UKSta18_Liang_Ex07

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1 The old lighthouse keeper and the sea

In a Scottish pub you meet a retired lighthouse keeper who is telling about the lighthouse, situated at position (x_0, y_0) miles of the coast (see gure) that he had been tending before retirement. The lighthouse is peculiar: instead of emitting a periodic signal in a given direction it emits short, focused light pulses randomly in every horizontal direction (or azimuth). No direction is preferred, and no correlation exists among the directions of the ashes. Since you got curious you take a walk along the coast, and you record your location when observing a ash. (see the recorded measurements in le *lighthouse.dat*) Now you want to infer the position of the lighthouse, given only the various positions of the light signal recordings. Since we want to restrict ourselves to a 1-parameter problem for this exercise, you can assume the x_0 coordinate of the lighthouse to be 1.25 miles.

- a: Translate the probability density distribution of the azimuth of the ashes into a probability density for observing a ash at location x. Hint: Applying simple trigonometry, relate the angle to position along the coast. Note, that the question has nothing to do with Bayesian estimation as such, it is related to the method of exact transformation between samples drawn from different PDFs.

Since all horizontal direction is equally possible, the PDF of can be seen as a uniform distribution in the domain of $[0, 2\pi)$, namely $\sim U(0, 2\pi) \Rightarrow p() = \chi_{[0,2\pi)} \frac{1}{2\pi}$

According to trigonometry we have $|x_0 - x| = y_0 \cdot \tan \Rightarrow \theta = \arctan \frac{|x_0 - x|}{y_0}$. Because of conservation of probability we get $p(x|y_0) = p()|\frac{d}{dx}| = \chi_{[0,2\pi)} \frac{1}{2\pi y_0} \cdot \frac{1}{1 + (\frac{|x_0 - x|}{y_0})^2}$ A Cauchy distribution

- b: Late during the very evening in the pub the lighthouse keeper tells stories of the early days when he in fact had to manually row out to the lighthouse which lies "at least 2miles off shore". You interpret this statement as a distance of (2.00.3) miles distributed like a Gaussian. What is the Bayesian estimate of y_0? Compare MAP, mean, and median of the posterior distribution! Compute also $_y$!

The likelihood is then a Gaussian $N(2.0, 0.3^2)$: $\frac{1}{0.3\sqrt{2\pi}} e^{-\frac{(y_0 - 2.0)^2}{2 \cdot 0.3^2}}$
We assume the prior is: $p(y_0|\{x_i\}) \propto \prod_{x_i} p(x_i|y_0)$

Thus the posterior is $P(y_0|\{x_k\}) \propto \frac{1}{0.3\sqrt{2\pi}} e^{-\frac{(y_0 - 2.0)^2}{2 \cdot 0.3^2}} \prod_{x_i} \chi_{[0,2\pi)} \frac{1}{2\pi y_0} \cdot \frac{1}{1 + (\frac{|x_0 - x_i|}{y_0})^2}$

Calculate MAP: $\frac{\partial \ln P(y_0|\{x_k\})}{\partial y_0} = -\frac{(y_0 - 2.0)}{0.3^2} - .. = 0 \Rightarrow$
Mean: $E(y_0|\{x_k\}) = \int_{-\infty}^{\infty} y_0 P(y_0|x) dy_0 =$

- c: Comparing the location of the maxima of the likelihood function and posterior, you start to wonder whether the lighthouse keeper did exaggerate the level of hardship of the early days. Redo the the previous estimate with largely uninformative priors ignoring the keeper's stories: a constant prior and $1/y$. What do you get for the MAP distances? What do you conclude?

Hints on the numerical treatment: since it is a 1D problem one can easily calculate the posterior a sufficiently ne – for simplicity equidistant – grid. A range of y between zero (excluding zero as such) and 3 miles is enough. Then plot the posterior(s). This helps to see what is going on.

Optional add-ons for the tireless ones: you may try and see what the quadratic approximation would give for y. Moreover, you may try to solve the problem by sampling the posterior with a – guess what – Markov chain.

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In [18]: x <- read.table("lighthouse.dat")
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