10. THE HUBBLE–LAW AND THE EXPANSION OF THE UNIVERSE

Astro Lab Landessternwarte Königstuhl
Max Camenzid & David Hiriart

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Abstract

The essential parameters of modern cosmological models are the Hubble constant H_0, the non–relativistic matter density parameter Ω_M, and the dark energy density parameter Ω_Λ. Besides the determination of the Hubble constant, the measurement of the two density parameters is a key issue in Cosmology. Different observing methods, including Supernovae of type Ia, have been used to fix these parameter for a concrete world model. We study the methods based on various cosmological Supernovae projects. Finally, we discuss the cosmological fundamental plane, where each point represents a concrete cosmological model.

1 Introduction

The Universe consists on the large–scale of galaxy clusters (Virgo, Coma), filaments (Great Wall), and Voids. With telescopes you could nowadays observe around 100 billion bright galaxies. These galaxies are the atoms of the Universe, whose distribution, motion and evolution one should understand. Many of these insights come from the last 20 years. Cosmology is a very young and successful discipline of Astronomy.

Hubble's law is the statement in Cosmology that the redshift in light coming from distant galaxies is proportional to their distance. The law was first formulated by Edwin Hubble and Milton Humason in 1929 after nearly a decade of observations. It is considered the first observational basis for the expanding space paradigm and today serves as one of the most often cited pieces of evidence in support of the Big Bang Theory. The most recent determination of the Hubble constant, using the satellite Planck as 2016, yields a value of 67.80±0.77 km s^{-1} /Mpc [1].

In the decade before Hubble made his observations, a number of physicists and mathematicians had established a consistent theory of the relationship between space and time by using Einstein's field equation of general relativity.

[1]Planck Collaboration: Ade, P.A.R. et al. 2016, A&A, 594, 12

Applying the most general principles to the question of the nature of the Universe yielded a dynamic solution that conflicted with the then prevailing notion of a static Universe.

However, a few scientists continued to pursue the dynamical universe and discovered that it could be characterized by a metric that came to be known after its discoverers, namely Friedmann, Lemaitre, Robertson, and Walker. When the metric was applied to the Einstein equations, the so–called Friedmann equations emerged which characterized the expansion of the universe based on a parameter known today as the scale factor which can be considered a scale invariant form of the proportionality constant of Hubble's Law. This idea of an expanding spacetime would eventually lead to the Big Bang Theory.

2 Redshift and Hubble–Law

Edwin Hubble did most of his professional astronomical observing work at Mount Wilson observatory, at the time the world's most powerful telescope. His observations of Cepheid variable stars in spiral nebulae enabled him to calculate the distances to these objects. Surprisingly, these objects were discovered to be at distances which place them well outside the Milky Way. The nebulae were first described as *island universes* and it was only later that the name *galaxy* would be applied to them.

For relatively nearby galaxies, the velocity v of a galaxy can be estimated from the galaxy's redshift z using the formula $v = zc$, where c is the speed of light.

$$z \equiv \frac{\lambda_o - \lambda_e}{\lambda_e} \quad , \tag{1}$$

λ_o is the observed wavelength and λ_e the emitted wavelength. $z > 0$ is a redshift and $z < 0$ a blueshift. $v = cz$ is the corresponding Doppler shift velocity.

The redshift z of most galaxies is positive, i.e. those galaxies move away from us. Figure 1 shows an example of the redshift in the galaxie NGC 2903 [2]. This fact had already been observed in 1912 by Slipher, and it was then carefully investigated by Hubble, who correlated the redshift with apparent magnitudes. In 1925, spectra of 43 galaxies (called nebulae at that time) were known. From those, 38 galaxies showed a positive redshift. In 1929, Hubble published this fundamental result that redshift of a galaxy increases with increasing distance

$$cz = H_0 D \quad . \tag{2}$$

This relation between the redshift z and the distance of cosmological objects is now known as the **Hubble–Law**

In vector form, the Hubble–law states that the velocity $\vec{v}$ of a galaxy G at position $\vec{r}$ with respect to the Galaxy is given by

$$\vec{v} = H_0 \vec{r} \quad . \tag{3}$$

[2] `www.astro.washington.edu/courses/labs/clearinghouse/labs/HubbleLaw/ngc2903_main.html`

Let us consider a the galaxy G from the point of view of a third galaxy G' at position $\vec{r}\,'$, which moves at velocity $\vec{v}\,' = H_o\vec{r}\,'$. Then, the velocity of G relative to G' is then

$$\vec{v} - \vec{v}\,' = H_0\vec{r} - H_0\vec{r}\,' = H_0(\vec{r} - \vec{r}\,') \quad . \tag{4}$$

This means that also galaxy G' observes that all galaxies move away. Since galaxies define the cosmological space, we call this motion **isotropic expansion of the space between the galaxies**. Only a relativistic theory of gravity can explain this fundamental fact about the expansion of the Universe.

Lab–Tasks I

1. Describe the positions of RR Lyrae and δ Cepheid stars in the Hertzsprung-Russell Diagram.
2. Describe the period–luminosity relation for δ Cepheid stars.
3. What is the observed wavelength of Lyα for a quasar with redshift $z = 6.4$?
4. Find out the maximum redshift observed for quasars (see Sloan Digital Sky Survey project).
5. What is the meaning of isotropy for the expansion of the Universe? How can one test the isotropy of the Universe?
6. Is the Universe really homogeneous?

3 The Hubble Space Telescope Key Project on the Extragalactic Distance Scale

The main goal of Hubble Space Telescope (HST) Key Project on the Extragalactic Distance Scale was to determine the Hubble Constant, H_0, to an accuracy of $\pm 10\%$. This goal has been achieved by the systematic observations of Cepheid variable stars in several galaxies using the HST. The Key Project team used the HST to observe 19 galaxies out to 108 million light–years (33 Mpc). They discovered almost 800 Cepheid variable stars, a special class of pulsating star used for accurate distance measurements. The spiral galaxy NGC 4603 is the most distant galaxy in which Cepheid variables have been found. It is associated with the Centaurus cluster, one of the most massive assemblages of galaxies in the nearby universe.

3.1 What are Cepheid Variables?

The structure of all stars, including the Sun and Cepheid variable stars, is determined by the opacity of matter in the star. If the matter is very opaque, then it takes a long time for the photons to diffuse out from the hot core of the

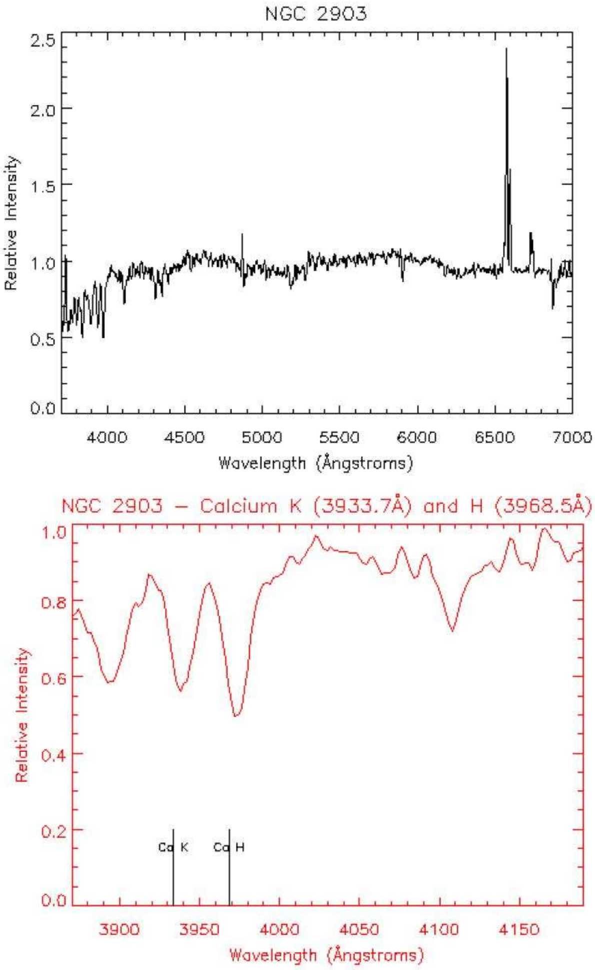

Figure 1: The global spectrum of the galaxy NGC 2903 (*top*). The spectrum contains characteristic absorption lines and hydrogen emission lines (Hα). For the measurements of the redshift, we can use especially Ca H and Ca K lines, which appear in absorption (*bottom*).

Figure 2: Henrietta Leavitt (1868–1929) discovered the period–luminosity relation for δ Cepheid stars.

star, and strong temperature and pressure gradients can be develop in the star. If the matter is nearly transparent, then photons move easily through the star and erase any temperature gradient. Cepheid stars oscillate between two states: when the star is in its compact state, the helium in a layer of its atmosphere is singly ionized. Photons scatter off of the bound electron in the singly ionized helium atoms , thus, the layer is very opaque and large temperature and pressure gradient build up across the layer. These large pressure cause the layer (and the whole star) to expand. When the star is in its expanded state, the helium in the layer is double ionized, so that the layer is more transparent to radiation and there is much weaker pressure gradient across the layer. Without the pressure gradient to support the star against gravity, the layer (and the whole star) contracts and the star returns to its compressed state.

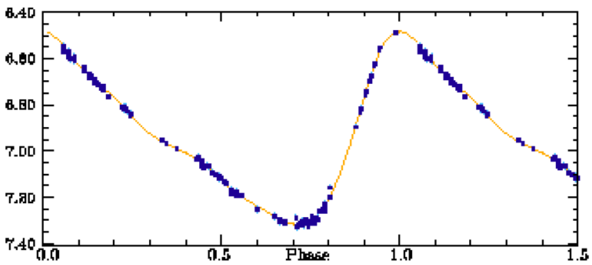

Figure 3: Light curve in the V-band of a δ Cepheid star as measured by HiPParCoS (**Hi**gh **P**recision **Par**allax **Co**llecting **S**atellite).

Cepheid variable stars have masses between five and twenty solar masses. The more massive stars are more luminous and have more extended envelopes. Because their envelopes are more extended and the density in their envelope is lower, their variability period, which is proportional to the inverse square root of the density in the layer, is longer. Brighter Cepheid have longer periods. This fundamental relationship between period and luminosity for δ Cepheid stars was found in 1912 by Henrietta Leavitt, and it has been extensively used by Hubble[3]. For given observed period P you can calculate from this relation the absolute luminosity and then derive with the mean apparent magnitude the distance of the star. For this purpose, not only one star is observed, but, if possible, a set of stars in the same galaxy. The method is be calibrated, e.g. with the Cepheid stars in the Magellanic clouds, given for the V–band

$$M_V = -2.760(\log P - 1.0) - 4.160 \quad , \tag{5}$$

and for the$I$–band

$$M_I = -2.962(\log P - 1.0) - 4.904 \quad . \tag{6}$$

The period P is taken in units of days. With this method, also the distance of the Andromeda galaxy is derived. Before HST, this method could be used for galaxies with distances upto 3 Mpc, with HST distances upto the Virgo galaxies could be derived (M 100 e.g.), i.e. upto distances of about 20 Mpc. Here, we have to take into account that galaxies in the Virgo cluster move at speeds of a hundred km/s to compensate gravity in the cluster. With the observations of a sample of 20 galaxies in the Virgo cluster, these uncertainties can be averaged out. The final result of the Hubble Key Project is published in Freedman et al. (2001) [4].

Lab–Tasks II: The Cepheid Method

1. Data in Table 3.1 give the lightcurve in the visual of the Cepheid Nr. 6 in M 101 (NGC 5457) as a function of the Julian date. Determine from these data the mean magnitude m_V, the period P, and using the relation from eq. (5) the distance modulus $DM = m_V - M_V$. (Keep in mind that data sampling might not had been taken at equally space time intervals. Is your estimate of P consistent with values in Figure 4?)
2. Make a plot for the data in Table 3.1: velocity V [in 1000km/s] vs. distance D [in Mpc].
3. Determine the best linear fit to these data.
4. The slope of this curve directly provides the Hubble constant H_0.
5. The inverse $T_0 = 1/H_0$ of the Hubble constant has the meaning of a time, and is therefore a measure for the age of the Universe (Why?). Determine the age T_0 with your data.

[3] Sterne und Weltraum, Heft 10/2003
[4] Freedman, W. et al., 2001, ApJ, 553, 47

Julian date	V [mag]
2449049.0327	23.37 ± 0.08
2449049.0938	23.49 ± 0.10
2449057.4598	23.82 ± 0.20
2449064.0828	23.83 ± 0.13
2449064.1136	23.94 ± 0.26
2449069.2661	23.69 ± 0.20
2449069.3293	23.60 ± 0.13
2449131.6589	23.23 ± 0.17
2449131.7228	23.32 ± 0.12
2449141.6263	23.58 ± 0.11
2449141.6936	23.60 ± 0.10
2449146.1096	23.74 ± 0.10
2449146.1770	23.74 ± 0.11
2449156.8860	23.90 ± 0.13
2449156.9499	23.85 ± 0.14
2449160.7658	23.60 ± 0.13
2449160.8304	23.64 ± 0.10
2449163.2450	23.48 ± 0.10
2449163.3054	23.47 ± 0.11
2449295.2633	23.77 ± 0.15
2449295.3195	23.88 ± 0.17
2449307.7036	23.10 ± 0.12
2449307.7661	23.18 ± 0.11
2449429.6016	23.58 ± 0.16

Table 1: HST light curve of the Cepheid Nr. 6 in the galaxy M 101. Data from HST Archive: http://www.ipac.caltech.edu.

Galaxy	N_{Ceph}	DM (mag)	Error (mag)	V_{Helio} [km/s]	V_{CMB} [km/s]
NGC 300	16	26.53	0.07	144	-57
NGC 925	73	29.80	0.04	553	398
NGC 1326A	17	31.04	0.09	1836	1787
NGC 1365	52	31.18	0.05	1636	1597
NGC 1425	29	31.60	0.05	1512	1477
NGC 2403	10	27.48	0.10	131	216
NGC 2541	34	30.25	0.05	559	736
NGC 2090	34	30.29	0.08	931	1057
NGC 3031	25	27.75	0.08	-34	65
NGC 3198	42	30.68	0.08	662	890
NGC 3351	49	29.85	0.09	778	1117
NGC 3368	11	29.97	0.06	897	1236
NGC 3621	69	29.08	0.06	805	1152
NGC 4321	52	30.78	0.07	1571	1856
NGC 4414	9	31.10	0.05	716	959
NGC 4496A	98	30.81	0.03	1730	2024
NGC 4548	24	30.88	0.05	486	763
NGC 4535	50	30.85	0.05	1961	2248
NGC 4536	39	30.80	0.04	1804	2097
NGC 4639	17	31.61	0.08	1010	1283
NGC 4725	20	30.38	0.06	1206	1446
NGC 5253	7	27.56	0.14	404	612
NGC 7331	13	30.81	0.09	816	508
IC 4182	18	28.28	0.06	321	513

Table 2: Mean distance moduli $DM = \langle m_V - M_V \rangle$ (in mag) of the Cepheid in local galaxies as measured with HST, together with the heliocentric velocity V_{Helio}, and Virgo–corrected expansion velocities V_{CMB}. Data: Freedman, W. et al. 2001, ApJ, 553, 47.

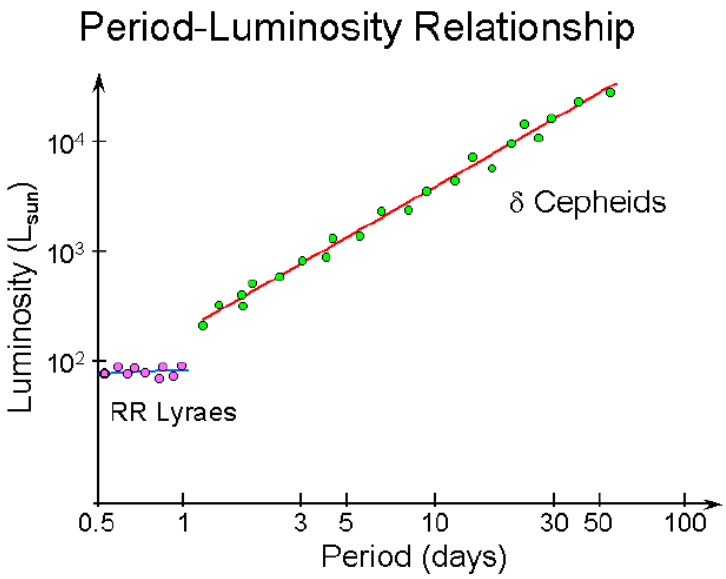

Figure 4: RR Lyrae stars and δ Cepheid stars with typical periods.

4 Supernova of Type Ia as Standard Candles

The peak brightness of SNe of type Ia vary considerably. This behavior has been investigated by means of Calan–Tololo calibration. Supernovae (Tabla 3). The lightcurves showed a specific trend: the brighter the Supernova, the slower it decays in brightness (see Fig. 5 top). Perlmutter and collaborators have then shown that this behavior of the lightcurve van be corrected by adjusting the decay time–scale. In this way, they found a universal lightcurve for Supernovae of type Ia (Fig. 5 bottom). The effective apparent magnitude of a supernova can then be calculated from the formula

$$m_B^{\mathrm{eff}} = m_{\mathrm{obs}}^{\mathrm{peak}} + \Delta_{\mathrm{corr}} - K_{BB} - A_E \tag{7}$$

Δ_{corr} is the correction factor, calculated from the observed lightcurve, A_E means the galactic extinction, and K_{BB} is the famous K-correction.

Lab–Tasks III: Hubble Constant from Local Supernovae

1. Why are Supernovae of type II not suitable for cosmological standard candles?
2. What is the difference in the spectra of Type I and Type II Supernovae?
3. What is the meaning of the K–Correction in Cosmology?

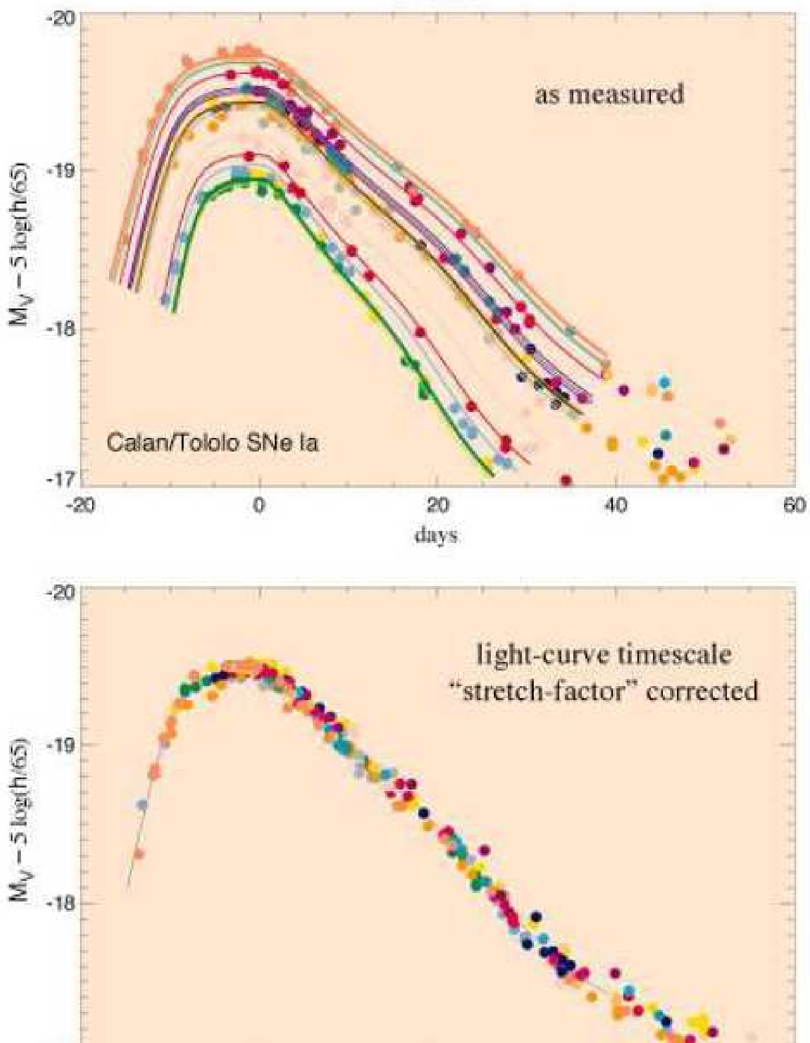

Figure 5: Low redshift SNe Type Ia lightcurves reveal a relationship between the peak brightness and the duration of the outburst: Brighter Supernovae decay slower, weaker ones faster (top). This fact can be used to correct the lightcurves to create a general template (bottom).

SN	z	σ_z	m_{obs}^{peak} [mag]	σ_{obs}^{peak} [mag]	A_E [mag]	K_{BB} [mag]	Δ_{corr} [mag]	σ_B [mag]
1990O	0.030	0.002	16.62	0.03	0.39	-0.00	0.03	0.20
1990af	0.050	0.002	17.92	0.01	0.16	0.01	-0.12	0.18
1992P	0.026	0.002	16.13	0.03	0.12	-0.01	0.06	0.24
1992ae	0.075	0.002	18.61	0.12	0.15	0.03	0.00	0.20
1992ag	0.026	0.002	16.59	0.04	0.38	-0.01	0.06	0.20
1992al	0.014	0.002	14.60	0.01	0.13	-0.01	-0.01	0.23
1992aq	0.101	0.002	19.29	0.12	0.05	0.05	-0.03	0.23
1992bc	0.020	0.002	15.20	0.01	0.07	-0.01	0.05	0.20
1992bg	0.036	0.002	17.41	0.07	0.77	0.00	0.03	0.21
1992bh	0.045	0.002	17.67	0.04	0.10	0.01	0.05	0.19
1992bl	0.043	0.002	17.31	0.07	0.04	0.01	-0.07	0.18
1992bo	0.018	0.002	15.85	0.02	0.11	-0.01	-0.14	0.21
1992bp	0.079	0.002	18.55	0.02	0.21	0.04	-0.03	0.18
1992br	0.088	0.002	19.71	0.07	0.12	0.04	-0.26	0.18
1992bs	0.063	0.002	18.36	0.05	0.09	0.03	0.00	0.18
1993B	0.071	0.002	18.68	0.08	0.31	0.03	-0.01	0.20
1993O	0.052	0.002	17.83	0.01	0.25	0.01	-0.04	0.18
1993ag	0.050	0.002	18.29	0.02	0.56	0.01	-0.02	0.20

Table 3: Supernovae of type Ia from the Calán–Tololo calibration data. z means redshift, σ_z the error redshift, m_{obs}^{peak} is the peak brightness in the B–band with the corresponding error. K_{BB} is the K–correction from the observed B–band to the intrinsic B–band and A_E the galactic extinction. Δ_{corr} corrects the B–band magnitude, as calculated from the lightcurve. σ_B is then the total error. Data adapted from Table 2 of Perlmutter, S. et al. 1999, ApJ, 517, 565.

4. Make a Hubble-Diagram with the data of Table 4 and Table 5: (z,$m_B^{\rm eff}$), linear in log$z$ (in the range 0.01≤ zle 1.0) and linear in the B–band magnitude $m_B^{\rm eff}$. Discuss the error bars.

5. Use the classical Hubble–law

$$m(z) - M_B = 5\log\left(\frac{d[z]}{[\mathrm{Mpc}]}\right) + 25 \quad , \tag{8}$$

with the distance

$$d[z] = \frac{c}{H_0} z \quad . \tag{9}$$

Since Supernovae are standard candles, we may write

$$m(z) = \mathcal{M} + 5\log z \quad , \tag{10}$$

with the constant

$$\mathcal{M} = M_B - 5\log\left(\frac{H_0\ [\mathrm{Mpc}]}{c}\right) + 25 \quad . \tag{11}$$

This represents a linear relation in the Hubble–diagram. Determine by using the data for $z \leq 0.1$ the constant $\mathcal{M}$ (called calibration constant). What is your value for H_0 according to your fit if the absolute magnitude of a Supernova M_B=-19.47±0.2? (SN–calibration by Tammann 1999) Estimate the error for your Hubble constant.

6. Discuss alternative methods to determine the Hubble constant.

5 Cosmology with Supernovae

Supernovae are essentially observable upto redshifts of $z = 2$, since they become as bright as their host galaxy. However, the spectrum gets redshifted by the expansion of the Universe. In Table 5, Supernovae of type Ia are listed as observed in the Supernova Project. With these data, you can extend the Hubble–diagram of local Supernovae towards redshifts of z=0.9. On the near future the Supernova/Acceleration Probe (SNAP), to detect SN at redshifts of $z \sim 1.7$.

5.1 Classical World Models

Besides the Hubble constant H_0, the density parameters Ω_M and Ω_Λ are the essential parameters of modern cosmological models. Ω_M contains al contributions from non-relativistic matter (Baryons and Dark matter), while Ω_Λ refers to the Dark Energy, or Cosmological Constant. The definition of these constants follows from the **Friedmann equation**[5]

$$\frac{\dot{R}}{R} \equiv H^2 = \frac{8\pi G}{3}\rho_M - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3} \quad , \tag{12}$$

[5] The derivation of the Friedmann equation can be found in any textbook of Cosmology.

SN	z	σ_z	m_{obs}^{peak} [mag]	σ_{obs}^{peak} [mag]	A_E [mag]	K_{BB} [mag]	Δ_{corr} [mag]	σ_B [mag]
1992bi	0.458	0.001	22.12	0.10	0.03	-0.72	0.30	0.46
1994F	0.354	0.001	22.08	0.10	0.11	-0.58	-0.17	0.33
1994G	0.425	0.001	21.52	0.21	0.03	-0.68	-0.04	0.49
1994H	0.374	0.001	21.28	0.06	0.10	-0.61	-0.07	0.22
1994al	0.420	0.001	22.37	0.06	0.42	-0.68	-0.08	0.25
1994am	0.372	0.001	21.82	0.07	0.10	-0.61	-0.06	0.20
1994an	0.378	0.001	22.14	0.08	0.21	-0.62	0.03	0.37
1995aq	0.453	0.001	22.60	0.07	0.21	-0.71	-0.07	0.25
1995ar	0.465	0.005	22.71	0.04	0.07	-0.71	-0.02	0.30
1995as	0.498	0.001	23.02	0.07	0.07	-0.71	0.05	0.25
1995at	0.655	0.001	22.62	0.03	0.07	-0.66	0.06	0.21
1995aw	0.400	0.030	21.75	0.03	0.12	-0.65	0.09	0.19
1995ax	0.615	0.001	22.53	0.07	0.11	-0.67	0.09	0.25
1995ay	0.480	0.001	22.64	0.04	0.35	-0.72	-0.04	0.24
1995az	0.450	0.001	22.44	0.07	0.61	-0.71	-0.02	0.23
1995ba	0.388	0.001	22.08	0.04	0.06	-0.63	-0.01	0.20
1996cf	0.570	0.010	22.70	0.03	0.13	-0.68	0.02	0.22
1996cg	0.490	0.010	22.46	0.03	0.11	-0.72	0.04	0.20
1996ci	0.495	0.001	22.19	0.03	0.09	-0.71	0.01	0.19
1996ck	0.656	0.001	23.08	0.07	0.13	-0.66	-0.05	0.28
1996cl	0.828	0.001	23.53	0.10	0.18	-1.22	0.07	0.54
1996cm	0.450	0.010	22.66	0.07	0.15	-0.71	-0.05	0.23
1996cn	0.430	0.010	22.58	0.03	0.08	-0.69	-0.06	0.22
1997F	0.580	0.001	22.90	0.06	0.13	-0.68	0.01	0.23
1997G	0.763	0.001	23.56	0.41	0.20	-1.13	-0.02	0.53
1997H	0.526	0.001	22.68	0.05	0.16	-0.70	-0.06	0.20
1997I	0.172	0.001	20.04	0.02	0.16	-0.33	-0.03	0.18
1997J	0.619	0.001	23.25	0.08	0.13	-0.67	0.00	0.28
1997K	0.592	0.001	23.73	0.10	0.07	-0.67	0.09	0.37
1997L	0.550	0.010	22.93	0.05	0.08	-0.69	-0.02	0.25
1997N	0.180	0.001	20.19	0.01	0.10	-0.34	0.01	0.17
1997O	0.374	0.001	22.97	0.07	0.09	-0.61	0.02	0.24
1997P	0.472	0.001	22.52	0.04	0.10	-0.72	-0.03	0.19
1997Q	0.430	0.010	22.01	0.03	0.09	-0.69	-0.03	0.18
1997R	0.657	0.001	23.28	0.05	0.11	-0.66	0.00	0.23
1997S	0.621	0.001	23.03	0.05	0.11	-0.67	0.10	0.21
1997ac	0.320	0.010	21.38	0.03	0.09	-0.55	0.03	0.18
1997af	0.579	0.001	22.96	0.07	0.09	-0.68	-0.06	0.22
1997ai	0.450	0.010	22.25	0.05	0.14	-0.71	0.02	0.30
1997aj	0.581	0.001	22.55	0.06	0.11	-0.68	-0.03	0.22
1997am	0.416	0.001	21.97	0.03	0.11	-0.67	0.05	0.20
1997ap	0.830	0.010	23.20	0.07	0.13	-1.23	0.02	0.22

Table 4: SCP SNe Ia data. z is redshift, σ_z the corresponding error, m_{obs}^{peak}, peak brightness in the B–band with corresponding uncertainty. A_E, K_{BB}, Δ_{corr}, and σ_B are in analogy to Table 3. Data adapted from Table 1 of Perlmutter, S. et al. 1999, ApJ, 517, 565.

ρ_M is the total mass–density of the Universe, $k = \pm\ 1{,}0$, the curvature of the 3–space. Λ is the Cosmological constant, introduced by Einstein. $R(t)$ is the expansion factor of a homogeneous and isotropic Universe, and $H(t) = \dot{R}(t)/R(t)$ the expansion velocity at time t, with its value for the present time t_0 equal to the Hubble constant,$H_0 = H(t_0)$.

The density parameters are defined in the following way:

- Matter density:
$$\Omega_M \equiv \frac{8\pi G}{3H_0^2}\rho_M \quad . \tag{13}$$

- Dark energy:
$$\Omega_\Lambda \equiv \frac{\Lambda c^2}{3H_0^2} \quad . \tag{14}$$

- Curvature parameter:
$$\Omega_k \equiv -\frac{kc^2}{R_0^2 H_0^2} \quad . \tag{15}$$

The Friedmann equation implies then the closure relation

$$\Omega_M + \Omega_\Lambda + \Omega_k = 1 \quad . \tag{16}$$

5.2 Luminosity Distance and the Hubble–Diagrams

The method to determine the cosmological model by means of the Supernova observations is based on the distance modulus

$$m(z) - M = 5\log\ d_L(z;\Omega_M,\Omega_\Lambda,H_0) + 25 \quad , \tag{17}$$

which relates the apparent magnitude $m(z)$ to the absolute magnitude M over the **luminosity distance** $d_L(z;\Omega_M,\Omega_\Lambda,H_0)$. Notice that the luminosity distance depends on two of the Ω parameters since the remaining third parameter is obtained from the closure relation given in eq. (16).

The expression for the luminosity distance given as a general integral over the redshift range

$$d_L(z;\Omega_M,\Omega_\Lambda,H_0) = \frac{c}{H_0}\frac{1+z}{\sqrt{|\Omega_k|}}\mathcal{S}(x[z]) \quad , \tag{18}$$

where
$$x[z] = \sqrt{|\Omega_k|}\int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_k(1+z')^2 + \Omega_\Lambda}} \quad . \tag{19}$$

$\mathcal{S}(x)$is the function $\sinh(x)$, x, or $\sin(x)$, depending on the curvature of 3–space, i.e. for $\Omega_k > 0$, $\Omega_k = 0$, $\Omega_k < 0$, respectively. This integral can always be done numerically.

Analytic expressions exist for the luminosity distance in the following special cases:

- deSitter (Flat) Universe ($\Omega_k = \Omega_M = 0$, $\Omega_\Lambda = 1$)

$$d_L(z; H_0) = \frac{c}{H_0} z(1+z) \quad . \tag{20}$$

- Standard Cold Dark Matter (SCDM): a Universe without Dark Energy ($\Omega_\Lambda = 0$). given by the Mattig-formula[6], parameterized by the Hubble constant H_0 and the deceleration parameter $q_0 = \Omega_M/2 - \Omega_\Lambda = \Omega_M/2$, $\Omega_k + \Omega_M = 1$

$$d_L(z; \Omega_M, H_0) = \frac{c}{H_0} \frac{1}{q_0^2} (q_0 z + (q_0 - 1)[\sqrt{1 + 2q_0 z} - 1]) \quad . \tag{21}$$

- Λ Cold Dark Matter Model(ΛCDM) with a flat Universe ($\Omega_k = 0$): Ue–Li Pen[7] has published an approximation for the case $\Omega_k = 0$, $\Omega_M + \Omega_\Lambda = 1$,

$$d_L(z; \Omega_M, H_0) = \frac{c}{H_0} (1+z)[\eta(0, \Omega_M) - \eta(z, \Omega_M)] \quad , \tag{22}$$

  where

$$\begin{aligned}\eta(z, \Omega_M) =& 2\sqrt{s^3 + 1}[(1+z)^4 - 0.1540 s(1+z)^3 + 0.4304 s^2 (1+z)^2 \\ &+ 0.19097 s^3 (1+z) + 0.066941 s^4]^{-1/8} \quad .\end{aligned} \tag{23}$$

  The parameter s is defined as follows, $s^3 = \Omega_\Lambda/\Omega_M = (1 - \Omega_M)/\Omega_M$. This formula is exact in the limit $\Omega_M \to 1$, $\Omega_M \to 0$, and for $z \gg 1$. In the range $0.2 < \Omega_M < 1$, the error is less than 0.4%, and for all parameters less than 4%.

Lab–Tasks IV: Extracting the best Cosmological model from your own Hubble diagram.

1. Calculate from the distance modulus $m_B^{\text{eff}} - M_B$, the luminosity distance d_L, for the Supernovae in Table 3 and 4. Plot this diagram with d_L in Gigaparsec as x–axis and cz in units of km/sec as y–axis. Test the linearity of the Hubble-relation for redshifts $z > 0.15$.

2. Show that for $z \ll 1$,the Mattig–formula converges towards the Hubble–law, independent of the deceleration parameter q_0. Notice that $q_0 < 1$.

3. Plot in your Hubble diagram theoretical curves for $\Omega_\Lambda = 0$. Is this compatible with the data? consider in particular SCDM with

$$d_L(z, H_0) = \frac{2c}{H_0}[1 + z - \sqrt{1+z}] \quad . \tag{24}$$

[6] Mattig, W. 1958 Astron.Nachr., 284, 109
[7] Ue-Li Penn 1999, ApJ Sup, 120, 49

4. Make a plot for the distance modulus DM as a function of the redshift for the redshift range $0.01 < z < 5.0$ (linear in redshift) for the following models: deSitter, SCDM, and ΛCDM.

6 The Cosmological Concordance Model

The term **Concordance Model** is used in Cosmology to indicate the currently accepted and most commonly used cosmological model. It is important to identify a concordance model, because the measurement of many astrophysical quantities (e.g. distance, size, luminosity, and surface brightness) depend upon the cosmological model used. Consequently, for ease of comparison if nothing else, the models assumed in different studies should at least be similar, if not identical.

The Cosmic Microwave Background (CMB) observations from COBE and BOOMERang proved that the Universe is practically flat, i.e. $\Omega_k = 0$ [8], so currently, the concordance model is the ΛCDM model (which includes cold dark matter and dark energy). In this model, the Universe is 13.8 billion years old, made up of 4% baryonic matter, 23% dark matter, and 73% dark energy. The Hubble constant for this model is 67.80 km/s/Mpc and the total density of the Universe is very close to the critical value for the re–collapse. These values were derived from Planck satellite observations of the cosmic microwave background radiation. The present state of the Universe is represented by one point in the Fundamental Plane of Cosmology $(\Omega_M, \Omega_\Lambda)$. Various constrains on possible models can be discussed with this plane.

Lab–Tasks IV: From the plot for the **Fundamental Plane of Cosmology** give the positions and discuss the following models:

1. A flat model with $\Omega_k = 0$.
2. The transition from decelerated to accelerated models, i.e. $q_0 = 0$.
3. Constrains from the Supernova projects.
4. Constrains from mass/luminosity observations in galaxy clusters.
5. Constrains from observations of the Cosmic Microwave Background (CMB) from the Cosmic Background Explorer (COBE) and the Balloon Observations Of Millimetric Extragalactic Radiation and Geophysics (BOOMERang).
6. Constrains from observations of the CMB from the Wilkinson Microwave Anisotropy Probe (WMAP).
7. Constrains from observations of the CMB from Planck Space Observatory.
8. How would you determine lines of constant ages on this plane?

[8] de Bernardis,P. 2000, Nature, 404, 955

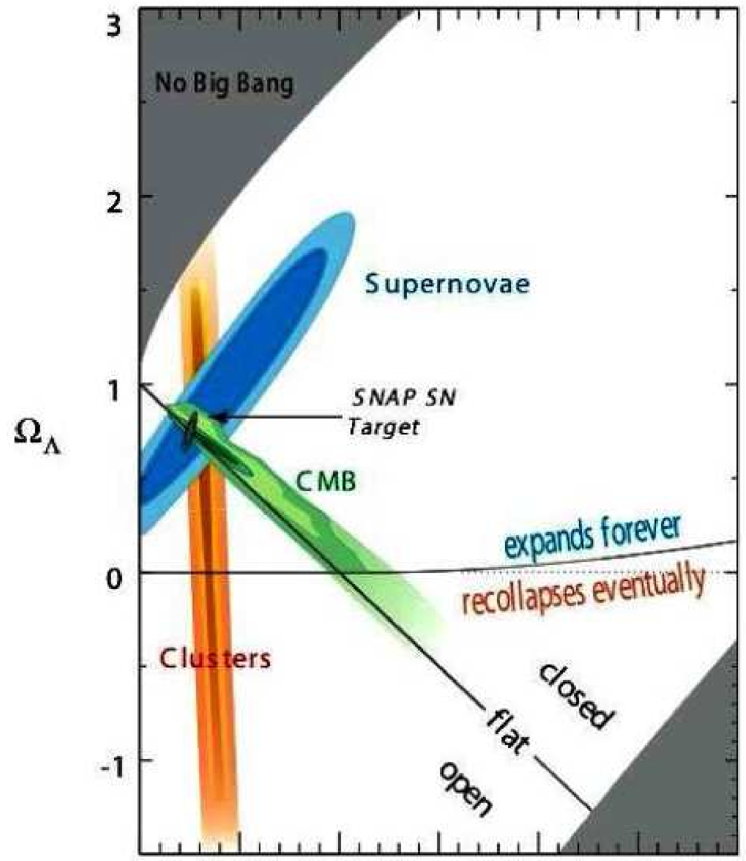

Figure 6: The fundamental plane of Cosmology. Each point $(\Omega_M, \Omega_\Lambda)$ in this plane represents the value of these parameters for a particular model. The graph shows the constrains of these values from observations of galaxy clusters, high-z Supernovae, and Cosmic Microwave Background (CMB) from COBE, BOOMERang, WMAP, and Planck experiments. The hue of the colors indicate the confidence of the measurements.