A1:Integration

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13th September 2023

1.

We will use this function for the sub-questions,

$$f(x) = e^{-x^2}$$

a. $[-\infty, \infty] \to [-1, 1]$

$$I = \int_{-\infty}^{\infty} f(x)dx$$

let, $y = \frac{2}{1+e^{-x}} - 1$

$\therefore x = -log(\frac{2}{y+1-1})$ and $dx = \frac{-1}{\frac{2}{y+1}-1}dy$

$$\therefore I = \int_{y=1}^{y=-1} e^{-[-log(\frac{2}{y+1}-1)]^2} \frac{-1}{\frac{2}{y+1}-1}dy$$

$$\therefore I = \int_{y=-1}^{y=1} e^{-[-log(\frac{2}{y+1}-1)]^2} \frac{1}{\frac{2}{y+1}-1}dy$$

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b. $[0, \infty] \to [0, 1]$

$$I = \int_{0}^{\infty} f(x)dx$$

Let, $y = e^{-x}$
so that, $x = -log(y)$ and $dx = \frac{-1}{y}dy$ substituting these values,

$$I = \int_{y=1}^{y=0} e^{-(-log(y))^2} (\frac{-1}{y}dy)$$

$$I = \int_{y=0}^{y=1} e^{-(-log(y))^2} (\frac{1}{y}dy)$$

---

c. $[0, \infty] \rightarrow [-1, 1]$

$$I = \int_0^\infty f(x)dx$$

Let, $y = 1 - 2e^{-x}$

$\therefore x = -log(\frac{1-y}{2})$ and $dx = \frac{-2}{1-y}dy$

$$I = \int_{y=-1}^{y=1} e^{-[-log(\frac{1-y}{2})]} \frac{-2}{1-y} dy$$

---

2. Direct MC in High Dimensions

In [1]:
```
import numpy as np
from scipy.special import gamma
import matplotlib.pyplot as plt
from typing import Union, Final
from IPython.display import Markdown, display
import warnings
warnings.filterwarnings("error")
```

Volume of HyperSphere

$$Vs = \frac{\pi^n/2}{\Gamma(n/2+1)} R^n$$

In [2]:
```
def Volume_of_HyperSphere(dimensions:int,radius:Union[int,float]) -> Union[int,float]:
    """Calculate the volume of hypersphere

    Args:
        dimensions (int): Number of dimensions
        radius (int or float): Radius of the sphere

    Returns:
        int or float: The Volume of Hypersphere
    """
    return ((np.pi**(dimensions/2))/gamma((dimensions/2)+1))*(radius**dimensions)
```

Volume of HyperCube

$$Vc = L^n$$

In [3]:
```
def Volume_of_HyperCube(dimensions:int,length:Union[int,float]) -> Union[int,float]:
    """Calculate the volume of hypercube

    Args:
        dimensions (int): Number of dimensions
        radius (int or float): Radius of the sphere

    Returns:
        int or float: The Volume of Hypersphere
    """
    return length**dimensions
```

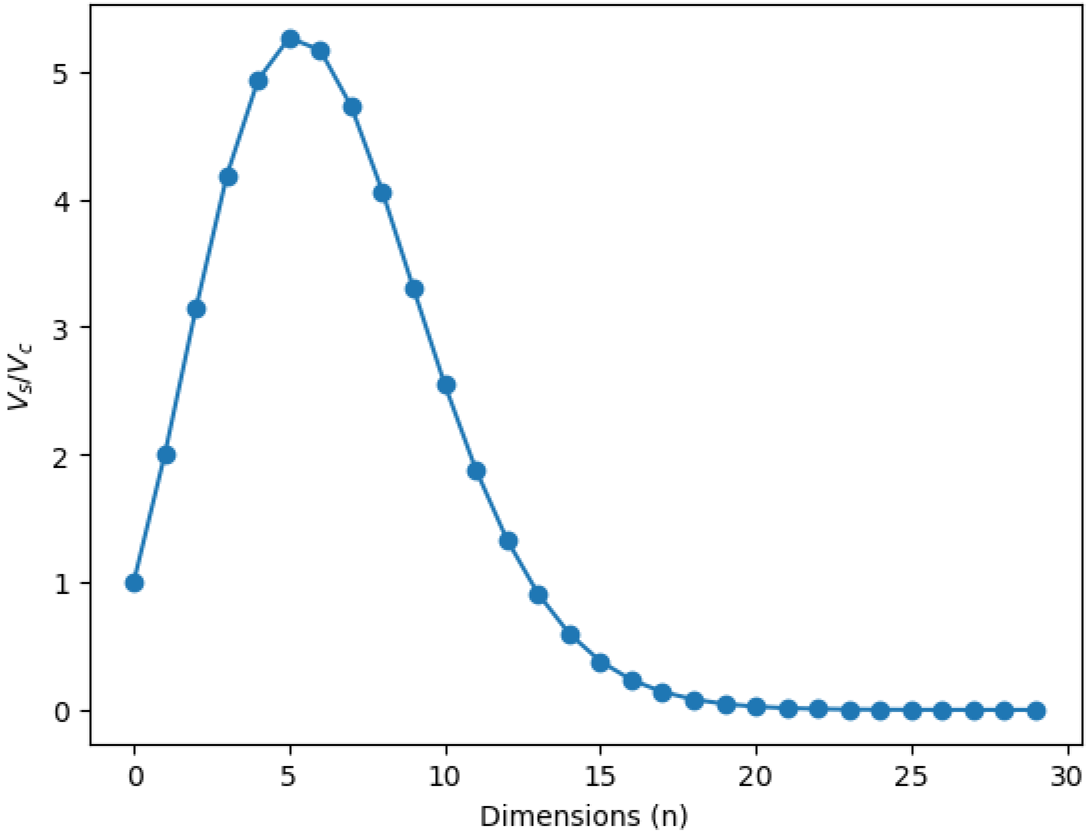
Plotting the ratio of volumes (V_s/V_c) w.r.t. dimensions (n)

In [4]:
```
def plot_ratio(radius:Union[int,float],dimensions:int) -> None:
    """Plot the ratio as a function of n

    Args:
        radius (int or float): Radius of the sphere
        dimensions (int): Number of dimensions

    Returns:
        None
    """
    try:
        dimensions_to_simulate = np.arange(0,dimensions,1)
        plt.plot(dimensions_to_simulate,
                 Volume_of_HyperSphere(dimensions_to_simulate,2)/Volume_of_HyperCube(dim
                 marker='o')
        plt.xlabel('Dimensions (n)')
        plt.ylabel('$V_s / V_c$')
        plt.show()
    except:
        display(Markdown('## Something went wrong!\n### Please check the values entered.
```

In [5]:
```
plot_ratio(1,30)
```



V_s/V_c
Dimensions (n)
0
1
2
3
4
5
0
5
10
15
20
25
30

3. Basic Integration

Using Monte Carlo integration to find the volume of a right circular cone of height H = 2, whose base is a unit circle centered at (0,0)

$$I = \int_{-1}^{1}\int_{-1}^{1} g(x_1, x_2)dx_1dx_2,$$

where,

$$g(x_1, x_2) = \begin{cases} H(1 - \sqrt{x_1^2 + x_2^2}), \text{ if } x_1^2 + x_2^2 \leq 1 \\ 0, \text{ otherwise.} \end{cases}$$

Since the domain size is 4, we need to multiply $\bar{g}$ by 4

In [6]:

```
# Declaring a constant value of height.
HEIGHT:Final[int] = 2
TRUE_VOLUME = np.pi*HEIGHT/3
```

In [7]:

```
def MC_method(number_of_points:int) -> float:

    try:
        # Generating random number between -1 to 1.
        x1 = np.random.uniform(-1,1,size=number_of_points)
        x2 = np.random.uniform(-1,1,size=number_of_points)

        # Defining the condition.
        condition = x1**2 + x2**2 <= 1

        # Creating an array which will hold the values of g.
        g = np.zeros(number_of_points)

        # Calculating the height for the points which satisfy the condition.
        for i,x in enumerate(condition):
            g[i] = HEIGHT*(1-np.sqrt(x1[i]**2+x2[i]**2)) if x else 0

        # Calculating the mean of g and multiplying it by domain size.
        return np.sum(g)*4/float(number_of_points)
    except:
        display(Markdown('## Something went wrong!'))
```

Calculating the volume of cone by using Monte Carlo method

In [8]:

```
MC_result = MC_method(1000)
display(Markdown(f'Volume using n = 1000 is <b>{MC_result}<b/>.'))
display(Markdown(f'The difference between True Volume ( {TRUE_VOLUME} ) <br/>and the vol
```

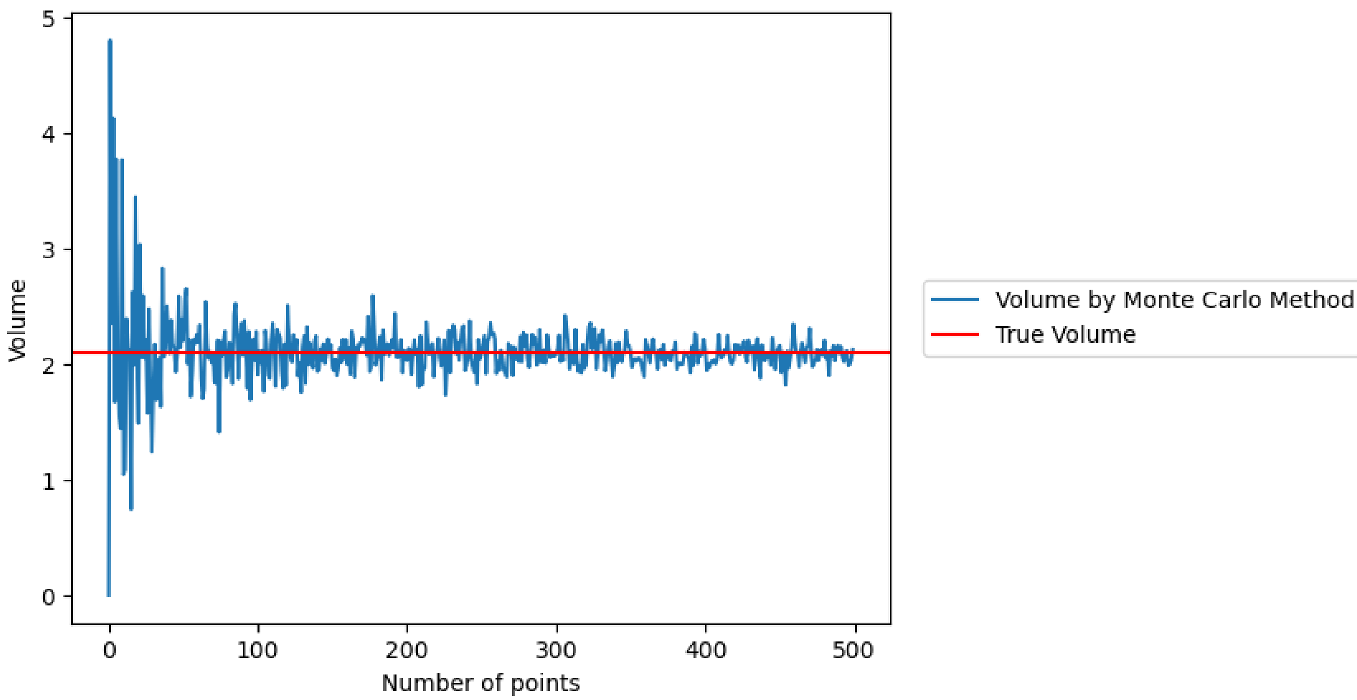
Volume using n = 1000 is **2.1177923096181495.**

The difference between True Volume (2.0943951023931953)
and the volume calculated by Monte Carlo method is 0.023397207224954197.

Observing effects of increasing the n in MC_method on the Volume.

In [9]:
```
def plot(iterations:int = 100) -> None:
    MC_results = np.zeros(iterations)
    for i in range(1,iterations):
        MC_results[i] = MC_method(i)
    plt.plot(MC_results,label='Volume by Monte Carlo Method')
    plt.axhline(TRUE_VOLUME,color='r',label='True Volume')
    plt.ylabel('Volume')
    plt.xlabel('Number of points')
    plt.legend(bbox_to_anchor=(1.04,0.5), loc="center left", borderaxespad=0)
    plt.show()
```

In [10]:
```
plot(500)
```



It can be seen from the above graph that the volume calculated by the Monte Carlo method gets closer to the true volume as the number of point increases.