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Numerical Methods 1
Gaussian Elimination

Calculating the determinate of a matrix using GE and GE with complete pivoting

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1 Title of a task

29. Computing det(A) and the growth factor using GE and GECP methods, where $A \in \mathbb{R}^{n \times n}$. Perform tests for various matrices, strictly dominant (row-wise, column-wise) and other.

2 Gaussian Elimination

Gaussian elimination (GE) is a method for solving systems of linear equations. The goal is to transform the matrix into an upper triangular form where all elements below the main diagonal are zeros. Here's a detailed explanation of how GE is performed:

First Column Elimination:

Use the first row to eliminate all elements below the main diagonal in the first column. For each row i (where $i > 1$), update the row as:

$$r_i := r_i - l_{i1} r_1$$

where

$$l_{i1} = \frac{a_{i1}}{a_{11}}$$

After this step, the matrix will look like:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & \cdots & a_{3n}^{(1)} & b_3^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{pmatrix}$$

Second Column Elimination:

Move to the second column and use the second row to eliminate all elements below the main diagonal in this column. For each row i (where $i > 2$), update the row as:

$$r_i := r_i - l_{i2} r_2$$

where

$$l_{i2} = \frac{a_{i2}^{(1)}}{a_{22}^{(1)}}$$

After this step, the matrix will look like:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & 0 & \cdots & a_{3n}^{(2)} & b_3^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{pmatrix}$$

General Step k:

For the k-th step, focus on the k-th column. Use the k-th row to eliminate all elements below the main diagonal in this column. For each row i (where $i > k$), update the row as:

$$r_i := r_i - l_{ik} r_k$$

where

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$$

After the k-th step, the matrix will have zeros below the main diagonal in the k-th column.

3 Gaussian Elimination with complete pivoting

The goal of Gaussian Elimination with complete pivoting is the same as Gaussian Elimination however each iteration includes additional steps.

Pivot Selection

At each step k of the elimination, the pivot element is chosen as the largest absolute value element in the submatrix $A_{k:n,k:n}^{(k-1)}$.

Finding the Pivot

Before the first step, the largest element in the entire matrix $A^{(0)}$ is found. Indices p and q are determined such that:

$$\left|a_{pq}^{(0)}\right| = \max_{i=1,\dots,n; j=1,\dots,n} \left|a_{ij}^{(0)}\right|$$

Before the k-th step, the largest element in the submatrix $A_{k:n,k:n}^{(k-1)}$ is found. Indices p and q are determined such that:

$$\left|a_{pq}^{(k-1)}\right| = \max_{i=k,\dots,n; j=k,\dots,n} \left|a_{ij}^{(k-1)}\right|$$

Row and Column Swapping

Once the pivot $a_{pq}^{(k-1)}$ is identified, swap the p-th row with the k-th row and the q-th column with the k-th column. This brings the pivot element to the k-th position on the main diagonal.

Elimination Phase

After selecting and positioning the pivot, perform the Gaussian elimination step as usual: For each row i (where $i > k$), update the row as:

$$r_i := r_i - l_{ik} r_k$$

where

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$$

This creates zeros below the pivot in column k.

4 Calculating the determinate

After transforming the matrix into upper triangular form, the determinant can be simply calculated by multiplying the elements located on the main diagonal as:

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

where a_{ii} are the elements on the main diagonal of the upper triangular matrix.
However, in GECP we need to keep track of the number of swaps during GECP to adjust the sign. If the number of swaps is odd the determinate must be multiplied by -1. Otherwise, the determinant sign remains unchanged if the number of swaps is even.

5 Calculating the Growth Factor

The growth factor for Gaussian Elimination (GE) and Gaussian Elimination with Complete Pivoting (GECP) is a measure of the stability of these methods. It is defined as the ratio of the largest element in the matrix after applying GE or GECP to the largest element in the original matrix. To calculate the growth factor, we follow these steps:

1. Identify the largest absolute value in the original matrix A, denoted as $\max_{i,j} |a_{ij}|$.
2. Perform GE or GECP on A and after each iteration find the largest element.
3. Then the formula of growth factor can be calculated using this formula:
   $\rho_n = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$
   where the $a_{ij}^{(k)}$ are the elements at the start of the k-th stage of Gaussian elimination.

6 Testing

Figure 1. shows the average error and the max error in the computation for random matrices of sizes from 20 to 500. For each matrix size, we created 10 matrices to test our function.
Figure 2. shows a maximal number of digits before the coma of a determinate of random matrices of sizes from 20 to 500. For each matrix size, we created 10 matrices to test our function. Figure 3. shows the average run time and the max run time of our functions for random matrices of sizes from 20 to 500. For each matrix size, we created 10 matrices to test our function
Figures 4, 5, and 6 show the same things as Figures 1, 2, and 3 but for row dominant matrices of sizes from 20 to 110 matrices (the reason for changing the size is the fact that the built-in Matlab function couldn't handle matrices of bigger size for row dominant

matrices)
Figures 7, 8, and 9 show the same things as Figures 1, 2, and 3 but for col dominant matrices of sizes from 20 to 110 matrices (the reason for changing the size is the fact that the built-in Matlab function couldn't handle matrices of bigger size for col dominant matrices)
Figure 10. shows the determinate calculated using the build-in Matlab function det() (true determinate), the determinate calculated using Gaussian Elimination (GE determinate), and the determinate calculated using Gaussian Elimination with Complete Pivoting (GECP determinate) for the specific matrices, for example, a row or a column dominate matrices.
Figure 11 shows the error between the true determinate and the determined calculated either using GE or GECP. This figure also shows the growth factor of GE and GECP.

We are confident that our algorithm computes the determinate correctly, as it has a very low error rate compared to the number of digits of the determinate. We also performed a test to see if the function return correct errors. To do that, a not-square matrix was used and matrices for which there would be division by zero during iterations.

N	average error GE	average error GECP	max error GE	max error GECP
20	2.56357435279853e-15	9.04224611852911e-17	1.11759559939806e-14	2.84494650060196e-16
30	4.27083368670367e-13	1.25510712933874e-14	2.62367905179417e-12	4.52970994047064e-14
40	1.02406616520057e-10	4.52189397037728e-12	4.76575223729014e-10	2.36468622460961e-11
60	0.000101525336503983	3.28496098518372e-06	0.000237941741943359	1.23083591461182e-05
80	36096.5625	838.3	189440	3200
100	6132381148774.4	228720430284.8	18979460481024	927712935936
200	1.98006525290664e+68	2.55464582524117e+66	7.45821649596722e+68	1.03206045835186e+67
300	7.73248588082637e+134	8.24580703060898e+132	2.44509540577861e+135	2.48431204593225e+133
400	7.16060931034408e+208	8.06787294665536e+205	4.13389877514764e+209	3.4237315681522e+206
500	1.18035586631561e+287	2.4434054835158e+285	7.78934308619455e+287	2.10319953992826e+286

Figure 1: Average error and max error for GE and GECP for random matrices of size N

N	number of digits in det
20	0
30	2
40	4
60	10
80	18
100	27
200	82
300	148
400	220
500	299

Figure 2: Maximal number of digits before coma of determinate of a random matrices of size N

N	average time GE(s)	average time GECP(s)	max time GE(s)	max time GECP(s)
20	0.0003411	0.000477	0.001815	0.002073
40	0.0003891	0.0006803	0.000534	0.000784
60	0.0009201	0.0015407	0.001235	0.00199
80	0.0034147	0.0049944	0.014957	0.017448
100	0.0032115	0.0047415	0.004036	0.006706
200	0.0309884	0.0459617	0.036263	0.052122
300	0.0897549	0.1287019	0.126637	0.181845
400	0.170391	0.2450596	0.197976	0.306011
500	0.2901536	0.385908	0.357941	0.41746
600	0.4792449	0.6558253	0.547288	0.78235

Figure 3: Average time and max time for GE and GECP for random matrices of size N

N	average error GE	average error GECP	max error GE	max error GECP
20	11468.8	92569.6	32768	360448
30	2.39807672958224e+19	3.93837985973699e+20	7.37869762948382e+19	1.91846138366579e+21
40	3.32306998946229e+36	1.78448858434125e+37	2.12676479325587e+37	6.11444878061061e+37
50	4.59748662259767e+53	1.42522085300528e+55	3.83123885216472e+54	4.59748662259767e+55
60	2.75628142105428e+73	5.6892475485864e+73	7.77412708502489e+73	2.12021647773406e+74
70	1.74696205330446e+93	4.1783681946573e+93	3.91110907456221e+93	9.38666177894931e+93
80	1.33712862824679e+113	3.24181545121703e+113	5.38699303466331e+113	1.50066234537049e+114
90	9.28422901736967e+133	6.33144669991876e+133	2.9527823174509e+134	1.70352826006783e+134
100	1.25027808946715e+155	1.87038920622699e+155	2.21228830844053e+155	8.04468475796556e+155
110	3.00753768191295e+176	6.93316581409407e+176	7.59798993325377e+176	2.72261305941593e+177

Figure 4: Average error and max error for GE and GECP for row dominate matrices of size N

N	number of digits in det
20	21
30	36
40	53
50	71
60	89
70	109
80	129
90	150
100	171
110	192

Figure 5: Maximal number of digits before coma of determinate of a row dominate matrices of size N

N	average time GE(s)	average time GECP(s)	max time GE(s)	max time GECP(s)
20	0.0001665	0.000266	0.000308	0.000369
30	0.0002047	0.0003863	0.000245	0.000413
40	0.0003443	0.000664	0.000385	0.000779
50	0.0005269	0.000902	0.000554	0.000959
60	0.0008584	0.0013664	0.001131	0.001589
70	0.0011409	0.0018947	0.001407	0.002123
80	0.0015666	0.0025457	0.001875	0.00322
90	0.0021149	0.0034266	0.002556	0.004885
100	0.0035343	0.005432	0.003932	0.006351
110	0.0039732	0.0058099	0.004173	0.006401

Figure 6: Average time and max time for GE and GECP for row dominate matrices of size N

N	average error GE	average error GECP	max error GE	max error GECP
20	11468.8	90931.2	32768	278528
30	2.21360928884515e+19	3.50488137400481e+20	1.47573952589676e+20	1.10680464442257e+21
40	1.06338239662793e+36	1.45550465538448e+37	7.9753679747095e+36	4.25352958651173e+37
50	3.37149018990496e+54	1.73555120003062e+55	2.45199286538542e+55	1.0420969677888e+56
60	3.95773742510358e+73	7.95081179150273e+73	9.1876047368476e+73	2.4029120080986e+74
70	2.15762850613349e+93	2.32059138424025e+93	1.04296241988326e+94	1.01688835938618e+94
80	2.3568094526652e+113	3.70355771133102e+113	1.23131269363733e+114	8.46527476875663e+113
90	6.64376021426453e+133	9.14226832903068e+133	1.58995970939664e+134	3.29348796946447e+134
100	1.83686968640214e+155	2.29273515602018e+155	5.63127933057589e+155	4.55865469618048e+155
110	5.87261305257739e+176	1.0906281383358e+177	1.2663316555423e+177	3.54572863551843e+177

Figure 7: Average error and max error for GE and GECP for col dominate matrices of size N

N	number of digits in det
20	21
30	36
40	53
50	71
60	89
70	109
80	129
90	149
100	171
110	192

Figure 8: Maximal number of digits before coma of determinate of a col dominate matrices of size N

N	average time GE(s)	average time GECP(s)	max time GE(s)	max time GECP(s)
20	0.0001694	0.0002766	0.000286	0.000366
30	0.0002089	0.0003859	0.000247	0.000453
40	0.0003349	0.0006226	0.000366	0.000802
50	0.0005681	0.0009733	0.000664	0.001087
60	0.0008005	0.0013331	0.000894	0.001415
70	0.0011207	0.001825	0.001175	0.001958
80	0.0015868	0.0024816	0.001722	0.00275
90	0.0021102	0.003198	0.002281	0.003481
100	0.0027042	0.0040755	0.00298	0.004418
110	0.0035231	0.0054754	0.003657	0.006903

Figure 9: Average time and max time for GE and GECP for col dominate matrices of size N

Matrix	True determinate	GE determinate	GECP determinate
$\begin{bmatrix} -5 & 5 & -67 & 15 & 23 & -25 & 48 & -66 \\ 16 & -34 & 83 & -67 & 29 & 72 & -11 & 93 \\ -17 & 76 & -10 & 13 & -58 & 94 & -49 & 80 \\ 1 & -43 & 99 & -7 & 35 & -25 & 17 & -88 \\ -14 & 20 & -6 & 6 & -52 & 66 & -97 & 71 \\ 29 & -21 & 13 & -32 & 9 & -68 & 10 & -55 \\ -28 & 30 & -59 & 65 & -22 & 12 & -30 & 12 \\ 17 & -19 & 81 & -14 & 50 & -9 & 33 & -65 \end{bmatrix}$	23936002070947	23936002070947	23936002070947
$\begin{bmatrix} 10 & -2 & 1 & -1 & 2 & 8 \\ -3 & 12 & 2 & -1 & 3 & -2 \\ 2 & -1 & 14 & 3 & -2 & -1 \\ -1 & 3 & -2 & 13 & 1 & 2 \\ 2 & -3 & -1 & 2 & 11 & -1 \\ -2 & 1 & -3 & -1 & 1 & 20 \end{bmatrix}$	5275450	5275450	5275450
$\begin{bmatrix} 38 & 12 & -5 & -7 & 7 & -6 \\ 10 & 52 & 6 & -12 & 20 & -1 \\ 2 & -1 & 90 & 12 & -30 & -41 \\ -5 & 7 & -12 & 130 & 56 & 48 \\ -8 & -3 & -8 & 2 & 30 & -5 \\ -4 & 1 & -23 & -14 & 1 & 60 \end{bmatrix}$	31907721826	31907721826	31907721826

Figure 10: Determinate calculated using different methods

Matrix	Growth GE	Growth GECP	Error GE	Error GECP
$\begin{bmatrix} -5 & 5 & -67 & 15 & 23 & -25 & 48 & -66 \\ 16 & -34 & 83 & -67 & 29 & 72 & -11 & 93 \\ -17 & 76 & -10 & 13 & -58 & 94 & -49 & 80 \\ 1 & -43 & 99 & -7 & 35 & -25 & 17 & -88 \\ -14 & 20 & -6 & 6 & -52 & 66 & -97 & 71 \\ 29 & -21 & 13 & -32 & 9 & -68 & 10 & -55 \\ -28 & 30 & -59 & 65 & -22 & 12 & -30 & 12 \\ 17 & -19 & 81 & -14 & 50 & -9 & 33 & -65 \end{bmatrix}$	5.3333	1.6846	0.0078	0.0234
$\begin{bmatrix} 10 & -2 & 1 & -1 & 2 & 8 \\ -3 & 12 & 2 & -1 & 3 & -2 \\ 2 & -1 & 14 & 3 & -2 & -1 \\ -1 & 3 & -2 & 13 & 1 & 2 \\ 2 & -3 & -1 & 2 & 11 & -1 \\ -2 & 1 & -3 & -1 & 1 & 20 \end{bmatrix}$	1.08	1	0	0
$\begin{bmatrix} 38 & 12 & -5 & -7 & 7 & -6 \\ 10 & 52 & 6 & -12 & 20 & -1 \\ 2 & -1 & 90 & 12 & -30 & -41 \\ -5 & 7 & -12 & 130 & 56 & 48 \\ -8 & -3 & -8 & 2 & 30 & -5 \\ -4 & 1 & -23 & -14 & 1 & 60 \end{bmatrix}$	1.0209	1	0	$3.81e-06$

Figure 11: Errors and Growth Factors (Growth) of GE and GECP