Faculty of Mathematics and Information Science

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Numerical Methods 1
Gaussian Elimination

Calculating the determinate of a matrix using GE and GE with complete pivoting

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1 Title of a task

29. Computing det(A) and the growth factor using GE and GECP methods, where $A \in \mathbb{R}^{n \times n}$. Perform tests for various matrices, strictly dominant (row-wise, column-wise) and other.

2 Gaussian Elimination

Gaussian elimination (GE) is a method for solving systems of linear equations. The goal is to transform the matrix into an upper triangular form where all elements below the main diagonal are zeros. Here's a detailed explanation of how GE is performed:

First Column Elimination:

Use the first row to eliminate all elements below the main diagonal in the first column. For each row i (where $i > 1$), update the row as:

$$r_i := r_i - l_{i1} r_1$$

where

$$l_{i1} = \frac{a_{i1}}{a_{11}}$$

After this step, the matrix will look like:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & \cdots & a_{3n}^{(1)} & b_3^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{pmatrix}$$

Second Column Elimination:

Move to the second column and use the second row to eliminate all elements below the main diagonal in this column. For each row i (where $i > 2$), update the row as:

$$r_i := r_i - l_{i2} r_2$$

where

$$l_{i2} = \frac{a_{i2}^{(1)}}{a_{22}^{(1)}}$$

After this step, the matrix will look like:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & 0 & \cdots & a_{3n}^{(2)} & b_3^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{pmatrix}$$

General Step k:

For the k-th step, focus on the k-th column. Use the k-th row to eliminate all elements below the main diagonal in this column. For each row i (where $i > k$), update the row as:

$$r_i := r_i - l_{ik} r_k$$

where

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$$

After the k-th step, the matrix will have zeros below the main diagonal in the k-th column.

3 Gaussian Elimination with complete pivoting

The goal of Gaussian Elimination with complete pivoting is the same as Gaussian Elimination however each iteration includes additional steps.

Pivot Selection

At each step k of the elimination, the pivot element is chosen as the largest absolute value element in the submatrix $A_{k:n,k:n}^{(k-1)}$.

Finding the Pivot

Before the first step, the largest element in the entire matrix $A^{(0)}$ is found. Indices p and q are determined such that:

$$\left|a_{pq}^{(0)}\right| = \max_{i=1,\ldots,n;j=1,\ldots,n} \left|a_{ij}^{(0)}\right|$$

Before the k-th step, the largest element in the submatrix $A_{k:n,k:n}^{(k-1)}$ is found. Indices p and q are determined such that:

$$\left|a_{pq}^{(k-1)}\right| = \max_{i=k,\ldots,n;j=k,\ldots,n} \left|a_{ij}^{(k-1)}\right|$$

Row and Column Swapping

Once the pivot $a_{pq}^{(k-1)}$ is identified, swap the p-th row with the k-th row and the q-th column with the k-th column. This brings the pivot element to the k-th position on the main diagonal.

Elimination Phase

After selecting and positioning the pivot, perform the Gaussian elimination step as usual: For each row i (where $i > k$), update the row as:

$$r_i := r_i - l_{ik} r_k$$

where

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$$

This creates zeros below the pivot in column k.

4 Calculating the determinate

After transforming the matrix into upper triangular form, the determinant can be simply calculated by multiplying the elements located on the main diagonal as:

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

where a_{ii} are the elements on the main diagonal of the upper triangular matrix.
However, in GECP we need to keep track of the number of swaps during GECP to adjust the sign. If the number of swaps is odd the determinate must be multiplied by -1. Otherwise, the determinant sign remains unchanged if the number of swaps is even.

5 Calculating the Growth Factor

The growth factor for Gaussian Elimination (GE) and Gaussian Elimination with Complete Pivoting (GECP) is a measure of the stability of these methods. It is defined as the ratio of the largest element in the matrix after applying GE or GECP to the largest element in the original matrix. To calculate the growth factor, we follow these steps:

1. Identify the largest absolute value in the original matrix A, denoted as $\max_{i,j} |a_{ij}|$.
2. Perform GE or GECP on A and after each iteration find the largest element.
3. Then the formula of growth factor can be calculated using this formula:
   $\rho_n = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$
   where the $a_{ij}^{(k)}$ are the elements at the start of the k-th stage of Gaussian elimination.

6 Testing

Figure 1. shows the average error and the max error in the computation for random matrices of sizes 6, 7 and 8. For each matrix size, we created 1000 matrices to test our function.
Figure 2. shows the average run time and the max run time of our functions for random matrices of sizes 6, 7 and 8. For each matrix size, we created 1000 matrices to test our function.
Figure 3. shows the determinate calculated using the build-in Matlab function det() (true determinate), the determinate calculated using Gaussian Elimination (GE determinate), and the determinate calculated using Gaussian Elimination with Complete Pivoting (GECP determinate) for the specific matrices, for example, a row or a column dominate matrices.

Figure 4 shows the error between the true determinate and the determined calculated either using GE or GECP. This figure also shows the growth factor of GE and GECP.

We are confident that our algorithm computes the determinate correctly, as it has a very low error rate. We also performed a test to see if the function return correct errors. To do that, a not-square matrix was used and matrices for which there would be division by zero during iterations.

N	average error GE	average error GECP	max error GE	max error GECP
6	1.89862756999571e-16	1.60350079297499e-17	3.94276625237389e-14	1.94289029309402e-16
7	1.46787262385187e-16	1.54142649165523e-17	4.24799084797201e-14	1.66533453693773e-16
8	1.55684127156781e-16	1.50693226072442e-17	2.23848717340047e-14	1.38777878078145e-16

Figure 1: Average error and max error for GE and GECP for matrices of size N

N	average time GE(s)	average time GECP(s)	max time GE(s)	max time GECP(s)
6	2.38899999999999e-05	4.5559e-05	0.000157	0.000181
7	2.89810000000001e-05	5.5615e-05	0.000157	0.000198
8	3.3898e-05	6.5275e-05	0.000176	0.000189

Figure 2: Average time and max time for GE and GECP for matrices of size N

Matrix	True determinate	GE determinate	GECP determinate
$\begin{bmatrix} -5 & 5 & -67 & 15 & 23 & -25 & 48 & -66 \\ 16 & -34 & 83 & -67 & 29 & 72 & -11 & 93 \\ -17 & 76 & -10 & 13 & -58 & 94 & -49 & 80 \\ 1 & -43 & 99 & -7 & 35 & -25 & 17 & -88 \\ -14 & 20 & -6 & 6 & -52 & 66 & -97 & 71 \\ 29 & -21 & 13 & -32 & 9 & -68 & 10 & -55 \\ -28 & 30 & -59 & 65 & -22 & 12 & -30 & 12 \\ 17 & -19 & 81 & -14 & 50 & -9 & 33 & -65 \end{bmatrix}$	23936002070947	23936002070947	23936002070947
$\begin{bmatrix} 10 & -2 & 1 & -1 & 2 & 8 \\ -3 & 12 & 2 & -1 & 3 & -2 \\ 2 & -1 & 14 & 3 & -2 & -1 \\ -1 & 3 & -2 & 13 & 1 & 2 \\ 2 & -3 & -1 & 2 & 11 & -1 \\ -2 & 1 & -3 & -1 & 1 & 20 \end{bmatrix}$	5275450	5275450	5275450
$\begin{bmatrix} 38 & 12 & -5 & -7 & 7 & -6 \\ 10 & 52 & 6 & -12 & 20 & -1 \\ 2 & -1 & 90 & 12 & -30 & -41 \\ -5 & 7 & -12 & 130 & 56 & 48 \\ -8 & -3 & -8 & 2 & 30 & -5 \\ -4 & 1 & -23 & -14 & 1 & 60 \end{bmatrix}$	31907721826	31907721826	31907721826

Figure 3: Determinate calculated using different methods

Matrix	Error GE	Error GECP	Growth GE	Growth GECP
$\begin{bmatrix} -5 & 5 & -67 & 15 & 23 & -25 & 48 & -66 \\ 16 & -34 & 83 & -67 & 29 & 72 & -11 & 93 \\ -17 & 76 & -10 & 13 & -58 & 94 & -49 & 80 \\ 1 & -43 & 99 & -7 & 35 & -25 & 17 & -88 \\ -14 & 20 & -6 & 6 & -52 & 66 & -97 & 71 \\ 29 & -21 & 13 & -32 & 9 & -68 & 10 & -55 \\ -28 & 30 & -59 & 65 & -22 & 12 & -30 & 12 \\ 17 & -19 & 81 & -14 & 50 & -9 & 33 & -65 \end{bmatrix}$	5.3333	1.6846	0.0078	0.0234
$\begin{bmatrix} 10 & -2 & 1 & -1 & 2 & 8 \\ -3 & 12 & 2 & -1 & 3 & -2 \\ 2 & -1 & 14 & 3 & -2 & -1 \\ -1 & 3 & -2 & 13 & 1 & 2 \\ 2 & -3 & -1 & 2 & 11 & -1 \\ -2 & 1 & -3 & -1 & 1 & 20 \end{bmatrix}$	1.08	1	0	0
$\begin{bmatrix} 38 & 12 & -5 & -7 & 7 & -6 \\ 10 & 52 & 6 & -12 & 20 & -1 \\ 2 & -1 & 90 & 12 & -30 & -41 \\ -5 & 7 & -12 & 130 & 56 & 48 \\ -8 & -3 & -8 & 2 & 30 & -5 \\ -4 & 1 & -23 & -14 & 1 & 60 \end{bmatrix}$	1.0209	1	0	$3.81e-06$

Figure 4: Errors and Growth Factors (Growth) of GE and GECP