscillations is reduced by $1/e$. Similarly, we define the phase velocity v, determined using the distance between the two ends of the rod divided by the time lag between oscillations measured at the two ends.

Using these two newly defined quantities, we get two expressions for thermal diffusivity D:

$$D_{\delta_n} = \frac{\pi}{T_p \delta_n^2}, \qquad\qquad D_v = \frac{T_p v^2}{4\pi}. \quad (6)$$

We experimentally determined the values for thermal diffusivity D using both these expressions.

4 Experimental procedure

For this experiment, we used a copper rod of length 50 ± 0.05 cm and diameter 3 ± 0.05 cm. On it, 4 thermocouples equidistantly placed 3 ± 0.05 cm apart were connected to it along, with cotton wrapped around it for maximum insulation. Each thermocouple was also connected with PhysTherms, which recorded temperature data from each thermocouple. All four PhysTherms were then connected to PhysLogger, a device that acquires, logs, plots and exports data via its own Desktop application, PhysLogger, on the monitor. The D.C. supply, PhysWatt, sends in square wave pulses of 10V to the cartridge heater of 40W, which then provides heat supply to one end of the copper rod.

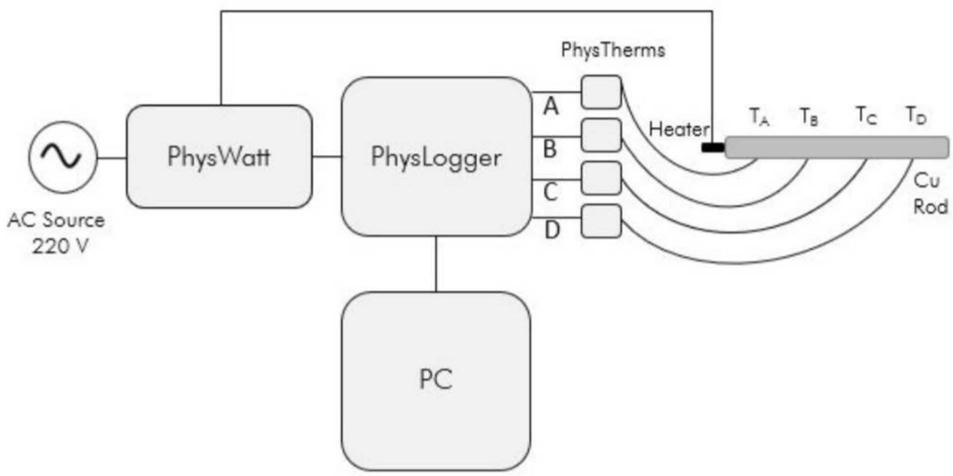

Figure 1: Schematic Setup of the Experiment.

The experiment starts when the whole apparatus is connected as shown in Figure (1), the PhysWatt is on, and the PhysLogger App has been set up. We set the sampling frequency to be 0.005 Hz, which implies that the time period was 200 s. The experiment runs till some time after the dynamic equilibrium has been reached. After that, the data was acquired through the PhysLogger App via the PhysLogger device, which was then saved into an .csv file and exported for further analysis.

5 Results and Discussion

The resulting data was analyzed in Python using VS Code. The data acquired from the PhysLogger for Temperature against time for all four thermocouples is shown in Figure (2). It comes in line intuitively, as the closer the thermocouple is to the heater, the higher their average temperature will be. However, the mean temperature at equilibrium is different for each thermocouple, which indicates that there is a constant temperature gradient throughout the rod. Moreover, Thermocouple A, being nearest to the heater, has sharper peaks, whereas Thermocouple D, being the farthest, has nearly sinusoidal peaks.

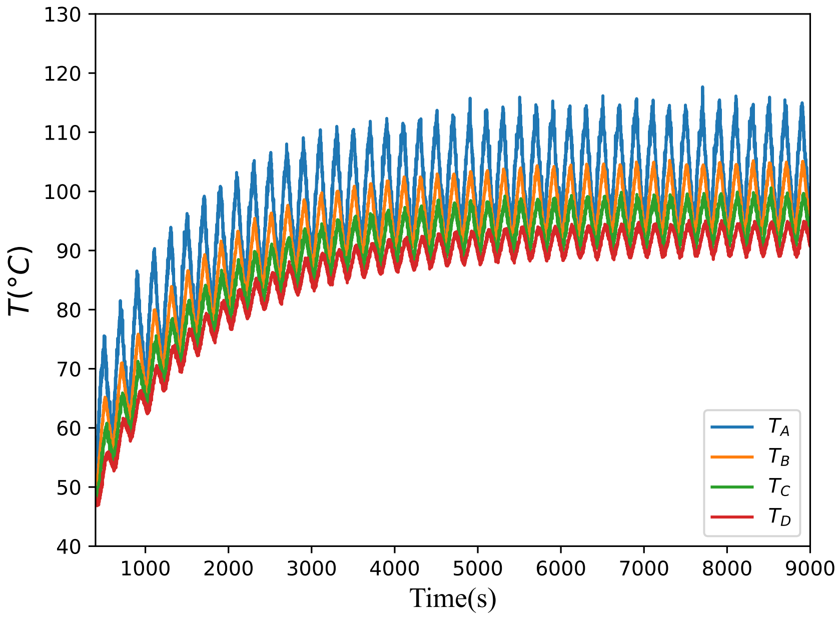

Figure 2: Temperature (in Kelvin) versus Time (in seconds).

Similarly, there is a small phase difference between each respective peak of the thermocouples. It corresponded to a time lag of 28.2 s. As the distance between the first and the fourth thermocouple was 9 cm, our phase velocity came out to be $3.19 \times 10^{-3}\ ms^{-1}$.

For each set of data corresponding to each thermocouple, we then computed their means. It was used to subtract it from each value in their respective data set so that an absolute FFT (Fast Fourier Transform) can be calculated and plotted, as shown in Figure (3). Doing this will remove the D.C. component of the signal in the plot. We can observe that not only odd harmonics are present, but the higher the harmonics have less amplitude. That means higher harmonics damp out quickly. It is affirmative with our theoretical predictions, as δ defined earlier as an inverse square root relation with ω_n.

To experimentally determine the value of the damping length for each odd harmonic, we took the logarithm of the spectral power density of the data and plotted it against the relative distance of the thermocouples, as shown in Figure (4). Spectral power density was calculated by computing the area under the peaks in Figure (3). The slope of this graph gives $c_1 = -1/\delta_n$, and it has a negative inverse relation with δ_n. The damping length for the first harmonic ω_1 was $\delta_1 = 40 \pm 10m$. The damping length for the third harmonic ω_3 is less than that of the first one, as its slope is more steeper.

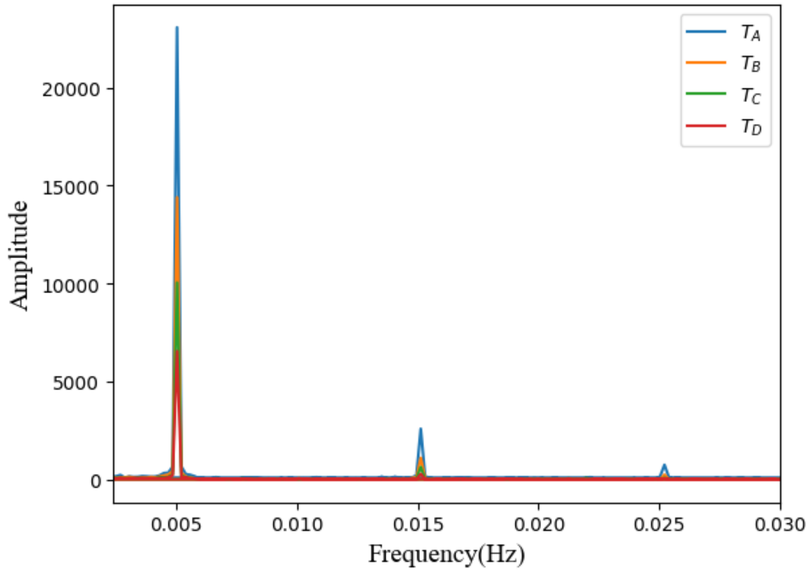

Figure 3: FFT Amplitude versus Frequency for each thermocouple. As shown, only odd harmonics are present in the graph.

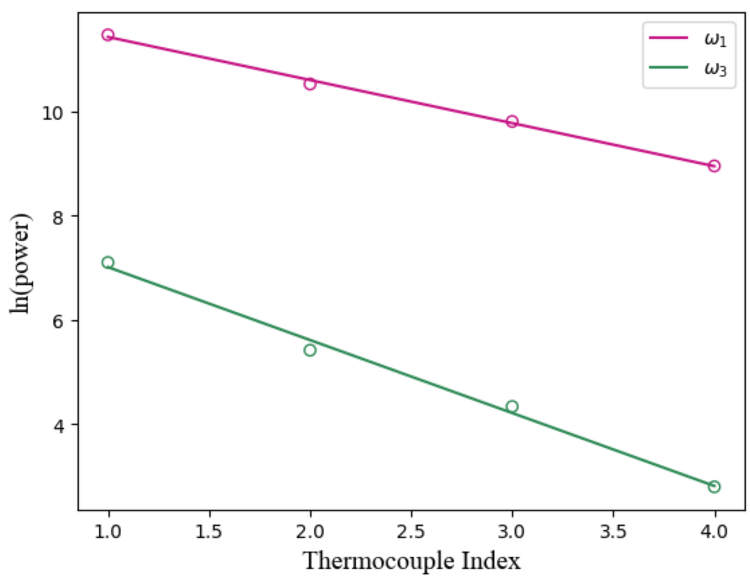

Figure 4: Logarithm of Spectral Power Density versus the relative distance of thermocouples (Thermocouple Index) for the first and third harmonics.

Therefore, the thermal diffusivity computed through phase velocity v is $D_v = (1.60 \pm 0.2)\text{x}10^{-4}\, m^2 s^{-1}$, whereas the thermal diffusivity computed through damping length δ_1 is $D_{\delta_n} = (9.66 \ \pm \ 0.2)\text{x}10^{-6} m^2 s^{-1}$. According to our experimental results, the value of D_v is closer to the actual value of thermal diffusivity in a copper rod.

References

[1] M.S. Anwar, "*Fourier analysis of thermal diffusive waves*", AAPT Physics Foundation, (2014).

[2] S. J. Blundell and K. M. Blundell, "*Concepts in Thermal Physics*", OUP.