Quantum Hall Effect

Course Project for PHY 633: Magnetism: Theory and Experiment

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Abstract

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Quantum Hall Effect(QHE) occurs in two-dimensional electron systems, subjected to low temperatures and strong magnetic fields. It has two further distinguished versions: Integer Quantum Hall Effect (IQHE) and Fractional Quantum Hall Effect (FQHE). This report first introduces the classical Hall effect, derived from the Drude Model, and then quantizes it. This will lead to the detailed derivation of IQHE and FQHE, and how Hall conductivity occurs in these frameworks. Additionally, special focus on the edge states for the former, and emphasis on the Laughlin states and existence of fractional charges of quasi-holes for the latter will be given.

Contents

1 Introduction

Quantum Hall Effect (QHE) is the quantum mechanical counterpart of the Hall Effect, that was observed in 1879. Unlike the latter, QHE works in the regime of low temperatures and strong magnetic fields in $2D$ electron systems. The Hall conductivity is calculated as:

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu, \tag{1.1}$$

where ν is either an integer greater than 0 or is fractional. The former explains the Integer Quantum Hall Effect (IQHE) and the latter the Fractional Quantum Hall Effect (FQHE). IQHE results in plateaux of Hall resistivity for certain ranges of magnetic field for a single value of ν. This corresponds to the quantization of Hall resistivity and longitudinal resistivity going to 0. On the other hand, FQHE has $\nu \in \mathbf{Q}$, which results in shorter but more such plateaus. Not all fractional values are allowed, as this phenomenon is caused due to electron-electron interactions. This is a fascinating open research question in condensed matter systems, where the physical interpretation for some fractional values of ν is yet to be understood.

2 Classical Hall Effect

Classically, the Hall Effect is set up as shown in Figure 1. It has a constant magnetic field directed in the z-axis, and electrons are restricted to move only in the $x-y$ plane. A constant current $\mathbf{I}$ is induced in the x direction, whereas the Hall Voltage V_H is induced in the y direction.

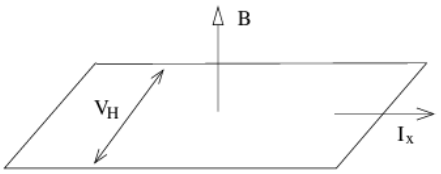

Figure 1: Classical set up of the Hall Effect (1).

Quantum mechanically, in the presence of magnetic field B, the charged particles move in cyclotron motion:

$$\begin{aligned} m\frac{d\mathbf{v}}{dt} &= -e\mathbf{v}\times\mathbf{B}, \\ &= -eBv_y\hat{\imath} + eBv_x\hat{\jmath}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \implies m\ddot{x} = -eB\dot{y}, &\quad m\ddot{y} = eB\dot{x}, \\ m\dot{v_x} = -eBv_y, &\quad m\dot{v_y} = eBv_x, \\ \implies v_x = c_1\cos(\omega t+\phi), &\quad v_y = c_1\sin(\omega t+\phi), \\ x(t) = X + R\sin(\omega t+\phi), &\quad y(t) = Y - R\cos(\omega t+\phi), \end{aligned} \tag{2.2}$$

where $R = c1/\omega$ and is the radius of the circle, ϕ is the arbitrary phase angle, and ω is the cyclotron frequency:

$$\omega = \omega_B = \frac{eB}{m}. \tag{2.3}$$

This causes the particle to move in the anticlockwise direction when $B > 0$, as ω_B is positive.

This is done in the absence of an external electric field. In the presence of one, an electric field term and a linear friction term - which captures the effect of an electron colliding with other particles or lattice structures - is also added to the expression in Eq. 2.1:

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau},$$

where τ is the scattering time for the electron. This is the Drude Model for electron transport. To get the equilibrium solutions,

$$\begin{aligned}\frac{d\mathbf{v}}{dt} = 0 &= -\frac{e}{m}\mathbf{E} - \frac{e}{m}\mathbf{v}\times\mathbf{B} - \frac{\mathbf{v}}{\tau},\\ \therefore \mathbf{v} + \frac{e\tau}{m}\mathbf{v}\times\mathbf{B} &= -\frac{e\tau}{m}\mathbf{E}.\end{aligned} \tag{2.4}$$

Since current density $\mathbf{J} = -ne\mathbf{v}$, where n is the density of charge carriers,

$$\begin{aligned}-ne\mathbf{v} - \frac{ne^2\tau}{m}\mathbf{v}\times\mathbf{B} &= \frac{ne^2\tau}{m}\mathbf{E}\\ \mathbf{J} + \frac{e\tau}{m}\mathbf{J}\times\mathbf{B} &= \frac{e^2n\tau}{m}\mathbf{E}\\ \implies \begin{pmatrix}1 & \omega_B\tau & 0\\ -\omega_B\tau & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{J} &= \frac{e^2n\tau}{m}\mathbf{E}.\end{aligned} \tag{2.5}$$

Since movement of the particles is restricted in the z-axis, the expression simplifies to

$$\begin{aligned}\begin{pmatrix}1 & \omega_B\tau\\ -\omega_B\tau & 1\end{pmatrix}\mathbf{J} &= \frac{e^2n\tau}{m}\mathbf{E},\\ \therefore\quad \mathbf{J} &= \frac{e^2n\tau}{m(\omega_B^2\tau^2+1)}\begin{pmatrix}1 & -\omega_B\tau\\ \omega_B\tau & 1\end{pmatrix}\mathbf{E},\end{aligned} \tag{2.6}$$

where the conductivity tensor σ is defined as

$$\sigma \equiv \frac{e^2n\tau}{m(\omega_B^2\tau^2+1)}\begin{pmatrix}1 & -\omega_B\tau\\ \omega_B\tau & 1\end{pmatrix}. \tag{2.7}$$

The off-diagonal elements, σ_{xy}, are responsible for the classical Hall effect. This calculation shows that to reach an equilibrium for a setup in Figure 1, an electric field $\mathbf{E}$ must be applied in the x direction. This, in turn, gives rise to J_x, which is the current density in the x direction. This gets deflected due to $\mathbf{B} = B\hat{z}$, in the y direction. The result is the build-up of charge along the edge of the conductor, giving rise to an induced E_y. This continues till this induced E_y is large enough to cancel the bending due to $\mathbf{B}$. Therefore, the electrons solely travel in the x direction, and E_y causes the Hall Voltage V_H as

$$V_H = LE_y,$$

where L is the length of the conductor along the y-axis.

Following from Eq. 2.7, the resistivity tensor, which is the inverse of the conductivity tensor σ, has two components:

$$\rho_{xx} = \frac{m}{ne^2\tau}, \quad \rho_{xy} = \frac{m\omega_B}{ne^2} = \frac{B}{ne}, \tag{2.8}$$

where ρ_{xx} is the longitudinal resistivity and ρ_{xy} is the transverse resistivity. We can also notice that the former does not depend upon the scattering time τ. This implies that the transverse resistivity, which brings about the classical Hall Effect, is equal to its transverse resistance R_{xy},

$$R_{xy} = \frac{V_y}{I_x} = \frac{LE_y}{LJ_x} = \frac{E_y}{J_x} = -\rho_{xy},$$

unlike the relationship between longitudinal resistance R_{xx} and longitudinal resistivity ρ_{xx}, as not only the latter depends upon τ, but also in the limit of it going to 0, the scattering processes become less relevant as τ goes to 0 as well (1).

3 Transition to Quantum Hall Effect

To quantize a Hall system, we must take the Hamiltonian $\hat{H}$ of the system into consideration. For the purposes of this report, we neglect the spin of the charged particle. The Hamiltonian would be

$$\begin{aligned} \hat{H} &= \frac{1}{2m}(\hat{\mathbf{p}} + e\mathbf{A})^2, \\ \hat{H} &= \frac{1}{2m}(\hat{\Pi})^2, \end{aligned} \tag{3.1}$$

where $\mathbf{A}$ is the magnetic vector potential in a suitable gauge and $\hat{\Pi} = \hat{\mathbf{p}} + e\mathbf{A}$ is the mechanical momentum. Since our particle is restricted to the $x-y$ plane and the applied constant magnetic field $\mathbf{B}$ in the z direction as in the classical case, the canonical commutation relations are as follows:

$$[\hat{x}_i, \hat{p}_i] = i\hbar\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{\Pi}_x, \hat{\Pi}_y] = -ie\hbar B. \tag{3.2}$$

Similarly, corresponding ladder operators are

$$\hat{a} = \frac{1}{\sqrt{2e\hbar B}}(\hat{\Pi}_x - i\hat{\Pi}_y), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\hat{\Pi}_x + i\hat{\Pi}_y), \tag{3.3}$$

to satisfy its commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Therefore, the Hamiltonian $\hat{H}$ can be further reduced down to a one similar to a one-dimensional harmonic oscillator

$$\begin{aligned} \hat{H} = \frac{1}{2m}(\Pi)^2 &= \hbar\omega_B \left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right], \\ \implies E_n &= \hbar\omega_B \left[n + \frac{1}{2}\right], \end{aligned} \tag{3.4}$$

where ω_B is the cyclotron frequency and $n \in \mathbf{N}$. We can construct its corresponding Hillbert Space by using these ladder operators:

$$\hat{a}|0>=0, \quad \hat{a}^\dagger|n>=\sqrt{n+1}|n+1>, \quad \hat{a}|n>=\sqrt{n}|n-1>. \tag{3.5}$$

One important thing to note is that in the presence of $\mathbf{B}$, energy levels of particles become equally spaced as $\hbar\omega_B \propto \mathbf{B}$ and has a degeneracy associated with it, unlike in the case of a harmonic oscillator. These are the Landau levels, and their existence plays an important part in both IQHE and FQHE (1).

4 Integer Quantum Hall Effect

IQHE, as the name suggests, allows integer values of the filling factor ν. It is because it does not take the interactions between electrons in the system into account, and the only way an electron knows about the presence of the other is through the Pauli Exclusion Principle. It was experimentally discovered, and the data in Figure 2 shows plateaux of constant transverse resistivity ρ_{xy} formed, for a range of magnetic fields, corresponding to each spike in longitudinal resistivity ρ_{xx} as ν increases by an integer.

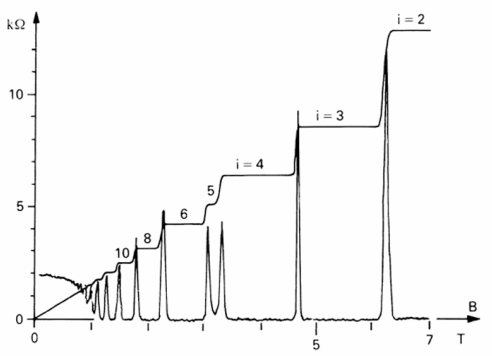

Figure 2: Experimental Results of the Integer Quantum Hall Effect (1).

4.1 Derivation

For its derivation, we prefer working in the Landau gauge, where $\mathbf{A} = (0, B\hat{x}, 0)$, as it preserves translational symmetry in both x and y directions, and no electric field is applied, that is, $\phi = 0$. This makes our system a free electron gas in the $x - y$ plane and restricted in the z directions. The canonical momentum shall be

$$\hat{\Pi} = \hat{\mathbf{p}} + e\mathbf{A},$$

Hence, our Schrodinger Equation turns out to be

$$\begin{aligned} \left[\frac{1}{2m^*}\hat{\Pi}^2 + \hat{V}(z)\right]\psi(x,y,z) &= E\psi(x,y,z), \\ \left[\frac{1}{2m^*}(\hat{\mathbf{p}} + e\mathbf{A})^2 + \hat{V}(x,y,z)\right]\psi(x,y,z) &= (E_{xy} + E_z)\psi(x,y,z) \end{aligned} \tag{4.1}$$

as movement of the electrons is only about $x - y$ plane, so there is no kinetic term for z. Here, m^* is the effective mass of the electron. Considering the potential of an infinite potential well, $V(x, y, z) = 0$, which means that the energies contributed along the z-axis is

$$E_z = \frac{n_z^2 \pi^2 \hbar^2}{2m^* L^2},$$

and the wavefunction $\psi_z(z)$ is sinusoidal,

$$\psi_z(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi z}{L}\right).$$

However, because of the Landau gauge, the Schrodinger Equation for x, y direction becomes

$$\begin{aligned} \frac{1}{2m^*}\left[(\hat{\mathbf{p}_y} + eB\hat{x})^2 + (\hat{\mathbf{p}}_x)^2\right]\psi_{xy}(x,y) &= E_{xy}\psi_{xy}(x,y), \\ \frac{1}{2m^*}\left[-\hbar^2\frac{\partial^2}{\partial x^2} + \frac{m^* e^2 B^2}{m^*}\left(\frac{\hbar\hat{k}_y}{eB} + \hat{x}\right)^2\right]\psi_{xy}(x,y) &= E_{xy}\psi_{xy}(x,y). \end{aligned} \tag{4.2}$$

Defining $\omega_c \equiv \frac{eB}{m^*}$, which is the cyclotron frequency, and $l_B^2 \equiv \frac{\hbar}{eB}$, which is the magnetic length, Eq. 4.2 reduces down to

$$\frac{1}{2m^*}\left[-\hbar^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}m^*\omega_c^2(\hat{x} + l_B^2 k_y)^2\right]\psi_{xy}(x,y) = E_{xy}\psi_{xy}(x,y). \tag{4.3}$$

We have transformed the Schrodinger Equation of our system to that of a Harmonic Oscillator in the x direction. Therefore the energies contributed along the x, y directions are

$$E_{xy} = \hbar\omega_c(n_x + \frac{1}{2}), \quad \text{where } n_x = 1, 2, 3, \ldots, \tag{4.4}$$

and the eigenstate becomes

$$\psi_{xy}(x, y) = u(x)e^{-ik_y y}, \tag{4.5}$$

where $u(x)$ is the Hermite polynomial multiplied with a Gaussian function solution to the Harmonic Oscillator. Since the canonical momentum does not depend upon $\hat{y}$, $\hat{p}_y$ commutes with $\hat{H}$, so the wavefunction along the y direction shall be planar.

The energy E of the system in the absence of the magnetic field is defined as $\frac{\hbar^2 k^2}{2m^*}$. Using this, the density of states per unit surface area of a $2D$ electron gas in the absence of magnetic field $n^0_{2D}(E)$ from $g^0_{2D}(k)$ for a one-dimensional harmonic oscillator would be

$$\begin{aligned} g^0_{2D}(k)dk \to \frac{1}{(2\pi)^2}\int d^2k &\to \frac{1}{(2\pi)^2}\int 2\pi k dk, \\ &= \frac{1}{2\pi}\int \sqrt{\left(\frac{2m^*E}{\hbar^2}\right)}\frac{2m^*}{\hbar^2}\sqrt{\left(\frac{\hbar^2}{2m^*E}\right)}dE, \\ g^0_{2D}(k)dk &= \frac{m^*}{\pi\hbar^2}dE, \\ g^0_{2D}(k)dk &= n^0_{2D}(E)dE. \end{aligned} \tag{4.6}$$

where $n^0_{2D}(k) \equiv \frac{m^*}{\pi\hbar^2}$.

Now, when we introduce a constant magnetic field. Here, the energy is split into two energy levels and the energy difference between them shall be $\Delta E = \pm g\mu_B B$. Therefore, to calculate the number of states per unit surface area n^B:

$$\begin{aligned} \text{density of states } N_B &= \frac{\Phi}{\Phi_0} = \frac{BA}{BL_y\Delta x_k} = \frac{A}{2\pi l_B^2}, \\ &= \frac{AeB}{2\pi\hbar} = \frac{m^*\omega_c A}{2\pi\hbar}, \end{aligned} \tag{4.7}$$

$$\implies \quad \text{density of states per unit surface area } n^B = \frac{m^*\omega_c}{2\pi\hbar} = \frac{eB}{h}. \tag{4.8}$$

Thus, the filling factor can be determined

$$\nu = \frac{n^0_{2D}}{n^B} = \frac{hn^0}{eB} \in \mathbf{Z}, \tag{4.9}$$

as it is assumed that the highest Landau level are fully filled. Generally, ν is a rational number which need not to be an integer. This is because the highest Landau level is not necessarily filled if one varies the magnetic field while keeping n^B constant. Moreover, $n^B \propto B$, and increase in magnetic field makes the Landau levels move up in energy. This causes a decrease in the degeneracy in the Landau levels as fewer electrons occupy top level. If magnetic field is large enough that all electrons are present in the lowest Landau level, then we have reached the magnetic quantum limit, where $\nu < 1$ (3).

Comparing the result of the classical transverse resistivity and that of its quantum counterpart, we use the definition of ν in Eq. 4.9, and the latter comes out to be

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}, \tag{4.10}$$

$$\therefore \sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu, \tag{4.11}$$

where the factor $2\pi\hbar/e^2$ is described as a "quanta" of resistivity.

4.2 Edge Modes

On the edge of a quantum Hall system, an interesting phenomenon occur with regards to the conductivity of IQHE. This is seen even in the classical case. Assuming that the particles are moving in the anticlockwise direction, the orbits near the edge bounce back and produce a current in a single direction, as shown in Figure 3. Particles restricted to move in such a way along the edge as said to be chiral, and since the chirality is opposite because the direction of the current is opposites on both sides in the absence of an external electric field, the net current turns out to be 0.

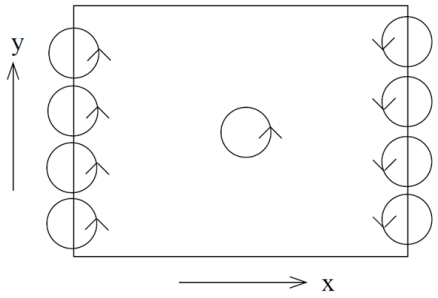

Figure 3: Classical depiction of edge states in Hall systems (2).

Quantum mechanically, working in the Landau gauge, the Hamiltonian $\hat{H}$ would be as described in Eq. 4.1, which can be simplified to

$$H_{k_y} = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_B^2(k_y l_B^2 + x)^2 + x\frac{\partial V}{\partial x}, \tag{4.12}$$

where we Taylor expand the potential $V(x)$ around its location $x = 0$, and ignored the higher order derivatives and constant $V(0)$:

$$V(x) \approx V(0) + x\left(\frac{\partial V}{\partial x}\right) + \dots. \tag{4.13}$$

Because of k_y, there is a drift velocity in the y direction

$$v_y = -\frac{1}{eB}\frac{\partial V}{\partial x}. \tag{4.14}$$

When we introduce a chemical potential difference $\Delta\mu$, as shown in Figure 4, this in turn introduces a current

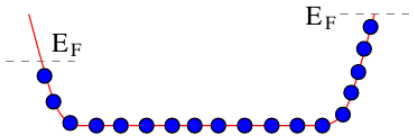

Figure 4: A chemical potential difference $\Delta\mu$ is introduced on the edge states (1).

I_y in the system,

$$\begin{aligned} I_y &= -e\int \frac{dk}{2\pi}v_y(k), \\ &= \frac{e}{2\pi l_B^2}\int dx \frac{1}{eB}\frac{\partial V}{\partial x} = \frac{e}{2\pi\hbar}\Delta\mu, \end{aligned} \tag{4.15}$$

where the Hall Voltage $V_H = \Delta\mu/e$. Thus, the expression for Hall conductivity σ_{xy} becomes

$$\sigma_{xy} = \frac{I_y}{V_H} = \frac{e^2}{2\pi\hbar}, \tag{4.16}$$

for a single filled Landau Level, which agrees with Eq. 4.10 (2; 5).

5 Fractional Quantum Hall Effect

As discussed above, it turns out that the filling factor ν is not always an integer. Experimentalists discovered that ν can take up fractional values, and their physical interpretation remains an open problem in condensed matter physics.

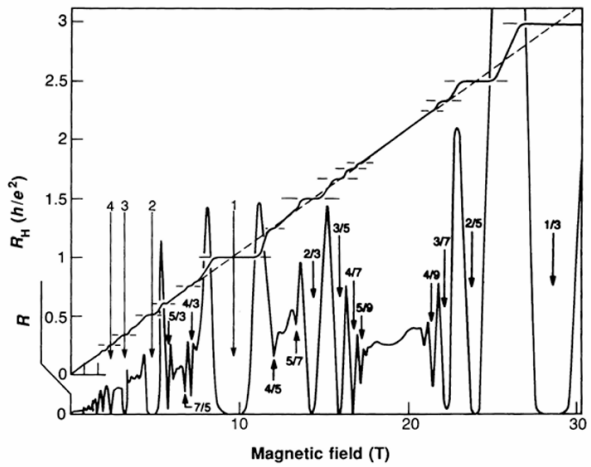

Figure 5: Experimental Results of the Fractional Quantum Hall Effect (1).

For this, the role of interactions between electrons must be considered, unlike in the case of IQHE. It is meant to lift the huge degeneracy of the ground state of the partially filled Landau level. Considering the Coulombic potential between two electrons:

$$V_{\text{Coulomb}} = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_i - \mathbf{r}_j|}, \tag{5.1}$$

we can employ perturbation theory to comprehend its dynamics. Unfortunately, it would come at a huge computational cost, both analytically and numerically, to diagonalize a macroscopically large matrix. Therefore, we use physical intuition and educated guesswork to understand the effect (1).

5.1 Derivation

For this derivation, we need to work in the symmetric gauge, that is,

$$\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B} = -\frac{yB}{2}\hat{\mathbf{x}} + \frac{xB}{2}\hat{\mathbf{y}}, \tag{5.2}$$

where translational symmetry is broken in both x and y directions but rotational symmetry is retained. Moreover, the general state in the Hilbert space coreesponding to this gauge is

$$|l, m> = \frac{\hat{a}^{\dagger n}\hat{b}^{\dagger n}}{\sqrt{l!m!}}|0, 0>,$$

where l is the Landau level and m is the angular momentum quantum number.

Now, we first proceed to develop some mathematics in this gauge before applying it to FQHE.

The lowering operator $\hat{a}$ is

$$\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2e\hbar B}}\left(\hat{\Pi}_x - i\hat{\Pi}_y\right), \\ &= \frac{1}{\sqrt{2e\hbar B}}\left(p_x - ip_y + e\left(A_x - iA_y\right)\right), \\ &= \frac{1}{\sqrt{2e\hbar B}}\left(-i\hbar\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{eB}{2}(-y - ix)\right),\end{aligned} \tag{5.3}$$

and working in the complex plane

$$z = x - iy, \qquad \bar{z} = x + iy, \tag{5.4}$$

and their corresponding derivatives

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{5.5}$$

the ladder operators reduces down to

$$\hat{a} = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right), \tag{5.6}$$

$$\hat{a}^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right), \tag{5.7}$$

where $f(z)$ is some complex analytic function. Therefore, the state in the lowest Landau level $\psi_{LLL}(z,\bar{z})$ that gives $\hat{a}\psi_{LLL}(z,\bar{z}) = 0$ is

$$\psi_{LLL}(z,\bar{z}) = f(z)e^{-|z|^2/4(l_B)^2}. \tag{5.8}$$

Similarly, we can construct the $\hat{b}$ and $\hat{b}^\dagger$ operators as

$$\hat{b} = -i\sqrt{2}\left(l_B\partial + \frac{\bar{z}}{4l_B}\right), \tag{5.9}$$

$$\hat{b}^\dagger = -i\sqrt{2}\left(l_B\bar{\partial} - \frac{z}{4l_B}\right). \tag{5.10}$$

Applying $\hat{b}^\dagger$ to $\psi_{LLL}(z,\bar{z})$ in Eq. 5.8, one gets

$$\begin{aligned}\hat{b}^\dagger\psi_{LLL}(z,\bar{z}) &= -i\sqrt{2}\left(l_B\bar{\partial} - \frac{z}{4l_B}\right)\left(f(z)e^{-|z|^2/4(l_B)^2}\right) \\ &= -i\sqrt{2}\left(\frac{-zf(z)}{4l_B}e^{-|z|^2/4(l_B)^2} - \frac{zf(z)}{4l_B}e^{-|z|^2/4(l_B)^2}\right) \\ &= \frac{iz}{\sqrt{2}l_B}\left(f(z)e^{-|z|^2/4(l_B)^2}\right),\end{aligned}$$

which means that each time we apply the creation operator, we get a factor of $z/\sqrt{2}l_B$. Therefore, the basis of our lowest Landau wavefunctions are:

$$\psi_{LLL,m}(z,\bar{z}) \sim \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4(l_B)^2}. \tag{5.11}$$

Using these, we shall apply them to interacting particles (1).

5.2 Laughlin States

Considering N particles in a system that have an arbitrary central potential between them. Since rotational symmetry is preserved in the symmetric gauge, the suitable quantum number that remains invariant is the angular momentum. The wavefunctions at the lowest Landau level are in a ring of radius $R = \sqrt{2m(N-1)}l_B$. When we apply the angular momentum operator $\hat{J}$ to $\psi_{LLL,m}$, where $\hat{J}$ is

$$\begin{aligned} \hat{J} &= i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \hbar(z\partial - \bar{z}\bar{\partial}), \\ \therefore \quad \hat{J}\psi_{LLL,m} &= \hbar m\psi_{LLL,m}. \end{aligned} \tag{5.12}$$

Neglecting the mixing between Landau levels since $\hbar\omega_B >> V$, our wavefunction turn out to be

$$\psi(z_1, \ldots, z_n) = f(z_1, \ldots, z_N)e^{-\sum_{i=1}^{N}|z_i|^2/4l_B^2}, \tag{5.13}$$

where $f(z)$ is some complex, antisymmetric analytic function. It must be antisymmetric considering this is a fermionic system.

For filling fractions described by $\nu = 1/m$, where m is the odd number angular momentum quantum number, Laughlin proposed that a single ground state wavefunction $\psi(z_i)$ is

$$\psi_{LLL,m}(z_i) \sim \prod_{i<j}(z_i - z_j)^m e^{-|z|^2/4(l_B)^2}, \tag{5.14}$$

which is antisymmetric under exchange of $z_i \leftrightarrow z_j$ when m is odd (1; 4). The term $(z_i - z_j)^m$ approaches 0 in the limit of the electrons coming very close to each other, whereas the exponential factor falls off to 0 when the electrons are further apart and away from the origin. To fully cement this proposed wavefunction as a good approximation to the actual experimentally verified wavefunction, we can show that it produces the filling factor $\nu = 1/m$. Taking the prefactor term with z_1 in it,

$$\prod_{i<j}(z_i - z_j)^m \sim \prod_{i=2}^{N}(z_1 - z_j)^m,$$

would give us m(N-1) powers of z_1. This implies the maximum angular momentum such a particle can have is $m(N-1)$ and the radius of the ring R it occupies is $\sqrt{2m(N-1)}l_B \approx \sqrt{2mN}l_B$. This approximation is taken to avoid constant terms not depending upon N. Consequently, the area occupied is $\pi R^2 \approx 2\pi mNl_B^2$. Recalling that the number of states in full Landau level $\mathcal{N}$ is

$$\begin{aligned} \mathcal{N} &= \frac{AB}{\Phi_0}, \\ &\sim \frac{(2\pi mN)(B)(e)}{2\pi\hbar}\left(\frac{\hbar}{eB}\right) \\ &= mN, \\ \implies N &= \frac{1}{m}\mathcal{N}, \\ N &= \nu\mathcal{N}, \end{aligned} \tag{5.15}$$

where $\nu = 1/m$. This is a remarkably successful proposal for small number of particles, as this produces a wavefunction which agrees 99.99% with one calculated numerically (1).

Consequently, the Hall Conductivity of FQHE becomes

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m}, \tag{5.16}$$

which aligns with the experimental results.

5.3 Fractional Charges for a Quasi-Hole

Looking at the expression in Eq. 5.14, we can deduce an expression for a quasi-hole at a position $\eta \in \mathbf{C}$,

$$\psi_{\text{hole}}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta)\prod_{k<l}(z_k - z_l)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}. \tag{5.17}$$

We can see that a "hole" is created at $z_i = \eta$, as ψ_{hole} goes to 0. To create M holes, Eq. 5.17 is modified to a more general form

$$\psi_{\text{M-holes}}(z;\eta) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}. \tag{5.18}$$

These quasi-holes have a fractional charge. A heuristic explanation for such an occurance is that if M holes are created at a single position η, then Eq. 5.18 would reduce down to

$$\psi_{\text{M-holes}}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta_j)^M\prod_{k<l}(z_k - z_l)^m e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}, \tag{5.19}$$

which essentially means that there is a deficit of a charge equivalent to that of one whole electron. In other words, all M-holes created at η have taken up a place of one electron. Therefore, one whole quasi-hole is equivalent to $(1/m)$th of an electron. So it must carry a charge equivalent to $e^* = +e/m$ (1).

6 Conclusion

Quantum Hall Effect is a very comprehensive area of study, both theoretically and experimentally. The mathematical derivation and the experimental data from different studies for both IQHE and FQHE serves as a starting point for our understanding of it. The various consequences of the Effect, like the conductivity by the edge modes and fractional charges, are detailed to provide a quantum mechanical insight into the overall framework.

There are many more remarkable phenomena occurring as a consequence of the mechanisms causing QHE. The role of disorder and topology in Hall systems are widely studied areas of physics, and it is predicted to play a key role in the production of topological quantum computers in the future. Moreover, the intuition behind the experimental data for certain FQHE systems remains an open problem left to be pondered over.

7 Bibliography

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