Basic Fraunhofer PSF theory

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1. Introduction

We look at Fraunhofer-approximation image formation theory, with a focus on high Strehl ratio point-spread function (PSF) found in laboratory testbeds and high order adaptive optics systems. We show how one can understand the idiosyncracies of a well-corrected PSF by inspecting the Fourier transform of the phase aberrations over the aperture. Speckles are found to be pinned to bright Airy rings, typically near the core, and halo speckles are predominantly symmetric. A power series expansion for the PSF is developed analytically and terms relevant to the high Strehl ratio PSF are examined. We demonstrate how some of these features affect the way the unaberrated PSF shape interacts with residual phase aberrations. Furthermore, with the underlying PSF suppressed by a coronagraph, residual speckles posses a high degree of symmetry.

2. Monochromatic scalar wave PSF formation

Perfect wave electric field strength:

$$Ee^{i(\kappa z-\omega t)} \tag{1}$$

$\omega/\kappa = c$, wavelength $\lambda = 2\pi/\kappa$. $\mathbf{x} = (x_1, x_2)$ is location in transverse plane, in units of λ.

Phase $\phi(\mathbf{x})$ is a phase difference from the reference phase associated with the wave $Ee^{i(\kappa z-\omega t)}$ at a given value of z and location in the transverse plane.

Aperture function $A(\mathbf{x})$

$$E_A = A(\mathbf{x})Ee^{i(\kappa z-\omega t+\phi(\mathbf{x}))} \tag{2}$$

Drop the common factor $Ee^{i(\kappa z-\omega t)}$.

$$E_A = A(\mathbf{x})e^{i\phi(\mathbf{x})} \tag{3}$$

Image plane with no phase errors (Fraunhofer approximation):

$$E_I = a(\mathbf{k}) \equiv \mathrm{FT}(A(\mathbf{k})), \tag{4}$$

With phase errors:

$$E_I = \mathrm{FT}(Ae^{i\phi(\mathbf{x})}), \tag{5}$$

3. Some comments

Because of the Fourier relationship between pupil and image fields, **k** *is also a spatial frequency vector for a given wavelength of light.*

Phase errors appears in exponent: non-linear relationship induces a many-to-one-ness in this association (frequency mixing or frequency-folding).

Complex-valued field a is called the *amplitude-spread function* (ASF), The PSF, aa^*, is real and non-negative everywhere. [Convention: change the case of a function to indicate its Fourier transform.]

The *Strehl ratio* (SR) of an imaging system is the ratio of the peak intensity of the actual image (with phase aberrations) to the peak of the image one would have obtained without phase aberrations (my def).

4. The PSF Expansion

Speckle pinning — Bloemhof et al. (2001)

First and second order speckle symmetries and properties — Sivaramakrishnan et al. (2002)

Full speckle expansion and more properties, symmetries, pinning — Perrin et al. (2003).

Taylor series expansion of $e^{i\phi}$, then FT for image.

Isolate the role of the FT(pupil) and $\mathrm{FT}(\phi(\mathbf{x})) \equiv \mathbf{\Phi}(\mathbf{k})$.

Spatial frequencies in ϕ mix in a “heterodyne” fashion (well-known in radio) to create structure near the origin.

5. The Second Order PSF Expansion

Aberrated aperture is $Ae^{i\epsilon\phi}$, where $A(\mathbf{x})$, $\phi(\mathbf{x})$, ϵ are real, define ϵ so

$$|\phi(\mathbf{x})|_{\text{max}} \equiv 1. \tag{6}$$

ϵ is the "small quantity" in the Perrin et al. (2003) expansion. The power to which ϵ is raised is the *order* of the term.

$e^{i\epsilon\phi}$ can be expanded in a convergent series for **any** finite value of ϵ:

$$e^{i\epsilon\phi} = 1 + i\epsilon\phi - \epsilon^2\phi^2/2 + \ldots. \tag{7}$$

High Strehl: $\epsilon \ll 1$ (if ϕ not wierd like a delta function).

Remember $\mathrm{FT}(fg) = F{*}G$ so $\mathrm{FT}(\phi\phi^*) = \Phi{*}\Phi^*$... so to second order in ϵ, the PSF is

$$\begin{aligned} p(\Phi) \;=\; & aa^* - i\epsilon\left[a(a^*{*}\Phi^*) - a^*(a{*}\Phi)\right] \\ & +\epsilon^2\left[(a{*}\Phi)(a^*{*}\Phi^*)\right. \\ & \left.-a(a^*{*}\Phi^*{*}\Phi^*) + a^*(a{*}\Phi{*}\Phi)\right]/2, \end{aligned} \tag{8}$$

$\Phi(\mathbf{k})$ the FT wavefront phase error $\phi(\mathbf{x})$, and $a(\mathbf{k})$ is the FT of $A(\mathbf{x})$ Choose ϕ to be zero-mean over aperture w.l.g. so $\Phi(\mathbf{0}) = 0$.

6. First order speckle pinning

$$
\begin{aligned}
p_0 &= aa^* && (9)\\
p_1 &= -i\varepsilon\Big[a(a^* * \Phi^*) - a^*(a * \Phi)\Big] && (10)
\end{aligned}
$$

p_1 has $|a|$ as a factor — **pinned** speckle amplified by perfect underlying ASF a.

$$
\begin{aligned}
p_0(\mathbf{0}) &= 1 && (11)\\
p_1(\mathbf{0}) &= 0 && (12)
\end{aligned}
$$

(first, normalization, and second, because a and Φ are Hermitian so p_1 is real, antisymmetric).

Because p_1 is **zero at the origin** so does not change SR.

Integral of p_0 over image plane is 1 (normalization).

Integral of p_1 over image plane is zero. No net power in its speckles.

7. Second order terms

Two second order terms:

$$\begin{aligned} p_{2,\ \mathrm{halo}} &= \varepsilon^2 (a * \Phi)(a^* * \Phi^*) \\ p_{2,\ \mathrm{Strehl}} &= -\varepsilon^2 \left[a(a^* * \Phi^* * \Phi^*) + a^*(a * \Phi * \Phi) \right] / 2. \end{aligned} \quad (13)$$

This truncated expansion for the PSF is valid when the aberration at any point in the pupil is significantly less than a radian.

Exercise for the reader...

$$p_{2,\ \mathrm{halo}}(\mathbf{0}) = 0, \quad (14)$$

$$p_{2,\ \mathrm{Strehl}}(\mathbf{0}) = -\sigma_\phi^2. \quad (15)$$

Halo term **free**, Strehl term **pinned**, reduces peak as per Maréchal ($SR = 1 - \sigma_\phi^2$). Both terms **symmetric**, halo term reflects **power spectrum** of phase aberration.

Integral of $p_2 = p_{2,\ \mathrm{halo}} + p_{2,\ \mathrm{Strehl}}$ over the image plane is zero. No net power in second order speckles.

8. Full expansion

Perrin et al. (2003) derive the infinite convergent series

$$p(\mathbf{k}) = p_0(\mathbf{k}) + p_1(\mathbf{k}) + p_2(\mathbf{k}) + \ldots \tag{16}$$

which describes the PSF p at any Strehl ratio. The n-th order term in the expansion is given by

$$p_n = i^n \varepsilon^n \sum_{r=0}^{n} \frac{(-1)^{n-r}}{r!(n-r)!} (a *^r \Phi)(a^* *^{n-r} \Phi^*). \tag{17}$$

The n-fold convolution operator $*^n$ is defined by *e.g.*,

$$x *^3 y \equiv x * y * y * y, \tag{18}$$

with $x *^0 y \equiv x$.

Integral of each p_n $(n \neq 0)$ over the image plane is zero.

Even terms symmetric, odd terms antisymmetric.

Every order has a pinned term (even p_0).

REFERENCES

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