CHAPTER

6

POLARIZATION OPTICS

Augustin Jean Fresnel (1788–1827) advanced a theory of light in which waves exhibit transverse vibrations. The equations describing the partial reflection and refraction of light are named in his honor. Fresnel also made important contributions to the theory of light diffraction.

The polarization of light at a fixed position is determined by the time course of the electric-field vector $\mathcal{E}(\mathbf{r}, t)$. In a simple medium, this vector lies in a plane tangential to the wavefront at that position. For monochromatic light, any two orthogonal components of the complex-amplitude vector $\mathbf{E}(\mathbf{r})$ in that plane vary sinusoidally with time, with amplitudes and phases that are generally different, so that the endpoint of the vector $\mathbf{E}(\mathbf{r})$ traces an ellipse. Since the wavefront generally has different directions at different positions, the plane, the orientation, and the shape of the ellipse also vary with position, as illustrated in Fig. 6.0-1(a).

For a plane wave, however, the wavefronts are parallel transverse planes and the polarization ellipses are the same everywhere, as illustrated in Fig. 6.0-1(b), although the field vectors are not necessarily parallel at any given time. The plane wave is therefore described by a single ellipse, and is said to be **elliptically polarized**. The orientation and ellipticity of the polarization ellipse determine the state of polarization of the plane wave, whereas the size of the ellipse is determined by the optical intensity. When the ellipse degenerates into a straight line or becomes a circle, the wave is said to be **linearly polarized** or **circularly polarized**, respectively.

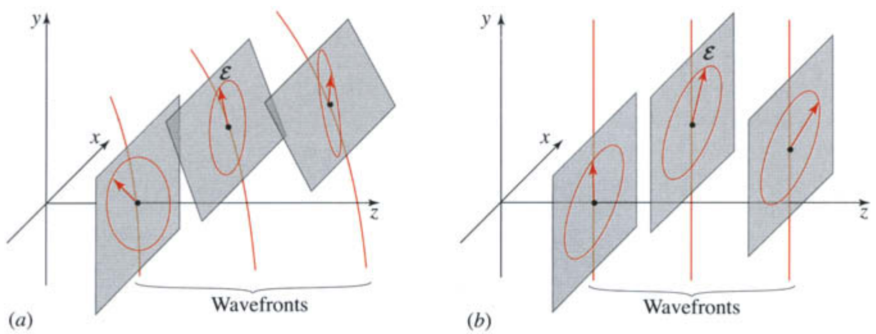


Figure 6.0-1 Time course of the electric field vector of monochromatic light at several positions: (a) arbitrary wave; (b) plane wave or paraxial wave traveling in the z direction.

In paraxial optics, light propagates along directions that lie within a narrow cone centered about the optical axis (the z axis). Waves are approximately transverse electromagnetic (TEM) and the electric-field vectors therefore lie approximately in transverse planes, and have negligible axial components. From the perspective of polarization, paraxial waves may be approximated by plane waves and described by a single polarization ellipse (or circle or line).

Polarization plays an important role in the interaction of light with matter as attested to by the following examples:

- The amount of light reflected at the boundary between two materials depends on the polarization of the incident wave.
- The amount of light absorbed by certain materials is polarization dependent.
- Light scattering from matter is generally polarization sensitive.
- The refractive index of anisotropic materials depends on the polarization. Waves with different polarizations travel at different velocities and undergo different phase shifts, so that the polarization ellipse is modified as the wave advances (e.g., linearly polarized light can be transformed into circularly polarized light). This property is used in the design of many optical devices.

- The polarization plane of linearly polarized light is rotated by passage through certain media, including those that are optically active, liquid crystals, and certain substances in the presence of an external magnetic field.

This Chapter

This chapter is devoted to a description of elementary polarization phenomena and a number of their applications. Elliptically polarized light is introduced in Sec. 6.1 using a matrix formalism that is convenient for describing polarization devices. Sec. 6.2 describes the effect of polarization on the reflection and refraction of light at the boundaries between dielectric media. The propagation of light through anisotropic media (crystals), optically active media, and liquid crystals are the subjects of Secs. 6.3, 6.4, and 6.5, respectively. Finally, basic polarization devices (polarizers, retarders, and rotators) are discussed in Sec. 6.6.

6.1 POLARIZATION OF LIGHT

A. Polarization

Consider a monochromatic plane wave of frequency ν and angular frequency $\omega = 2\pi\nu$ traveling in the z direction with velocity c. The electric field lies in the x–y plane and is generally described by

$$\mathcal{E}(z,t) = \mathrm{Re}\left\{\mathbf{A}\exp\left[j\omega\left(t-\frac{z}{c}\right)\right]\right\}, \tag{6.1-1}$$

where the complex envelope

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}, \tag{6.1-2}$$

is a vector with complex components A_x and A_y. To describe the polarization of this wave, we trace the endpoint of the vector $\mathcal{E}(z,t)$ at each position z as a function of time.

Polarization Ellipse

Expressing A_x and A_y in terms of their magnitudes and phases, $A_x = \mathfrak{a}_x\exp(j\varphi_x)$ and $A_y = \mathfrak{a}_y\exp(j\varphi_y)$, and substituting into (6.1-2) and (6.1-1) we obtain

$$\mathcal{E}(z,t) = \mathcal{E}_x\hat{\mathbf{x}} + \mathcal{E}_y\hat{\mathbf{y}}, \tag{6.1-3}$$

where

$$\mathcal{E}_x = \mathfrak{a}_x\cos\left[\omega\left(t-\frac{z}{c}\right)+\varphi_x\right] \tag{6.1-4a}$$

$$\mathcal{E}_y = \mathfrak{a}_y\cos\left[\omega\left(t-\frac{z}{c}\right)+\varphi_y\right] \tag{6.1-4b}$$

are the x and y components of the electric-field vector $\mathcal{E}(z,t)$. The components $\mathcal{E}_x$ and $\mathcal{E}_y$ are periodic functions of $t - z/c$ that oscillate at frequency ν. Equations (6.1-4) are the parametric equations of the ellipse

$$\frac{\mathcal{E}_x^2}{\mathfrak{a}_x^2} + \frac{\mathcal{E}_y^2}{\mathfrak{a}_y^2} - 2\cos\varphi\,\frac{\mathcal{E}_x\mathcal{E}_y}{\mathfrak{a}_x\mathfrak{a}_y} = \sin^2\varphi, \tag{6.1-5}$$

where $\varphi = \varphi_y - \varphi_x$ is the phase difference.

At a fixed value of z, the tip of the electric-field vector rotates periodically in the x–y plane, tracing out this ellipse. At a fixed time t, the locus of the tip of the electric-field vector follows a helical trajectory in space that lies on the surface of an elliptical cylinder (see Fig. 6.1-1). The electric field rotates as the wave advances, repeating its motion periodically for each distance corresponding to a wavelength $\lambda = c/\nu$.

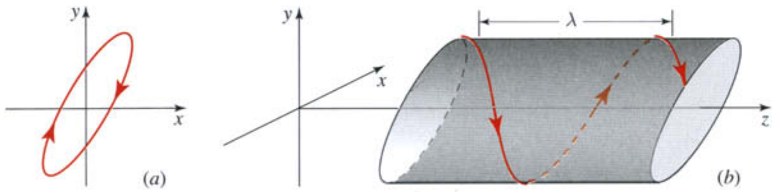


Figure 6.1-1 (*a*) Rotation of the endpoint of the electric-field vector in the x–y plane at a fixed position z. (*b*) Snapshot of the trajectory of the endpoint of the electric-field vector at a fixed time t.

The state of polarization of the wave is determined by the orientation and shape of the polarization ellipse, which is characterized by the two angles defined in Fig. 6.1-2: the angle ψ determines the direction of the major axis, whereas the angle χ determines the ellipticity, namely the ratio of the minor to major axes of the ellipse b/a. These angles depend on the ratio of the complex-envelope magnitudes $R = \mathfrak{a}_y/\mathfrak{a}_x$, and on the phase difference $\varphi = \varphi_y - \varphi_x$, in accordance with the following relations:

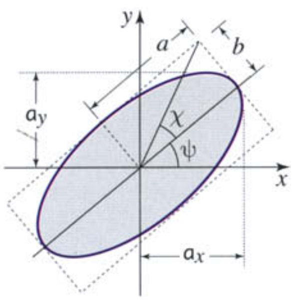


Figure 6.1-2 Polarization ellipse.

$$\tan 2\psi = \frac{2R}{1-R^2}\cos\varphi\,, \qquad R = \frac{\mathfrak{a}_y}{\mathfrak{a}_x}\,, \tag{6.1-6}$$

$$\sin 2\chi = \frac{2R}{1+R^2}\sin\varphi\,, \qquad \varphi = \varphi_y - \varphi_x\,. \tag{6.1-7}$$

Equations (6.1-6) and (6.1-7) may be derived by finding the angle ψ that achieves a transformation of the coordinate system of $\mathcal{E}_x$ and $\mathcal{E}_y$ in (6.1-5) such that the rotated ellipse has no cross term. The size of the ellipse is determined by the intensity of the wave, which is proportional to $|A_x|^2 + |A_y|^2 = \mathfrak{a}_x^2 + \mathfrak{a}_y^2$.

Linearly Polarized Light

If one of the components vanishes ($\mathfrak{a}_x = 0$, for example), the light is **linearly polarized (LP)** in the direction of the other component (the y direction). The wave is also linearly polarized if the phase difference $\varphi = 0$ or π, since (6.1-4) gives $\mathcal{E}_y = \pm(\mathfrak{a}_y/\mathfrak{a}_x)\mathcal{E}_x$, which is the equation of a straight line of slope $\pm\mathfrak{a}_y/\mathfrak{a}_x$ (the + and − signs correspond to $\varphi = 0$ or π, respectively). In these cases the elliptical cylinder in Fig. 6.1-1(*b*) collapses into a plane as illustrated in Fig. 6.1-3. The wave is therefore also said to have **planar polarization**. If $\mathfrak{a}_x = \mathfrak{a}_y$, for example, the plane of polarization makes an angle 45° with the x axis. If $\mathfrak{a}_x = 0$, the plane of polarization is the y–z plane.

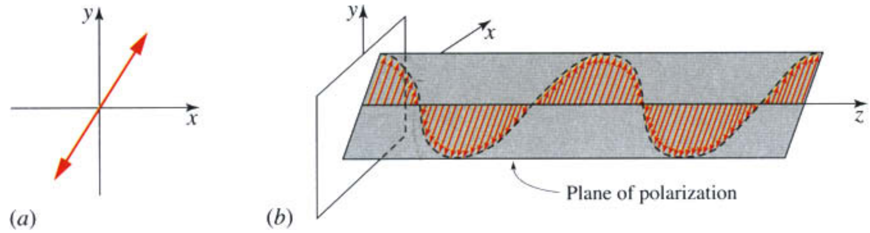


Figure 6.1-3 Linearly polarized light (also called plane polarized light). (a) Time course at a fixed position z. (b) A snapshot (fixed time t).

Circularly Polarized Light

If $\varphi = \pm\pi/2$ and $\mathfrak{a}_x = \mathfrak{a}_y = \mathfrak{a}_0$, (6.1-4) gives $\mathcal{E}_x = \mathfrak{a}_0 \cos[\omega(t - z/c) + \varphi_x]$ and $\mathcal{E}_y = \mp\mathfrak{a}_0 \sin[\omega(t - z/c) + \varphi_x]$, from which $\mathcal{E}_x^2 + \mathcal{E}_y^2 = \mathfrak{a}_0^2$, which is the equation of a circle. The elliptical cylinder in Fig. 6.1-1(b) becomes a circular cylinder and the wave is said to be circularly polarized. In the case $\varphi = +\pi/2$, the electric field at a fixed position z rotates in a clockwise direction when viewed from the direction toward which the wave is approaching. The light is then said to be **right circularly polarized (RCP)**. The case $\varphi = -\pi/2$ corresponds to counterclockwise rotation and **left circularly polarized (LCP)** light.† In the right circular case, a snapshot of the lines traced by the endpoints of the electric-field vectors at different positions is a right-handed helix (like a right-handed screw pointing in the direction of the wave), as illustrated in Fig. 6.1-4. For left circular polarization, a left-handed helix is followed.

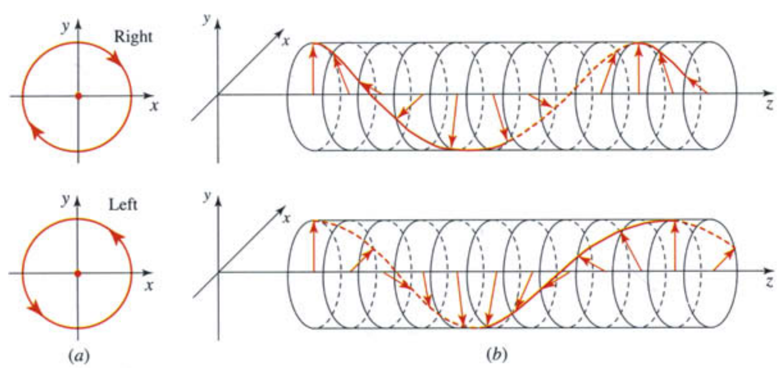


Figure 6.1-4 Trajectories of the endpoint of the electric-field vector of a circularly polarized plane wave. (a) Time course at a fixed position z. (b) A snapshot at a fixed time t. The sense of rotation in (a) is opposite that in (b) because the traveling wave depends on $t - z/c$.

Poincaré Sphere and Stokes Parameters

As indicated above, the state of polarization of a light wave can be described by two real parameters: the magnitude ratio $\mathrm{R} = \mathfrak{a}_y/\mathfrak{a}_x$ and the phase difference $\varphi = \varphi_y - \varphi_x$. These are sometimes lumped into a single complex number $\mathrm{R}\exp(j\varphi)$, called the **complex polarization ratio**. Alternatively, we may characterize the state of

† This convention is used in most optics textbooks. The opposite designation is often used in the engineering literature: in the case of right (left) circularly polarized light, the electric-field vector at a fixed position rotates counterclockwise (clockwise) when viewed from the direction toward which the wave is approaching.

polarization by the two angles ψ and χ, which represent the orientation and ellipticity of the polarization ellipse, respectively, as defined in Fig. 6.1-2.

The **Poincaré sphere** (see Fig. 6.1-5) is a geometrical construct in which the state of polarization is represented by a point on the surface of a sphere of unit radius, with coordinates ($r = 1$, $\theta = 90° - 2\chi$, $\phi = 2\psi$) in a spherical coordinate system. Each point on the sphere represents a polarization state. For example, points on the equator ($\chi = 0°$) represent states of linear polarization, with the two points $2\psi = 0°$ and $2\psi = 180°$ representing linear polarization along the x and y axes, respectively. The north and south poles ($2\chi = \pm 90°$) represent right-handed and left-handed circular polarization, respectively. Other points on the sphere represent states of elliptical polarization.

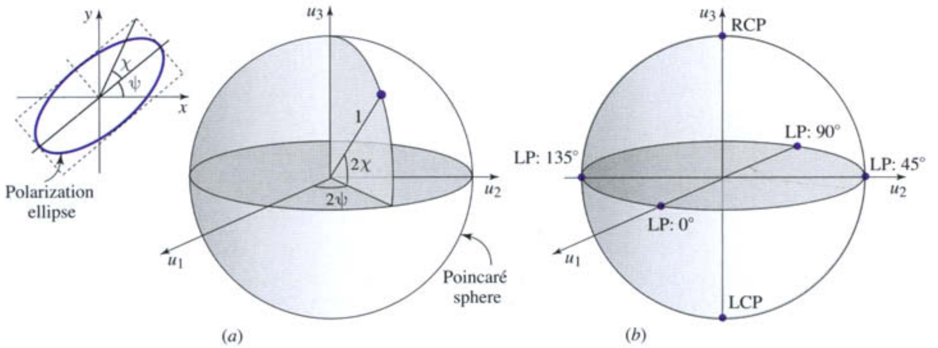


Figure 6.1-5 (a) The orientation and ellipticity of the polarization ellipse are represented geometrically as a point on the Poincaré sphere. (b) Points on the Poincaré sphere representing linearly polarized (LP) light at various angles with the x direction, as well as right-circularly polarized (RCP) and left-circularly polarized (LCP) light.

The two real quantities (R, φ), or equivalently the angles (χ, ψ), describe the state of polarization but contain no information about the intensity of the wave. Another representation that does contain such information is the **Stokes vector**. This is a set of four real numbers $(\mathrm{S}_0, \mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3)$, called the **Stokes parameters**. The first of these, $\mathrm{S}_0 = \mathrm{a}_x^2 + \mathrm{a}_y^2$, is proportional to the optical intensity whereas the other three, $(\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3)$, are the Cartesian coordinates of the point on the Poincaré sphere, $(u_1, u_2, u_3) = (\cos 2\chi \cos 2\psi,\ \cos 2\chi \sin 2\psi,\ \sin 2\chi)$, multiplied by S_0, so that

$$\mathrm{S}_1 = \mathrm{S}_0 \cos 2\chi \cos 2\psi \tag{6.1-8a}$$

$$\mathrm{S}_2 = \mathrm{S}_0 \cos 2\chi \sin 2\psi \tag{6.1-8b}$$

$$\mathrm{S}_3 = \mathrm{S}_0 \sin 2\chi. \tag{6.1-8c}$$

Using (6.1-6) and (6.1-7), together with a few trigonometric identities, the Stokes parameters in (6.1-8) may be expressed in terms of the field parameters $(\mathrm{a}_x, \mathrm{a}_y, \varphi)$, and in terms of the components of the complex envelope (A_x, A_y), as:

$$\mathrm{S}_0 = \mathrm{a}_x^2 + \mathrm{a}_y^2 = |A_x|^2 + |A_y|^2 \tag{6.1-9a}$$

$$\mathrm{S}_1 = \mathrm{a}_x^2 - \mathrm{a}_y^2 = |A_x|^2 - |A_y|^2 \tag{6.1-9b}$$

$$\mathrm{S}_2 = 2\mathrm{a}_x \mathrm{a}_y \cos\varphi = 2\,\mathrm{Re}\{A_x^* A_y\} \tag{6.1-9c}$$

$$\mathrm{S}_3 = 2\mathrm{a}_x \mathrm{a}_y \sin\varphi = 2\,\mathrm{Im}\{A_x^* A_y\}\,. \tag{6.1-9d}$$

Stokes Parameters

Since $\mathrm{S}_1^2 + \mathrm{S}_2^2 + \mathrm{S}_3^2 = \mathrm{S}_0^2$, only three of the four components of the Stokes vector are independent; they completely define the intensity and the state of polarization of the light. A generalization of the Stokes parameters suitable for describing partially coherent light is presented in Sec. 11.4.

We conclude that there are a number of equivalent representations for describing the state of polarization of an optical field: (1) the polarization ellipse, (2) the Poincaré sphere, and (3) the Stokes vector. Yet another equivalent representation, the Jones vector, is introduced in the following section.

B. Matrix Representation

The Jones Vector

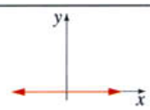
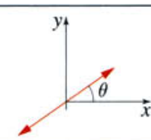
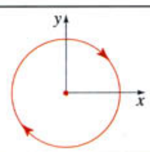
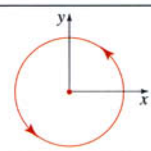
As indicated above, a monochromatic plane wave of frequency ν traveling in the z direction is completely characterized by the complex envelopes $A_x = \mathrm{a}_x \exp(j\varphi_x)$ and $A_y = \mathrm{a}_y \exp(j\varphi_y)$ of the x and y components of the electric-field vector. These complex quantities may be written in the form of a column matrix known as the **Jones vector**:

$$\mathbf{J} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}. \tag{6.1-10}$$

Given $\mathbf{J}$, we can determine the total intensity of the wave, $I = (|A_x|^2 + |A_y|^2)/2\eta$, and use the ratio $\mathrm{R} = \mathrm{a}_y/\mathrm{a}_x = |A_y|/|A_x|$ and the phase difference $\varphi = \varphi_y - \varphi_x = \arg\{A_y\} - \arg\{A_x\}$ to determine the orientation and shape of the polarization ellipse, as well as the Poincaré sphere and the Stokes parameters.

The Jones vectors for some special polarization states are provided in Table 6.1-1. The intensity in each case has been normalized so that $|A_x|^2 + |A_y|^2 = 1$ and the phase of the x component is taken to be $\varphi_x = 0$.

Table 6.1-1 Jones vectors of linearly polarized (LP) and right- and left-circularly polarized (RCP, LCP) light.

LP in x direction	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$		LP at angle θ	$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$	
RCP	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$		LCP	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix}$	

Orthogonal Polarizations

Two polarization states represented by the Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ are said to be orthogonal if the inner product between $\mathbf{J}_1$ and $\mathbf{J}_2$ is zero. The inner product is defined by

$$(\mathbf{J}_1, \mathbf{J}_2) = A_{1x}A_{2x}^* + A_{1y}A_{2y}^*, \tag{6.1-11}$$

where A_{1x} and A_{1y} are the elements of $\mathbf{J}_1$ and A_{2x} and A_{2y} are the elements of $\mathbf{J}_2$. An example of orthogonal Jones vectors are the linearly polarized waves in the x and

y directions, or any other pair of orthogonal directions. Another example is provided by right and left circularly polarized waves.

Expansion of Arbitrary Polarization as a Superposition of Two Orthogonal Polarizations

An arbitrary Jones vector $\mathbf{J}$ can always be analyzed as a weighted superposition of two orthogonal Jones vectors, say $\mathbf{J}_1$ and $\mathbf{J}_2$, that form the expansion basis; thus $\mathbf{J} = \alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2$. If $\mathbf{J}_1$ and $\mathbf{J}_2$ are normalized such that $(\mathbf{J}_1, \mathbf{J}_1) = (\mathbf{J}_2, \mathbf{J}_2) = 1$, the expansion coefficients are the inner products $\alpha_1 = (\mathbf{J}, \mathbf{J}_1)$ and $\alpha_2 = (\mathbf{J}, \mathbf{J}_2)$.

EXAMPLE 6.1-1. ***Expansions in Linearly Polarized and Circularly Polarized Bases.*** Using the x and y linearly polarized vectors $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as an expansion basis, the expansion coefficients for a Jones vector of components A_x and A_y with $|A_x|^2 + |A_y|^2 = 1$ are, by definition, $\alpha_1 = A_x$ and $\alpha_2 = A_y$. The same polarization state may be expanded in other bases.

- In a basis of linearly polarized vectors at angles 45° and 135°, i.e., $\mathbf{J}_1 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\mathbf{J}_2 = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$, the expansion coefficients α_1 and α_2 are:

$$A_{45} = \frac{1}{\sqrt{2}}(A_x + A_y), \quad A_{135} = \frac{1}{\sqrt{2}}(A_y - A_x). \tag{6.1-12}$$

- Similarly, if the right and left circularly polarized waves $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\j\end{smallmatrix}\right]$ and $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\-j\end{smallmatrix}\right]$ are used as an expansion basis, the coefficients α_1 and α_2 are:

$$A_R = \frac{1}{\sqrt{2}}(A_x - jA_y), \quad A_L = \frac{1}{\sqrt{2}}(A_x + jA_y). \tag{6.1-13}$$

  For example, a linearly polarized wave with a plane of polarization that makes an angle θ with the x axis (i.e., $A_x = \cos\theta$ and $A_y = \sin\theta$) is equivalent to a superposition of right and left circularly polarized waves with coefficients $\frac{1}{\sqrt{2}}e^{-j\theta}$ and $\frac{1}{\sqrt{2}}e^{j\theta}$, respectively. A linearly polarized wave therefore equals a weighted sum of right and left circularly polarized waves.

EXERCISE 6.1-1

Measurement of the Stokes Parameters. Show that the Stokes parameters defined in (6.1-9) for light with Jones vector components A_x and A_y are given by

$$\mathrm{S}_0 = |A_x|^2 + |A_y|^2 \tag{6.1-14a}$$

$$\mathrm{S}_1 = |A_x|^2 - |A_y|^2 \tag{6.1-14b}$$

$$\mathrm{S}_2 = |A_{45}|^2 - |A_{135}|^2 \tag{6.1-14c}$$

$$\mathrm{S}_3 = |A_R|^2 - |A_L|^2, \tag{6.1-14d}$$

where A_{45} and A_{135} are the coefficients of expansion in a basis of linearly polarized vectors at angles 45° and 135° as in (6.1-12), and A_R and A_L are the coefficients of expansion in a basis of the right and left circularly polarized waves set forth in (6.1-13). Suggest a method of measuring the Stokes parameters of light with arbitrary polarization.

Matrix Representation of Polarization Devices

Consider the transmission of a plane wave of arbitrary polarization through an optical system that maintains the plane-wave nature of the wave, but alters its polarization, as illustrated schematically in Fig. 6.1-6. The system is assumed to be linear, so that the principle of superposition of optical fields is obeyed. Two examples of such systems are the reflection of light from a planar boundary between two media, and the transmission of light through a plate with anisotropic optical properties.

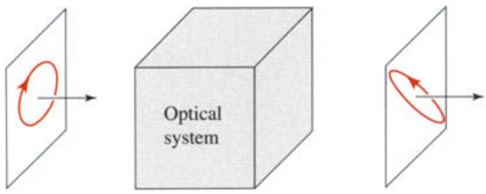


Figure 6.1-6 An optical system that alters the polarization of a plane wave.

The complex envelopes of the two electric-field components of the input (incident) wave, A_{1x} and A_{1y}, and those of the output (transmitted or reflected) wave, A_{2x} and A_{2y}, are in general related by the weighted superpositions

$$\begin{aligned} A_{2x} &= T_{11}A_{1x} + T_{12}A_{1y} \\ A_{2y} &= T_{21}A_{1x} + T_{22}A_{1y}, \end{aligned} \tag{6.1-15}$$

where T_{11}, T_{12}, T_{21}, and T_{22} are constants describing the device. Equations (6.1-15) are general relations that all linear optical polarization devices must satisfy.

The linear relations in (6.1-15) may conveniently be written in matrix notation by defining a 2×2 matrix $\mathbf{T}$ with elements T_{11}, T_{12}, T_{21}, and T_{22} so that

$$\begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix}. \tag{6.1-16}$$

If the input and output waves are described by the Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$, respectively, then (6.1-16) may be written in the compact matrix form

$$\mathbf{J}_2 = \mathbf{T}\mathbf{J}_1. \tag{6.1-17}$$

The matrix $\mathbf{T}$, called the **Jones matrix**, describes the optical system, whereas the vectors $\mathbf{J}_1$ and $\mathbf{J}_2$ describe the input and output waves.

The structure of the Jones matrix $\mathbf{T}$ of a given optical system determines its effect on the polarization state and intensity of the wave. The following is a compilation of the Jones matrices of some systems with simple characteristics. Physical devices that have such characteristics will be discussed subsequently in this chapter.

Linear polarizers. The system represented by the Jones matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \tag{6.1-18}$$

Linear Polarizer Along x Direction

transforms a wave of components (A_{1x}, A_{1y}) into a wave of components $(A_{1x}, 0)$ by eliminating the y component, thereby yielding a wave polarized along the x direction, as illustrated in Fig. 6.1-7. The system is a **linear polarizer** with its transmission axis pointing in the x direction.

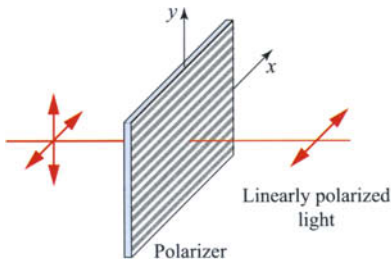


Figure 6.1-7 The linear polarizer. The lines in the polarizer represent the field direction that is permitted to pass.

Wave retarders. The system represented by the matrix

$$\boxed{\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\Gamma} \end{bmatrix}} \tag{6.1-19}$$

Wave-Retarder (Fast Axis Along x Direction)

transforms a wave with field components (A_{1x}, A_{1y}) into another with components $(A_{1x}, e^{-j\Gamma}A_{1y})$, thereby delaying the y component by a phase Γ while leaving the x component unchanged. It is therefore called a **wave retarder**. The x and y axes are called the fast and slow axes of the retarder, respectively.

The simple application of matrix algebra permits the results illustrated in Fig. 6.1-8 to be understood:

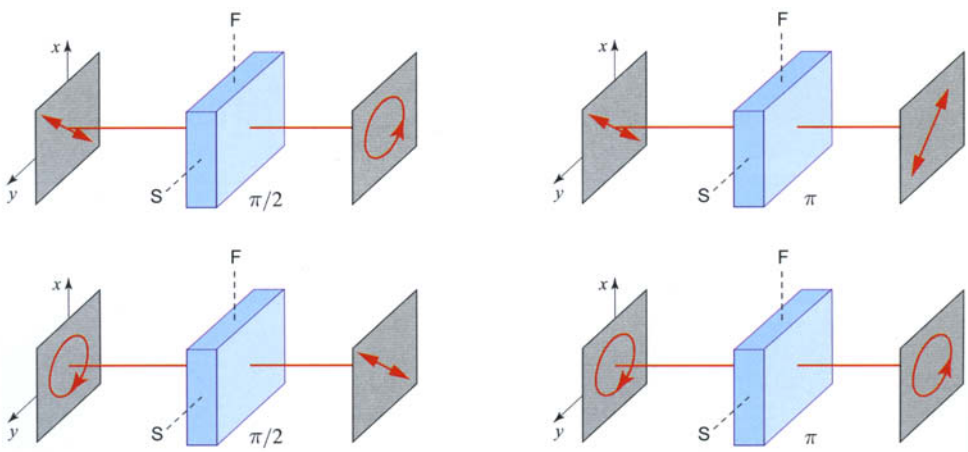


Figure 6.1-8 Operations of quarter-wave ($\pi/2$) and half-wave (π) retarders on several particular states of polarization are shown in (a) and (b), respectively. F and S represent the fast and slow axes of the retarder, respectively.

- When $\Gamma = \pi/2$, the retarder (called a **quarter-wave retarder**) converts the linearly polarized wave $\begin{bmatrix}1\\1\end{bmatrix}$ into the left circularly polarized wave $\begin{bmatrix}1\\-j\end{bmatrix}$, and converts the right circularly polarized wave $\begin{bmatrix}1\\j\end{bmatrix}$ into the linearly polarized wave $\begin{bmatrix}1\\1\end{bmatrix}$.
- When $\Gamma = \pi$, the retarder (called a **half-wave retarder**) converts the linearly polarized wave $\begin{bmatrix}1\\1\end{bmatrix}$ into the linearly polarized wave $\begin{bmatrix}1\\-1\end{bmatrix}$, thereby rotating the plane of polarization by 90°. The half-wave retarder converts the right circularly polarized wave $\begin{bmatrix}1\\j\end{bmatrix}$ into the left circularly polarized wave $\begin{bmatrix}1\\-j\end{bmatrix}$.

Polarization rotators. While a wave retarder can transform a wave with one form of polarization into another, a **polarization rotator** always maintains the linear polarization of a wave but rotates the plane of polarization by a particular angle. The Jones matrix

$$\boxed{\mathbf{T} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}} \tag{6.1-20}$$

Polarization Rotator

represents a device that converts a linearly polarized wave $\begin{bmatrix}\cos\theta_1\\ \sin\theta_1\end{bmatrix}$ into another linearly polarized wave $\begin{bmatrix}\cos\theta_2\\ \sin\theta_2\end{bmatrix}$, where $\theta_2 = \theta_1 + \theta$. It therefore rotates the plane of polarization of a linearly polarized wave by an angle θ.

Cascaded Polarization Devices

The action of cascaded optical systems on polarized light may be conveniently determined by using conventional matrix multiplication formulas. A system characterized by the Jones matrix $\mathbf{T}_1$ followed by another characterized by $\mathbf{T}_2$ are equivalent to a single system characterized by the product matrix $\mathbf{T} = \mathbf{T}_2\mathbf{T}_1$. The matrix of the system through which light is first transmitted must stand to the right in the matrix product since it is the first to affect the input Jones vector.

EXERCISE 6.1-2

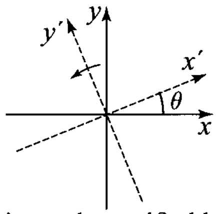
Cascaded Wave Retarders. Show that two cascaded quarter-wave retarders with parallel fast axes are equivalent to a half-wave retarder. What is the result if the fast axes are orthogonal?

Coordinate Transformation

The elements of the Jones vectors and Jones matrices are dependent on the choice of the coordinate system. However, if these elements are known in one coordinate system, they can be determined in another coordinate system by using matrix methods. If $\mathbf{J}$ is the Jones vector in the x–y coordinate system, then in a new coordinate system x'–y', with the x' direction making an angle θ with the x direction, the Jones vector $\mathbf{J}'$ is given by

$$\mathbf{J}' = \mathbf{R}(\theta)\,\mathbf{J}, \tag{6.1-21}$$

where $\mathbf{R}(\theta)$ is the matrix



$$\boxed{\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.} \tag{6.1-22}$$

Coordinate Transformation

This can be verified by relating the components of the electric field in the two coordinate systems.

The Jones matrix $\mathbf{T}$, which represents an optical system, is similarly transformed into $\mathbf{T}'$, in accordance with the matrix relations

$$\mathbf{T}' = \mathbf{R}(\theta)\,\mathbf{T}\,\mathbf{R}(-\theta) \tag{6.1-23}$$

$$\mathbf{T} = \mathbf{R}(-\theta)\,\mathbf{T}'\,\mathbf{R}(\theta), \tag{6.1-24}$$

where $\mathbf{R}(-\theta)$ is given by (6.1-22) with $-\theta$ replacing θ. The matrix $\mathbf{R}(-\theta)$ is the inverse of $\mathbf{R}(\theta)$, so that $\mathbf{R}(-\theta)\,\mathbf{R}(\theta)$ is a unit matrix. Equation (6.1-23) can be obtained by using the relation $\mathbf{J}_2 = \mathbf{T}\mathbf{J}_1$ and the transformation $\mathbf{J}'_2 = \mathbf{R}(\theta)\,\mathbf{J}_2 = \mathbf{R}(\theta)\,\mathbf{T}\,\mathbf{J}_1$. Since $\mathbf{J}_1 = \mathbf{R}(-\theta)\,\mathbf{J}'_1$, we have $\mathbf{J}'_2 = \mathbf{R}(\theta)\,\mathbf{T}\,\mathbf{R}(-\theta)\,\mathbf{J}'_1$; since $\mathbf{J}'_2 = \mathbf{T}'\,\mathbf{J}'_1$, (6.1-23) follows.

EXERCISE 6.1-3

Jones Matrix of a Polarizer. Show that the Jones matrix of a linear polarizer with a transmission axis making an angle θ with the x axis is

$$\boxed{\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}.} \tag{6.1-25}$$

Linear Polarizer at Angle θ

Derive (6.1-25) using (6.1-18), (6.1-22), and (6.1-24).

Normal Modes

The normal modes of a polarization system are the states of polarization that are not changed when the wave is transmitted through the system (see Appendix C). These states have Jones vectors satisfying

$$\mathbf{T}\mathbf{J} = \mu\mathbf{J}, \tag{6.1-26}$$

where μ is constant. The normal modes are therefore the eigenvectors of the Jones matrix $\mathbf{T}$, and the values of μ are the corresponding eigenvalues. Since the matrix $\mathbf{T}$ is of size 2×2 there are only two independent normal modes, $\mathbf{T}\mathbf{J}_1 = \mu_1\mathbf{J}_1$ and $\mathbf{T}\mathbf{J}_2 = \mu_2\mathbf{J}_2$. If the matrix $\mathbf{T}$ is a Hermitian, i.e., if $T_{12} = T_{21}^*$, the normal modes are orthogonal: $(\mathbf{J}_1, \mathbf{J}_2) = 0$. The normal modes are usually used as an expansion basis, so that an arbitrary input wave $\mathbf{J}$ may be expanded as a superposition of normal modes: $\mathbf{J} = \alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2$. The response of the system may then be easily evaluated since $\mathbf{T}\mathbf{J} = \mathbf{T}(\alpha_1\mathbf{J}_1 + \alpha_2\mathbf{J}_2) = \alpha_1\mathbf{T}\mathbf{J}_1 + \alpha_2\mathbf{T}\mathbf{J}_2 = \alpha_1\mu_1\mathbf{J}_1 + \alpha_2\mu_2\mathbf{J}_2$ (see Appendix C).

EXERCISE 6.1-4

Normal Modes of Simple Polarization Systems.

(a) Show that the normal modes of the linear polarizer are linearly polarized waves.

(b) Show that the normal modes of the wave retarder are linearly polarized waves.

(c) Show that the normal modes of the polarization rotator are right and left circularly polarized waves.

What are the eigenvalues of the systems described above?

6.2 REFLECTION AND REFRACTION

In this section we examine the reflection and refraction of a monochromatic plane wave of arbitrary polarization incident at a planar boundary between two dielectric media. The media are assumed to be linear, homogeneous, and isotropic with impedances η_1 and η_2, and refractive indexes n_1 and n_2. The incident, refracted, and reflected waves are labeled with the subscripts 1, 2, and 3, respectively, as illustrated in Fig. 6.2-1.

As shown in Sec. 2.4A, the wavefronts of these waves are matched at the boundary if the angles of reflection and incidence are equal, $\theta_3 = \theta_1$, and if the angles of refraction and incidence satisfy Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{6.2-1}$$

To relate the amplitudes and polarizations of the three waves, we associate with each wave an x–y coordinate system in a plane normal to the direction of propagation (Fig. 6.2-1). The electric-field envelopes of these waves are described by the Jones vectors

$$\mathbf{J}_1 = \begin{bmatrix} A_{1x} \\ A_{1y} \end{bmatrix}, \qquad \mathbf{J}_2 = \begin{bmatrix} A_{2x} \\ A_{2y} \end{bmatrix}, \qquad \mathbf{J}_3 = \begin{bmatrix} A_{3x} \\ A_{3y} \end{bmatrix}. \tag{6.2-2}$$

We proceed to determine the relations between $\mathbf{J}_2$ and $\mathbf{J}_1$ and between $\mathbf{J}_3$ and $\mathbf{J}_1$. These relations are written in the form of matrices $\mathbf{J}_2 = \mathbf{t}\mathbf{J}_1$ and $\mathbf{J}_3 = \mathbf{r}\mathbf{J}_1$, where $\mathbf{t}$ and $\mathbf{r}$ are 2×2 Jones matrices describing the transmission and reflection of the wave, respectively.

The elements of the transmission and reflection matrices may be determined by imposing the boundary conditions required by electromagnetic theory, namely the continuity at the boundary of the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and the normal components of $\mathbf{D}$ and $\mathbf{B}$. The electric field associated with each wave is orthogonal to the magnetic field; the ratio of their envelopes is the characteristic impedance, which is η_1 for the incident and reflected waves and η_2 for the transmitted wave. The result is a set of equations that are solved to obtain relations between the components of the electric fields of the three waves.

The algebra involved is reduced substantially if we observe that the two normal modes for this system are linearly polarized waves with polarizations along the x and y directions. This may be proved if we show that an incident, a reflected, and a refracted wave with their electric field vectors pointing in the x direction are self-consistent with the boundary conditions, and similarly for three waves linearly polarized in the y direction. This is indeed the case. The x and y polarized waves are therefore uncoupled.

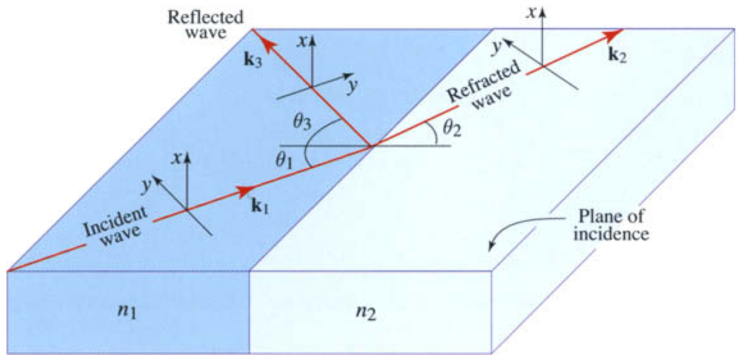


Figure 6.2-1 Reflection and refraction at the boundary between two dielectric media.

The x-polarized mode is called the **transverse electric (TE)** polarization or the **orthogonal** polarization, since the electric fields are orthogonal to the plane of incidence. The y-polarized mode is called the **transverse magnetic (TM)** polarization since the magnetic field is orthogonal to the plane of incidence, or the **parallel** polarization since the electric fields are parallel to the plane of incidence. The orthogonal and parallel polarizations are also called the s (for the German *senkrecht*, meaning "perpendicular") and p (for "parallel") polarizations, respectively. The y axes in Fig. 6.2-1 are arbitrarily defined such that their components parallel to the boundary between the dielectrics all point in the same direction.

The independence of the x and y polarizations implies that the Jones matrices $\mathbf{t}$ and $\mathbf{r}$ are diagonal,

$$\mathbf{t} = \begin{bmatrix} \mathrm{t}_x & 0 \\ 0 & \mathrm{t}_y \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} \mathrm{r}_x & 0 \\ 0 & \mathrm{r}_y \end{bmatrix} \tag{6.2-3}$$

so that

$$E_{2x} = \mathrm{t}_x E_{1x}, \qquad E_{2y} = \mathrm{t}_y E_{1y} \tag{6.2-4}$$

$$E_{3x} = \mathrm{r}_x E_{1x}, \qquad E_{3y} = \mathrm{r}_y E_{1y}. \tag{6.2-5}$$

The coefficients t_x and t_y are the complex amplitude transmittances for the TE and TM polarizations, respectively; r_x and r_y are the analogous complex amplitude reflectances.

Applying the boundary conditions (i.e., equating the tangential components of the electric fields and the tangential components of the magnetic fields at both sides of the boundary) in each of the TE and TM cases, we obtain the following expressions for the reflection and transmission coefficients:

$$\boxed{\begin{aligned} \mathrm{r}_x &= \frac{\eta_2 \sec\theta_2 - \eta_1 \sec\theta_1}{\eta_2 \sec\theta_2 + \eta_1 \sec\theta_1}, \qquad \mathrm{t}_x = 1 + \mathrm{r}_x, \\ \mathrm{r}_y &= \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_2 \cos\theta_2 + \eta_1 \cos\theta_1}, \qquad \mathrm{t}_y = (1 + \mathrm{r}_y)\frac{\cos\theta_1}{\cos\theta_2}. \end{aligned}}$$

(6.2-6) TE Polarization

(6.2-7) TM Polarization

Reflection & Transmission

The characteristic impedance $\eta = \sqrt{\mu/\epsilon}$ is complex if ϵ and/or μ are complex, as is the case for lossy or conductive media. For nonlossy, nonmagnetic, dielectric media, $\eta = \eta_o/n$ is real, where $\eta_o = \sqrt{\mu_o/\epsilon_o}$ and n is the refractive index. In this case, the reflection and transmission coefficients in (6.2-6) and (6.2-7) yield the **Fresnel equations**:

$$\boxed{\begin{aligned} \mathfrak{r}_x &= \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}, \quad && \mathfrak{t}_x = 1 + \mathfrak{r}_x, \\ \mathfrak{r}_y &= \frac{n_1 \sec\theta_1 - n_2 \sec\theta_2}{n_1 \sec\theta_1 + n_2 \sec\theta_2}, \quad && \mathfrak{t}_y = (1 + \mathfrak{r}_y)\,\frac{\cos\theta_1}{\cos\theta_2}. \end{aligned}}$$

(6.2-8)
TE Polarization

(6.2-9)
TM Polarization
Fresnel Equations

Given n_1, n_2, and θ_1, the reflection coefficients can be determined from the Fresnel equations by first determining θ_2 using Snell's law, (6.2-1), from which

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - (n_1/n_2)^2 \sin^2\theta_1}\,. \qquad (6.2\text{-}10)$$

Since the quantities under the square-root signs in (6.2-10) can be negative, the reflection and transmission coefficients are in general complex. The magnitudes $|\mathfrak{r}_x|$ and $|\mathfrak{r}_y|$, and the phase shifts $\varphi_x = \arg\{\mathfrak{r}_x\}$ and $\varphi_y = \arg\{\mathfrak{r}_y\}$, are plotted in Figs. 6.2-2 to 6.2-5 for the two polarizations, as functions of the angle of incidence θ_1. Plots are provided for external reflection ($n_1 < n_2$) as well as for internal reflection ($n_1 > n_2$).

TE Polarization

The dependence of the reflection coefficient $\mathfrak{r}_x$ on θ_1 for the TE-polarized wave is given by (6.2-8):

External reflection ($n_1 < n_2$). The reflection coefficient $\mathfrak{r}_x$ is always real and negative, corresponding to a phase shift $\varphi_x = \pi$. The magnitude $|\mathfrak{r}_x| = (n_2 - n_1)/(n_1 + n_2)$ at $\theta_1 = 0$ (normal incidence) and increases to unity at $\theta_1 = 90°$ (grazing incidence), as shown in Fig. 6.2-2.

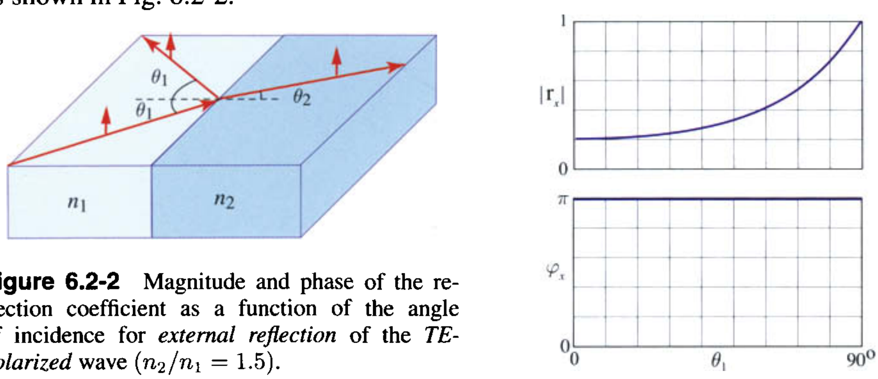


Figure 6.2-2 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *external reflection* of the *TE-polarized* wave ($n_2/n_1 = 1.5$).

Internal reflection ($n_1 > n_2$). For small θ_1 the reflection coefficient is real and positive. Its magnitude is $(n_1 - n_2)/(n_1 + n_2)$ when $\theta_1 = 0°$, and increases gradually to a value of unity, which is attained when θ_1 equals the critical angle $\theta_c = \sin^{-1}(n_2/n_1)$. For $\theta_1 > \theta_c$, the magnitude of $\mathfrak{r}_x$ remains at unity, which corresponds to total internal reflection. This may be shown by using (6.2-10) to write† $\cos\theta_2 =$

† The choice of the minus sign for the square root is consistent with the derivation that leads to the Fresnel equation.

$-\sqrt{1-\sin^2\theta_1/\sin^2\theta_c} = -j\sqrt{\sin^2\theta_1/\sin^2\theta_c - 1}$, and substituting into (6.2-8). Total internal reflection is accompanied by a phase shift $\varphi_x = \arg\{r_x\}$ given by

$$\boxed{\tan\frac{\varphi_x}{2} = \sqrt{\frac{\cos^2\theta_c}{\cos^2\theta_1} - 1}} \quad (6.2\text{-}11)$$

TE-Reflection Phase Shift

The phase shift φ_x increases from 0 at $\theta_1 = \theta_c$ to π at $\theta_1 = 90°$, as illustrated in Fig. 6.2-3. This phase plays an important role in dielectric waveguides (see Sec. 8.2).

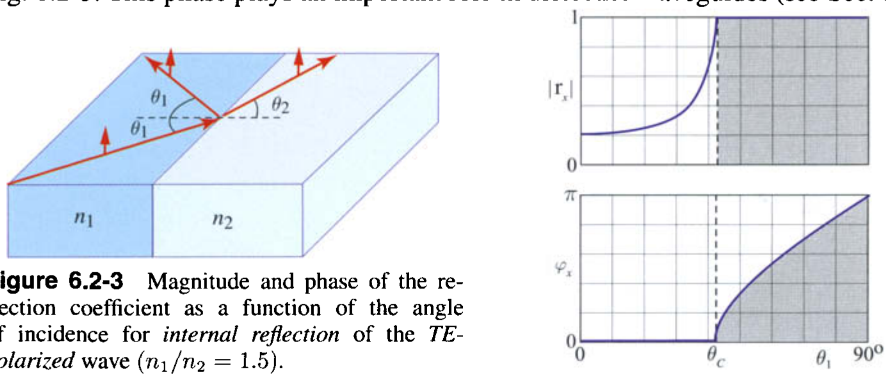


Figure 6.2-3 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *internal reflection* of the *TE-polarized* wave ($n_1/n_2 = 1.5$).

TM Polarization

Similarly, the dependence of the reflection coefficient r_y on θ_1 for the TM-polarized wave is provided by (6.2-9):

External reflection ($n_1 < n_2$). The reflection coefficient r_y is always real, as shown in Fig. 6.2-4. It assumes a negative value of $(n_1 - n_2)/(n_1 + n_2)$ at $\theta_1 = 0$ (normal incidence). Its magnitude then decreases until it vanishes when $n_1 \sec\theta_1 = n_2 \sec\theta_2$, at an angle $\theta_1 = \theta_B$, known as the **Brewster angle**:

$$\boxed{\theta_B = \tan^{-1}(n_2/n_1)} \quad (6.2\text{-}12)$$

Brewster Angle

(see Prob. 6.2-5 for other properties of the Brewster angle). For $\theta_1 > \theta_B$, r_y reverses sign (φ_y goes from π to 0) and its magnitude gradually increases until it approaches unity at $\theta_1 = 90°$. The absence of reflection of the TM wave at the Brewster angle is useful for making polarizers (see Sec. 6.6).

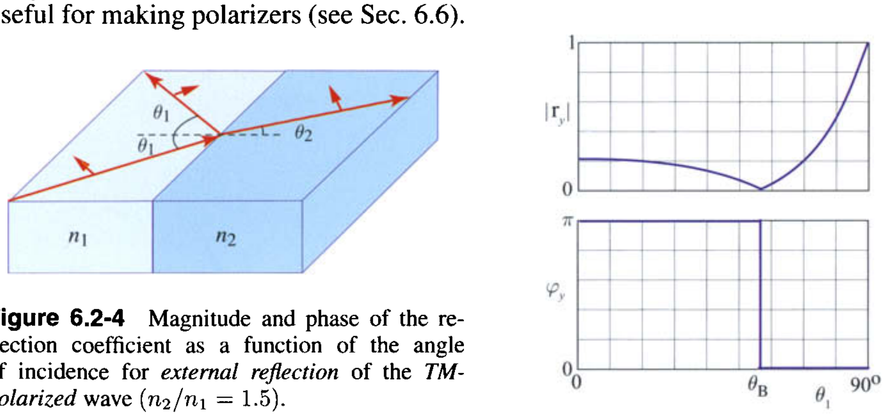


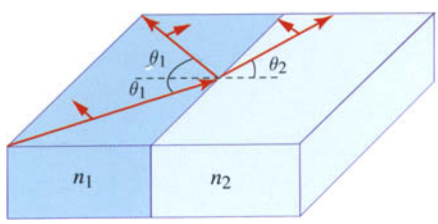
Figure 6.2-4 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *external reflection* of the *TM-polarized* wave ($n_2/n_1 = 1.5$).

Internal reflection $(n_1 > n_2)$. At $\theta_1 = 0°$, r_y is positive and has magnitude $(n_1 - n_2)/(n_1 + n_2)$, as illustrated in Fig. 6.2-5. As θ_1 increases, the magnitude decreases until it vanishes at the Brewster angle $\theta_B = \tan^{-1}(n_2/n_1)$. As θ_1 increases beyond θ_B, r_y becomes negative and its magnitude increases until it reaches unity at the critical angle θ_c. For $\theta_1 > \theta_c$ the wave undergoes total internal reflection accompanied by a phase shift $\varphi_y = \arg\{\mathrm{r}_y\}$ given by

$$\boxed{\tan\frac{\varphi_y}{2} = \frac{-1}{\sin^2\theta_c}\sqrt{\frac{\cos^2\theta_c}{\cos^2\theta_1} - 1}\,.} \tag{6.2-13}$$

TM-Reflection Phase Shift

At normal incidence, evidently, the reflection coefficient is $\mathrm{r} = (n_1 - n_2)/(n_1 + n_2)$, whether the reflection is TE or TM, or external or internal.



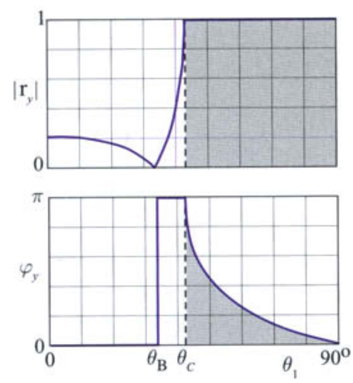


Figure 6.2-5 Magnitude and phase of the reflection coefficient as a function of the angle of incidence for *internal reflection* of the *TM-polarized wave* $(n_1/n_2 = 1.5)$.

EXERCISE 6.2-1

Brewster Windows. At what angle is a TM-polarized beam of light transmitted through a glass plate of refractive index $n = 1.5$ placed in air $(n = 1)$ without suffering reflection losses at either surface? Such plates, known as Brewster windows (Fig. 6.2-6), are used in lasers, as described in Sec. 15.2D.

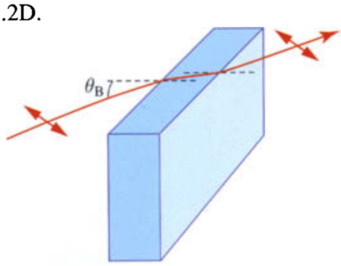


Figure 6.2-6 The Brewster window transmits TM-polarized light with no reflection loss.

Power Reflectance and Transmittance

The reflection and transmission coefficients r and t represent ratios of complex amplitudes. The power reflectance $\mathcal{R}$ and power transmittance $\mathcal{T}$ are defined as the ratios of power flow (along a direction normal to the boundary) of the reflected and transmitted waves to that of the incident wave. Because the reflected and incident waves propagate in the same medium and make the same angle with the normal to the surface, it follows that

$$\mathcal{R} = |\mathrm{r}|^2. \qquad (6.2\text{-}14)$$

For both TE and TM polarizations, and for both external and internal reflection, the power reflectance at normal incidence is therefore

$$\boxed{\mathcal{R} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2.} \qquad (6.2\text{-}15)$$

Power Reflectance at Normal Incidence

At the boundary between glass ($n = 1.5$) and air ($n = 1$), for example, $\mathcal{R} = 0.04$, so that 4% of the light is reflected at normal incidence. At the boundary between GaAs ($n = 3.6$) and air ($n = 1$), $\mathcal{R} \approx 0.32$, so that 32% of the light is reflected at normal incidence. However, at oblique angles the reflectance can be much greater or much smaller than 32%, as illustrated in Fig. 6.2-7.

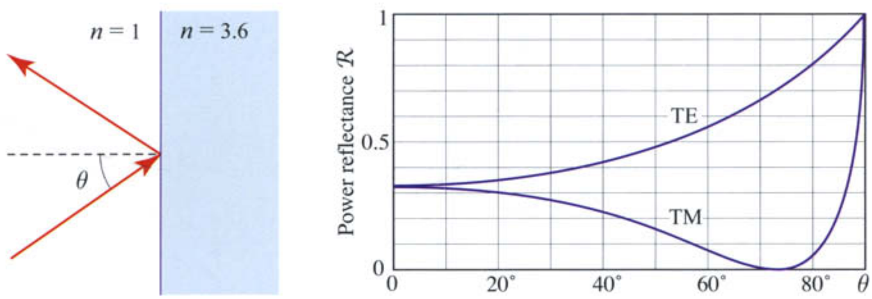


Figure 6.2-7 Power reflectance of TE- and TM-polarization plane waves at the boundary between air ($n = 1$) and GaAs ($n = 3.6$), as a function of the angle of incidence θ.

The power transmittance $\mathcal{T}$ is determined by invoking the conservation of power, so that in the absence of absorption loss the transmittance is simply

$$\mathcal{T} = 1 - \mathcal{R}. \qquad (6.2\text{-}16)$$

It is important to note, however, that $\mathcal{T}$ is generally *not* equal to $|\mathrm{t}|^2$ since the power travels at different angles and with different impedances in the two media. For a wave traveling at an angle θ in a medium of refractive index n, the power flow in the direction normal to the boundary is $(|\mathcal{E}|^2/2\eta)\cos\theta = (|\mathcal{E}|^2/2\eta_o)\, n\cos\theta$. It follows that

$$\mathcal{T} = \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1}\,|\mathrm{t}|^2. \qquad (6.2\text{-}17)$$

Reflectance from a plate. The power reflectance at normal incidence from a plate with two surfaces is described by $\mathcal{R}(1 + \mathcal{T}^2)$ since the power reflected from the far surface involves a double transmission through the near surface. For a glass plate in air, the overall reflectance is $\mathcal{R}(1 + \mathcal{T}^2) = 0.04[1 + (0.96)^2] \approx 0.077$, so that about 7.7% of the incident light power is reflected. However, this calculation does not include interference effects, which are washed out when the light is incoherent (see Sec.11.2), nor does it account for multiple reflections inside the plate. Optical transmission and reflectance from multiple boundaries in layered media are described in detail in Sec. 7.1.

EXERCISE 6.2-2

Reflectance of a Conductive Medium. The equations for the reflection coefficients set forth in (6.2-6) and (6.2-7) can be used to determine the intensity reflectance $\mathcal{R}$ at the boundary between a dielectric medium and a conductive medium.

(a) Show that $\mathcal{R} \approx 1$ if the conductivity of the conductive medium σ is infinite.

(b) Show that at normal incidence, and for $\sigma \gg \epsilon_o\omega$, the relation $\mathcal{R} = 1 - 2\sqrt{2\epsilon_o\omega/\sigma}$, known as the **Hagen–Rubens relation**, emerges. Use this relation to determine the reflectance of copper at the wavelengths $\lambda_o = 1.06$ μm and 10.6 μm. Assume that the conductivity of copper is $\sigma = 0.58 \times 10^8$ $(\Omega\text{-m})^{-1}$.

(c) Show that if the conductive medium is described by the Drude model, (5.5-39), then $\mathcal{R} = 1$ at frequencies below the plasma frequency.

6.3 OPTICS OF ANISOTROPIC MEDIA

A dielectric medium is said to be anisotropic if its macroscopic optical properties depend on direction. The macroscopic properties of a material are, of course, ultimately governed by its microscopic properties: the shape and orientation of the individual molecules and the organization of their centers in space. Optical materials have different kinds of positional and orientational types of order, which may be described as follows (see Fig. 6.3-1):

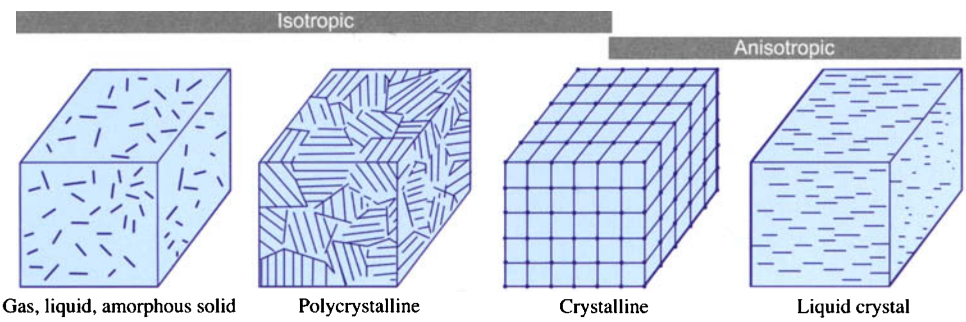


Figure 6.3-1 Positional and orientational order in different types of materials.

- If the molecules are located at totally random positions in space, and are themselves isotropic or oriented along random directions, the medium is isotropic. *Gases*, *liquids*, and *amorphous solids* follow this prescription.
- If the structure takes the form of disjointed crystalline grains that are randomly oriented with respect to each other, the material is said to be *polycrystalline*. The individual grains are, in general, anisotropic, but their averaged macroscopic behavior is isotropic.
- If the molecules are organized in space according to a regular periodic pattern and they are oriented in the same direction, as in *crystals*, the medium is, in general, anisotropic.

- If the molecules are anisotropic and their orientations are not totally random, the medium is anisotropic, even if their positions are totally random. This is the case for *liquid crystals*, which have orientational order but lack complete positional order.

A. Refractive Indexes

Permittivity Tensor

In a linear anisotropic dielectric medium (a crystal, for example), each component of the electric flux density $\mathbf{D}$ is a linear combination of the three components of the electric field,

$$D_i = \sum_j \epsilon_{ij} E_j. \tag{6.3-1}$$

The indexes $i, j = 1, 2, 3$ refer to the x, y, and z components, respectively, as described in Sec. 5.2B. The dielectric properties of the medium are therefore characterized by a 3×3 array of nine coefficients, $\{\epsilon_{ij}\}$, that form the **electric permittivity tensor** $\boldsymbol{\epsilon}$, which is a tensor of second rank. The **material equation** (6.3-1) is usually written in the symbolic form

$$\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}. \tag{6.3-2}$$

For most dielectric media, the electric permittivity tensor is symmetric, i.e., $\epsilon_{ij} = \epsilon_{ji}$. This means that the relation between the vectors $\mathbf{D}$ and $\mathbf{E}$ is reciprocal, i.e., their ratio remains the same if their directions are exchanged. This symmetry is obeyed for dielectric nonmagnetic materials that do not exhibit optical activity, and in the absence of an external magnetic field (see Sec. 6.4). With this symmetry, the medium is characterized by only six independent numbers in an arbitrary coordinate system. For crystals of certain symmetries, even fewer coefficients suffice since some vanish and some are related.

Geometrical Representation of Vectors and Tensors

A *vector*, such as the electric field $\mathbf{E}$, for example, describes a physical variable with magnitude and direction. It is represented *geometrically* by an arrow pointing in that particular direction, whose length is proportional to the magnitude of the vector [Fig. 6.3-2(a)]. A vector, which is a tensor of first rank, is represented *numerically* by three numbers: its projections on the three axes of a particular coordinate system. Though these components depend on the choice of the coordinate system, the magnitude and direction of the vector in physical space are independent of the choice of the coordinate system. A scalar, which is described by a single number, is a tensor of zero rank.

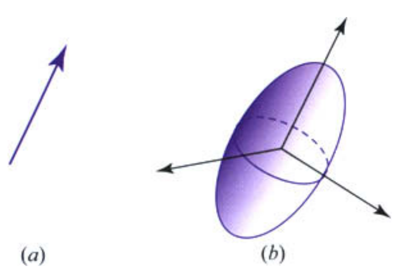


Figure 6.3-2 Geometrical representation of (a) a vector and (b) a symmetric second-rank tensor.

A second-rank *tensor* is a rule that relates two vectors. In a given coordinate system, it is represented *numerically* by nine numbers. Changing the coordinate system yields a different set of nine numbers, but the physical nature of the rule is unchanged. A useful *geometrical* representation [Fig. 6.3-2(*b*)] of a *symmetric* second-rank tensor (the dielectric tensor $\boldsymbol{\epsilon}$, for example), is a quadratic surface (an ellipsoid) defined by

$$\sum_{ij} \epsilon_{ij}\, x_i x_j = 1, \tag{6.3-3}$$

which is known as the **quadric representation**. This surface is invariant to the choice of the coordinate system; if the coordinate system is rotated, both x_i and ϵ_{ij} are altered but the ellipsoid remains intact in physical space. The ellipsoid has six degrees of freedom and carries all information about the symmetric second-rank tensor. In the principal coordinate system, ϵ_{ij} is diagonal and the ellipsoid assumes a particularly simple form:

$$\epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 = 1. \tag{6.3-4}$$

Its principal axes are those of the tensor, and its axes have half-lengths $1/\sqrt{\epsilon_1}$, $1/\sqrt{\epsilon_2}$, and $1/\sqrt{\epsilon_3}$.

Principal Axes and Principal Refractive Indexes

The elements of the permittivity tensor depend on how the coordinate system is chosen relative to the crystal structure. However, a coordinate system can always be found for which the off-diagonal elements of ϵ_{ij} vanish, so that

$$D_1 = \epsilon_1 E_1, \qquad D_2 = \epsilon_2 E_2, \qquad D_3 = \epsilon_3 E_3, \tag{6.3-5}$$

where $\epsilon_1 = \epsilon_{11}$, $\epsilon_2 = \epsilon_{22}$, and $\epsilon_3 = \epsilon_{33}$. According to (6.3-1), **E** and **D** are parallel along these particular directions so that if, for example, **E** points in the x direction, then so too must **D**. This coordinate system defines the **principal axes** and principal planes of the crystal. Throughout the remainder of this chapter, the coordinate system x, y, z, which is equivalently denoted x_1, x_2, x_3, is assumed to lie along the principal axes of the crystal. This choice simplifies all analyses without loss of generality. The permittivities ϵ_1, ϵ_2, and ϵ_3 correspond to refractive indexes

$$n_1 = \sqrt{\epsilon_1/\epsilon_o}, \qquad n_2 = \sqrt{\epsilon_2/\epsilon_o}, \qquad n_3 = \sqrt{\epsilon_3/\epsilon_o}, \tag{6.3-6}$$

respectively, where ϵ_o is the permittivity of free space; these are known as the **principal refractive indexes**.

Biaxial, Uniaxial, and Isotropic Crystals

Crystals in which the three principal refractive indexes are different are termed **biaxial**. For crystals with certain symmetries, namely a single axis of threefold, fourfold, or sixfold symmetry, two of the refractive indexes are equal ($n_1 = n_2$) and the crystal is called **uniaxial**. In this case, the indexes are usually denoted $n_1 = n_2 = n_o$ and $n_3 = n_e$, which are known as the **ordinary** and **extraordinary** indexes, respectively, for reasons that will become clear shortly. The crystal is said to be **positive uniaxial** if $n_e > n_o$, and **negative uniaxial** if $n_e < n_o$. The z axis of a uniaxial crystal is called the **optic axis**. In certain crystals with even greater symmetry (those with cubic unit cells, for example), all three indexes are equal and the medium is optically isotropic.

Impermeability Tensor

The relation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$ can be inverted and written in the form $\mathbf{E} = \boldsymbol{\epsilon}^{-1}\mathbf{D}$, where $\boldsymbol{\epsilon}^{-1}$ is the inverse of the tensor $\boldsymbol{\epsilon}$. It is also useful to define the **electric impermeability tensor** $\boldsymbol{\eta} = \epsilon_o\boldsymbol{\epsilon}^{-1}$ (not to be confused with the impedance of the medium η), so that $\epsilon_o\mathbf{E} = \boldsymbol{\eta}\mathbf{D}$. Since $\boldsymbol{\epsilon}$ is symmetric, so too is $\boldsymbol{\eta}$. Both tensors, $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$, share the same principal axes. In the principal coordinate system, $\boldsymbol{\eta}$ is diagonal with principal values $\epsilon_o/\epsilon_1 = 1/n_1^2$, $\epsilon_o/\epsilon_2 = 1/n_2^2$, and $\epsilon_o/\epsilon_3 = 1/n_3^2$. Either tensor, $\boldsymbol{\epsilon}$ or $\boldsymbol{\eta}$, fully describes the optical properties of the crystal.

Index Ellipsoid

The **index ellipsoid** (also called the **optical indicatrix**) is the quadric representation of the electric impermeability tensor $\boldsymbol{\eta} = \epsilon_o\boldsymbol{\epsilon}^{-1}$:

$$\sum_{ij} \eta_{ij}\, x_i x_j = 1, \qquad i,j = 1,2,3. \tag{6.3-7}$$

If the principal axes were to be used as the coordinate system, we would obtain

$$\boxed{\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1\,,} \tag{6.3-8}$$

Index Ellipsoid

with principal values $1/n_1^2$, $1/n_2^2$, and $1/n_3^2$, and axes of half-lengths n_1, n_2, and n_3.

The optical properties of the crystal (the directions of the principal axes and the values of the principal refractive indexes) are therefore completely described by the index ellipsoid (Fig. 6.3-3). For a uniaxial crystal, the index ellipsoid reduces to an ellipsoid of revolution; for an isotropic medium it becomes a sphere.

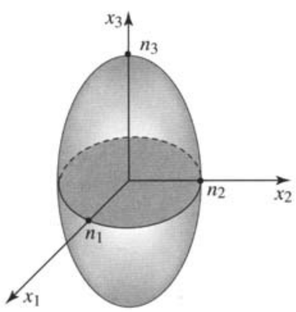


Figure 6.3-3 The index ellipsoid. The coordinates (x_1, x_2, x_3) are the principal axes while (n_1, n_2, n_3) are the principal refractive indexes of the crystal.

B. Propagation Along a Principal Axis

The rules that govern the propagation of light in crystals under general conditions are rather complex. However, they become relatively simple if the light is a plane wave traveling along one of the principal axes of the crystal. We begin with this case.

Normal Modes

Let x–y–z be a coordinate system that coincides with the principal axes of a crystal. A plane wave traveling in the z direction and linearly polarized along the x direction [Fig. 6.3-4(a)] travels with phase velocity c_o/n_1 (wavenumber $k = n_1 k_o$) without changing its polarization. The reason for this is that the electric field has only one component, E_1 pointed along the x direction, so that **D** is also in the x direction with $D_1 = \epsilon_1 E_1$; the wave equation derived from Maxwell's equations therefore provides a velocity of light given by $1/\sqrt{\mu_o \epsilon_1} = c_o/n_1$. Similarly, a plane wave traveling in the z direction and linearly polarized along the y direction [Fig. 6.3-4(b)] travels with phase velocity c_o/n_2, thereby experiencing a refractive index n_2. Thus, the normal modes for propagation in the z direction are linearly polarized waves in the x and y directions. These waves are said to be normal modes because their velocities and polarizations are maintained as they propagate (see Appendix C). Other cases in which the wave propagates along one of the principal axes and is linearly polarized along another are treated similarly [Fig. 6.3-4(c)].

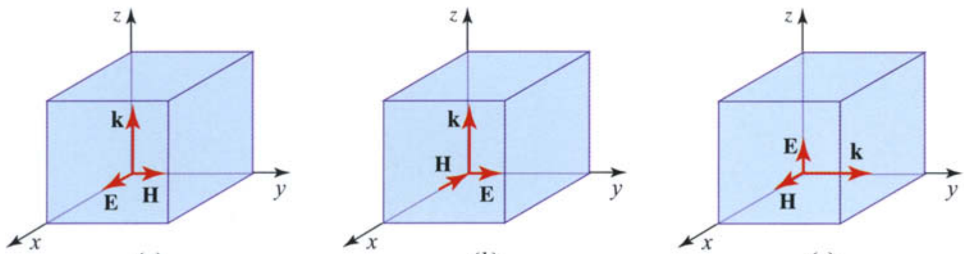


Figure 6.3-4 A wave traveling along a principal axis and polarized along another principal axis has phase velocity c_o/n_1, c_o/n_2, or c_o/n_3, when the electric field vector points in the x, y, or z directions, respectively. (a) $k = n_1 k_o$; (b) $k = n_2 k_o$; (c) $k = n_3 k_o$.

Polarization Along an Arbitrary Direction

We now consider a wave traveling along one principal axis (the z axis, for example) that is linearly polarized along an arbitrary direction in the x–y plane. This case is addressed by analyzing the wave as a sum of the normal modes, namely the linearly polarized waves in the x and y directions. These two components travel with different phase velocities, c_o/n_1 and c_o/n_2, respectively. They therefore undergo different phase shifts, $\varphi_x = n_1 k_o d$ and $\varphi_y = n_2 k_o d$, respectively, after propagating a distance d. Their phase retardation is thus $\varphi = \varphi_y - \varphi_x = (n_2 - n_1) k_o d$. Recombination of the two components yields an elliptically polarized wave, as explained in Sec. 6.1 and illustrated in Fig. 6.3-5. Such a crystal can therefore serve as a **wave retarder**, a device in which two orthogonal polarizations travel at different phase velocities so that one is retarded with respect to the other (see Fig. 6.1-8).

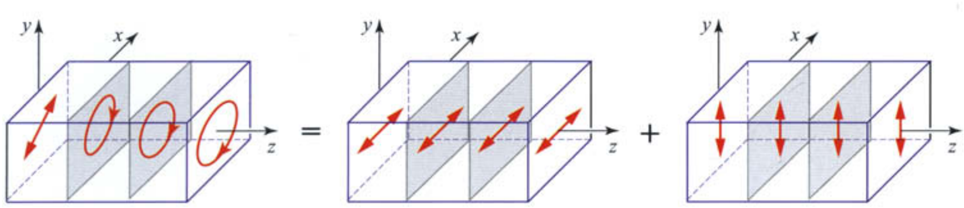


Figure 6.3-5 A linearly polarized wave at 45° in the $z = 0$ plane (a) is analyzed as a superposition of two linearly polarized components in the x and y directions (normal modes), which travel at velocities c_o/n_1 and c_o/n_2 [(b) and (c), respectively]. As a result of phase retardation, the wave is converted from plane polarization to elliptical polarization (a). It is therefore clear that the initial linearly polarized wave is not a normal mode of the system.

C. Propagation in an Arbitrary Direction

We now consider the general case of a plane wave traveling in an anisotropic crystal in an arbitrary direction defined by the unit vector $\hat{\mathbf{u}}$. We demonstrate that the two normal modes are linearly polarized waves. The refractive indexes n_a and n_b, and the directions of polarization of these modes, may be determined by use of a procedure based on the index ellipsoid:

Index-Ellipsoid Construction for Determining Normal Modes

Figure 6.3-6 illustrates a geometrical construction for determining the polarizations and refractive indexes n_a and n_b of the normal modes of a wave traveling in the direction of the unit vector $\hat{\mathbf{u}}$ in an anisotropic material characterized by the index ellipsoid:

$$\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = 1.$$

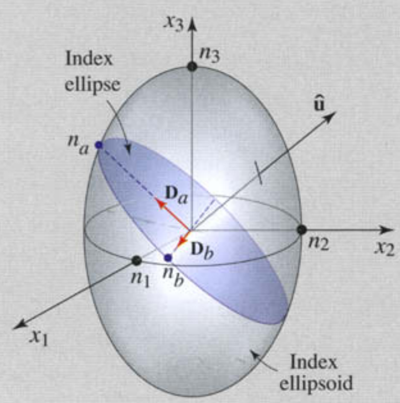


Figure 6.3-6 Determination of the normal modes from the index ellipsoid.

- Draw a plane passing through the origin of the index ellipsoid, normal to $\hat{\mathbf{u}}$. The intersection of the plane with the ellipsoid is an ellipse called the **index ellipse**.
- The half-lengths of the major and minor axes of the index ellipse are the refractive indexes n_a and n_b of the two normal modes.
- The directions of the major and minor axes of the index ellipse are the directions of the vectors $\mathbf{D}_a$ and $\mathbf{D}_b$ for the normal modes. These directions are orthogonal.
- The vectors $\mathbf{E}_a$ and $\mathbf{E}_b$ may be determined from $\mathbf{D}_a$ and $\mathbf{D}_b$ with the help of (6.3-5).

☐ **Proof of the Index-Ellipsoid Construction for Determining the Normal Modes.** To determine the normal modes (see Sec. 6.1B) for a plane wave traveling in the direction $\hat{\mathbf{u}}$, we cast Maxwell's equations (5.3-2)–(5.3-5), and the material equation $\mathbf{D} = \epsilon\mathbf{E}$ given in (6.3-2), as an eigenvalue problem. Since all fields are assumed to vary with the position $\mathbf{r}$ as $\exp(-j\mathbf{k}\cdot\mathbf{r})$, where $\mathbf{k} = k\hat{\mathbf{u}}$, Maxwell's equations (5.4-3) and (5.4-4) reduce to

$$\mathbf{k} \times \mathbf{H} = -\omega\mathbf{D} \tag{6.3-9}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mu_o\mathbf{H} \tag{6.3-10}$$

Substituting (6.3-10) into (6.3-9) leads to

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = -\omega^2 \mu_o \mathbf{D}. \tag{6.3-11}$$

Using $\mathbf{E} = \boldsymbol{\epsilon}^{-1}\mathbf{D}$, we obtain

$$\mathbf{k} \times (\mathbf{k} \times \boldsymbol{\epsilon}^{-1}\mathbf{D}) = -\omega^2 \mu_o \mathbf{D}. \tag{6.3-12}$$

This is an eigenvalue equation that $\mathbf{D}$ must satisfy. Working with $\mathbf{D}$ is convenient since we know that it lies in a plane normal to the wave direction $\hat{\mathbf{u}}$.

We now simplify (6.3-12) by using $\boldsymbol{\eta} = \epsilon_o \boldsymbol{\epsilon}^{-1}$, $\mathbf{k} = k\hat{\mathbf{u}}$, $n = k/k_o$, and $k_o^2 = \omega^2 \mu_o \epsilon_o$ to obtain

$$-\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\eta}\, \mathbf{D}) = \frac{1}{n^2}\mathbf{D}. \tag{6.3-13}$$

The operation $-\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\eta}\mathbf{D})$ may be interpreted as a projection of the vector $\boldsymbol{\eta}\mathbf{D}$ onto a plane normal to $\hat{\mathbf{u}}$. We may therefore rewrite (6.3-13) in the form

$$\mathbf{P}_u \boldsymbol{\eta}\, \mathbf{D} = \frac{1}{n^2}\mathbf{D}, \tag{6.3-14}$$

where $\mathbf{P}_u$ is an operator representing projection. Equation (6.3-14) is an eigenvalue equation for the operator $\mathbf{P}_u \boldsymbol{\eta}$, with eigenvalue $1/n^2$ and eigenvector $\mathbf{D}$. The two eigenvalues, $1/n_a^2$ and $1/n_b^2$, and two corresponding eigenvectors, $\mathbf{D}_a$ and $\mathbf{D}_b$, represent the two normal modes.

The eigenvalue problem (6.3-14) has a simple geometrical interpretation. The tensor $\boldsymbol{\eta}$ is represented geometrically by its quadric representation, the index ellipsoid. The operator $\mathbf{P}_u \boldsymbol{\eta}$ represents projection onto a plane normal to $\hat{\mathbf{u}}$. Solving the eigenvalue problem in (6.3-14) is thus equivalent to finding the principal axes of the ellipse formed by the intersection of the plane normal to $\hat{\mathbf{u}}$ with the index ellipsoid. This is precisely the construction set forth in Fig. 6.3-6 for determining the normal modes. ■

Special Case: Uniaxial Crystals

In uniaxial crystals ($n_1 = n_2 = n_o$ and $n_3 = n_e$) the index ellipsoid of Fig. 6.3-6 is an ellipsoid of revolution. For a wave whose direction of travel $\hat{\mathbf{u}}$ forms an angle θ with the optic axis, the index ellipse has half-lengths n_o and $n(\theta)$, where

$$\boxed{\frac{1}{n^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2},} \tag{6.3-15}$$

Refractive Index of Extraordinary Wave

so that the normal modes have refractive indexes $n_b = n_o$ and $n_a = n(\theta)$. The first mode, called the **ordinary wave**, has a refractive index n_o regardless of θ. In accordance with the ellipse shown in Fig. 6.3-7, the second mode, called the **extraordinary wave**, has a refractive index $n(\theta)$ that varies from n_o when $\theta = 0°$, to n_e when $\theta = 90°$. The vector $\mathbf{D}$ of the ordinary wave is normal to the plane defined by the optic axis (z axis) and the direction of wave propagation $\mathbf{k}$, and the vectors $\mathbf{E}$ and $\mathbf{D}$ are parallel. The extraordinary wave, on the other hand, has a vector $\mathbf{D}$ that is normal to $\mathbf{k}$ and lies in the k–z plane, and $\mathbf{E}$ is not parallel to $\mathbf{D}$, as shown in Fig. 6.3-7.

D. Dispersion Relation, Rays, Wavefronts, and Energy Transport

We now examine other properties of waves in anisotropic media including the dispersion relation (the relation between ω and $\mathbf{k}$).

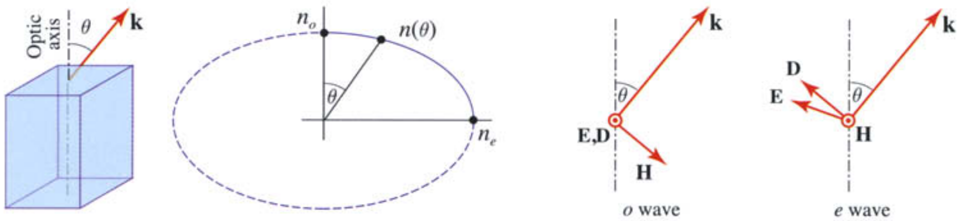


Figure 6.3-7 Variation of the refractive index $n(\theta)$ of the extraordinary wave with θ (the angle between the direction of propagation and the optic axis) in a uniaxial crystal, and directions of the electromagnetic fields of the ordinary (o) and extraordinary (e) waves. The circle with a dot at the center located at the origin signifies that the direction of the vector is out of the plane of the paper, toward the reader.

The optical wave is characterized by the wavevector $\mathbf{k}$, the field vectors $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, and $\mathbf{B}$, and the complex Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ (direction of power flow). These vectors are related by (6.3-9) and (6.3-10). It follows from (6.3-9) that $\mathbf{D}$ is normal to both $\mathbf{k}$ and $\mathbf{H}$. Equation (6.3-10) similarly indicates that $\mathbf{H}$ is normal to both $\mathbf{k}$ and $\mathbf{E}$. These geometrical conditions are illustrated in Fig. 6.3-8, which also shows the complex Poynting vector $\mathbf{S}$, which is orthogonal to both $\mathbf{E}$ and $\mathbf{H}$. Thus, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{k}$, and $\mathbf{S}$ lie in one plane to which $\mathbf{H}$ and $\mathbf{B}$ are normal. In this plane $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{S} \perp \mathbf{E}$; but $\mathbf{D}$ is not necessarily parallel to $\mathbf{E}$, and $\mathbf{S}$ is not necessarily parallel to $\mathbf{k}$.

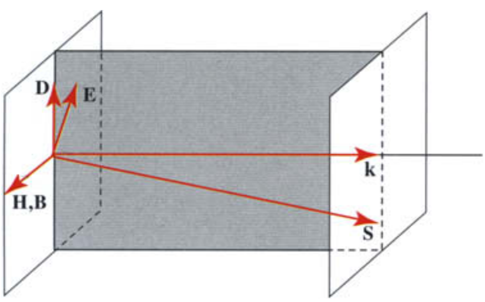


Figure 6.3-8 The vectors $\mathbf{D}$, $\mathbf{E}$, $\mathbf{k}$, and $\mathbf{S}$ all lie in one plane to which $\mathbf{H}$ and $\mathbf{B}$ are normal. $\mathbf{D} \perp \mathbf{k}$ and $\mathbf{E} \perp \mathbf{S}$.

Using the relation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$ in (6.3-11), we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2 \mu_o \boldsymbol{\epsilon}\mathbf{E} = \mathbf{0}. \tag{6.3-16}$$

This vector equation, which $\mathbf{E}$ must satisfy, translates to three linear homogeneous equations for the components E_1, E_2, and E_3 along the principal axes, written in the matrix form

$$\begin{bmatrix} n_1^2 k_o^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_o^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_o^2 - k_1^2 - k_2^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{6.3-17}$$

where (k_1, k_2, k_3) are the components of $\mathbf{k}$, $k_o = \omega / c_o$, and (n_1, n_2, n_3) are the principal refractive indexes given by (6.3-6). The condition for these equations to have a nontrivial solution is obtained by setting the determinant of the matrix to zero. The result is an equation that relates ω to k_1, k_2, and k_3 and that takes the form $\omega = \omega(k_1, k_2, k_3)$, where $\omega(k_1, k_2, k_3)$ is a nonlinear function. This relation, known as the **dispersion relation**, is the equation of a surface in the k_1, k_2, k_3 space, known as the **normal surface** or the **k surface**. The intersection of the direction $\hat{\mathbf{u}}$ with the **k** surface determines the vector **k** whose magnitude $k = n\omega/c_o$ provides the refractive index n. There are two intersections corresponding to the two normal modes associated with each direction.

The **k** surface is a centrosymmetric surface comprising two sheets, each corresponding to a solution (a normal mode). It can be shown that the **k** surface intersects each of the principal planes in an ellipse and a circle, as illustrated in Fig. 6.3-9. For biaxial crystals $(n_1 < n_2 < n_3)$, the two sheets meet at four points, defining two optic axes. In the uniaxial case $(n_1 = n_2 = n_o,\, n_3 = n_e)$, the two sheets become a sphere and an ellipsoid of revolution that meet at only two points, thereby defining a single optic axis (the z axis). In the isotropic case $(n_1 = n_2 = n_3 = n)$, the two sheets degenerate into a single sphere.

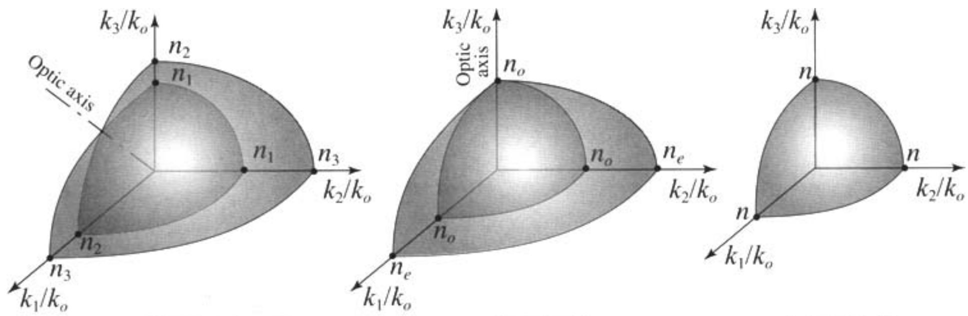


Figure 6.3-9 One octant of the k surface for (a) a biaxial crystal $(n_1 < n_2 < n_3)$; (b) a uniaxial crystal $(n_1 = n_2 = n_o, n_3 = n_e)$; and (c) an isotropic crystal $(n_1 = n_2 = n_3 = n)$.

The intersection of the direction $\hat{\mathbf{u}} = (u_1, u_2, u_3)$ with the **k** surface corresponds to a wavenumber k that satisfies

$$\sum_{j=1,2,3} \frac{u_j^2\, k^2}{k^2 - n_j^2\, k_o^2} = 1. \tag{6.3-18}$$

This is a fourth-order equation in k (or second order in k^2). It has four solutions, $\pm k_a$ and $\pm k_b$, of which only the two positive values are meaningful, since the negative values represent a reversed direction of propagation. The problem is therefore solved: the wavenumbers of the normal modes are k_a and k_b and the refractive indexes are $n_a = k_a/k_o$ and $n_b = k_b/k_o$.

To determine the directions of polarization of the two normal modes, we determine the components $(k_1, k_2, k_3) = (ku_1, ku_2, ku_3)$ and the elements of the matrix in (6.3-17) for each of the two wavenumbers $k = k_a$ and $k = k_b$. We then solve two of the three equations in (6.3-17) to determine the ratios E_1/E_3 and E_2/E_3, from which we determine the direction of the corresponding electric field **E**.

The nature of waves in anisotropic media is best explained by examining the **k** surface $\omega = \omega(k_1, k_2, k_3)$ obtained by equating the determinant of the matrix in (6.3-17) to zero, as illustrated in Fig. 6.3-9. The variation of the phase velocity $c = \omega/k$ with the direction $\hat{\mathbf{u}}$ can be determined from the **k** surface: the distance from the origin to the **k** surface in the direction of $\hat{\mathbf{u}}$ is inversely proportional to the phase velocity.

The group velocity may also be determined from the **k** surface. In analogy with the group velocity $v = d\omega/dk$ that governs the propagation of light pulses (wavepackets), as discussed in Sec. 5.6, the group velocity for *rays* (localized beams or spatial wavepackets) is the vector $\mathbf{v} = \nabla_k \omega(\mathbf{k})$, the gradient of ω with respect to $\mathbf{k}$. Since the $\mathbf{k}$ surface is the surface $\omega(k_1, k_2, k_3) =$ constant, $\mathbf{v}$ must be normal to the $\mathbf{k}$ surface. Thus, rays travel along directions normal to the $\mathbf{k}$ surface. The wavefronts are perpendicular to the wavevector $\mathbf{k}$ since the phase of the wave is $\mathbf{k} \cdot \mathbf{r}$. The wavefront normals are therefore parallel to the wavevector $\mathbf{k}$.

The complex Poynting vector $\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ is also normal to the $\mathbf{k}$ surface. This can be demonstrated by choosing a value for ω and considering two vectors $\mathbf{k}$ and $\mathbf{k} + \Delta\mathbf{k}$ that lie on the $\mathbf{k}$ surface. By taking the differential of (6.3-9) and (6.3-10), and using certain vector identities, it can be shown that $\Delta\mathbf{k} \cdot \mathbf{S} = 0$, so that $\mathbf{S}$ is normal to the $\mathbf{k}$ surface. Consequently, $\mathbf{S}$ is also parallel to the group velocity vector $\mathbf{v}$.

If the $\mathbf{k}$ surface is a sphere, as it is for isotropic media, the vectors $\mathbf{v}$, $\mathbf{S}$, and $\mathbf{k}$ are all parallel, indicating that rays are parallel to the wavevector $\mathbf{k}$ and energy flows in the same direction, as illustrated in Fig. 6.3-10(a). On the other hand, if the $\mathbf{k}$ surface is not normal to the wavevector $\mathbf{k}$, as illustrated in Fig. 6.3-10(b), the rays and the direction of energy transport are not orthogonal to the wavefronts. Rays then have the "extraordinary" property of traveling at an oblique angle to their wavefronts [Fig. 6.3-10(b)].

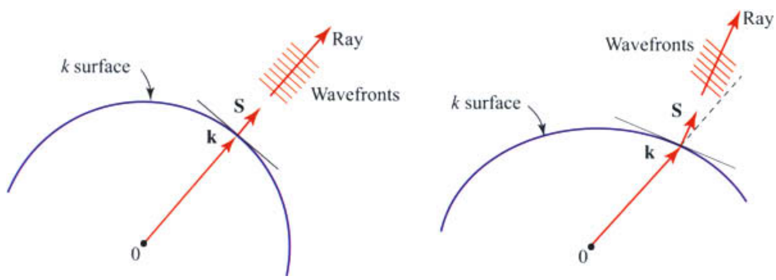


Figure 6.3-10 Rays and wavefronts for (a) a spherical $\mathbf{k}$ surface, and (b) a nonspherical $\mathbf{k}$ surface.

Special Case: Uniaxial Crystals

In uniaxial crystals ($n_1 = n_2 = n_o$ and $n_3 = n_e$), the equation of the $\mathbf{k}$ surface $\omega = \omega(k_1, k_2, k_3)$ simplifies to

$$(k^2 - n_o^2 k_o^2)\left(\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} - k_o^2\right) = 0. \qquad (6.3\text{-}19)$$

This equation has two solutions: a sphere, corresponding to the leftmost factor being zero:

$$k = n_o k_o, \qquad (6.3\text{-}20)$$

and an ellipsoid of revolution, corresponding to the rightmost factor being zero:

$$\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} = k_o^2. \qquad (6.3\text{-}21)$$

Because of symmetry about the z axis (optic axis), there is no loss of generality in assuming that the vector $\mathbf{k}$ lies in the y–z plane. Its direction is then characterized by

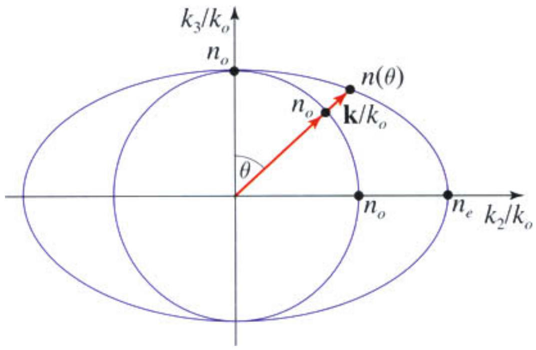


Figure 6.3-11 Intersection of the $\mathbf{k}$ surfaces with the y–z plane for a positive uniaxial crystal ($n_e > n_o$).

the angle θ it makes with the optic axis. It is thus convenient to draw the k-surfaces only in the y–z plane, as a circle and an ellipse, as shown in Fig. 6.3-11.

Given the direction $\hat{\mathbf{u}}$ of the vector $\mathbf{k}$, the wavenumber k is determined by finding the intersection with the $\mathbf{k}$ surfaces. The two solutions define the two normal modes, the ordinary and extraordinary waves. The ordinary wave has wavenumber $k = n_o k_o$ regardless of the direction of $\hat{\mathbf{u}}$, whereas the extraordinary wave has wavenumber $n(\theta)k_o$, where $n(\theta)$ is given by (6.3-15), thereby confirming earlier results obtained from the index-ellipsoid geometrical construction. The directions of the rays, wavefronts, energy flow, and field vectors $\mathbf{E}$ and $\mathbf{D}$ for the ordinary and extraordinary waves in a uniaxial crystal are illustrated in Fig. 6.3-12.

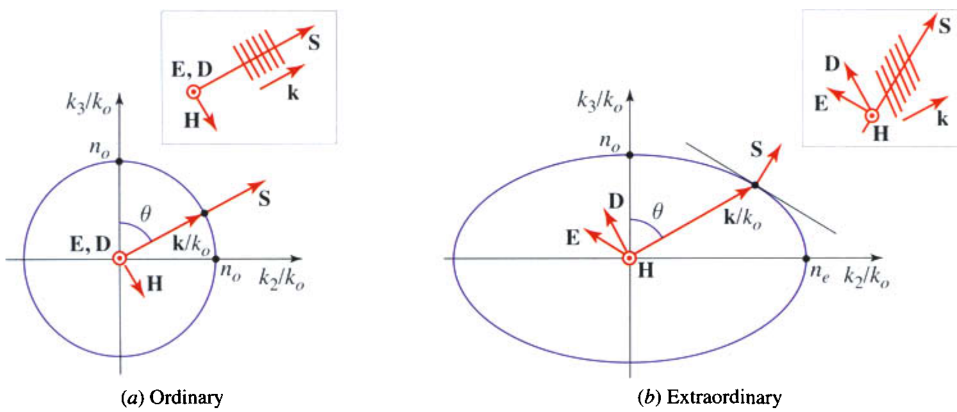


Figure 6.3-12 The normal modes for a plane wave traveling in a direction $\mathbf{k}$ that makes an angle θ with the optic axis z of a uniaxial crystal are: (a) An ordinary wave of refractive index n_o polarized in a direction normal to the k–z plane. (b) An extraordinary wave of refractive index $n(\theta)$ [given by (6.3-15)] polarized in the k–z plane along a direction tangential to the ellipse (the $\mathbf{k}$ surface) at the point of its intersection with $\mathbf{k}$. This wave is "extraordinary" in the following ways: $\mathbf{D}$ is not parallel to $\mathbf{E}$ but both lie in the k–z plane and $\mathbf{S}$ is not parallel to $\mathbf{k}$ so that power does not flow along the direction of $\mathbf{k}$; the rays are therefore not normal to the wavefronts so that the wave travels "sideways."

E. Double Refraction

Refraction of Plane Waves

We now examine the refraction of a plane wave at the boundary between an isotropic medium (say air, $n = 1$) and an anisotropic medium (a crystal). The key principle that governs the refraction of waves for this configuration is that the wavefronts of the incident and refracted waves must be matched at the boundary. Because the anisotropic medium supports two modes with distinctly different phase velocities, and therefore different indexes of refraction, an incident wave gives rise to two refracted waves with different directions and different polarizations. The effect is known as **double refraction** or **birefringence**.

The phase-matching condition requires that Snell's law be obeyed, i.e.,

$$k_o \sin\theta_1 = k\sin\theta, \tag{6.3-22}$$

where θ_1 and θ are the angles of incidence and refraction, respectively. In an anisotropic medium, however, the wavenumber $k = n(\theta)k_o$ is itself a function of θ, so that

$$\sin\theta_1 = n(\theta_a + \theta)\sin\theta, \tag{6.3-23}$$

where θ_a is the angle between the optic axis and the normal to the surface, so that $\theta_a + \theta$ is the angle the refracted ray makes with the optic axis. Equation (6.3-23) is a modified version of Snell's law. To solve (6.3-22), we draw the intersection of the **k** surface with the plane of incidence and search for an angle θ for which (6.3-22) is satisfied. Two solutions, corresponding to the two normal modes, are expected. The polarization state of the incident light governs the distribution of energy among the two refracted waves.

Take, for example, a uniaxial crystal and a plane of incidence parallel to the optic axis. The **k** surfaces intersect the plane of incidence in a circle and an ellipse (Fig. 6.3-13). The two refracted waves that satisfy the phase-matching condition are determined by satisfying (6.3-23):

- An ordinary wave of orthogonal polarization (TE) at an angle $\theta = \theta_o$ for which

$$\sin\theta_1 = n_o \sin\theta_o\,; \tag{6.3-24}$$

- An extraordinary wave of parallel polarization (TM) at an angle $\theta = \theta_e$, for which

$$\sin\theta_1 = n(\theta_a + \theta_e)\sin\theta_e\,, \tag{6.3-25}$$

where $n(\theta)$ is given by (6.3-15).

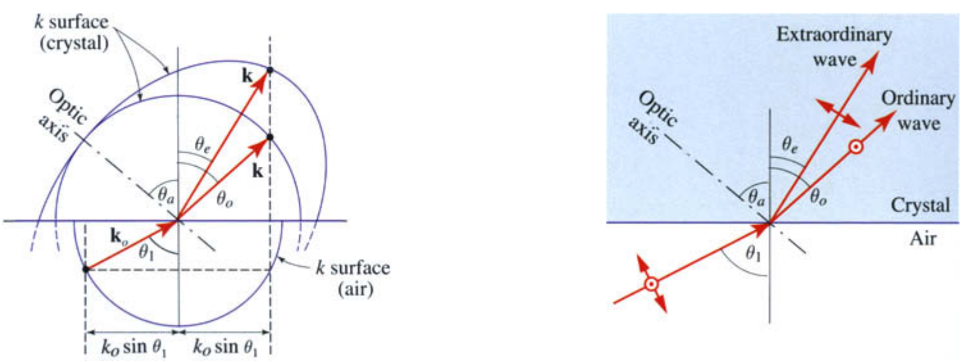


Figure 6.3-13 Determination of the angles of refraction by matching projections of the **k** vectors in air and in a uniaxial crystal.

If the incident wave carries the two polarizations, the two refracted waves will emerge, as shown in Fig. 6.3-13.

Refraction of Rays

The analysis immediately above dealt with the refraction of plane waves. The refraction of rays is different in an anisotropic medium, since rays do not necessarily travel in directions normal to the wavefronts. In air, before entering the crystal, the wavefronts are normal to the rays. The refracted wave must have a wavevector that satisfies the phase-matching condition, so that Snell's law (6.3-23) is applicable, with the angle of refraction θ determining the direction of $\mathbf{k}$. However, since the direction of $\mathbf{k}$ is not the direction of the ray, Snell's law is not applicable to rays in anisotropic media.

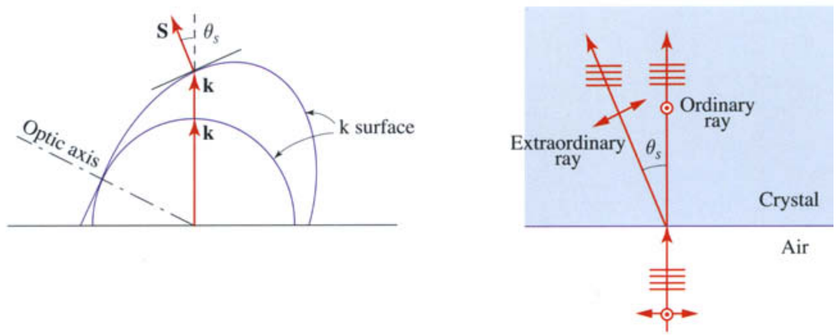


Figure 6.3-14 Double refraction at normal incidence.

An example that dramatizes the deviation from Snell's law is that of normal incidence into a uniaxial crystal whose optic axis is neither parallel nor perpendicular to the crystal boundary. The incident wave has a $\mathbf{k}$ vector normal to the boundary. To ensure phase matching, the refracted waves must also have wavevectors in the same direction. Intersections with the $\mathbf{k}$ surface yield two points corresponding to two waves. The ordinary ray is parallel to $\mathbf{k}$. But the extraordinary ray points in the direction of the normal to the $\mathbf{k}$ surface, at an angle θ_s with the normal to the crystal boundary, as illustrated in Fig. 6.3-14. Thus, normal incidence creates oblique refraction. The principle of phase matching is maintained, however: wavefronts of both refracted rays are parallel to the crystal boundary and to the wavefront of the incident ray.

When light rays are transmitted through a plate of anisotropic material as described above, the two rays refracted at the first surface refract again at the second surface, creating two laterally separated rays with orthogonal polarizations, as illustrated in Fig. 6.3-15.

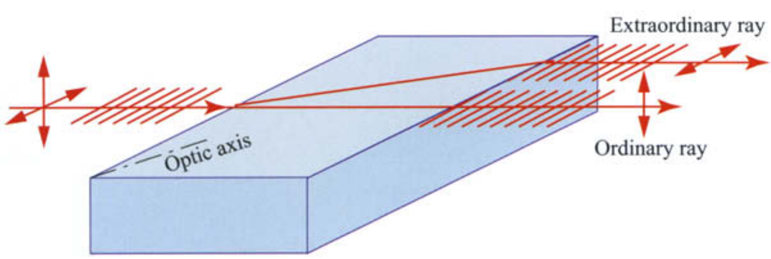


Figure 6.3-15 Double refraction through an anisotropic plate. The plate serves as a polarizing beamsplitter.

6.4 OPTICAL ACTIVITY AND MAGNETO-OPTICS

A. Optical Activity

Certain materials act as natural polarization rotators, a property known as **optical activity**. Their normal modes are waves that are circularly, rather than linearly polarized; waves with right- and left-circular polarizations travel at different phase velocities.

We demonstrate below that an optically active medium with right- and left-circular-polarization phase velocities c_o/n_+ and c_o/n_- acts as a polarization rotator with an angle of rotation $\pi(n_- - n_+)d/\lambda_o$ that is proportional to the thickness of the medium d. The rotatory power (rotation angle per unit length) of the optically active medium is therefore

$$\boxed{\rho = \frac{\pi}{\lambda_o}(n_- - n_+).} \qquad (6.4\text{-}1)$$

Rotatory Power

The direction in which the polarization plane rotates is the same as that of the circularly polarized component with the greater phase velocity (smaller refractive index). If $n_+ < n_-$, ρ is positive and the rotation is in the same direction as the electric field vector of the right circularly polarized wave [clockwise when viewed from the direction toward which the wave is approaching, as illustrated in Fig. 6.4-1(a)]. Such materials are said to be **dextrorotatory**, whereas those for which $n_+ > n_-$ are termed **levorotatory**.

☐ **Derivation of the Rotatory Power.** Equation (6.4-1) may be derived by decomposing the incident linearly polarized wave into a sum of right and left circularly polarized components of equal amplitudes (see Exercise 6.1B),

$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \tfrac{1}{2}e^{-j\theta}\begin{bmatrix}1\\ j\end{bmatrix} + \tfrac{1}{2}e^{j\theta}\begin{bmatrix}1\\ -j\end{bmatrix}, \qquad (6.4\text{-}2)$$

where θ is the initial angle of the plane of polarization. After propagating a distance d through the medium, the phase shifts encountered by the right and left circularly polarized waves are $\varphi_+ = 2\pi n_+ d/\lambda_o$ and $\varphi_- = 2\pi n_- d/\lambda_o$, respectively, resulting in a Jones vector

$$\tfrac{1}{2}e^{-j\theta}e^{-j\varphi_+}\begin{bmatrix}1\\ j\end{bmatrix} + \tfrac{1}{2}e^{j\theta}e^{-j\varphi_-}\begin{bmatrix}1\\ -j\end{bmatrix} = e^{-j\varphi_o}\begin{bmatrix}\cos(\theta-\varphi/2)\\ \sin(\theta-\varphi/2)\end{bmatrix}, \qquad (6.4\text{-}3)$$

where $\varphi_o = \frac{1}{2}(\varphi_+ + \varphi_-)$ and $\varphi = \varphi_- - \varphi_+ = 2\pi(n_- - n_+)d/\lambda_o$. This Jones vector represents a linearly polarized wave with the plane of polarization rotated by an angle $\varphi/2 = \pi(n_- - n_+)d/\lambda_o$, as provided in (6.4-1). ■

Optical activity occurs in materials with an intrinsically helical structure. Examples include selenium, tellurium, tellurium oxide (TeO_2), quartz (α-SiO_2), and cinnabar (HgS). Optically active liquids consist of so-called chiral molecules, which come in distinct left- and right-handed mirror-image forms. Many organic compounds, such as amino acids and sugars, exhibit optical activity. Almost all amino acids are levorotatory, whereas common sugars come in both forms: dextrose (d-glucose) and levulose (fructose) are dextrorotatory and levorotatory, respectively, as their names imply. The rotatory power and sense of rotation for solutions of such substances are therefore sensitive to both the concentration and structure of the solute. A saccharimeter is used to determine the optical activity of sugar solutions, from which the sugar concentration is calculated.

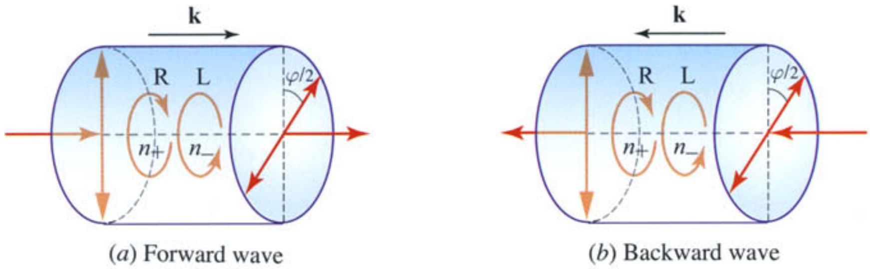


Figure 6.4-1 (a) The rotation of the plane of polarization by an optically active medium results from the difference in the velocities for the two circular polarizations. In this illustration, the right circularly polarized wave (R) is faster than the left circularly polarized wave (L), i.e., $n_+ < n_-$, so that ρ is positive and the material is dextrorotatory. (b) If the wave in (a) is reflected after traversing the medium, the plane of polarization rotates in the opposite direction so that the wave retraces itself.

Material Equations

A time-varying magnetic flux density $\mathbf{B}$ applied to an optically active structure induces a circulating current, by virtue of its helical character, that sets up an electric dipole moment (and hence a polarization) proportional to $j\omega\mathbf{B} = -\nabla \times \mathbf{E}$. The optically active medium is therefore spatially dispersive; i.e., the relation between $\mathbf{D}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ is not local. $\mathbf{D}(\mathbf{r})$ at position $\mathbf{r}$ is determined not only by $\mathbf{E}(\mathbf{r})$, but also by $\mathbf{E}(\mathbf{r}')$ at points $\mathbf{r}'$ in the immediate vicinity of $\mathbf{r}$, since it is dependent on the spatial derivatives contained in $\nabla \times \mathbf{E}(\mathbf{r})$. For a plane wave, we have $\mathbf{E}(\mathbf{r}) = \mathbf{E}\exp(-j\mathbf{k}\cdot\mathbf{r})$ and $\nabla \times \mathbf{E} = -j\mathbf{k} \times \mathbf{E}$, so that the dielectric permittivity tensor is dependent on the wavevector $\mathbf{k}$. Spatial dispersiveness is analogous to temporal dispersiveness, which has its origin in the noninstantaneous response of the medium (see Sec. 5.2). While the permittivity of a medium exhibiting temporal dispersion depends on the frequency ω, that of a medium exhibiting spatial dispersion depends on the wavevector $\mathbf{k}$.

An optically active medium is described by the $\mathbf{k}$-dependent material equation

$$\mathbf{D} = \epsilon\mathbf{E} + j\epsilon_o\xi\mathbf{k} \times \mathbf{E}, \tag{6.4-4}$$

where ξ is a quantity (called a pseudoscalar) that changes sign depending on the handedness of the coordinate system. This relation is a first-order approximation of the $\mathbf{k}$ dependence of the permittivity tensor, under appropriate symmetry conditions.[†] The first term represents the response of an isotropic dielectric medium whereas the second term accounts for the optical activity, as will be shown subsequently. This $\mathbf{D}$-$\mathbf{E}$ relation is often written in the form

$$\mathbf{D} = \epsilon\mathbf{E} + j\epsilon_o\mathbf{G} \times \mathbf{E}, \tag{6.4-5}$$

where $\mathbf{G} = \xi\mathbf{k}$ is known as the **gyration vector**. In such media the vector $\mathbf{D}$ is clearly not parallel to $\mathbf{E}$ since the vector $\mathbf{G} \times \mathbf{E}$ in (6.4-5) is perpendicular to $\mathbf{E}$.

Normal Modes of the Optically Active Medium

We proceed to show that the two normal modes of the medium described by (6.4-5) are circularly polarized waves, and we determine the velocities c_o/n_+ and c_o/n_- in terms of the constant $G = \xi k$.

[†] See, for example, L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, Pergamon, 2nd revised ed. 1984, Chapter 12.

We assume that the wave propagates in the z direction, so that $\mathbf{k} = (0, 0, k)$ and thus $\mathbf{G} = (0, 0, G)$. Equation (6.4-5) may then be written in matrix form as

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \epsilon_o \begin{bmatrix} n^2 & -jG & 0 \\ jG & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \tag{6.4-6}$$

where $n^2 = \epsilon/\epsilon_o$. The diagonal elements in (6.4-6) correspond to propagation in an isotropic medium with refractive index n, whereas the off-diagonal elements, proportional to G, represent the optical activity.

To prove that the normal modes are circularly polarized, consider the two circularly polarized waves with electric-field vectors $\mathbf{E} = (E_0, \pm jE_0, 0)$. The $+$ and $-$ signs correspond to right and left circularly polarized waves, respectively. Substitution in (6.4-6) yields $\mathbf{D} = (D_0, \pm jD_0, 0)$, where $D_0 = \epsilon_o(n^2 \pm G)E_0$. It follows that $\mathbf{D} = \epsilon_o n_\pm^2 \mathbf{E}$, where

$$\boxed{n_\pm = \sqrt{n^2 \pm G}\,.} \tag{6.4-7}$$

Hence, for either of the two circularly polarized waves the vector $\mathbf{D}$ is parallel to the vector $\mathbf{E}$. Equation (6.3-11) is satisfied if the wavenumber $k = n_\pm k_o$. Thus, the right and left circularly polarized waves propagate without changing their state of polarization, with refractive indexes n_+ and n_-, respectively. They are therefore the normal modes for this medium.

EXERCISE 6.4-1

Rotatory Power of an Optically Active Medium. Show that if $G \ll n$, the rotatory power of an optically active medium (rotation of the polarization plane per unit length) is approximately given by

$$\rho \approx -\frac{\pi G}{\lambda_o n}\,. \tag{6.4-8}$$

The rotatory power is strongly dependent on the wavelength. Since G is proportional to k, as indicated by (6.4-5), it is inversely proportional to the wavelength λ_o. Thus, the rotatory power in (6.4-8) is inversely proportional to λ_o^2. Moreover, the refractive index n is itself wavelength dependent. By way of example, the rotatory power ρ of quartz is ≈ 31 deg/mm at $\lambda_o = 500$ nm and ≈ 22 deg/mm at $\lambda_o = 600$ nm; for silver thiogallate ($AgGaS_2$), ρ is ≈ 700 deg/mm at 490 nm and ≈ 500 deg/mm at 500 nm.

B. Magneto-Optics: The Faraday Effect

Many materials act as polarization rotators in the presence of a static magnetic field, a property known as the **Faraday effect**. The angle of rotation is then proportional to the thickness of the material, and the rotatory power ρ (rotation angle per unit length) is proportional to the component of the magnetic flux density B in the direction of the wave propagation,

$$\rho = \mathfrak{V}B, \tag{6.4-9}$$

where $\mathfrak{V}$ is called the **Verdet constant**.

The sense of rotation is governed by the direction of the magnetic field: for $\mathfrak{V} > 0$, the rotation is in the direction of a right-handed screw pointing in the direction of the magnetic field [Fig. 6.4-2(*a*)]. In contrast to optical activity, however, the sense of rotation does not reverse with the reversal of the direction of propagation of the wave. Thus, when a wave travels through a Faraday rotator and then reflects back onto itself, traveling once more through the rotator in the opposite direction, it undergoes twice the rotation [Fig. 6.4-2(*b*)]. Materials that exhibit the Faraday effect include glasses, yttrium iron garnet (YIG), terbium gallium garnet (TGG), and terbium aluminum garnet (TbAlG). The Verdet constant of TbAlG is $\mathfrak{V} \approx -1.16$ min/Oe-cm at $\lambda_o = 500$ nm. Thin films of these ferrimagnetic materials are used to make compact devices.

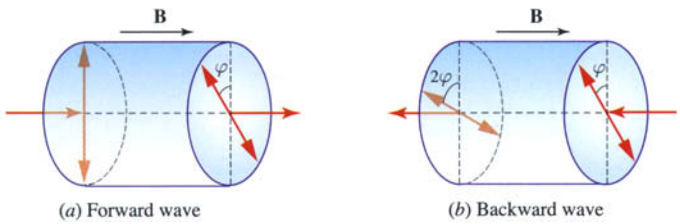


Figure 6.4-2 *(a)* Polarization rotation in a medium exhibiting the Faraday effect. *(b)* The sense of rotation is invariant to the direction of travel of the wave.

Material Equations

In magneto-optic materials, the electric permittivity tensor ϵ is altered by the application of a *static* magnetic field $\mathbf{H}$, so that $\epsilon = \epsilon(\mathbf{H})$. This effect originates from the interaction of the static magnetic field with the motion of the electrons in the material in response to an *optical* electric field $\mathbf{E}$. For the Faraday effect, in particular, the material equation is

$$\mathbf{D} = \epsilon\mathbf{E} + j\epsilon_o\mathbf{G} \times \mathbf{E} \qquad (6.4\text{-}10)$$

with

$$\mathbf{G} = \gamma\mathbf{B}. \qquad (6.4\text{-}11)$$

Here, $\mathbf{B} = \mu\mathbf{H}$ is the static magnetic flux density, and γ is a constant of the medium known as the **magnetogyration coefficient**.

Equation (6.4-10) is identical to (6.4-5) so that the vector $\mathbf{G} = \gamma\mathbf{B}$ in Faraday rotators plays the role of the gyration vector $\mathbf{G} = \xi\mathbf{k}$ in optically active media. For the Faraday effect, however, $\mathbf{G}$ does not depend on $\mathbf{k}$, so that reversing the direction of propagation does not reverse the sense of rotation of the plane of polarization. This property is useful for constructing optical isolators, as explained in Sec. 6.6C.

With this analogy, and using (6.4-8), we conclude that the rotatory power of the Faraday medium is $\rho \approx -\pi G/\lambda_o n = -\pi\gamma B/\lambda_o n$, from which the Verdet constant (rotatory power per unit magnetic flux density) is seen to be

$$\boxed{\mathfrak{V} \approx -\frac{\pi\gamma}{\lambda_o n}.} \qquad (6.4\text{-}12)$$

The Verdet constant is clearly a function of the wavelength λ_o.

6.5 OPTICS OF LIQUID CRYSTALS

Liquid Crystals

A liquid crystal comprises a collection of elongated organic molecules that are typically cigar-shaped. The molecules lack positional order (like liquids) but possess orientational order (like crystals). There are three types (phases) of liquid crystals, as illustrated in Fig. 6.5-1:

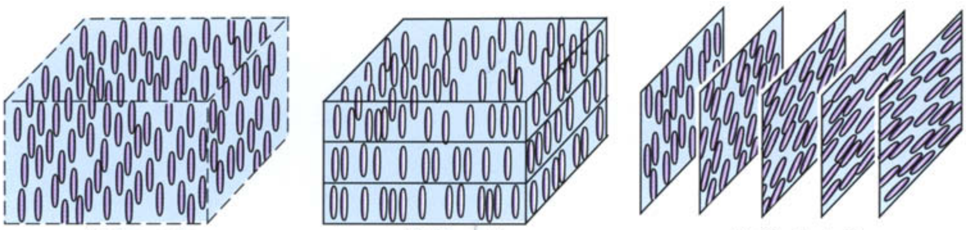


Figure 6.5-1 Molecular organizations of different types of liquid crystals.

- In **nematic liquid crystals** the orientations of the molecules tend to be the same but their positions are totally random.
- In **smectic liquid crystals** the orientations of the molecules are the same, but their centers are stacked in parallel layers within which they have random positions; they therefore have positional order only in one dimension.
- The **cholesteric liquid crystal** is a distorted form of its nematic cousin in which the orientations undergo helical rotation about an axis.

Liquid crystallinity is a *fluid* state of matter. The molecules are able to change orientation when subjected to a force. When a thin layer of liquid crystal is placed between two parallel glass plates that are rubbed together, for example, the molecules orient themselves along the direction of rubbing.

Twisted nematic liquid crystals are nematic liquid crystals on which a twist (similar to the twist that exists naturally in the cholesteric phase) is externally imposed. This can be achieved, for example, by placing a thin layer of nematic liquid crystal between two glass plates that are polished in perpendicular directions, as schematized in Fig. 6.5-2. This section is devoted to a discussion of the optical properties of twisted nematic liquid crystals, which are widely used in photonics, e.g., for liquid-crystal displays. The electro-optic properties of twisted nematic liquid crystals, and their use as optical modulators and switches, are described in Chapter 20.

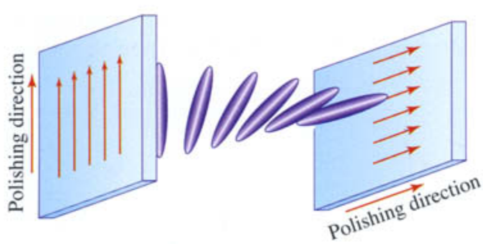


Figure 6.5-2 Molecular orientations of the twisted nematic liquid crystal.

Optical Properties of Twisted Nematic Liquid Crystals

The twisted nematic liquid crystal is an optically *inhomogeneous* and *anisotropic* medium that acts locally as a uniaxial crystal, with the optic axis parallel to the elongated direction. The optical properties are conveniently analyzed by considering the material to be divided into thin layers perpendicular to the axis of twist, each of which acts as a uniaxial crystal; the optic axis is taken to rotate gradually, in a helical fashion, along the axis of twist (Fig. 6.5-3). The cumulative effects of these layers on the transmitted wave is then calculated. We show that, under certain conditions, the twisted nematic liquid crystal acts as a polarization rotator in which the plane of polarization rotates in alignment with the molecular twist.

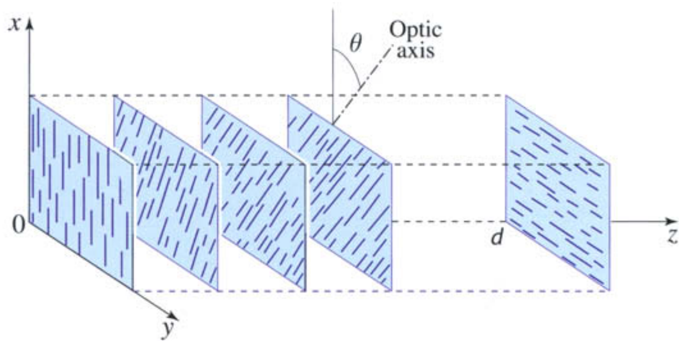


Figure 6.5-3 Propagation of light in a twisted nematic liquid crystal. In this diagram the angle of twist is 90°.

Consider the propagation of light along the axis of twist (the z axis) of a twisted nematic liquid crystal and assume that the twist angle θ varies linearly with z,

$$\theta = \alpha z, \tag{6.5-1}$$

where α is the twist coefficient (degrees per unit length). The optic axis is therefore parallel to the x–y plane and makes an angle θ with the x direction. The ordinary and extraordinary refractive indexes are n_o and n_e, respectively (typically, $n_e > n_o$), and the phase-retardation coefficient (retardation per unit length) is

$$\beta = (n_e - n_o)k_o. \tag{6.5-2}$$

The liquid crystal cell is completely characterized by the twist coefficient α and the retardation coefficient β.

In practice, $\beta \gg \alpha$ so that many cycles of phase retardation are introduced before the optic axis rotates appreciably. We show below that if this condition is satisfied, and the incident wave at $z = 0$ is linearly polarized in the x direction, then the wave maintains its linearly polarized state but the plane of polarization rotates in alignment with the molecular twist, so that the angle of rotation is $\theta = \alpha z$ and the total rotation in a crystal of length d is the angle of twist αd. The liquid crystal cell then serves as a polarization rotator with rotatory power α. The polarization-rotation property of the twisted nematic liquid crystal is useful for making display devices, as explained in Sec. 20.3.

☐ **Proof that the Twisted Nematic Liquid Crystal Acts as a Polarization Rotator.** We proceed to show that the twisted nematic liquid crystal acts as a polarization rotator if $\beta \gg \alpha$. We divide the overall width of the cell d into N incremental layers of equal widths $\Delta z = d/N$. The mth layer, located at the distance $z = z_m = m\Delta z$, $m = 1, 2, \ldots, N$, is a wave retarder whose slow axis (the optic axis) makes an angle $\theta_m = m\Delta\theta$ with the x axis, where $\Delta\theta = \alpha\Delta z$. It therefore has a Jones matrix [see (6.1-24)]

$$\mathbf{T}_m = \mathbf{R}(-\theta_m)\,\mathbf{T}_r\,\mathbf{R}(\theta_m), \tag{6.5-3}$$

where

$$\mathbf{T}_r = \begin{bmatrix} \exp(-jn_e k_o \Delta z) & 0 \\ 0 & \exp(-jn_o k_o \Delta z) \end{bmatrix} \tag{6.5-4}$$

is the Jones matrix of a wave retarder whose axis is along the x direction and $\mathbf{R}(\theta)$ is the coordinate rotation matrix in (6.1-22).

It is convenient to rewrite $\mathbf{T}_r$ in terms of the phase-retardation coefficient $\beta = (n_e - n_o)k_o$,

$$\mathbf{T}_r = \exp(-j\varphi\Delta z) \begin{bmatrix} \exp(-j\beta\Delta z/2) & 0 \\ 0 & \exp(j\beta\Delta z/2) \end{bmatrix}, \tag{6.5-5}$$

where $\varphi = (n_o + n_e)k_o/2$. Since multiplying the Jones vector by a constant phase factor does not affect the state of polarization, we simply ignore the prefactor $\exp(-j\varphi\Delta z)$ in (6.5-5).

The overall Jones matrix of the device is the product

$$\mathbf{T} = \prod_{m=N}^{1} \mathbf{T}_m = \prod_{m=N}^{1} \mathbf{R}(-\theta_m)\,\mathbf{T}_r\,\mathbf{R}(\theta_m). \tag{6.5-6}$$

Using (6.5-3) and noting that $\mathbf{R}(\theta_m)\,\mathbf{R}(-\theta_{m-1}) = \mathbf{R}(\theta_m - \theta_{m-1}) = \mathbf{R}(\Delta\theta)$, we obtain

$$\mathbf{T} = \mathbf{R}(-\theta_N)\;[\mathbf{T}_r\mathbf{R}(\Delta\theta)]^{N-1}\;\mathbf{T}_r\,\mathbf{R}(\theta_1). \tag{6.5-7}$$

Substituting from (6.5-5) and (6.1-22), we obtain

$$\mathbf{T}_r\,\mathbf{R}(\Delta\theta) = \begin{bmatrix} \exp(-j\beta\Delta z/2) & 0 \\ 0 & \exp(j\beta\Delta z/2) \end{bmatrix} \begin{bmatrix} \cos\alpha\Delta z & \sin\alpha\Delta z \\ -\sin\alpha\Delta z & \cos\alpha\Delta z \end{bmatrix}. \tag{6.5-8}$$

Using (6.5-7) and (6.5-8), the Jones matrix $\mathbf{T}$ of the device can, in principle, be determined in terms of the parameters α, β, and $d = N\Delta z$.

For $\alpha \ll \beta$, we may assume that the incremental rotation matrix $\mathbf{R}(\Delta\theta)$ is approximately the identity matrix, whereupon

$$\begin{aligned} \mathbf{T} \approx \mathbf{R}(-\theta_N)\,[\mathbf{T}_r]^N\,\mathbf{R}(\theta_1) &= \mathbf{R}(-\alpha N\Delta z) \begin{bmatrix} \exp(-j\beta\Delta z/2) & 0 \\ 0 & \exp(j\beta\Delta z/2) \end{bmatrix}^N \\ &= \mathbf{R}(-\alpha N\Delta z) \begin{bmatrix} \exp(-j\beta N\Delta z/2) & 0 \\ 0 & \exp(j\beta N\Delta z/2) \end{bmatrix}, \end{aligned} \tag{6.5-9}$$

so that

$$\mathbf{T} = \mathbf{R}(-\alpha d) \begin{bmatrix} \exp(-j\beta d/2) & 0 \\ 0 & \exp(j\beta d/2) \end{bmatrix}. \tag{6.5-10}$$

This Jones matrix represents a wave retarder of retardation βd with the slow axis along the x direction, followed by a polarization rotator with rotation angle αd. If the original wave is linearly polarized along the x direction, the wave retarder imparts only a phase shift; the device then simply rotates the polarization by an angle αd equal to the twist angle. A wave linearly polarized along the y direction is rotated by the same angle. ■

6.6 POLARIZATION DEVICES

This section offers a brief description of a number of devices that are used to modify the state of polarization of light. The basic principles underlying the operation of these devices have been set forth earlier in this chapter.

A. Polarizers

A linear polarizer is a device that transmits the component of the electric field that lies along the direction of its transmission axis while blocking the orthogonal component. The blocking action may be achieved by selective absorption, selective reflection from isotropic media, or selective reflection/refraction in anisotropic media.

Polarization by Selective Absorption (Dichroism)

The absorption of light by certain anisotropic media, called **dichroic materials**, depends on the direction of the incident electric field (Fig. 6.6-1). These materials generally have anisotropic molecular structures whose response is sensitive to the direction of the electric field. The most common dichroic material is **Polaroid H-sheet**, invented in 1938 and still in common use. It is fabricated from a sheet of iodine-impregnated polyvinyl alcohol that is heated and stretched in a particular direction. The analogous device in the infrared is the **wire-grid polarizer**, which comprises a planar configuration of closely spaced fine wires stretched in a single direction. The component of the incident electric field in the direction of the wires is absorbed whereas the component perpendicular to the wires passes through.

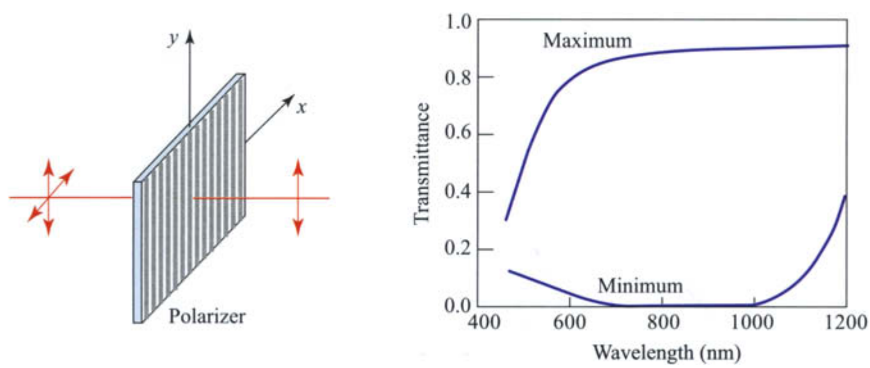


Figure 6.6-1 Power transmittances of a typical dichroic polarizer with the plane of polarization of the light aligned for maximum and minimum transmittance, as indicated.

Polarization by Selective Reflection

The reflectance of light at the boundary between two isotropic dielectric materials is dependent on its polarization, as discussed in Sec. 6.2. At the Brewster angle of incidence, in particular, the reflectance of TM-polarized light vanishes so that it is totally refracted (Fig. 6.2-4). At this angle, therefore, only TE-polarized light is reflected, so that the reflector serves as a polarizer.

Polarization by Selective Refraction (Polarizing Beamsplitters)

When light enters an anisotropic crystal, the ordinary and extraordinary waves refract at different angles and gradually separate from each other (see Sec. 6.3E and Fig. 6.3-15). This provides an effective means for obtaining polarized light from unpolarized

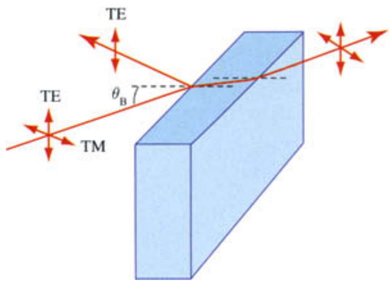
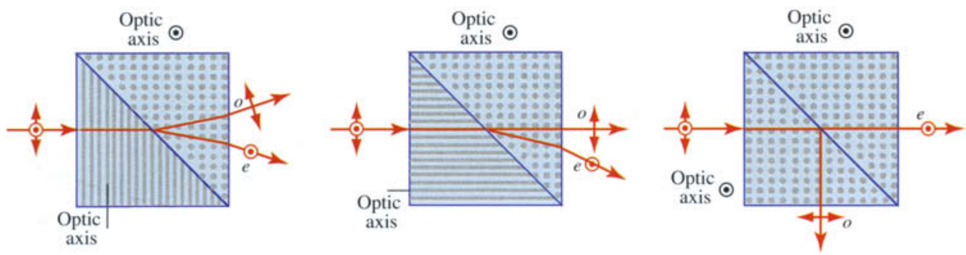


Figure 6.6-2 Brewster-angle polarizer.

light, and it is commonly used. These devices usually consist of two cemented prisms comprising anisotropic (uniaxial) materials, often with different orientations, as illustrated by the examples in Fig. 6.6-3. These prisms therefore serve as **polarizing beamsplitters**.



(*a*) Wollaston prism (*b*) Rochon prism (*c*) Glan–Thompson prism

Figure 6.6-3 Polarizing beamsplitters. The directions and polarizations of the waves that exit differ for the three prisms. In this illustration, the crystals are negative uniaxial (e.g., calcite). The Glan–Thompson device has the merit of providing a large angular separation between the emerging waves.

B. Wave Retarders

A wave retarder serves to convert a wave with one form of polarization into another form. It is characterized by its retardation Γ and its fast and slow axes (see Sec. 6.1B). The normal modes are linearly polarized waves polarized along the directions of the axes. The velocities of the two waves differ so that transmission through the retarder imparts a relative phase shift Γ to these modes.

Wave retarders are often constructed from anisotropic crystals in the form of plates. As explained in Sec. 6.3B, when light travels along a principal axis of a crystal (say the z axis), the normal modes are linearly polarized waves pointing along the two other principal axes (the x and y axes). These modes experience the principal refractive indexes n_1 and n_2, and thus travel at velocities c_o/n_1 and c_o/n_2, respectively. If $n_1 < n_2$, the x axis is the fast axis. If the plate has thickness d, the phase retardation is $\Gamma = (n_2 - n_1)k_o d = 2\pi(n_2 - n_1)d/\lambda_o$. The retardation is thus directly proportional to the thickness d of the plate and inversely proportional to the wavelength λ_o (note, however, that $n_2 - n_1$ is itself wavelength dependent).

The refractive indexes of a thin sheet of mica, for example, are 1.599 and 1.594 at $\lambda_o = 633$ nm, so that $\Gamma/d \approx 15.8\pi$ rad/mm. A sheet of thickness 63.3 μm yields $\Gamma \approx \pi$ and thus serves as a half-wave retarder.

Light Intensity Control via a Wave Retarder and Two Polarizers

Consider a wave retarder of retardation Γ placed between two crossed polarizers whose axes are oriented at 45° with respect to the axes of the retarder, as illustrated in Fig. 6.6-4. The power (or intensity) transmittance of this system is

$$\mathcal{T} = \sin^2(\Gamma/2), \tag{6.6-1}$$

which may be established by making use of Jones matrices or by examining the polarization ellipse of the retarded light as a function of Γ, and then determining the component that lies in the direction of the output polarizer, as illustrated in Fig. 6.6-4. If $\Gamma = 0$ no light is transmitted through the system since the polarizers are orthogonal. On the other hand, if $\Gamma = \pi$ all of the light is transmitted since the retarder then rotates the plane of polarization 90° whereupon it matches the transmission axis of the second polarizer.

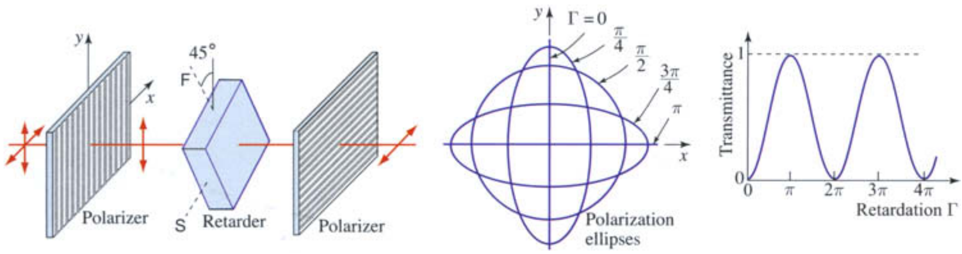


Figure 6.6-4 Controlling light intensity by means of a wave retarder with variable retardation Γ placed between two crossed polarizers.

The intensity of the transmitted light is thus readily controlled by altering the retardation Γ. This can be achieved, for example, by deliberately changing the indexes n_1 and n_2 by application of an external DC electric field to the retarder. This is the basic principle that underlies the operation of electro-optic modulators, as discussed in Chapter 20.

Furthermore, since Γ depends on d, slight variations in the thickness of a sample can be monitored by examining the pattern of the transmitted light. Moreover, since Γ is wavelength dependent, the transmittance of the system is frequency sensitive. Though it can be used as a filter, the selectivity is not sharp. Other configurations using wave retarders and polarizers can be used to construct narrowband transmission filters.

C. Polarization Rotators

A polarization rotator serves to rotate the plane of polarization of linearly polarized light by a fixed angle, while maintaining its linearly polarized nature. Optically active media and materials exhibiting the Faraday effect act as polarization rotators, as discussed in Sec. 6.4. The twisted nematic liquid crystal also acts as a polarization rotator under certain conditions, as shown in Sec. 6.5.

If a polarization rotator is placed between two polarizers, the amount of light transmitted depends on the rotation angle. The intensity of the light can therefore be controlled (modulated) if the angle of rotation is externally changed (e.g., by varying the magnetic flux density applied to a Faraday rotator or by changing the molecular orientation of a liquid crystal by means of an applied electric field). Electro-optic modulation of light and liquid-crystal display devices are discussed in Chapter 20.

D. Nonreciprocal Polarization Devices

A device whose effect on the polarization state is invariant to reversal of the direction of propagation is said to be **reciprocal**. If a wave is transmitted through such a device in one direction and the emerging wave is retransmitted in the opposite direction, then it retraces the changes in the polarization state and arrives at the input in the very same initial polarization state. Devices that do not have this directional invariance are called **nonreciprocal**. All of the polarization systems described in this chapter are reciprocal, with the exception of the Faraday rotator (see Sec. 6.4B). A number of useful nonreciprocal polarization devices are obtained by combining the Faraday rotator with other reciprocal polarization components.

Optical Isolator

An **optical isolator** is a device that transmits light in only one direction, thereby acting as a "one-way valve." Optical isolators are useful for preventing reflected light from returning back to the source. Such feedback can have deleterious effects on the operation of certain devices, such as semiconductor lasers.

An optical isolator is constructed by placing a Faraday rotator between two polarizers whose axes make a 45° angle with respect to each other. The magnetic flux density applied to the rotator is adjusted so that it rotates the polarization by 45° in the direction of a right-handed screw pointing in the z direction [Fig. 6.6-5(a)]. Light traveling through the system in the forward direction (from left to right) thus crosses polarizer A, rotates 45°, and is thence transmitted through polarizer B. Linearly polarized light with the polarization plane at 45° but traveling through the system in the backward direction [from right to left in Fig. 6.6-5(b)] successfully crosses polarizer B. However, on passing through the Faraday rotator, the plane of polarization rotates an additional 45° and is therefore blocked by polarizer A. Since the backward light might be generated by reflection of the forward wave from subsequent surfaces, the isolator serves to protect its source from reflected light.

Note that the Faraday rotator is a necessary component of the optical isolator. An optically active, or liquid-crystal, polarization rotator cannot be used in its place. In those *reciprocal* components, the sense of rotation is such that the polarization of the reflected wave retraces that of the incident wave so that the light would be transmitted back through the polarizers to the source.

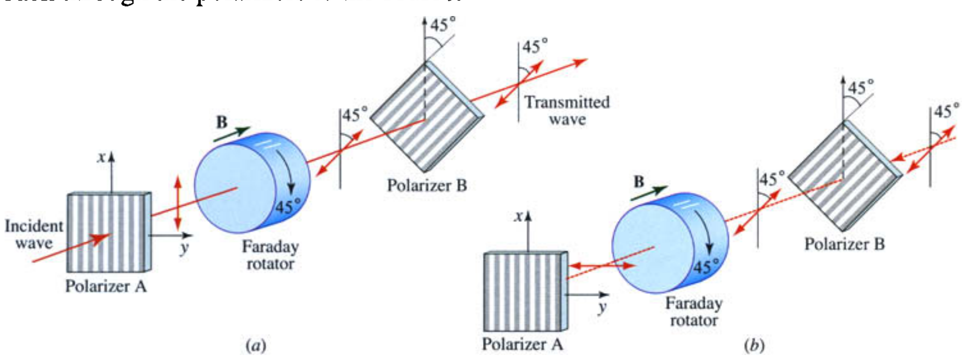


Figure 6.6-5 An optical isolator that makes use of a Faraday rotator transmits light in one direction. (a) A wave traveling in the forward direction is transmitted. (b) A wave traveling in the backward (or reverse) direction is blocked.

Faraday-rotator isolators constructed from yttrium iron garnet (YIG) or terbium gallium garnet (TGG) offer attenuations of the backward wave of up to 90 dB, over a relatively wide wavelength range. Thin films of these materials placed in permanent magnetic fields are used to make very compact optical isolators.

Nonreciprocal Polarization Rotation

A combination of a 45° Faraday rotator followed by a half-wave retarder is another useful nonreciprocal device. As illustrated in Fig. 6.6-6(a), the state of polarization of a forward linearly polarized wave, with the plane of polarization oriented at 22.5° with the fast axis of the retarder, maintains its state of polarization upon transmission through the device (since it undergoes 45° rotation by the Faraday rotator, followed by $-45°$ rotation by the retarder). However, for a wave traveling in the reverse direction, the plane of polarization is rotated by $45° + 45° = 90°$, as can be readily seen in Fig. 6.6-6(b). The device may therefore be used in combination with a polarizing beamsplitter to direct the backward wave away from the source of the forward wave and to access it independently. The system can be useful in implementing nonreciprocal interconnects, such as **optical circulators**, as described in Sec. 23.1.

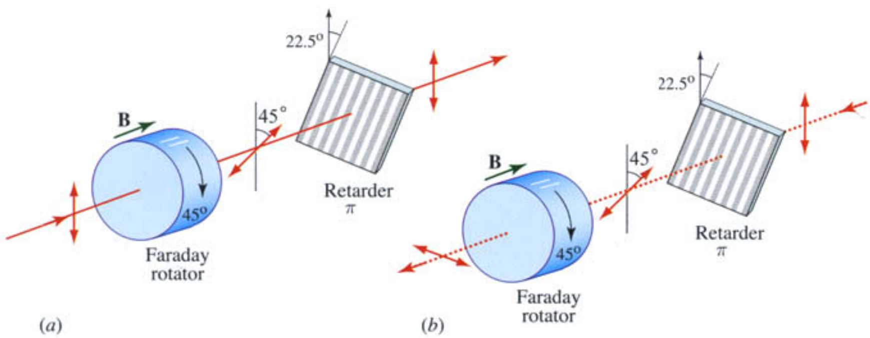


Figure 6.6-6 A nonreciprocal device that maintains the polarization state of a linearly polarized forward wave (a), but rotates the plane of polarization of the backward wave (b) by 90°.

READING LIST

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See also the general reading lists in Chapters 1 and 5.

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PROBLEMS

6.1-5 **Orthogonal Polarizations.** Show that if two elliptically polarized states are orthogonal, the major axes of their ellipses are perpendicular and the senses of rotation are opposite.

6.1-6 **Rotating a Polarization Rotator.** Show that the Jones matrix of a polarization rotator is invariant to rotation of the coordinate system.

6.1-7 **Half-Wave Retarder.** Consider linearly polarized light passed through a half-wave retarder. If the polarization plane makes an angle θ with the fast axis of the retarder, show that the transmitted light is linearly polarized at an angle $-\theta$, i.e., it is rotated by an angle 2θ. Why is the half-wave retarder not equivalent to a polarization rotator?

6.1-8 **Wave Retarders in Tandem.** Write the Jones matrices for:

(a) A $\pi/2$ wave retarder with the fast axis along the x direction.

(b) A π wave retarder with the fast axis at 45° to the x direction.

(c) A $\pi/2$ wave retarder with the fast axis along the y direction.

If these three retarders are placed in tandem, with (c) following (b) following (a), show that the resulting device introduces a 90° rotation. What happens if the order of the three retarders is reversed?

6.1-9 **Reflection of Circularly Polarized Light.** Show that circularly polarized light changes handedness (right becomes left, and vice versa) upon reflection from a mirror.

6.1-10 **Anti-Glare Screen.** A self-luminous object is viewed through a glass window. An anti-glare screen is used to eliminate glare caused by reflection of background light from the window surfaces. Show that such a screen may be made of a combination of a linear polarizer and a quarter-wave retarder whose axes are at 45° with respect to the transmission axis of the polarizer. Can the screen be regarded as an optical isolator?

6.2-3 **Derivation of Fresnel Equations.** Derive the reflection equation (6.2-6), which is used to derive the Fresnel equation (6.2-8) for TE polarization. How would you go about obtaining the reflection coefficient if the incident light took the form of a beam rather than a plane wave?

6.2-4 **Reflectance of Glass.** A plane wave is incident from air ($n = 1$) onto a glass plate ($n = 1.5$) at an angle of incidence of 45°. Determine the power reflectances of the TE and TM waves. What is the average reflectance for unpolarized light (light carrying TE and TM waves of equal intensities)?

6.2-5 **Refraction at the Brewster Angle.** Use the condition $n_1 \sec\theta_1 = n_2 \sec\theta_2$ and Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, to derive (6.2-12) for the Brewster angle. Also show that at the Brewster angle, $\theta_1 + \theta_2 = 90°$, so that the directions of the reflected and refracted waves are orthogonal, and hence the electric field of the refracted TM wave is parallel to the direction of the reflected wave. The reflection of light may be regarded as a scattering process in which the refracted wave acts as a source of radiation generating the reflected wave. At the Brewster angle, this source oscillates in a direction parallel to the direction of propagation of the reflected wave, so that radiation cannot occur and no TM light is reflected.

6.2-6 **Retardation Associated with Total Internal Reflection.** Determine the phase retardation between the TE and TM waves that is introduced by total internal reflection at the boundary between glass ($n = 1.5$) and air ($n = 1$) at an angle of incidence $\theta = 1.2\,\theta_c$, where θ_c is the critical angle.

6.2-7 **Goos–Hänchen Shift.** Consider two TE plane waves undergoing total internal reflection at angles θ and $\theta + d\theta$, where $d\theta$ is an incremental angle. If the phase retardation introduced between the reflected waves is written in the form $d\varphi = \xi\, d\theta$, find an expression for the coefficient ξ. Sketch the interference patterns of the two incident waves and the two reflected waves and verify that they are shifted by a lateral distance proportional to ξ. When the incident wave is a beam (composed of many plane-wave components), the reflected beam is displaced laterally by a distance proportional to ξ. This is known as the Goos–Hänchen effect.

6.2-8 **Reflection from an Absorptive Medium.** Use Maxwell's equations and appropriate boundary conditions to show that the complex amplitude reflectance at the boundary between free space and a medium with refractive index n and absorption coefficient α, at normal incidence, is $\mathrm{r} = [(n - j\alpha c/2\omega) - 1]/[(n - j\alpha c/2\omega) + 1]$.

6.3-1 **Maximum Retardation in Quartz.** Quartz is a positive uniaxial crystal with $n_e = 1.553$ and $n_o = 1.544$. (a) Determine the retardation per mm at $\lambda_o = 633$ nm when the crystal is oriented such that retardation is maximized. (b) At what thickness(es) does the crystal act as a quarter-wave retarder?

6.3-2 **Maximum Extraordinary Effect.** Determine the direction of propagation in quartz ($n_e = 1.553$ and $n_o = 1.544$) at which the angle between the wavevector **k** and the Poynting vector **S** (which is also the direction of ray propagation) is maximum.

6.3-3 **Double Refraction.** An unpolarized plane wave is incident from free space onto a quartz crystal ($n_e = 1.553$ and $n_o = 1.544$) at an angle of incidence 30°. The optic axis lies in the plane of incidence and is perpendicular to the direction of the incident wave before it enters the crystal. Determine the directions of the wavevectors and the rays of the two refracted components.

6.3-4 **Lateral Shift in Double Refraction.** What is the optimum geometry for maximizing the lateral shift between the refracted ordinary and extraordinary beams in a positive uniaxial crystal? Indicate all pertinent angles and directions.

6.3-5 **Transmission Through a $LiNbO_3$ Plate.** Examine the transmission of an unpolarized He–Ne laser beam ($\lambda_o = 633$ nm) normally incident on a $LiNbO_3$ plate ($n_e = 2.29$, $n_o = 2.20$) of thickness 1 cm, cut such that its optic axis makes an angle 45° with the normal to the plate. Determine the lateral shift at the output of the plate and the retardation between the ordinary and extraordinary beams.

*6.3-6 **Conical Refraction.** When the wavevector **k** points along an optic axis of a biaxial crystal an unusual situation occurs. The two sheets of the **k** surface meet and the surface can be approximated by a conical surface. Consider a ray normally incident on the surface of a biaxial crystal for which one of its optic axes is also normal to the surface. Show that multiple refraction occurs with the refracted rays forming a cone. This effect is known as conical refraction. What happens when the conical rays refract from the parallel surface of the crystal into air?

6.6-1 **Circular Dichroism.** Certain materials have different absorption coefficients for right and left circularly polarized light, a property known as **circular dichroism**. Determine the Jones matrix for a device that converts light with any state of polarization into right circularly polarized light.

6.6-2 **Polarization Rotation by a Sequence of Linear Polarizers.** A wave that is linearly polarized in the x direction is transmitted through a sequence of N linear polarizers whose transmission axes are inclined by angles $m\theta$ ($m = 1, 2, \ldots, N$; $\theta = \pi/2N$) with respect to the x axis. Show that the transmitted light is linearly polarized in the y direction but its amplitude is reduced by the factor $\cos^N \theta$. What happens in the limit $N \to \infty$? *Hint:* Use Jones matrices and note that

$$\mathbf{R}[(m+1)\theta]\,\mathbf{R}(-m\theta) = \mathbf{R}(\theta),$$

where $\mathbf{R}(\theta)$ is the coordinate transformation matrix.