Project Report for Quantum Field Theory (Spring 2024)

IR Divergences and Resummation

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Abstract: Infrared divergences are a phenomenon that occur when the momentum integral of a Feynman diagram diverges as $k \to 0$. This report calculates and explores how these divergences arise in bremsstrahlung and electron scattering cases. Both classical and quantum approaches are employed to bremsstrahlung which produces infinite soft photons. Furthermore, vertex corrections are studied in the scattering case. Lastly, the solution to these infinite contributions is discussed: resummation of the bremsstrahlung and vertex correction Feynman diagrams.

Keywords: infrared divergence, soft bremsstrahlung, electron scattering, vertex correction, scattering cross-section, quantum electrodynamics, Feynman diagram

Contents

1 Introduction

Infrared divergences occur when the Feynman diagram diverges due to infinite contributions from those particles that have very low energy that approaches 0. Moreover, they occur in those quantum electrodynamical processes that involve massless particles.

A phenomenon where infrared divergences occur is in bremsstrahlung ("braking radiation"). An infinite number of soft photons are emitted and are not being detected. However, even though calculations show that their contribution to the transition amplitude between any states with a finite number of photons goes to 0, they cannot be ignored as the sum over all infinite soft photons gives the finite transition amplitudes.

Similarly, when taking account of the radiative corrections to the electron scattering from a very heavy particle, as shown in Fig 1, the vertex correction gives both ultraviolet and infrared divergences. Both these divergences arise because of the integration over the undetermined loop momentum k, with the former occurring at $k \to \infty$, and the latter occurring at $k \to 0$. To deal with the former, parametrization either by using a hard cutoff momentum or Pauli-Villars regularization cancels out the divergent pieces so that only finite pieces remain.

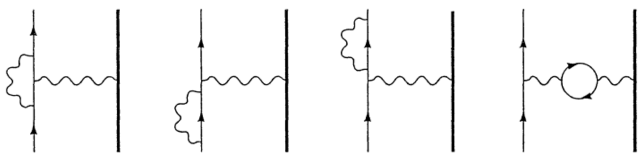

Figure 1: The next leading terms after the tree-level diagram. The first one is the vertex correction. The second and third are known as the external legs corrections and the fourth one is the vacuum polarization. For the purposes of this report, the focus shall be only on the vertex correction.

For infrared divergences, adding the two Bremsstrahlung diagrams, shown in Fig 2, to the vertex corrections cancels out the infrared divergent parts and gives a finite result. Interestingly, the case where these two diagrams are not added and the case where they are added are physically equivalent, since the soft photons in the latter cannot be detected.

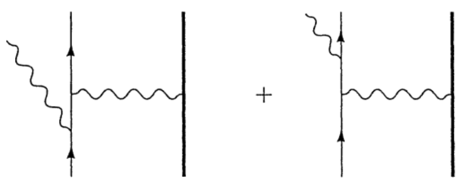

Figure 2: Bremsstralung diagrams that will be added to the vertex correction to cancel out the infrared divergences.

Therefore, by making use of several identities, like the Ward and Gordon Identity, and some algebraic manipulations, the report solves the infrared divergences that occur in the vertex function and bremsstrahlung, to the first loop correction [1].

2 Soft Bremsstrahlung

In computing the intensity of low-frequency bremsstrahlung radiation using both classical and quantum formalism, it can be shown how both end up with equivalent results. In both cases, the radiation with a frequency less than the reciprocal of the scattering time of the electron is being considered.

2.1 Classical Computation

Consider an ideal case where a charged particle, with 4-momentum p, charge e and at rest at $\mathbf{x} = 0$, receives a sudden "kick" at $t = 0$ by a low-frequency photon, which changes its 4-momentum from p to p'. This causes a sudden acceleration and a "kink" in the diagram as shown in Fig 3, which generates a radiation field.

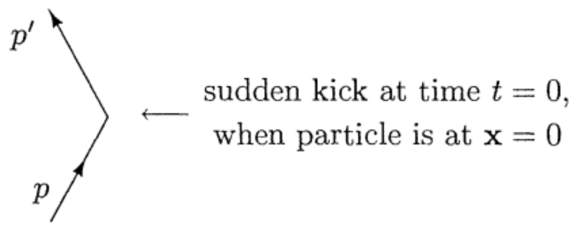

Figure 3

For such a particle, its trajectory is $y^\mu(t) = (t, \mathbf{0})$. Consequently, its 4-velocity is $v^\mu(t) = d(y^\mu(t))/dt = (1, \mathbf{0})$. Therefore, its current density would be

$$\begin{aligned} j^\mu(x) &= \int dt\ v^\mu(t)\ e\ \delta^{(4)}(x^\mu - y^\mu(t)), \\ &= \int dt\ (1, \mathbf{0})\ e\ \delta^{(4)}(x^\mu - y^\mu(t)), \\ &= (1, \mathbf{0})\ e\ \delta^{(3)}(\mathbf{x}), \end{aligned} \tag{2.1}$$

where $y^\mu(t) = (t, \mathbf{0})$. Extending this to an arbitrary trajectory $y^\mu(\tau)$ of the particle, the current density $j^\mu(x)$ becomes

$$\begin{aligned} j^\mu(x) &= e \int d\tau\ v^\mu(\tau)\ \delta^{(4)}(x^\mu - y^\mu(\tau)), \\ &= e \int d\tau\ \frac{d(y^\mu(\tau))}{d\tau}\ \delta^{(4)}(x^\mu - y^\mu(\tau)). \end{aligned} \tag{2.2}$$

This expression is independent of how $y^\mu(\tau)$ is parametrized. For this process, the trajectory is

$$y^\mu(\tau) = \begin{cases} (p^\mu/m)\tau & \tau < 0; \\ (p'^\mu/m)\tau & \tau > 0. \end{cases} \tag{2.3}$$

Hence, the integral of the current density can be split, and using (2.3),

$$j^\mu(x) = e\int_0^\infty d\tau\ \frac{p'}{m}\ \delta^{(4)}\Big(x - \frac{p'}{m}\tau\Big)\ +\ e\int_{-\infty}^0 d\tau \frac{p}{m}\delta^{(4)}\Big(x - \frac{p}{m}\tau\Big). \tag{2.4}$$

To convert $j^\mu(x)$ to momentum space, we would need to take its Fourier transform. To make our integral converge, we also insert $e^{-\epsilon\tau}$ to the first integral term and $e^{\epsilon\tau}$ to the second integral term.

$$\begin{aligned}
\widetilde{j}^\mu(k) &= \int d^4x e^{ik\cdot x} j^\mu(x), \\
&= e\int_0^\infty d\tau\ \frac{p'^\mu}{m}\ e^{i(\frac{kp'}{m}+i\epsilon)\tau}\ +\ e\int_{-\infty}^0 d\tau \frac{p^\mu}{m}\ e^{i(\frac{kp}{m}i\epsilon)\tau}, \\
&= -ie\ \frac{p'^\mu}{k\cdot p' + i\epsilon m}\ e^{i(kp'/m+i\epsilon)\tau}\Big|_0^\infty\ + ie\ \frac{p^\mu}{k\cdot p - i\epsilon m}\ e^{i(kp'/m-i\epsilon)\tau}\Big|_{-\infty}^0, \\
\therefore\ \widetilde{j}^\mu(k) &= ie\Big(\frac{p'^\mu}{k\cdot p' + i\epsilon'} - \frac{p^\mu}{k\cdot p - i\epsilon'}\Big).
\end{aligned} \tag{2.5}$$

We choose the Lorentz Gauge, where $\partial_\mu A^\mu = 0$, and solve the Maxwell Equations. Therefore, we must solve for $\partial_\nu\partial^\nu A^\mu = j^\mu(k)$. Working in Fourier Space and using (2.5):

$$\begin{aligned}
\partial_\nu\partial^\nu \widetilde{A}^\mu(k) &= \widetilde{j}^\mu(k), \\
\partial^2 \widetilde{A}^\mu(k) &= \int d^4x e^{ik\cdot x} j^\mu(x), \\
\therefore\ \widetilde{A}^\mu(k) &= -\frac{1}{k^2}\widetilde{j}^\mu(k).
\end{aligned} \tag{2.6}$$

Therefore, using (2.5) and (2.6), we get

$$\begin{aligned}
A^\mu(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\widetilde{A}^\mu(k), \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{-e^{-ik\cdot x}}{k^2}\widetilde{j}^\mu(k), \\
\therefore\ A^\mu(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\frac{-ie}{k^2}\Big(\frac{p'^\mu}{k\cdot p' + i\epsilon'} - \frac{p^\mu}{k\cdot p - i\epsilon'}\Big).
\end{aligned} \tag{2.7}$$

This integral can be solved in the complex plane as a contour integral. The poles are shown in Fig 8.

At $t < 0$, before the sudden change in trajectory, our contour is closed anticlockwise which contains only $k\cdot p = i\epsilon$ pole. This also corresponds to $k^0 = (\mathbf{k}\cdot\mathbf{p})/p^0$. Using the residue theorem,

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i((\mathbf{k}\cdot\mathbf{p})/p^0)t}\frac{(2\pi i)(-ie)}{(2\pi)k^2}\Big[\frac{-p^\mu}{p}\Big]. \tag{2.8}$$

If we choose a reference frame where the particle is initially at rest, then $p^\mu = (p^0, \mathbf{0})$. Consequently, such a reference frame also fixes k to $k^\mu = (0, \mathbf{k})$. (2.8) reduces to

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\frac{e}{|\mathbf{k}|^2}(1, \mathbf{0}), \tag{2.9}$$

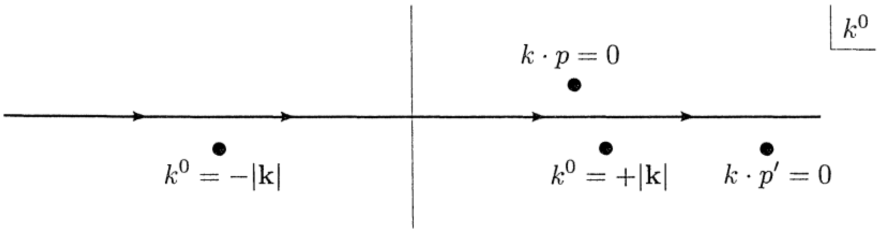

Figure 4: Poles of the integral in (2.7) in the complex plane. Poles $k^0 \pm -|\mathbf{k}|$ are below the real line so that the radiation field satisfies the retarded Green's Function.

which is the Coulomb potential for an unaccelerated charge. This result is reasonable because $t < 0$, there was no scattering, hence no radiation.

For $t > 0$, that is the time after the scattering took place, we close the contour clockwise. This time, we get the rest of the three poles. The contribution of the pole $k \cdot p' = 0$ is calculated similar to that of $k \cdot p = 0$, and we can see that this will be the Coulomb potential of the outgoing particle. Therefore, the only two poles responsible for radiation are the poles $k^0 = \pm|\mathbf{k}|$. Taking the pole $k^0 = \mathbf{k}$, the radiation potential becomes

$$
\begin{aligned}
A^\mu_{\text{rad}}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{(-2\pi i)(-ie)}{(2\pi)} \left(\frac{1}{k + |\mathbf{k}|}\bigg|_{k^0=|\mathbf{k}|} \right) \left\{ e^{-ik\cdot x} \left(\frac{p'^\mu}{k \cdot p' + i\epsilon'} + \frac{p^\mu}{k \cdot p - i\epsilon'} \right) \right\}, \\
A^\mu_{\text{rad}}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{|\mathbf{k}|} \left\{ e^{-ik\cdot x} \left(\frac{p'^\mu}{k \cdot p' + i\epsilon'} + \frac{p^\mu}{k \cdot p - i\epsilon'} \right) \right\}, \\
&= \text{Re} \int \frac{d^3k}{(2\pi)^3} A^\mu(\mathbf{k}) e^{-ik\cdot x},
\end{aligned} \tag{2.10}
$$

where $A^\mu(\mathbf{k})$ is the momentum-space potential amplitude given by

$$
A^\mu(k) = \frac{-e}{|\mathbf{k}|} \left(\frac{p'^\mu}{k \cdot p' + i\epsilon'} + \frac{p^\mu}{k \cdot p - i\epsilon'} \right). \tag{2.11}
$$

For the energy radiated by $A^\mu_{\text{rad}}(x)$,

$$
\text{Energy}_{\text{rad}} = \frac{1}{2} \int d^3x \; (|\text{E}(x)|^2 + |\text{B}(x)|^2). \tag{2.12}
$$

We can write $\text{E}(x)$ and $\text{B}(x)$ in terms of the real parts of its complex Fourier integrals which have their respective momentum-space amplitudes $\mathcal{E}(\mathbf{k})$ and $\mathcal{B}(\mathbf{k})$,

$$
\begin{aligned}
\text{E}(x) &= \text{Re} \int \frac{d^3k}{(2\pi)^3} \mathcal{E}(\mathbf{k}) e^{-ik\cdot x}, \\
\text{B}(x) &= \text{Re} \int \frac{d^3k}{(2\pi)^3} \mathcal{B}(\mathbf{k}) e^{-ik\cdot x}.
\end{aligned} \tag{2.13}
$$

Using $\text{E} = -\nabla\phi - \partial_t A$, the expression for momentum operator $\hat{k} = i\nabla$, and $\text{E}(x)$ and $\text{B}(x)$ are transverse, the momentum-space amplitudes for $\mathcal{E}(\mathbf{k})$ and $\mathcal{B}(\mathbf{k})$ for the radiation fields is

$$\mathcal{E}(k) = -i\mathbf{k}A^0(\mathbf{k}) + ik^0\mathbf{A}(\mathbf{k}),$$
$$\mathcal{B}(k) = \hat{k} \times \mathcal{E}(k). \tag{2.14}$$

In computing (2.12), we use the properties of $\mathcal{E}(\mathbf{k})$ and $\mathcal{B}(\mathbf{k})$ and get

$$\begin{aligned}\text{Energy}_{\text{rad}} &= \frac{1}{4}\int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}[\mathcal{E}^*(\mathbf{k})\mathcal{E}(\mathbf{k}')e^{i(k-k')x} + \mathcal{B}^*(\mathbf{k})\mathcal{B}(\mathbf{k}')e^{i(k-k')x}],\\ &= \frac{1}{4}\int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\left[\mathcal{E}^*(\mathbf{k})\mathcal{E}(\mathbf{k}')e^{i(k-k')x} + |\hat{k}|^2|\mathcal{E}^*(\mathbf{k})||\mathcal{E}(\mathbf{k}')|\sin\left(\frac{\pi}{2}\right)\right],\\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\mathcal{E}^*(\mathbf{k})\mathcal{E}(\mathbf{k}).\end{aligned} \tag{2.15}$$

Since $\mathcal{E}(\mathbf{k'})$ is transverse, we can add two transverse polarization vectors ϵ_λ, where $\lambda = 1,\ 2$. The integral then becomes

$$\begin{aligned}\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\mathcal{E}^*(\mathbf{k})\mathcal{E}(\mathbf{k}) &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}|\epsilon_\lambda(\mathbf{k})\cdot\mathcal{E}(\mathbf{k})|^2,\\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|\mathbf{k}|^2\sum_{\lambda=1,2}|\epsilon_\lambda(\mathbf{k})\cdot\mathcal{A}(\mathbf{k})|^2,\\ \text{Energy}_{\text{rad}} &= \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\frac{e^2}{2}\left|\epsilon_\lambda(\mathbf{k})\cdot\left(\frac{\mathbf{p}'}{k\cdot p' + i\epsilon'} - \frac{\mathbf{p}}{k\cdot p - i\epsilon'}\right)\right|^2,\end{aligned} \tag{2.16}$$

where we used (2.11) for the explicit expression for $\mathcal{A}(\mathbf{k})$. We can replace $\epsilon(\mathbf{k}), p$', and p to their 4-momentum counterparts to make Energy$_{\text{rad}}$ into a Lorentz-invariant scalar. By replacing e^μ with k^μ, we get the Ward identity, which is

$$k_\mu \mathcal{M}^\mu = 0,$$

where it means that the longitudinal polarization of the photon is not possible physically. A manifestation of this identity is that we can replace the $\sum \epsilon^*_\mu \epsilon_\nu$ by $-g_{\mu\nu}$. The proof of this substitution is provided in Appendix A. The formula for radiated energy reduces down to

$$\begin{aligned}\text{Energy}_{\text{rad}} &= \int \frac{d^3k}{(2\pi)^3}\frac{e^2}{2}(-g_{\mu\nu})\left(-\frac{2p\cdot p'}{(k\cdot p')(k\cdot p)} + \frac{p'^2}{(k\cdot p')^2} + \frac{p^2}{(k\cdot p)^2}\right),\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{e^2}{2}\left(\frac{2p\cdot p'}{(k\cdot p')(k\cdot p)} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right).\end{aligned} \tag{2.17}$$

We choose a frame where $p^0 = p'^0 = E$. Therefore, the momenta are $k^\mu = (k, \mathbf{k})$, $p^\mu = E(1, \mathbf{v})$, and $p'^\mu = E(1, \mathbf{v}')$. In this frame, Energy$_{\text{rad}}$ can be written with this structure:

$$\text{Energy}_{\text{rad}} = \frac{e^2}{(2\pi)^2}\int dk\ \mathcal{I}(\mathbf{v}, \mathbf{v}'), \tag{2.18}$$

where $\mathcal{I}(\mathbf{v},\mathbf{v}')$ is the differential intensity:

$$\begin{aligned}\mathcal{I}(\mathbf{v},\mathbf{v}') &= \int \frac{k^2 d\Omega_k}{4\pi}\left(\frac{2E^2(1-\mathbf{v}\cdot\mathbf{v}')}{k^2E^2(1-\hat{k}\cdot\mathbf{v})(1-\hat{k}\cdot\mathbf{v}')} - \frac{m^2}{E^2(1-\hat{k}\cdot\mathbf{v}')^2} - \frac{m^2}{E^2(1-\hat{k}\cdot\mathbf{v})^2}\right),\\ &= \int \frac{d\Omega_k}{4\pi}\left(\frac{2(1-\mathbf{v}\cdot\mathbf{v}')}{(1-\hat{k}\cdot\mathbf{v})(1-\hat{k}\cdot\mathbf{v}')} - \frac{m^2/E^2}{(1-\hat{k}\cdot\mathbf{v}')^2} - \frac{m^2/E^2}{(1-\hat{k}\cdot\mathbf{v})^2}\right). \end{aligned} \tag{2.19}$$

We can see here that since $\mathcal{I}(\mathbf{v},\mathbf{v}')$ does not depend upon k, calculating the integral in (2.18) trivial but divergent, because of the setup of the problem, where the particle has a sudden change in momentum. For a relativistic particle, individual photons take away some of the particle's energy; therefore, imposing a cutoff on the integral. Nevertheless, in the case of low-frequency limit, we impose a hard cutoff on the maximum value of frequency k_{max}. This, in turn, gives the radiated energy

$$\begin{aligned}\text{Energy}_{\text{rad}} &= \int_0^{k_{\text{max}}} dk \frac{e^2}{(2\pi)^2}\mathcal{I}(\mathbf{v},\mathbf{v}'),\\ &= \frac{\alpha}{\pi}\cdot k_{\text{max}}\cdot \mathcal{I}(\mathbf{v},\mathbf{v}'),\end{aligned} \tag{2.20}$$

where it peaks when $\hat{k}$ is parallel to either v or v'.

Considering the ultrarelativistic limit, the two peaks in energy comes from the first term in (2.17). To solve $\mathcal{I}(\mathbf{v},\mathbf{v}')$, define the θ as the angle between $\hat{k}$ and $\mathbf{v}$ at one peak and $\hat{k}$ and $\mathbf{v}'$ at the other peak. We take the limit of $\theta = 0$, break the integral into its two peaks, and integrate over the small region around $\theta = 0$.

$$\begin{aligned}\mathcal{I}(\mathbf{v},\mathbf{v}') &\approx \frac{(2\pi)}{4\pi}\left(\int \sin\theta \; d\theta \frac{(1-\mathbf{v}\cdot\mathbf{v}')}{(1-|\mathbf{v}|\cos\theta)(1-\mathbf{v}\cdot\mathbf{v}')} + \int \sin\theta \; d\theta \frac{(1-\mathbf{v}\cdot\mathbf{v}')}{(1-\mathbf{v}\cdot\mathbf{v}')(1-|\mathbf{v}'|\cos\theta)}\right),\\ &\approx \int_{\hat{k}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{v}'}^{\cos\theta=1} d(\cos\theta)\frac{(1-\mathbf{v}\cdot\mathbf{v}')}{(1-|\mathbf{v}|\cos\theta)(1-\mathbf{v}\cdot\mathbf{v}')} + \int_{\hat{k}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{v}'}^{\cos\theta=1} d(\cos\theta)\frac{(1-\mathbf{v}\cdot\mathbf{v}')}{(1-\mathbf{v}\cdot\mathbf{v}')(1-|\mathbf{v}'|\cos\theta)},\\ &\approx -\log(1-|\mathbf{v}|) + \log(1-\mathbf{v}\cdot\mathbf{v}') - \log(1-|\mathbf{v}'|) + \log(1-\mathbf{v}\cdot\mathbf{v}'),\\ &\approx \log\left(\frac{1-\mathbf{v}\cdot\mathbf{v}'}{1-|\mathbf{v}|}\right) + \log\left(\frac{1-\mathbf{v}\cdot\mathbf{v}'}{1-|\mathbf{v}'|}\right),\\ &= \log\left(\frac{(E^2-\mathbf{p}\cdot\mathbf{p}')^2}{E^2(E-|\mathbf{p}|)^2}\right),\\ &\approx 2\log\left(\frac{p\cdot p'}{(E^2-|\mathbf{p}|^2)/2}\right)\\ &\approx 2\log\left(\frac{-q^2}{m^2}\right).\end{aligned} \tag{2.21}$$

where $q^2 = (p'-p)^2$ and $E >> m$. Hence, the energy radiated at low-limit frequencies, following from (2.20), is

$$\text{Energy}_{\text{rad}} = \frac{2\alpha}{\pi}\int_0^{k_{\text{max}}} dk \log\left(\frac{-q^2}{m^2}\right). \tag{2.22}$$

This is the energy by photons, each of energy k. Thus, the total number of photons will be

$$\text{Number of Photons} = \frac{2\alpha}{\pi}\int_0^{k_{\text{max}}} dk \frac{1}{k}\log\left(\frac{-q^2}{m^2}\right). \tag{2.23}$$

We notice that in solving the integral (2.23), the integral will have an infrared divergence, that is, the divergence at low k. We shall see in the next section that using the quantum approach for soft Bremsstrahlung, we conclude the same expression as (2.23).

2.2 Quantum Computation

After having discussed soft photon radiation in the classical framework in much detail, the same process shall now be analyzed with the help of quantum theory.

Consider the emission of a single photon when an electron collides with a heavy particle. To calculate the total amplitude of scattering for this process, we will make use of the Feynman rules for Quantum Electrodynamics (as stated in appendix B).

Moreover, for the total amplitude, we consider the sum of all possible Feynman diagrams, in this case: two (as shown in figure 5).

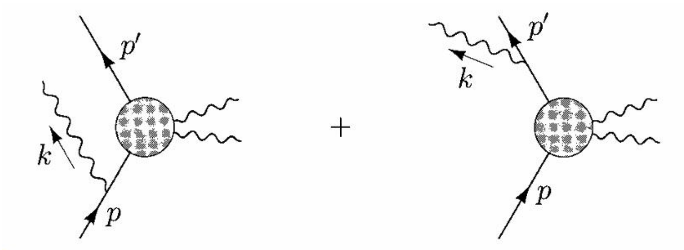

Figure 5: Radiation of soft photon during electron scattering process.

Hence, the expression is:

$$\iota\mathcal{M}_1 = (-\iota e\gamma^\mu)(\epsilon^*_\mu(k))(\bar{u}(p'))(u(p))\frac{\iota(\not{p} - \not{k} + m)}{(p-k)^2 - m^2}(\mathcal{M}_o(p', p-k)). \tag{2.24}$$

$$\iota\mathcal{M}_2 = (-\iota e\gamma^\mu)(\epsilon^*_\mu(k))(\bar{u}(p'))(u(p))\frac{\iota(\not{p}' + \not{k} + m)}{(p'+k)^2 - m^2}(\mathcal{M}_o(p'+k, p)). \tag{2.25}$$

$$\iota\mathcal{M} = \iota\mathcal{M}_1 + \iota\mathcal{M}_2. \tag{2.26}$$

where in equation (2.24) - the expression for the amplitude of the first Feynman diagram (on the left), $-\iota e\gamma^\mu$ represents the vertex of the diagram, $\epsilon^*_\mu(k)$ is the external photon line (the emitted photon), $\bar{u}(p')$ and $u(p)$ are the external momenta lines, and $\frac{\iota(\not{p}-\not{k}+m)}{(p-k)^2-m^2}$ is the fermion propagator associated with the internal line. The same explanations would also hold for equation (2.25) - the expression for the amplitude of the second Feynman diagram (on the right). Additionally, $\mathcal{M}_o$ is the part of the amplitude that depicts the electron's interaction with the external field in both equations (2.24) and (2.25) and, equation (2.26) is the sum of both equations.

Upon simplifying:

$$\iota\mathcal{M} = -\iota e\bar{u}(p')[\mathcal{M}_o(p', p-k)\frac{\iota(\not{p} - \not{k} + m)}{(p-k)^2 - m^2}\gamma^\mu \epsilon^*_\mu(k) + \gamma^\mu \epsilon^*_\mu(k)\frac{\iota(\not{p}' + \not{k} + m)}{(p'+k)^2 - m^2}\mathcal{M}_o(p'+k, p)]u(p). \tag{2.27}$$

which appears complicated, hence, will be further simplified.

Connecting with the classical case where it was assumed that the radiated photon is soft, thus: $|\mathbf{k}| << |\mathbf{p'} - \mathbf{p}|$, where $|\mathbf{k}|$ is the difference in the momentum of the electron before and after scattering, which is minimal. Since $\mathbf{k}$ is small, we can approximate:

$$\mathcal{M}_o(p', p-k) \approx \mathcal{M}_o(p'+k, p) \approx \mathcal{M}_o(p', p). \tag{2.28}$$

and $\not{k}$ may be safely ignored in equation (2.27).

Furthermore, the numerators in equation (2.27) can be simplified with the help of some Dirac algebra:

$$\begin{aligned}(\not{p} + m)\gamma^\mu \epsilon^*_\mu u(p) &= \not{p}\gamma^\mu \epsilon^*_\mu u(p) + m\gamma^\mu \epsilon^*_\mu u(p), \\ &= [2p^\mu - \gamma^\mu \not{p}]\,\epsilon^*_\mu u(p) + m\gamma^\mu \epsilon^*_\mu u(p), \\ &= 2p^\mu \epsilon^*_\mu u(p) - m\gamma^\mu \epsilon^*_\mu u(p) + m\gamma^\mu \epsilon^*_\mu u(p).\end{aligned} \tag{2.29}$$

where in the second line, $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$ and in the last line, $(\not{p} - m)u(p) = 0$.

A similar calculation holds for the second numerator term $\bar{u}(p')\gamma^\mu \epsilon^*_\mu(\not{p}' + m)$ as:

$$\bar{u}(p')\gamma^\mu \epsilon^*_\mu(\not{p}' + m) = \bar{u}(p')2p'^\mu \epsilon^*_\mu. \tag{2.30}$$

After having simplified the numerators, the denominators can be written as:

$$\begin{aligned}(p-k)^2 - m^2 &= p^2 + k^2 - 2p.k - m^2, \\ &= -2p \cdot k.\end{aligned} \tag{2.31}$$

here we note that $(p^2 + k^2) = m^2$ results from the solution of the Klein-Gordon equation, which the Dirac field obeys.

Similarly,

$$\begin{aligned}(p'+k)^2 - m^2 &= p'^2 + k^2 + 2p'.k - m^2, \\ &= 2p' \cdot k.\end{aligned} \tag{2.32}$$

With this knowledge, equation (2.27) becomes:

$$\begin{aligned}\iota\mathcal{M} &= \bar{u}(p')u(p)\mathcal{M}_o(p', p)\left[e\left(\frac{-2p^\mu \epsilon^*_\mu}{2(p \cdot k)} + \frac{2p^\mu \epsilon^*_\mu}{2(p' \cdot k)}\right)\right], \\ \iota\mathcal{M} &= \bar{u}(p')u(p)\mathcal{M}_o(p', p)\left[e\left(\frac{p' \cdot \epsilon^*}{p' \cdot k} - \frac{p \cdot \epsilon^*}{p \cdot k}\right)\right].\end{aligned} \tag{2.33}$$

which is the much simplified amplitude for elastic electron scattering in the soft photon approximation.

After having computed the scattering amplitude, the cross-section of this scattering shall now be examined. The scattering cross-section is proportional to the amplitude: $|\mathcal{M}_{fi}|^2$ [2]. Moreover, in the case of elastic scattering, just an additional phase-space integration over k (photon momentum) is added. Summation over the two polarizations of the photon is also performed as:

$$d\sigma(p \to p' + \gamma) = \left|\bar{u}(p')u(p)\mathcal{M}_o(p',p)\right|^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\sum_{\lambda=1,2} e^2 \left|\left(\frac{p'\cdot\epsilon^{(\lambda)}}{p'\cdot k} - \frac{p\cdot\epsilon^{(\lambda)}}{p\cdot k}\right)\right|^2,$$

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\sum_{\lambda=1,2} e^2 \left|\left(\frac{p'\cdot\epsilon^{(\lambda)}}{p'\cdot k} - \frac{p\cdot\epsilon^{(\lambda)}}{p\cdot k}\right)\right|^2. \tag{2.34}$$

where in equation (2.34), $|\bar{u}(p')u(p)\mathcal{M}_o(p',p)|^2 = d\sigma(p \to p')$ is just the cross section for the elastic scattering and the rest of the expression is related to the emission of the photon.

Thus, the differential probability of radiating a photon that has momentum k, as an electron scatters (from p' to p) is:

$$d(prob) = \frac{d^3k}{(2\pi)^3}\sum_{\lambda}\frac{e^2}{2k}\left|\epsilon_\lambda\left(\frac{\mathbf{p}'}{p'\cdot k} - \frac{\mathbf{p}}{p\cdot k}\right)\right|^2. \tag{2.35}$$

where we can multiply equation (2.35) with photon energy k to compute the expected energy radiated due to one photon and thus, recover the classical expression (2.16).

As stated, equation (2.35) is the probability for radiating a single photon, not the total number of photons radiated in the whole process. This becomes a problem as it is not the final answer required and it gets worse when integration over k is carried out. In that case, this expression can only be integrated up to the energy of the soft-photon approximation i.e., $|q| = |p' - p|$.

As equation (2.35) is similar to equation (2.16), therefore, the total probability is:

$$\text{Probability} \approx \frac{\alpha}{\pi}\int_0^{|\mathbf{q}|} dk\frac{1}{k}\mathcal{I}(\mathbf{v}',\mathbf{v}). \tag{2.36}$$

where we note that, in comparison, just the limits are different.

Furthermore, we note from earlier that $\mathcal{I}(\mathbf{v}',\mathbf{v})$ is k-independent, therefore, the integral diverges in the lower limit. To solve this problem of infrared divergence, it is assumed that the radiated photon has a very small mass μ which acts as a lower bound for the integral. Then, equation (2.34) becomes:

$$\begin{aligned} d\sigma(p \to p' + \gamma) &= d\sigma(p \to p') \cdot \text{Probability}, \\ &= d\sigma(p \to p') \cdot \frac{2\alpha}{2\pi}\int_\mu^{|\mathbf{q}|} dk\frac{1}{k}\cdot\mathcal{I}(\mathbf{v}',\mathbf{v}), \\ &= d\sigma(p \to p') \cdot \frac{\alpha}{2\pi}\log(k^2)\Big|_\mu^{|q|}\mathcal{I}(\mathbf{v}',\mathbf{v}), \\ &\approx d\sigma(p \to p') \cdot \frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\log\left(\frac{-q^2}{m^2}\right). \end{aligned} \tag{2.37}$$

where from equation (2.21): $\mathcal{I}(\mathbf{v}', \mathbf{v}) \approx 2\log\left(\frac{-q^2}{m^2}\right)$ and, q^2 is negative.

Equation (2.37) is the scattering cross section for the emission of a single soft photon. Moreover, it is observed that the classical and quantum result are strikingly similar, and that the classical result is the limiting case of the quantum result as the quantum computation was reduced to limitations, whereas, the classical computation started with them.

3 The Electron Vertex Function

After thoroughly having discussed QED radiative corrections due to the emission of a single soft photon (bremsstrahlung), the correction to electron scattering coming from the presence of an additional virtual photon shall now be looked at.

3.1 Formal Structure

To study the correction to electron scattering, Feynman diagrams containing loops shall be dealt with - the result of which can be complicated to compute, therefore, it would be useful to discuss the formal structure of the result beforehand and, how to interpret it.

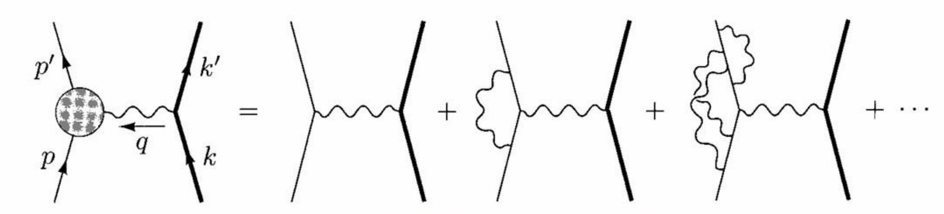

Figure 6: Sum of vertex correction diagrams.

Figure (6) shows electron scattering to different order of corrections, where the gray circle depicts the sum of all lowest-order vertex corrections. Moreover, the sum of vertex diagrams is: $-\iota e\Gamma^\mu(p', p)$. Thus, the total amplitude for this process can be written as:

$$\iota\mathcal{M} = \bar{u}(p')u(p)\left(-\iota e\Gamma^\mu\right)\left[\frac{-\iota g_{\mu\nu}}{q^2 + \iota\epsilon}\right]\bar{u}(k')u(k)(-\iota e\gamma^\nu),$$
$$\iota\mathcal{M} = \iota e^2\left(\bar{u}(p')\Gamma^\mu(p,p')u(p)\right)\frac{1}{q^2}\left(\bar{u}(k')\gamma_\mu u(k)\right). \tag{3.1}$$

where $\bar{u}(p'), u(p), \bar{u}(k')$ and $u(k)$ are the external momenta lines, $\frac{-\iota g_{\mu\nu}}{q^2+\iota\epsilon}$ is the photon propagator and, $-\iota e\gamma^\nu$ is the vertex.

In general, the function $\Gamma^\mu(p', p)$ appears in the amplitude as a result of scattering an electron from an electromagnetic field. The QED interaction Hamiltonian is then:

$$\Delta H_{int} = \int d^3x e A^{cl}_\mu \bar{\psi}(x)\gamma^\mu\psi(x). \tag{3.2}$$

where $\bar{\psi}(x)\gamma^\mu\psi(x) = j^\mu(x)$ is the electromagnetic current and A^{cl}_μ is a fixed classical potential.

To leading order, the T-matrix for scattering from this field can be computed as [3]:

$$\begin{aligned}
\langle p' \left| iT \right| p\rangle &= \langle p' \left| T(exp(-\iota \int d^4x H_I)) \right| p\rangle, \\
&= -\iota e \int d^4x \langle p' \left| T\left(\bar{\psi}(x)\gamma^\mu \psi(x)\right) \right| A_\mu^{cl} |p\rangle, \\
&= -\iota e \int d^4x A_\mu^{cl} \langle p' \left| \bar{\psi}(x)\gamma^\mu \psi(x) \right| p\rangle, \\
&= -\iota e \int d^4x A_\mu^{cl} \bar{u}(p')\gamma^\mu u(p) e^{\iota x(p'-p)}, \\
\langle p' \left| iT \right| p\rangle &= -\iota e \bar{u}(p')\gamma^\mu u(p) A_\mu^{cl}(p'-p).
\end{aligned} \tag{3.3}$$

where $\psi(x) = u(p)e^{-\iota px}$ is the Dirac field and, $\langle p' \left| iT \right| p\rangle$ is equal to $\iota\mathcal{M}(2\pi)\delta(E_{p'} - E_p)$, which is the T-matrix.

The vertex corrections then modify equation (3.3) to:

$$\iota\mathcal{M}(2\pi)\delta(E_{p'} - E_p) = -\iota e \bar{u}(p')\Gamma^\mu(p',p)\gamma^\mu u(p) \cdot A_\mu^{cl}(p'-p). \tag{3.4}$$

Furthermore, the form of $\Gamma^\mu(p', p)$ can be restricted due to Lorentz invariance, discrete symmetries of QED and the Ward identity:

To lowest order, $\Gamma^\mu(p', p) = \gamma^\mu$. In general, Γ^μ is some function of the associated momenta in the Feynman diagram, in this case: p, p', γ^μ and, some constants such as m, e and pure numbers. Additionally, Γ^μ transforms like a vector (just as γ^μ), and so, it must be a linear combination of the vectors listed above. With the benefit of hindsight, we write:

$$\Gamma^\mu = \gamma^\mu \cdot A + (p'^\mu + p^\mu) \cdot B + (p'^\mu - p^\mu) \cdot C. \tag{3.5}$$

where A, B and C are coefficients dependent on different combinations of p', p and q such as: $p^2, p'^2, q^2, p \cdot p', \not{p}, \not{p}'$ and etc.

Here, it is noted that p^2 and p'^2 are easy to change for the mass of the electron and, $\not{p}u(p) = m \cdot u(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p') \cdot m$ are just ordinary numbers, whereas, $p \cdot p'$ allows us to exchange dependencies between p, p' and q as:

$$\begin{aligned}
q^2 &= (p'-p)^2 = p^2 + p'^2 - 2p \cdot p', \\
q^2 &= -2p \cdot p' + 2m^2.
\end{aligned} \tag{3.6}$$

therefore, A, B and C must only be functions of q^2 (and of constants such as m).

For further interpretation and restriction of vectors, the Ward identity is applied on equation (3.5): $q_\mu \Gamma^\mu = 0$. Hence:

$$\begin{aligned}
\bar{u}(p') q_\mu \gamma^\mu u(p) &= \bar{u}(p')\not{q} u(p), \\
&= \left(\bar{u}(p')\not{p}' u(p) - \bar{u}(p')\not{p} u(p)\right), \\
&= \left(\bar{u}(p') m u(p) - \bar{u}(p') m u(p)\right), \\
&= 0.
\end{aligned} \tag{3.7}$$

$$q_\mu \cdot (p'^\mu + p^\mu) = (p'_\mu - p_\mu) \cdot (p'^\mu + p^\mu),$$
$$= p'^2 - p^2 = m^2 - m^2 = 0. \tag{3.8}$$

$$q_\mu \cdot (p'^\mu - p^\mu) = q_\mu q^\mu = q^2 \neq 0. \tag{3.9}$$

therefore, coefficients A and B are non-zero and C is zero.

Equation (3.5) can also be rewritten with the help of the Gordon identity (proof in appendix C) given by:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{\iota\sigma^{\mu\nu}q_\nu}{2m}\right]u(p). \tag{3.10}$$

Equation (3.10) can then be reexpressed as:

$$\bar{u}(p')\gamma^\mu u(p) - \bar{u}(p')\frac{\iota\sigma^{\mu\nu}q_\nu}{2m}u(p) = \bar{u}(p')\frac{p'^\mu + p^\mu}{2m}u(p),$$
$$2m\gamma^\mu - \iota\sigma^{\mu\nu}q_\nu = p'^\mu + p^\mu. \tag{3.11}$$

Substituting the result of equation (3.11) in equation (3.5), we get the final form as:

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{\iota\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2). \tag{3.12}$$

where F_1 and F_2 are unknown functions of q^2 known as form factors, which are evaluated in the following subsection.

3.2 Evaluation

Knowing the final form of the answer, the contribution of a single loop to the electron-vertex function shall now be calculated. The Feynman diagram of concern is given in figure (7):

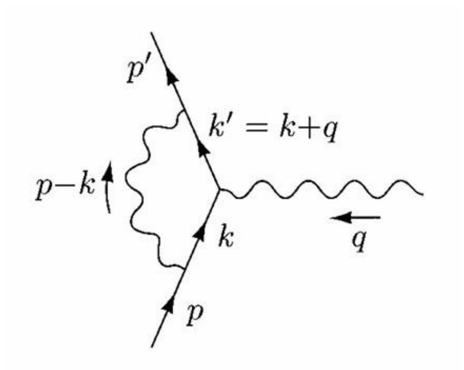

Figure 7: Feynman diagram depicting one-loop contribution to the vertex function.

To order α-correction, the function $\Gamma^{\mu}(p',p)$ is: $-\iota e\Gamma^{\mu} \approx -ie\left(\gamma^{\mu} + \delta\Gamma^{\mu}\right)$ where:

$$\begin{aligned}
\delta\Gamma^{\mu}(p',p) &= \int \frac{d^4k}{(2\pi)^4} \frac{-\iota g_{\nu\rho}}{(k-p)^2+\iota\epsilon} \bar{u}(p')(-\iota e\gamma^{\nu}) \frac{\iota(\not{k}'+m)}{k'^2-m^2+\iota\epsilon} \gamma^{\mu} \frac{\iota(\not{k}+m)}{k^2-m^2+\iota\epsilon} (-\iota e\gamma^{\rho}) u(p), \\
&= \int \frac{d^4k}{(2\pi)^4} (-2\gamma^{\mu}) \frac{\bar{u}(p')(\iota e^2)(\not{k}'+m)(\not{k}+m)u(p)}{((k-p)^2)(k'^2-m^2+\iota\epsilon)(k^2-m^2+\iota\epsilon)}, \\
&= 2\iota e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}'\gamma^{\mu}\not{k} + m^2\gamma^{\mu} - 2m(k'+k)^{\mu}\right]u(p)}{((k-p)^2+\iota\epsilon)(k'^2-m^2+\iota\epsilon)(k^2-m^2+\iota\epsilon)}.
\end{aligned} \tag{3.13}$$

where in the second line, $\gamma^{\nu}\gamma^{\mu}\gamma_{\nu} = -2\gamma^{\mu}$.

This integral can be evaluated with the help of a method known as Feynman parameters, where the goal is to squeeze the three denominator factors of equation (3.13) into a single quadratic polynomial raised to the third power, and then, shift k by a constant to complete the square and evaluate the remaining integral. The proof of this is given in Appendix D.

Rewriting equation (3.13) in terms of Feynman parameters x, y, and z, the denominator reduces down to:

$$\frac{1}{((k-p)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} = \int_0^1 dxdydz\ \delta(x+y+z-1)\ \frac{2}{D^3}, \tag{3.14}$$

where D^3 is the new denominator and

$$D = x(k^2-m^2) + y(k'^2-m^2) + z(k-p)^2 + (x+y+z)i\epsilon. \tag{3.15}$$

Since (3.14) has a Dirac Delta Function and $k' = k+q$ as previously stated,

$$\begin{aligned}
D &= k^2x + k^2z - m^2(x+y) - 2kpz + p^2z + k^2y + 2kqy + q^2y, \\
&= k^2(x+y+z) - m^2(x+y) + yq^2 + zp^2 + 2k(qy-pz) + (x+y+z)i\epsilon, \\
&= k^2 + 2k(qy-pz) + yq^2 + zp^2 - (1-z)m^2 + i\epsilon.
\end{aligned} \tag{3.16}$$

Defining the shifted loop momentum $l = k + yq - zp$ and its square $l^2 = k^2 + 2k(yq - zp) + (l-k)^2$, k is shifted to complete the square,

$$\begin{aligned}
D &= l^2 - (yq-zp)^2 + yq^2 + zp^2 - (1-z)m^2 + i\epsilon, \\
&= l^2 - (y^2-y)q^2 - (z^2-z)p^2 + 2yqzp - (1-z)m^2 + i\epsilon, \\
&= l^2 + y(x+z)q^2 + z(x+y)p^2 + 2yqzp - (1-z)m^2 + i\epsilon, \\
&= l^2 + xyq^2 + xzp^2 + yz(p+q)^2 - (1-z)m^2 + i\epsilon.
\end{aligned}$$

For an on-shell electron momenta, $p^2 = (p+q)^2 = p'^2 = m^2$ as the photon is massless,

$$\begin{aligned}
D &= l^2 + xyq^2 - m^2((1-z) - (x+y)z) + i\epsilon, \\
&= l^2 + xyq^2 - m^2((1-z) - (1-z)z) + i\epsilon, \\
&= l^2 + xyq^2 - m^2(1-z)^2 + i\epsilon, \\
&= l^2 - \Delta + i\epsilon,
\end{aligned} \tag{3.17}$$

where $\Delta = -xyq^2 + m^2(1-z)^2$.

Since q^2 is less than 0 as it is caused by a space-like separated photon, Δ is positive and can be considered the mass term.

Now the numerator in (3.13) must also be expressed in terms of l. Since the denominator depends only on the magnitude of l, the integral in (3.13) can be solved using the following identities:

$$\int \frac{d^4l}{(2\pi)^4}\frac{l^\mu}{D^3} = 0, \tag{3.18}$$

$$\int \frac{d^4l}{(2\pi)^4}\frac{l^\mu l^\nu}{D^3} = \int \frac{d^4l}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}l^2}{D^3}, \tag{3.19}$$

where in (3.19),

$$\begin{aligned} l^2 &= l_\mu l^\mu, \\ &= g_{\mu\nu}l^\mu l^\nu, \\ g^{\mu\nu}l^2 &= g^{\mu\nu}g_{\mu\nu}l^\mu l^\nu = (4)l^\mu l^\nu, \\ \implies l^\mu l^\nu &= \frac{1}{4}g^{\mu\nu}l^2. \end{aligned} \tag{3.20}$$

In (3.19), the integral vanishes when $\mu \neq \nu$ because of symmetry in the integral. Moreover, since (3.18) is an odd function with symmetric limits (from $-\infty$ to $+\infty$), the integral is 0. Therefore, the numerator in the integral (3.13) reduces down to

$$\begin{aligned} &\implies \bar{u}(p')[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu]u(p) \\ &= \bar{u}(p')[(\not{l} - y\not{p} + z\not{p})\gamma^\mu(\not{l} + (1-y)\not{p} + z\not{p}) + m^2\gamma^\mu - 2m(2l^\mu - 2yq^\mu + q^\mu + 2zp^\mu)]u(p), \\ &= \bar{u}(p')[\gamma^\sigma l_\sigma\gamma^\mu\gamma^\nu l_\nu - \not{l}(1-y)\not{q} + z\not{l}\gamma^\mu\not{p} - y\not{q}\gamma^\mu\not{l} + z\not{p}\gamma^\mu\not{l} + \\ &+ (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 4ml^\mu - 2m((1-2y)q^\mu + 2zp^\mu)]u(p), \\ &= \bar{u}(p')\Bigg[\frac{\gamma^\sigma\gamma^\mu\gamma_\sigma l^2}{4} - \not{l}(1-y)\not{q} + z\not{l}\gamma^\mu\not{p} - y\not{q}\gamma^\mu\not{l} + z\not{p}\gamma^\mu\not{l} + \\ &+ (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 4ml^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\Bigg]u(p), \\ &= \bar{u}(p')\Bigg[-\frac{1}{2}\gamma^\mu l^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\Bigg]u(p). \end{aligned} \tag{3.21}$$

The terms in (3.21) can be rearranged in such a way that these are proportional to γ^μ and $i\sigma^{\mu\nu}q_\nu$. An advantage of this would be that the coefficients of these two terms will correspond to form factors F_1 and F_2 respectively. A way to achieve this is to use the expression for Γ^μ

$$\Gamma^\mu = \gamma^\mu \cdot A + (p'^\mu + p^\mu) \cdot B + q^\mu \cdot C, \tag{3.22}$$

as in (3.5). Using the anticommutation relation $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}$ and the Dirac equation $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p')m$, which also implies $\bar{u}(p')\not{q}u(p) = 0$, the numerator can be calculated. For the second term in the numerator,

$$
\begin{aligned}
\implies \bar{u}(p')&[(-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p})]u(p) \\
&= \bar{u}(p')[(-y\not{q} + z(\not{p}' - \not{q}))\gamma^\mu((1-y)\not{p} + z\not{p})]u(p), \\
&= \bar{u}(p')[(-(y+z)\not{q} + mz)\gamma^\mu((1-y)\not{p} + z\not{p})]u(p), \\
&= \bar{u}(p')[((x-1)(1-y)\not{q}\gamma^\mu\not{q} + mz(x-1)\not{q}\gamma^\mu + \\
&+ mz(1-y)\gamma^\mu\not{q} + m^2z^2\gamma^\mu]u(p), \\
&= \bar{u}(p')[((1-x)(1-y)\gamma^\mu q^2 + m^2z^2 + \\
&+ 2mz(1-y)q^\mu + mz(x+y-2)\not{q}\gamma^\mu]u(p).
\end{aligned} \tag{3.23}
$$

The term $\not{q}\gamma^\mu$, sandwiched between $\bar{u}(p')$ and $u(p)$, gives

$$
\begin{aligned}
\bar{u}(p')[\not{q}\gamma^\mu]u(p) &= \bar{u}(p')[(\not{p}' - \not{p})\gamma^\mu]u(p), \\
&= \bar{u}(p')[(m - \not{p})\gamma^\mu]u(p), \\
&= \bar{u}(p')[(m\gamma^\mu - 2p^\mu + \gamma^\mu\not{p})]u(p), \\
&= \bar{u}(p')[2m\gamma^\mu - 2p^\mu]u(p), \\
&= 2\bar{u}(p')[m\gamma^\mu - p^\mu]u(p).
\end{aligned} \tag{3.24}
$$

Therefore, using $\not{q}\gamma^\mu\not{q} = -\gamma^\mu q^2$ and (3.24) in (3.23), the second terms becomes

$$
\implies \bar{u}(p')[\gamma^\mu(1-x)(1-y)q^2 + m^2(-2z-z^2) + 2mz(1-y)q^\mu + 2mz(1+z)p^\mu]u(p). \tag{3.25}
$$

Plugging (3.25) into (3.21),

$$
\begin{aligned}
\text{Numerator} = \bar{u}(p')\Bigg[\gamma^\mu \cdot \bigg(-\frac{1}{2}\cdot l^2(1-x)(1-y)q^2 + (1-2z-z^2)m^2\bigg) + 2m[(1-y)z + \\
-(1-2y)]q^\mu + 2mz(z-1)p^\mu\Bigg].
\end{aligned} \tag{3.26}
$$

By adding and subtracting $mz(z-1)p^\mu$ and $mz(z-1)p'^\mu$ to (3.26), and rearranging the terms so that the numerator has the form in (3.22), it becomes

$$
\begin{aligned}
\implies \bar{u}(p')\Bigg[\gamma \cdot \bigg(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)\bigg) + (p'^\mu + p^\mu)\cdot mz(z-1) + \\
+ q^\mu \cdot m(z-2)(x-y)\Bigg]u(p).
\end{aligned} \tag{3.27}
$$

Therefore the overall integral for $\delta\Gamma^\mu(p', p)$ turns out to be

$$
\begin{aligned}
\implies \delta\Gamma^\mu(p', p) = 2ie^2 \int_{-\infty}^{\infty} \frac{d^4l}{(2\pi)^4} \int_0^1 dxdydz\ \delta(x+y+z-1)\frac{2}{D^3} \times \\
\times \bar{u}(p')\Bigg[\gamma^\mu \cdot \bigg(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2 + \\
+ (p'+p)\cdot mz(z-1) + q^\mu \cdot m(z-2)(x-y)\Bigg]u(p).
\end{aligned} \tag{3.28}
$$

Using the Ward Identity, which implies that

$$q^\mu \mathcal{M}_\mu = q^\mu \Gamma^\mu = 0,$$

the integral can be written in terms of its form factors. Moreover, the denominator is symmetric under the exchange of x and y, as it gives a minus sign to the coefficient of q^μ when exchanged. This makes the q^μ term disappear when integrating over dx and dy. Moreover, using the Gordon Identity in (3.10) where $(p' + p)$ is exchanged with $i\sigma^{\mu\nu}q_\nu$, the overall integral now becomes

$$\begin{aligned}\implies \delta\Gamma(p', p) = 2ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 & dxdydz\ \delta(x+y+z-1) \frac{2}{D^3} \times \\ & \times \bar{u}(p') \Bigg[\gamma^\mu \cdot \Bigg(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2 \Bigg) + \\ & + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(2m^2 z(z-1)) \Bigg] u(p'). \end{aligned} \tag{3.29}$$

Now, only integration over the momentum integral is needed to be done. For that, using the Wick rotation, whereby the 4-dimensional integral in the Minkowski space can be changed to the one in Euclidean space, would simplify the calculations. Consider l in the Minkowski space-time as

$$l^0 = il_E^0 \quad ; \quad \mathbf{l} = \mathbf{l}_E,$$

where l_E is the Euclidean 4-momentum. Equating l^0 to an imaginary quantity causes a "rotation" from the real to the complex axis in the complex plane. This conversion relates l^μ and l_E^μ as

$$\begin{aligned} l^2 &= (l^0)^2 - (l^1)^2 - (l^2)^2 - (l^3)^2, \\ &= (il_E^0)^2 - (l_E^1)^2 - (l_E^2)^2 - (l_E^3)^2, \\ &= -(l_E^0)^2 - (l_E^1)^2 - (l_E^2)^2 - (l_E^3)^2, \\ &= -(l_E)^2. \end{aligned} \tag{3.30}$$

The poles in integral (3.29) in the l^0-plane are shown in Fig 8.

Using $dl^0 = idl_E^0$ and (3.30), the terms without l^2 in the numerator in (3.29) has this factor

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dl^4}{(2\pi)^4} \frac{1}{[l^2 - \Delta]^m} &= \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^4l_E \frac{1}{[-l_E^2 + \Delta]^m}, \\ &= \frac{i}{(-1)^m (2\pi)^4} \int_{-\infty}^{\infty} d^4l_E \frac{1}{[l_E^2 + \Delta]^m}, \\ &= \frac{i(-1)^m}{(2\pi)^4} \int d\Omega \int_0^\infty dl_E \frac{l_E^3}{[l_E^2 + \Delta]^m}, \\ &= \frac{i(-1)^m}{8\pi^2} \int_0^\infty dl_E \frac{l_E^3}{[l_E^2 + \Delta]^m}, \end{aligned} \tag{3.31}$$

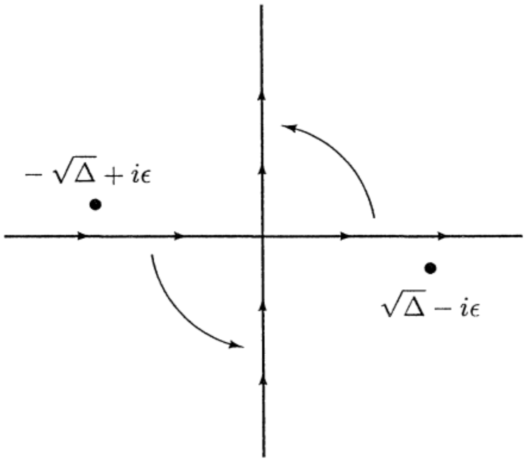

Figure 8: Poles of the integral in (3.29).

where $\int d\Omega$ is the surface area of the 4-dimensional unit sphere, which equals to $2\pi^2$. Taking $\beta = l_E^2 + \Delta$ and $d\beta = 2l_E dl_E$, (3.31) becomes

$$\begin{aligned} \frac{i(-1)^m}{8\pi^2}\int_0^\infty dl_E \frac{l_E^3}{[l_E^2+\Delta]^m} &= \frac{i(-1)^m}{(4\pi)^2}\int_0^\infty dl_E \frac{\beta-\Delta}{\beta^m} \\ \therefore \int_{-\infty}^\infty \frac{dl^4}{(2\pi)^4}\frac{1}{[l^2-\Delta]^m} &= \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}. \end{aligned} \tag{3.32}$$

Similarly, for the l^2 term in the numerator of the integral in (3.29), the integration over dl_E gives

$$\int_{-\infty}^\infty \frac{dl^4}{(2\pi)^4}\frac{l^2}{[l^2-\Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}, \tag{3.33}$$

which is convergent only for $m > 3$. However, in this case, $m = 3$ which makes the integral divergent. This poses a problem, as this makes the Wick rotation unjustfied.

To fix this, Pauli-Villars regularization is employed, where the photon propagator is replaced by

$$\frac{1}{(k-p)^2+i\epsilon} \to \frac{1}{(k-p)^2+i\epsilon} - \frac{1}{(k-p)^2-\Lambda^2+i\epsilon},$$

where the second term is the propagator of a fictitious photon whose contribution is subtracted from the actual photon, making the result finite. Λ is the very large mass of the said fictitious photon. Therefore, when we calculate the new $\delta\Gamma^\mu(p',p)$, the numerator in (3.29) remains the same, whereas the denominator is changed to $D^3 = (l^2-\Delta_\Lambda)^3$, where

$$\Delta \to \Delta_\Lambda = -xyq^2 + (1-z)^2m^2 + z\Lambda^2.$$

Therefore, the integral (3.33) can now be Wick rotated. Taking β as before and

$u = l_E^2 + \Delta_\Lambda$ and $du = 2l_E dl_E$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dl^4}{(2\pi)^4}\left(\frac{l^2}{[l^2-\Delta]^3} - \frac{l^2}{[l^2-\Delta_\Lambda]^3}\right) &= \frac{i}{8\pi^2}\left[\int_0^\infty dl_E \frac{l_E^4}{(l_E^2+\Delta)^3} - \int_0^\infty dl_E \frac{l_E^4}{(l_E^2+\Delta_\Lambda)^3}\right], \\
&= \frac{i}{(4\pi)^2}\left[\int_0^\infty d\beta \frac{(\beta+\Delta)^2}{\beta^3} + \int_0^\infty du \frac{(u+\Delta)^2}{u^3}\right], \\
&= \frac{i}{(4\pi)^2}\left[\lim_{l_E\to\infty} \log\left(\frac{l_E^2-\Delta}{l_E^2-\Delta_\Lambda}\right) + \log\left(\frac{\Delta_\Lambda}{\Delta}\right)\right] + O(\Lambda^{-2}), \\
&= \frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right) + O(\Lambda^{-2}), \\
&= \frac{i}{(4\pi)^2}\log\left(\frac{-xyq^2+(1-z)^2m^2}{\Delta} + \frac{z\Lambda^2}{\Delta}\right) + O(\Lambda^{-2}), \\
&= \frac{i}{(4\pi)^2}\log\left(\frac{z\Lambda^2}{\Delta} + 1\right) + O(\Lambda^{-2}), \\
&\approx \frac{i}{(4\pi)^2}\log\left(\frac{z\Lambda^2}{\Delta}\right) + O(\Lambda^{-2}). \qquad (3.34)
\end{aligned}$$

Consequently, using (3.32) and (3.34), we get the explicit expression for the one-loop vertex correction in terms of form factors

$$\begin{aligned}
\implies \delta\Gamma^\mu(p', p) = \frac{\alpha}{2\pi}\int_0^1 & dxdydz\ \delta(x+y+z-1)\times \\
&\times \bar{u}(p')\left(\gamma^\mu\left[\log\left(\frac{z\Lambda^2}{\Delta}\right) + \frac{1}{\Delta}\Big((1-x)(1-y)q^2 + (1+4z-z^2)m^2\Big)\right] + \right. \\
&\left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}2m^2 z(1-z)\right]\right)u(p), \qquad (3.35)
\end{aligned}$$

where the coefficient of γ^μ is $F_1(q^2)$ and the coefficient of $i\sigma^{\mu\nu}q_\nu/2m$ is $F_2(q^2)$.

Using the Pauli-Villars regularization, we have parametrized the ultraviolet divergence in (3.35). This divergence occurs at $F_1(0)$, which for an electron should be 1. To deal with this, the following substitution is made

$$\delta F_1(q^2) \to \delta F_1(q^2) - \delta F_1(0),$$

where $\delta F_1(0)$ is the first order correction to $F_1(q^2)$. It compensates for the omission of the external leg corrections diagrams.

From the $1/\Delta$ term, there is an infrared divergence present in $F_1(q^2 = 0)$ as well. For example, at $q = 0$

$$\begin{aligned}
\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{1-4z+z^2}{\Delta(q^2=0)} &= \int_0^1\int_0^{1-z} \frac{-2+(1-z)(3-z)}{m^2(1-z)^2}, \\
&= \int^1 \frac{-2}{m^2(1-z)} + \text{ finite terms},
\end{aligned}$$

this term from (3.35) is infrared divergent. To rectify this, the photon is assumed to have a very small mass μ, and it also appears in the photon propagator as $(k-p)^2 - \mu^2$. This gives an additional term $z\mu^2$ to Δ. Thus, $F_1(q^2)$ becomes

$$
\begin{aligned}
F_1(q^2) &= 1 + \delta F_1(q^2) - \delta F_1(0), \\
&= 1 + \frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\times \\
&\times \left[\log\left(\frac{z\Lambda^2}{m^2(1-z)^2 - q^2xy}\right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + z\mu^2} + \right. \\
&\left. - \log\left(\frac{z\Lambda^2}{m^2(1-z)^2}\right) - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + z\mu^2}\right] + O(\alpha^2), \\
\therefore F_1(q^2) &= 1 + \frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\times \\
&\times \left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + z\mu^2} + \right. \\
&\left. - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + z\mu^2}\right] + O(\alpha^2).
\end{aligned}
\tag{3.36}
$$

Moreover, from (3.35) the second form factor $F_2(q^2)$ is

$$
F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\left[\frac{2m^2z(1-z)}{m^2(1-z)^2 - q^2xy}\right] + O(\alpha^2). \tag{3.37}
$$

Unlike for $F_1(q^2)$, $F_2(q^2)$ does not have neither ultraviolet divergence nor infrared divergence. Interestingly, for $q^2 = 0$,

$$
\begin{aligned}
F_2(q^2 = 0) &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\left[\frac{2m^2z(1-z)}{m^2(1-z)^2}\right], \\
&= \frac{\alpha}{\pi}\int_0^1 dz \int_0^{1-z} dy \frac{z}{1-z}, \\
&= \frac{\alpha}{2\pi} = a_e,
\end{aligned}
\tag{3.38}
$$

where a_e is the correction to the g-factor of the electron:

$$
a_e \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx 0.0011614.
$$

4 Infrared Divergence

The infrared divergence in equation (3.36) shall now be our main focus. As $\mu \to 0$, the dominant part contributing to this IR divergence is:

$$F_1(q^2) \approx \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\left[\frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy+\mu^2 z} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2+\mu^2 z}\right] \tag{4.1}$$

To interpret this, it is noted that the divergence occurs when $z \approx 1$ i.e., in the corner of the Feynman-parameter space, where $x \approx y \approx 0$. Thus, we can set $z = 1$ and $x = y = 0$ in the numerators and, set $z = 1$ only in the μ^2 terms in the denominators of equation (4.1) as:

$$\begin{aligned} F_1(q^2) &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\left[\frac{-2m^2+q^2}{m^2(1-z)^2-q^2xy+\mu^2} - \frac{-2m^2}{m^2(1-z)^2+\mu^2}\right], \\ &= \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dy\left[\frac{-2m^2+q^2}{m^2(1-z)^2-q^2y(1-z-y)+\mu^2} - \frac{-2m^2}{m^2(1-z)^2+\mu^2}\right]. \end{aligned} \tag{4.2}$$

where the x-integral has been evaluated.

Next, the following variable changes are made:

$$y = (1-z)\xi$$
$$\omega = (1-z)$$

Therefore, equation (4.2) becomes:

$$F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 (1-z)d\xi\int_0^1 \frac{d(\omega^2)}{2(1-z)}\left[\frac{-2m^2+q^2}{m^2\omega^2-q^2(1-z)\xi(1-z-y)+\mu^2} - \frac{-2m^2}{m^2\omega^2+\mu^2}\right]. \tag{4.3}$$

Simplifying the denominator of the first term in equation (4.3):

$$\begin{aligned} &m^2\omega^2 - q^2(1-z)\xi(1-z-y)+\mu^2, \\ &= m^2\omega^2 - q^2\left(\frac{y}{1-z}\right)\left(\frac{1-z-y}{1-z}\right)(1-z)^2+\mu^2, \\ &= m^2\omega^2 - q^2\left(\frac{y}{1-z}\right)\left(1-\frac{y}{1-z}\right)(1-z)^2+\mu^2, \\ &= m^2\omega^2 - q^2\xi(1-\xi)\omega^2+\mu^2. \end{aligned} \tag{4.4}$$

Therefore, equation (4.3) becomes:

$$F_1(q^2) = \frac{\alpha}{4\pi}\int_0^1 d\xi\int_0^1 d(\omega^2)\left[\frac{-2m^2+q^2}{m^2\omega^2-q^2\xi(1-\xi)\omega^2+\mu^2} - \frac{-2m^2}{m^2\omega^2+\mu^2}\right]. \tag{4.5}$$

Performing the (ω^2) integral in equation (4.5) as:

$$\int_0^1 d(\omega^2)\frac{-2m^2+q^2}{m^2\omega^2-q^2\xi(1-\xi)\omega^2+\mu^2} = \int_0^1 \frac{d(u)}{m^2\omega^2-q^2\xi(1-\xi)}\frac{-2m^2+q^2}{u}. \tag{4.6}$$

where $u = m^2\omega^2 - q^2\xi(1-\xi)\omega^2 + \mu^2$, then:

$$\begin{aligned}\int_0^1 \frac{d(u)}{m^2\omega^2 - q^2\xi(1-\xi)} \frac{-2m^2+q^2}{u} &= \frac{-2m^2+q^2}{m^2 - q^2\xi(1-\xi)} \log(m^2\omega^2 - q^2\xi(1-\xi)\omega^2 + \mu^2)\Big|_0^1, \\ &= \frac{-2m^2+q^2}{m^2 - q^2\xi(1-\xi)} \log\left(\frac{m^2\omega^2 - q^2\xi(1-\xi)\omega^2 + \mu^2}{\mu^2}\right).\end{aligned} \tag{4.7}$$

And:

$$\int_0^1 d(\omega^2)\frac{2m^2}{m^2\omega^2+\mu^2} = \int_0^1 \frac{d(u)}{m^2}\frac{2m^2}{u}. \tag{4.8}$$

where $u = m^2\omega^2 + \mu^2$, then:

$$\int_0^1 \frac{du}{m^2}\frac{2m^2}{u} = 2\log(m^2\omega^2+\mu^2)\Big|_0^1 = 2\log\left(\frac{m^2+\mu^2}{\mu^2}\right). \tag{4.9}$$

Therefore, equation (4.5) is then:

$$F_1(q^2) = \frac{\alpha}{4\pi}\int_0^1 d\xi\left[\frac{-2m^2+q^2}{m^2-q^2\xi(1-\xi)}\log\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right) + 2\log\left(\frac{m^2}{\mu^2}\right)\right]. \tag{4.10}$$

As $\mu \to 0$, the details inside the numerator can be ignored as anything proportional to m^2 or q^2 is effectively the same. Hence, equation (4.10) becomes:

$$F_1(q^2) = 1 - \frac{\alpha}{2\pi} f_{IR}(q^2)\log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) + \mathcal{O}(\alpha^2). \tag{4.11}$$

where f_{IR} is the coefficient of the divergent logarithm as:

$$f_{IR}(q^2) = \int \frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)} d\xi - 1. \tag{4.12}$$

Moreover, as q^2 is negative and $\xi(1-\xi)$ has a maximum value of $1/4$:

$$\begin{aligned}\frac{d}{d\xi}(\xi(1-\xi)) &= 0, \\ 1 - 2\xi = 0 \to \xi &= \frac{1}{2}, \\ \to \xi(1-\xi) &= \frac{1}{4}.\end{aligned} \tag{4.13}$$

hence, f_{IR} is positive.

Now, the question is: how does this divergence affect the electron scattering cross-section? First, it is noted that:

$$\frac{d\sigma}{d\Omega} \propto \iota\mathcal{M} \propto \Gamma^\mu \propto F_1(q^2).$$

Therefore:

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{d\sigma}{d\Omega}\right)_0 \cdot \left[1 - \frac{\alpha}{2\pi} f_{IR}(q^2)\log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]. \tag{4.14}$$

where equation (4.14) is the cross-section for electron scattering ($\mathbf{p} \to \mathbf{p'}$) and, the first factor is the tree-level result. Adding to that, in the correction to order α, the cross-section is not only infinite but, also negative - which is horribly wrong.

To understand this divergence better, solving the coefficient of the divergent logarithm, f_{IR}, in the limit $-q^2 \to \infty$ gives:

$$f_{IR}(q^2) = \int_0^1 d\xi \frac{-q^2/2}{m^2 - q^2\xi(1-\xi)} \simeq \frac{1}{2}\int_0^1 d\xi \frac{-q^2}{-q^2\xi + m^2} + (\text{equal contribution from } \xi \approx 1\ ). \tag{4.15}$$

Computing the integral as:

$$\begin{aligned} \frac{1}{2}\int_0^1 d\xi \frac{-q^2}{-q^2\xi + m^2} &= \frac{1}{2}\int_0^1 \frac{du}{-q^2}\frac{-q^2}{u} = \frac{1}{2}\log(-q^2\xi + m^2)\Big|_0^1, \\ &= \frac{1}{2}\log\left(\frac{-q^2+m^2}{m^2}\right). \end{aligned} \tag{4.16}$$

Hence:

$$f_{IR}(q^2) = \log\left(\frac{-q^2}{m^2}\right) \tag{4.17}$$

where in equation (4.15), the integral receives equal contributions from ξ and $1-\xi$, therefore, the integral has been split in some sense. In equation (4.16), the substitution $u = -q^2\xi + m^2$ has been made and in equation (4.17), $-q^2 >> m^2$ and both contributions have been added.

Therefore, the form factor, from equation (4.11), in this limit is:

$$F_1(-q^2 \to \infty) = 1 - \frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2). \tag{4.18}$$

It appears that equation (2.37) and (4.18) are similar in the sense that both expressions contain the double logarithm of $-q^2$. As for the cross-sections of the elastic electron scattering (equation (4.18)) and soft bremsstrahlung (equation (2.37)):

$$\begin{aligned} \frac{d\sigma}{d\Omega}(p \to p') &= \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]; \\ \frac{d\sigma}{d\Omega}(p \to p' + \gamma) &= \left(\frac{d\sigma}{d\Omega}\right)_0 \left[+\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]. \end{aligned} \tag{4.19}$$

Upon observing, the solution to our problem becomes quite clear. The separate cross-sections may be divergent but, their sum is finite as it is independent of μ^2.

Adding to this, the elastic cross-section and the soft bremsstrahlung cross-section cannot be measured individually; only their sum is observable! In any real experiment, a photon detector can only detect photons down to some minimum limiting energy, E_l. Thus, the probability that the scattering process occurs and no photon is detected is:

$$\frac{d\sigma}{d\Omega}(p \to p') + \frac{d\sigma}{d\Omega}(p \to p' + \gamma(k < E_l)) \equiv \left(\frac{d\sigma}{d\Omega}\right)_{measured}. \tag{4.20}$$

where:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 [1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) + \frac{\alpha}{2\pi} \mathcal{I}(\mathbf{v}', \mathbf{v}) \log\left(\frac{E_l^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)]. \quad (4.21)$$

We further note that $\mathcal{I}(\mathbf{v}', \mathbf{v}) = 2f_{IR}$ in the limit $-q^2 >> m^2$. If the same result holds for general q^2, then equation (4.21) becomes:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \left[\log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) + \log\left(\frac{E_l^2}{\mu^2}\right)\right] + \mathcal{O}(\alpha^2)\right],$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{-q^2 \text{ or } m^2}{E_l^2}\right) + \mathcal{O}(\alpha^2)\right]. \quad (4.22)$$

which only depends on experimental conditions and not on μ^2. Therefore, we get a cross-section independent of any divergence. Additionally, the value of $\mathcal{I}(\mathbf{v}', \mathbf{v})$ for arbitrary values of $-q^2$ still happens to be $2f_{IR}$ as calculated in (2.21).

Lastly, for practicality and knowledge of dependence on q^2, the total calculation is considered in the limit $-q^2 >> m^2$, and hence equation (4.22) is:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_l^2}\right) + \mathcal{O}(\alpha^2)\right]. \quad (4.23)$$

which is the required cross-section as measured in experiments.

5 Conclusion

The paper began with an introduction to IR divergences which arise in Feynman diagrams and what cause them, followed by a definition to both bremsstrahlung and elastic electron scattering processes. Next, a thorough calculation of QED radiative corrections due to the emission of soft photons in both classical and quantum theory frameworks was performed and it was found that the classical result arises as a limiting case of the quantum result. Moreover, an infrared divergence was stumbled upon in both cases (as μ (the cutoff) $\to 0$).

With this concept covered, the complicated study of the one-loop correction to the electron-vertex function (/correction to electron scattering due to virtual photon) became our next focus; the function was discussed and computed explicitly along with its form factors, in which another infrared divergence was encountered.

It was then discovered that the solution to the IR divergences in both the bremsstrahlung and elastic electron scattering cases lied in the fact that the scattering cross-sections for both cases are identical and that their summation was the answer to our problem (also known as renormalization); it is finite and measurable. Even in real experiments, we find that both cross-sections are only observable as a sum and never individually.

Appendices

A Sum Over External Photon Polarizations

Assuming $k^\mu = (k, 0, 0, k)$, our Ward Identity reduces down to

$$k\mathcal{M}^0(k) - k\mathcal{M}^3(k) = 0,$$

which implies that $\mathcal{M}^0 = \mathcal{M}^3$. As $\mathcal{M}(k) = \epsilon_\mu \mathcal{M}^\mu(k)$,

$$\begin{aligned}\sum_\epsilon \epsilon^*_\mu \epsilon_\nu \mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k) &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2, \\ &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2, \\ &= -g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k). \end{aligned} \tag{A.1}$$

Therefore, we can replace $\sum_\epsilon \epsilon^*_\mu \epsilon_\nu$ with $-g_{\mu\nu}$, where we some over external photon polarizations. Since $k_\mu \mathcal{M}^\mu$ is a Lorentz scalar, and therefore will hold for all Lorentz frames, this manifestation of the Ward Identity will hold true for all other frames.

B Feynman Rules for Quantum Electrodynamics

The Feynman Rules provide a straightforward method for computing the amplitude of a Feynman diagram. Following are the QED Feynman Rules:

1. Vertex: $-\iota e \gamma^\mu$

2. Photon Propagator: $\frac{-\iota g_{\mu\nu}}{q^2 + \iota\epsilon}$

3. Fermion Propagator: $\frac{\iota(\not{p} + m)}{p^2 - m^2 + \iota\epsilon}$

4. External Photon Lines: $\epsilon(p)$ or $\epsilon^*(p)$ (depends on associated momentum)

C Gordon Identity

The Gordon identity as stated in equation (3.10) is given by [4]:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{\iota\sigma^{\mu\nu}q_\nu}{2m}\right]u(p). \tag{C.1}$$

To prove this, expand the right-hand side of equation (C.1) as:

$$
\begin{aligned}
\text{R.H.S} &= \bar{u}(p')\frac{1}{2m}\left(p'^{\mu} + p^{\mu} - \frac{1}{2}\gamma^{\mu}\gamma^{\nu}(p'-p)_{\nu} + \frac{1}{2}\gamma^{\nu}\gamma^{\mu}(p'-p)_{\nu}\right)u(p), \\
&= \bar{u}(p')\frac{1}{2m}\left(p'^{\mu} + p^{\mu} - \frac{1}{2}\gamma^{\mu}\gamma^{\nu}(p'-p)_{\nu} + \frac{1}{2}(2g^{\mu\nu} - \gamma^{\mu}\gamma^{\nu})(p'-p)_{\nu}\right)u(p), \\
&= \bar{u}(p')\frac{1}{2m}\left(p'^{\mu} + p^{\mu} - \frac{1}{2}\gamma^{\mu}\gamma^{\nu}(p'-p)_{\nu} + g^{\mu\nu}(p'-p)_{\nu} - \frac{1}{2}\gamma^{\mu}\gamma^{\nu}(p'-p)_{\nu}\right)u(p), \\
&= \bar{u}(p')\frac{1}{2m}\left(p'^{\mu} + p^{\mu} - \gamma^{\mu}\gamma^{\nu}(p'-p)_{\nu} + p^{\mu} - p'^{\mu}\right)u(p), \\
&= \bar{u}(p')\frac{1}{2m}\left(2p^{\mu} - \gamma^{\mu}\not{p}' + \gamma^{\mu}\not{p}\right)u(p), \\
&= \bar{u}(p')\frac{p^{\mu}}{m}u(p), \\
&= \bar{u}(p')\gamma^{\mu}u(p). \checkmark
\end{aligned}
\tag{C.2}
$$

where in the first line, $\imath\sigma^{\mu\nu} = \frac{\imath}{2}[\gamma^{\mu}, \gamma^{\nu}]$ [5], in the second, $\gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} - \gamma^{\mu}\gamma^{\nu}$, in the second last line, $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p}' = m\bar{u}(p')$, and in the last, $mu(p) = \not{p}u(p) = \gamma^{\mu}p_{\mu}u(p)$.

D Feynman Parameters

The Feynman parametrized form for n denominators is:

$$
\frac{1}{A_1...A_n} = \int_0^1 d\alpha_1...d\alpha_n\delta(1-\alpha_1-...-\alpha_n)\frac{(n-1)!}{[\alpha_1 A_1 + ... + \alpha_n A_n]^n}. \tag{D.1}
$$

The proof is as follows [6]:

Reexpressing all the factors in the denominator in Schwinger parametrized form as:

$$
\frac{1}{A_i} = \int_0^{\infty} ds_i e^{-\imath s_i A_i}. \tag{D.2}
$$

for $i = 1, ..., n$.

Therefore:

$$
\frac{1}{A_1...A_n} = \int_0^{\infty} ds_1 ... \int_0^{\infty} ds_n e^{-(s_1 A_1 + ... + s_n A_n)}. \tag{D.3}
$$

Next, the following substitutions are made:

$$
\alpha = s_1 + ... + s_n. \tag{D.4}
$$

$$
\alpha_i = \frac{s_i}{s_1 + ... + s_n}. \tag{D.5}
$$

for $i = 1, ..., n-1$.

Equation (D.3) then becomes:

$$
\frac{1}{A_1...A_n} = \alpha^{n-1}\int_0^1 d\alpha_1...d\alpha_{n-1}\int_0^{\infty} d\alpha\left(e^{-\alpha(\alpha_1 A_1 + ... + \alpha_{n-1}A_{n-1} + (1-\alpha_1-...-\alpha_{n-1})A_n)}\right) \tag{D.6}
$$

Performing the integration over α as:

$$\begin{aligned}
\int_0^\infty d\alpha(\alpha^{n-1})e^{-\alpha x} &= \frac{\partial^{n-1}}{\partial(-x)^{n-1}}\left(\int_0^\infty e^{-\alpha x}d\alpha\right), \\
&= \frac{\partial^{n-1}}{\partial(-x)^{n-1}}\left(\frac{1}{x}\right), \\
&= \frac{(n-1)!}{x^n}.
\end{aligned} \tag{D.7}$$

where $x = \alpha_1 A_1 + ... + \alpha_{n-1}A_{n-1} + (1 - \alpha_1 - ... - \alpha_{n-1})A_n$.

Therefore, equation (D.6) becomes:

$$\frac{1}{A_1...A_n} = (n-1)!\int_0^1 d\alpha_1...d\alpha_{n-1}\frac{1}{[\alpha_1 A_1 + ... + \alpha_{n-1}A_{n-1} + (1 - \alpha_1 - ... - \alpha_{n-1})A_n]^n}. \tag{D.8}$$

after adding an extra integral:

$$\frac{1}{A_1...A_n} = (n-1)!\int_0^1 d\alpha_1...\int_0^1 d\alpha_n\frac{\delta(1-\alpha_1-...-\alpha_n)}{[\alpha_1 A_1 + ... + \alpha_n A_n]^n}. \tag{D.9}$$

which is the final form of the Feynman parameterization.

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References

[1] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Perseus Books Publishing, L.L.C (1995).

[2] M. Kotlarchyk, *Scattering Theory*, Elsevier, Oxford (1999), https://doi.org/10.1006/rwsp.2000.0274.

[3] A. Pokraka, *Rutherford Scattering*.

[4] J.L. Bourjaily, *Gordon Identity Proof*.

[5] D. O'Brien, *Proof of Gordon Decomposition identity*.

[6] R. Feynman, *Space-Time Approach to Quantum Electrodynamics*, vol. 76, Physical Review (1949).