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Implementation of Polar Codes over AWGN and Binary Symmetric Channel

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Abstract

Objectives: Forward Error Correction (FEC) is a mechanism that has been used for years to improve the reliability of digital communications over wireless channels as they are subjected to interference, fading and noise. The goal of FEC is to add some redundant data to the input stream which facilitates the receiver to recognize and correct the errors with no need of sender to transmit the data again. In this paper we have computed bit error rate using polar codes in an Additive White Gaussian Noise Channel (AWGN) and Binary Symmetric Channel (BSC) at different block lengths. **Methods/ Analysis:** Shannon limit states that it is the maximum amount of information that any given channel can carry. Latterly, there have been inventions in Forward Error Correction (FEC) area that allows today's systems to approach the Shannon limit. One such advancement in FEC is Polar Codes which has mathematically proved to attain the Shannon capacity for Binary Discrete Memoryless Channels using the phenomenon of channel polarization which states that by recursively combining and splitting N individual channels, some channels will become completely noiseless and some become noisy. These noiseless channels are then chosen to send information bits. **Findings:** We have implemented polar codes over BI-AWGN and BSC and have calculated their bit error rate's at different channel ranges. The decoder used for decoding polar codes is successive cancellation decoder and it's seen that the performance is not that good at small block lengths but as the block length increases the performance improves when compared with the un-coded data stream. **Applications:** As polar codes are the first codes that achieve the Shannon capacity, so they seem to be the best solution for discrete memoryless channels which encounter packet failures and other noisy disturbances. Also, apart from channel encoding, polar codes found its applications in lossy and lossless source coding, coding for secrecy, communication over multiple access channels.

Keywords: Additive White Gaussian Noise (AWGN), Binary Discrete Memoryless Channel (B-DMC), Binary Erasure Channel (BEC), Binary Symmetric Channel (BSC), Bit Error Rate (BER), Forward Error Correction (FEC), Log Likelihood Ratio (LLR)

1. Introduction

Polar codes are a new type of FEC codes that are discovered by Arikan[1] and have attracted much attention recently because of the fact that they can provably attain the Shannon capacity of the class of symmetric Binary Discrete Memoryless Channels (B-DMC) with a low encoding and decoding complexity. Polar Codes are build using the phenomenon of 'channel polarization' which states that by recursively combining and splitting N independent copies B-DMC channels, the channels polarize i.e. some channels become noisy and some become noiseless and these error-free channels achieve symmetric capacity of DMC. The BER performance analysis of OFDM, incorporating dissimilar channel encoding technique for MPSK is presented over Rician channel[2]. The fountain codes have performance near to Shannon limit. FEC is required at physical layer but at the upper layers, erasure correcting codes like Fountain codes are beneficiary[3]. But it's seen in papers[4–6] that polar codes when concatenated and compared with other codes like BCH, LDPC and RS outperforms conventional codes. Various applications of polar codes involve multiple access channels[7], wiretap channels[8], and quantum channels.

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2. Polar Codes

A new type of linear block codes which have proved to achieve the capacity of symmetric Binary Discrete Memoryless Channels like BEC and BSC[9] are known as polar codes. Let B-DMC be represented as W: X →Y, with input alphabet X = {0, 1}, output alphabet Y and their transition probabilities being {W (y|x), x ∈ X, y ∈ Y}.

Mutual information between X and Y is given by equation (1).

$$I(W) \triangleq \sum_{y\in Y} \frac{\sum_{x\in X} \frac{1}{2} P(y\,|\,x)\log\left(P(y\,|\,x)\right)}{\sum_{x\in X} \frac{1}{2} P(y\,|\,x)} \tag{1}$$

where the Bhattacharyya parameter is given by equation (2).

$$Z(W) \triangleq \sum_{y\in Y} \sqrt{W(y\,|\,0)W(y\,|\,1)} \tag{2}$$

Assume W to be a symmetric channel, then its capacity is given by I(W). Note that I(W) = 1 only if Z(W) = 0, also I(W) = 0 only if Z(W) = 1. If W is a BSC with crossover probability p, then Z(W) = p and I(W) = 1-Z(W) = 1-p.

Polar codes are described as:

$$x_1^N = u_1^N G_N$$

where N is always set to 2^n, n ≥ 0. Here G_N represents the generator matrix of polar codes, which is taken from the n^{th} Kronecker power of G_2, where matrix of G_2 is given as, $G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

2.1 Channel Polarization

It means that by recursively combining and splitting N individual channels we can create N different channels $W_N^{(i)}\,(1 < i < N)$, in a way that as we make N large, then some of these channels become noisy and some become noiseless. These error-free channels, approaches the capacity of binary symmetric channels and they are used to send the information bits the rest of the channels are fixed with a value that is known to sender and receiver. Channel Polarization consists of two parts:

2.1.1 Channel Combining

Here N independent copies of B-DMC are combined in an iterative order which in-turn form a vector channel given by:

W_N: $X^N \to Y^N$ where N = 2^n, n ≥ 0.

First copy is initialized as $W_1 \triangleq$ W (n = 0). In next recursion two copies are combined to form

W_2: $X^2 \to \gamma^2$

as shown in Figure 1 with transition probability given by:

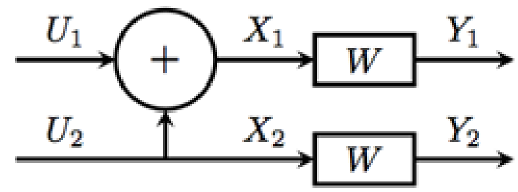

Figure 1. Construction of channel W_2.

$W_2\,(y_1,y_2|u_1,u_2) = W(y_1|u_1 \oplus u_2)W(y_2|\,u_2)$

Simmilarly, for n = 0, N = 2^n = 4, we combine two independent copies of W_2 and construct

W_4: $X^4 \to Y^4$

in a recursive way from W_2.

In the Figure 2, R_4 performs the permutation operation i.e. mapping the input {1, 2, 3,4} to {1,3,5,7}. Thus we have,

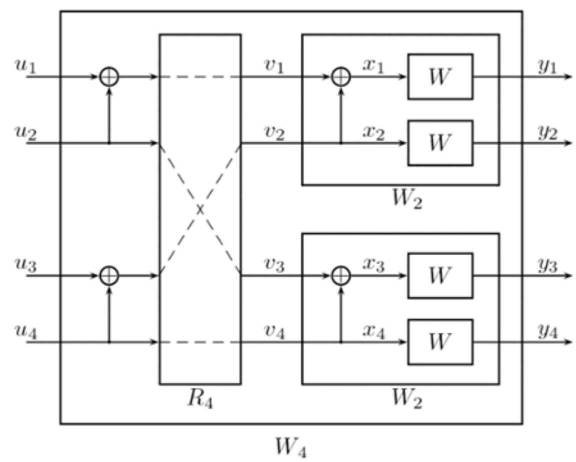

Figure 2. Channel W_4.

$$W_4 [\![\, (y)_1^4 \,|\, u_1^4 \big) = W_4\left(y_1^4 \,|\, u_1^4 G_4\right)$$

where the matrix G_4 is given by, $G_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

In general, for any n ≥ 0 W_N can be constructed using equation (3).

$$W_{N/2} : W_N\left(y_1^N \,|\, u_1^N\right) = W^N : W_N\left(y_1^N \,|\, u_1^N G_N\right) \tag{3}$$

where $G_N = B_N F^{\otimes n}, F \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and B_N represents permutation matrix also known as bit-reversal which is calculated iteratively as:

$B_N = R_N\,(I_2 \oplus B_{N/2})$

where I_2 is the 2-D unit matrix and the recursion is initialized with $B_2 = I_2$.

2.1.2 Channel Splitting

In this phase the synthesized channel W_N are splitted to construct polarized channels

$$W_N^{(i)} : X \rightarrow Y^N x\, X^{i-1}, 1 \leq i \leq N.$$

where the transition probabilities are given as:

$$W_N^{(i)}\left(y_1^N, u_1^{i-1} \mid u_i\right) \triangleq u_{i+1}^N \sum_{\in X^{N-i}} \frac{1}{2^{N-1}} W_N\left(y_1^N \mid u_1^N\right)$$

2.2 Non-Systematic Encoding

A block code given by (N, K) is a code with K inputs and N outputs and so is for polar codes. The input to G_N is a row vector (u_1, u_2,……u_N). We choose K bits of G_N's input and place our information bits onto these input's and the rest input's(N-K) are set frozen i.e. they are held constant with a value known to decoder. Let A be any arbitrary subset of {1,…..,N}. Polar codes perhaps can be determined by three variables (N, K, A, u_{Ac}) where $u_{Ac} \in X^{N-K}$ (frozen bits). The encoding is done as follows:

$x_1^N = u_1^N G_N = u_A G_N (A) \oplus u_{Ac}\, G_N (A^c)$

where G_N (A) indicates the submatrix of G_N which is created by rows with indices in A.

2.3 Successive Cancellation Decoding

The encoder maps the input bits u_1^N into the codeword x_1^N which is then transmitted in the channel W^N and corresponding y_1^N are received. The duty of decoder is to estimate $\widehat{u}_1^N$ of u_1^N from the received y_1^N. Now if u_i is a frozen bit then the decoder sets its value to a known value. The log-likelihood ratio (LLR) for $W_N^{(i)}\left(y_1^N, \widehat{u}1^{i-1} \mid u_i\right)$ is defined as follows:

$$L_N^{(i)}\left(y^N, \widehat{u}1^{i-1} \mid u_i\right) \triangleq \begin{Bmatrix} W_N^{(i)}\left(y_1^N, \widehat{u}_1^{i-1} \mid u_i = 0\right. \\ W_N^{(i)}\left(y_1^N, \widehat{u}_1^{i-1} \mid u_i = 1\right. \end{Bmatrix}$$

and the decisisons for the estimation of bits $(\widehat{u}_i)$ are taken using following rule:

$$\widehat{u}_i = \{\blacksquare\left(0, L((y_1(N), \widehat{u}_1^{(i-1)} \dashv| u_{i>0}, i \in A @ 1,\ \ L((y_1 \widehat{u}_1^{(i-1)} \dashv| u_{i<0},\right.$$

3. Channel Models

In this paper, we examine two memoryless noisy channels. In general, a DMC can be described by discrete input and output alphabets X and Y respectively, with transition probabilities p(y|x) for $x \in X$ and $y \in Y$. It is assumed that the channel's output relies only on the current state, so as to get memoryless channel. We analyze the behavior of polar codes over following channels:

3.1 Additive White Gaussian Noise (AWGN) Channel

AWGN channel's output can be represented as y = x + n, where $x \in \{+1, -1\}$ and n is random noise variable having zero mean and variance $[\![(\sigma]\!]^2)$. The transition probability for AWGN channel is given by equation (4).

$$P(y \mid x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-([\![y-1)]\!]^2}{2\sigma^2}\right) \quad (4)$$

The channel LLR is calculated as follows:

$$L(y) = \log\frac{P(y \mid x=+1)}{P(y \mid x=+1)} = \log\left(\frac{\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-([\![y-1)]\!]^2}{2\sigma^2}\right)}{\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-([\![y+1)]\!]^2}{2\sigma^2}\right)}\right)$$

$$L(y) = \log\left(\exp\left(\left(\frac{-([\![y-1)]\!]^2}{2\sigma^2} + \frac{-([\![y+1)]\!]^2}{2\sigma^2}\right)\right)\right) = \frac{2y}{\sigma^2}$$

So, the received LLR for AWGN channel can be computed by equating, L(y) = $L(y) = \frac{2y}{\sigma^2}$.

3.2 Binary Symmetric Channel (BSC)

For BSC, the channel input and output alphabets are defined as X = {0, 1} and Y = {0, 1} respectively, where the transmitted bit it flipped with a probability of 'p' known as crossover probability by the channel. Transitional probabilities, $P(Y = 0 \mid X = 1) = P(Y = 1 \mid X = 0) = p$ and $P(Y = 0 \mid X = 0) = P(Y = 1 \mid X = 1) = 1 - p$

4. Results

BER is described as the errors which occur in per unit time in a transmission system. BER can be written as:

BER = number of error bits/total number of bits sent

For calculating BER in MATLAB using Polar Codes in Gaussian channels and BSC, the following steps are performed repeadly:

- K random bits are generated.
- Those bits are given as input to the encoder which gives N bits.
- Then normally distributed noise is added.

- The LLRs of the results is computed.
- These LLRs are fed to the SC decoder.
- SC decoder produces an estimate of input data.
- At the end the output bits are compared with the input bits and the errors are counted.

Polar codes have been implemented over AWGN and BSC and the BER have been calculated and comparison at various block length's (N), with code rate = 1/2(K/N),has been done as shown in figure's 3 and 4 respectively. It's seen that BER decreases as the N increases. So, as a whole polar codes perform well in both the channels and are more likely to be useful at higher data rates.

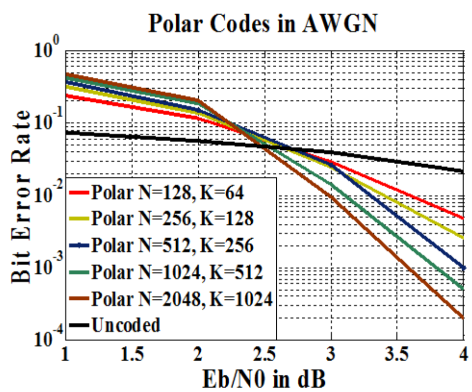

Figure 3. BER vs. Eb/N0 for AWGN.

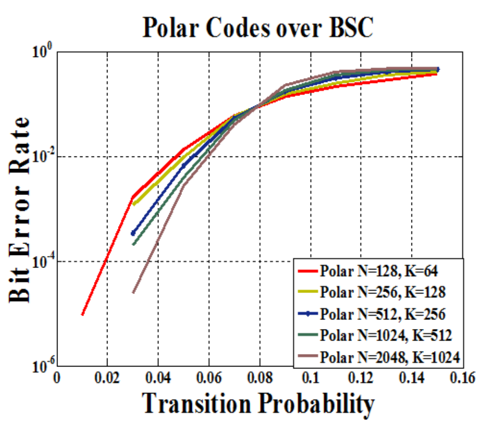

Figure 4. BER vs. Transition probability for BSC.

5. Conclusion

We discussed a recent subject in communication theory i.e. channel polarization and polar codes, proposed by Arikan. BER estimation for polar codes over different channel models like AWGN and BSC have been evaluated at different block lengths. We know that as length of N increases the channels polarize i.e. some become noisy and some noise-less. From the plots it's clear that the BER of polar code decreases as N increases, so they perform well at higher data rates and are used in respective applications. Further research is needed to test polar codes in more realistic models of RF environments.

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