Applied Computational Science

Homework: Linear Algebra.

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Q.1 a)

To show A^TA symmetric,

$$A^T\mathrm{A} = (A^TA)^T$$

$$\mathrm{A}^\mathrm{T}\left(\mathrm{A}^\mathrm{T}\right)^\mathrm{T} = \mathrm{A}^\mathrm{T}\,\mathrm{A}$$

Since transpose of the term A^TA is gives the original matrix as result.

For Positive Semi-definite,

We need to show $x^T\mathrm{Ax} \geq 0$,

$$x^T(A^TA)\mathrm{x} = (Ax)^T(Ax)$$

$$\left||Ax|\right|^2$$

Since square of any real number is always positive.

$$\left||Ax|\right|^2 \geq 0$$

To demonstrate that A^TA is positive definite,

Assume A is invertible,

$$x^T(A^TA)x = (Ax)^T(Ax) = \|Ax\|^2$$

And, we know, $\mathrm{Ax} \neq 0$ for $\mathrm{x} \neq 0$. Therefore, $\left||Ax|\right|^2 > 0$ which means $A^T\mathrm{A}$ is positive definite.

Also, let us assume that $A^T\mathrm{A}$ is positive definite, then for a non-zero vector x,

$$x^T(A^TA)\mathrm{x} > 0$$

If A was invertible, then the value of x should be 0, which is not possible since x is a non-zero vector, A must be invertible.

Q.1 b)

The given matrix is,

$$\mathrm{H} = \mathrm{I} - \left(\frac{2}{x^T x}\right) \mathrm{x}\boldsymbol{x^T}$$

Transpose of H will be,

$$H^T = \left(I - \frac{2}{x^T x} x x^T\right)^T$$

$$H^T = I^T - \left(\frac{2}{x^T x} x x^T\right)^T$$

$$H^T = I - \frac{2}{x^T x} x x^T$$

Since, xx^T is symmetric, the term $\frac{2}{x^T x} x x^T$ will be symmetric, and subtracting a symmetric matrix from the identity matrix results in a symmetric matrix. Therefore,

$$H^T = H$$

Since H is orthogonal, its inverse will be equal to the transpose.

$$H^{-1} = H^T$$

Therefore,

$$H^{-1} = I - \frac{2}{x^T x} x^T x$$

Q.1 c)

If A is symmetric then,

$$A = A^T$$

$$A^{-1} = (A^T)^{-1}$$

$$A^{-1} = (A^{-1})^T$$

Hence, A^{-1}will also be symmetric.

Q2. a)

Let A be a strictly upper triangular matrix of size $n * n$,

$$\mathrm{A} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The eigenvector of A will be,

$$\mathrm{v} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Where, x_1 is non-zero.

Q2. B)

The equation relating $B, X \; and \; \Lambda$, where X is an orthogonal matrix, can be given by,

$$B = X\Lambda X^{-1}$$

This method is much cheaper for computing B^{100} than carrying out 99 matrix multiplications.

So to compute B^{100},

$$B^{100} = X\Lambda^{100}X^{T}$$

Q2. C)

If C is an $n * n$ upper triangular matrix of 1's, to solve the linear system $Cx = b$, in which b is given and x is unknown, we can find the values of x by,

$$x_1 = b_1$$

$$x_2 = b_2 - x_1$$

$$x_3 = b_3 - x_2$$

And so on …,

$$x_i = b_i - \sum_{j=1}^{i-1} b_j x_{j-1}$$

Q.3 a)

We have the matrix,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

For A^2,

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

For A^3,

$$A^3 = A^2.A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For $n = 3$, $A^{n-2} + A^2 - I$ will be,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Which is equal to A^3, hence $A^n = A^{n-2} + A^2 - I$ holds true for $n = 3$.

For $n \geq 3$,

We know that,

$$A^n = A^{n-1}.A$$

Assuming the given relation is true,

$$A^n = (A^{n-3} + A^2 - I).A$$

$$A^n = A^{n-2} + A^3 - A$$

We know the value of A^3,

$$A^n = A^{n-2} + (A + A^2 - I) - A$$

$$A^n = A^{n-2} + A^2 - I$$

Therefore, the given relation holds true for $n \geq 3$.

Q.3 b)

Value of A^{100}will be,

$$\begin{aligned} A^{100} &= A^{98} + A^2 - I \\ &= (A^{96} + A^2 - I) + A^2 - I \\ &= A^{96} + 2A^2 - 2I \\ &\vdots \\ &= A^2 + 49A^2 - 49I \\ &= 50A^2 - 49I \end{aligned}$$

Since we know A^2,

$$A^{100} = 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 50 & 0 & 0 \\ 50 & 50 & 0 \\ 50 & 0 & 50 \end{bmatrix} - \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 0 & 50 & 1 \end{bmatrix}$$