Measures of closeness to cordiality for graphs

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Abstract

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1 Introduction

Let $f : V(G) \to \{0,1\}$, and let $\overline{f} : E(G) \to \{0,1\}$ be the induced mapping given by

$$\overline{f}(xy) = |f(x) - f(y)|.$$

We say that f is a *cordial* labelling of G if the number of vertices labelled 0 and the number of vertices labelled 1 differ by at most 1, and the number of edges labelled 0 and the number of edges labelled 1 differ at most by 1.

For $i \in \{0,1\}$, let $v_i(f)$ denote the number of vertices labelled i and let $e_i(f)$ denote the number of edges labelled i. Let

$$\Delta_v(f) = |v_0(f) - v_1(f)| \;\text{ and }\; \Delta_e(f) = |e_0(f) - e_1(f)|.$$

We introduce two measures of closeness to cordiality of a graph.

$$\mathscr{D}_1(G) = \min\Big\{\Delta_v(f) + \Delta_e(f) | f : V(G) \to \{0,1\}\Big\}. \tag{1}$$

$$\mathscr{D}_2(G) = \min\Big\{\Delta_e(f) | f : V(G) \to \{0,1\}, \Delta_v(f) \le 1\Big\}. \tag{2}$$

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2 The Tree

Theorem 2.1. *Let $\mathcal{T}_n$ denote a tree with n vertices.*

$$\mathscr{D}_1\big(\mathcal{T}_n\big) = 1 \quad \textit{and} \quad \mathscr{D}_2\big(\mathcal{T}_n\big) = \begin{cases} 0 & \textit{if } n \equiv 1 \pmod 2 \\ 1 & \textit{if } n \equiv 0 \pmod 2 \end{cases}$$

3 The complete graphs $\mathcal{K}_n$

Theorem 3.1. *If $a^2 \le n < (a+1)^2$, then*

$$\mathscr{D}_1\big(\mathcal{K}_n\big) = \begin{cases} a + \frac{1}{2}(n - a^2) & \textit{if } n \in \{a^2 + 2t : 0 \le t \le a\}; \\ 2a - 1 & \textit{if } n = a^2 + 1; \\ a + 1 + \frac{1}{2}\big((a+1)^2 - n\big) & \textit{if } n \in \{a^2 + 2t + 1 : 1 \le t \le a - 1\}. \end{cases}$$

$$\mathscr{D}_2\big(\mathcal{K}_n\big) = \left\lfloor \tfrac{n}{2} \right\rfloor.$$

4 The complete r-partite graphs $\mathcal{K}_{n_1,\ldots,n_r}$

Theorem 4.1. *If $n_1, \ldots, n_r$ are positive integers of which s are odd, then*

$$\mathscr{D}_1\big(\mathcal{K}_{n_1,\ldots,n_r}\big) \le \begin{cases} \frac{1}{2}\left(s + 1 - (2a-1)^2\right) & \textit{if } (2a)^2 \le s \le (2a+1)^2 \textit{ and } s \equiv 0 \pmod 2 \\ \frac{1}{2}\left(s + 1 - (2a-2)^2\right) & \textit{if } (2a)^2 \le s \le (2a+1)^2 \textit{ and } s \equiv 1 \pmod 2 \\ \frac{1}{2}\left(s + 1 - (2a-1)^2\right) & \textit{if } (2a+1)^2 \le s \le (2a+2)^2 \textit{ and } s \equiv 0 \pmod 2 \\ \frac{1}{2}\left(s + 1 - 4a^2\right) & \textit{if } (2a+1)^2 \le s \le (2a+2)^2 \textit{ and } s \equiv 1 \pmod 2 \end{cases}$$

$$\mathscr{D}_2\big(\mathcal{K}_{n_1,\ldots,n_r}\big) = \left\lfloor \tfrac{s}{2} \right\rfloor.$$

5 The cycles $\mathcal{C}_n$

Theorem 5.1.

$$\mathscr{D}_1\big(\mathcal{C}_n\big) = \begin{cases} 0 & \textit{if } n \equiv 0 \pmod 4; \\ 2 & \textit{if } n \not\equiv 0 \pmod 4. \end{cases} \quad \text{and} \quad \mathscr{D}_2\big(\mathcal{C}_n\big) = \begin{cases} 0 & \textit{if } n \equiv 0 \pmod 4; \\ 1 & \textit{if } n \equiv 1, 3 \pmod 4; \\ 2 & \textit{if } n \equiv 2 \pmod 4. \end{cases}$$

Corollary 5.1. *An Eulerian graph G with $4k+2$ edges is not cordial for all $k \in \mathbb{Z}^+$.*

6 The wheel graphs $\mathcal{W}_n$

Theorem 6.1.

$$\mathscr{D}_1(\mathcal{W}_n) = \begin{cases} 0 & \textit{if } n \equiv 2 \pmod 4; \\ 1 & \textit{if } n \equiv 1,3 \pmod 4; \\ 2 & \textit{if } n \equiv 0 \pmod 4. \end{cases} \quad \text{and} \quad \mathscr{D}_2(\mathcal{W}_n) = \begin{cases} 0 & \textit{if } n \not\equiv 0 \pmod 4; \\ 2 & \textit{if } n \equiv 0 \pmod 4. \end{cases}$$

7 The fan graphs $\mathcal{F}_{m,n}$

Theorem 7.1.

$$\mathscr{D}_1(\mathcal{F}_{m,n}) = \begin{cases} 1 & \textit{if } n \equiv 0,2 \pmod 4 \textit{ and } m \equiv 0,2 \pmod 4; \\ 2 & \textit{if } n \equiv 0,2 \pmod 4 \textit{ and } m \equiv 1,3 \pmod 4; \\ 1 & \textit{if } n \equiv 1,3 \pmod 4 \textit{ and } m \equiv 0,2 \pmod 4; \\ 1 & \textit{if } n \equiv 1,3 \pmod 4 \textit{ and } m \equiv 1,3 \pmod 4. \end{cases} \tag{3}$$

$$\mathscr{D}_2(\mathcal{F}_{m,n}) = \begin{cases} 1 & \textit{if } n \equiv 0,2 \pmod 4 \textit{ and } m \equiv 0,2 \pmod 4; \\ 1 & \textit{if } n \equiv 0,2 \pmod 4 \textit{ and } m \equiv 1,3 \pmod 4; \\ 0 & \textit{if } n \equiv 1,3 \pmod 4 \textit{ and } m \equiv 0,2 \pmod 4; \\ 1 & \textit{if } n \equiv 1,3 \pmod 4 \textit{ and } m \equiv 1,3 \pmod 4. \end{cases} \tag{4}$$

8 Bound on $\mathscr{D}_1(G_1 \vee G_2)$ and $\mathscr{D}_2(G_1 \vee G_2)$

Theorem 8.1. *$G_1 \vee G_2$ is a graph join between G_1 and G_2. It is defined as $G_1 \vee G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2) \cup V(G_1) \times V(G_2))$. Then,*

$$\mathscr{D}_1(G_1 \vee G_2) \leq \mathscr{D}_1(G_1) + \mathscr{D}_1(G_2) + \mathscr{D}_1(G_1)\mathscr{D}_1(G_2) \tag{5}$$

$$\mathscr{D}_2(G_1 \vee G_2) \leq \mathscr{D}_2(G_1) + \mathscr{D}_2(G_2) + 1 \tag{6}$$