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The structure of the high Strehl ratio point-spread function

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ABSTRACT

Notes on the expansion of the point-spread function (PSF) in terms of the phase aberrations over the pupil of an imaging optical system. We use general considerations based on Fourier theory, and a power series expansion of the PSF developed by Bloemhof et al. (2001); Sivaramakrishnan et al. (2002); Perrin et al. (2003).

1. Monochromatic PSF theory

A plane monochromatic wave travelling in the z direction in a homogenous medium without loss of energy can be characterized by a complex amplitude E representing the transverse (*e.g.,* electric) field strength of the wave. The full spatio-temporal expression for the field strength is $Ee^{i(\kappa z - \omega t)}$, where $\omega/\kappa = c$, the speed of the wave. We do not use the term *field* to denote image planes — the traditional optics usage — we always use it to denote electromagnetic fields or scalar simplifications of them. The wavelength of the wave is $\lambda = 2\pi/\kappa$. The time-averaged intensity of a wave at a point, taken over one period, $T = 2\pi/\omega$, is proportional to EE^* (* denotes complex conjugation). The phase of the complex number E represents a phase difference from the reference phase associated with the wave $Ee^{i(\kappa z - \omega t)}$ at a given value of z. A real, positive E corresponds to an electric field oscillating in phase with our reference wave. A purely imaginary positive value of E indicates that the electric field lags by a quarter cycle from the reference travelling wave. Transmission through passive, linear filters such as apertures, mirrors, apodizers, as well as the atmosphere itself, is represented by multiplying the field strength by the transmission of these objects which modify the wave. Again, such multiplicative modification is accomplished using complex numbers to represent phase changes forced on the wave reflecting off or passing through such objects.

We assume that Fourier optics describes our astronomical telescope. That is, the image field strength is given by the Fourier transform of the aperture (or pupil — we use the two terms interchangeably) illumination function (a function which which includes the optical aberrations of the system). Our imaging system creates a focussed image by imparting a quadratic phase delay across a flat incoming wavefront (see *e.g.,* Born & Wolf (1999) for the basis of this assumption).

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A telescope aperture is described by a transmission function $A(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2)$ is the location in the aperture, in units of the wavelength of the monochromatic light under consideration. The corresponding aperture illumination describing the electric field strength in the pupil (in response to an unaberrated, unit field strength, monochromatic incident wave) is $E_A = A(\mathbf{x})$ (from this point onwards we drop the common factor $Ee^{i(\kappa z - \omega t)}$ when describing fields).

The field strength in the image plane, $E_I = a(\mathbf{k})$, is the Fourier transform of E_A, where $\mathbf{k} = (k_x, k_y)$ is the image plane coordinate in radians. *Because of the Fourier relationship between pupil and image fields,* $\mathbf{k}$ *is also a spatial frequency vector for a given wavelength of light.*

We refer to this complex-valued field a as the *amplitude-spread function* (ASF), by analogy with the PSF of an optical system (following Krist et al. (2000)). The PSF, aa^*, is real and non-negative everywhere. Our convention is to change the case of a function to indicate its Fourier transform.

The *Strehl ratio* (SR) of an imaging system is the ratio of the peak intensity of the actual image (with aberrations) to the peak of the image one would have obtained if the optics had no aberrations.

2. The PSF Expansion

Here we review the expansion of the PSF in terms of the Fourier transform of the wavefront's phase aberrations over the aperture of the optical system, following Bloemhof et al. (2001); Sivaramakrishnan et al. (2002); Perrin et al. (2003). The latter reference presents a complete derivation, and more discussion.

From here on we write the aperture illumination function as $Ae^{i\varepsilon\phi}$, where $A(\mathbf{x})$ and $\phi(\mathbf{x})$ are real. ε is a 'small' real number which we take to be positive (or zero in the case without any phase aberrations).

At any location in the pupil plane $e^{i\varepsilon\phi}$ can be expanded in an absolutely convergent series for any finite value of ϕ:

$$e^{i\varepsilon\phi} = 1 + i\varepsilon\phi - \varepsilon^2\phi^2/2 + \ldots. \tag{1}$$

Truncating the above expansion above the second order in ε, we calculate the PSF of the image to be

$$\begin{aligned} p(\Phi) \quad = \quad & aa^* - i\varepsilon[a(a^* * \Phi^*) - a^*(a * \Phi)] \\ & + \varepsilon^2 (a * \Phi)(a^* * \Phi^*) \\ & - \varepsilon^2 [a(a^* * \Phi^* * \Phi^*) + a^*(a * \Phi * \Phi)]/2, \end{aligned}$$

where a is the Fourier transform of the real aperture illumination function A, $\Phi(\mathbf{k})$ the Fourier transform of the AO-corrected wavefront phase error $\phi(\mathbf{x})$, and $*$ denotes the convolution operator. This truncated expansion for the PSF is valid when the aberration at any point in the pupil is significantly less than a radian (see Perrin et al. (2003) for further discussion).

Perrin et al. (2003) derive the infinite convergent series $p(\mathbf{k}) = p_0(\mathbf{k}) + p_1(\mathbf{k}) + p_2(\mathbf{k}) + \ldots$ which describes the PSF p at any Strehl ratio. The n-th order term in the expansion is given by

$$p_n = i^n \varepsilon^n \sum_{r=0}^{n} \frac{(-1)^{n-r}}{r!(n-r)!} (a *^r \Phi)(a^* *^{n-r} \Phi^*). \tag{2}$$

The n-fold convolution operator $*^n$ is defined by *e.g.,*

$$x *^3 y \equiv x * y * y * y, \tag{3}$$

with $x *^0 y \equiv x$.

3. Spatially-filtered Wavefront Sensing (SFWFS)

On ground-based AO systems, anti-aliasing SFWFS AO (Poyneer & Macintosh 2004) makes use of optical arrangements that implement approximate Fourier filtering techniques to exclude information outside the wavefront sensing system's spatial frequency bandpass (details on the nature of the approximations involved are discussed in the cited reference, as well as in a discussion of the AO control radius in Perrin et al. (2003)).

We use the framework of spatial frequency (in cycles per aperture diameter) rather than Zernike or Karhunen-Loeve modes, because it is easier to understand the theory of PSF formation in terms of a Fourier decomposition of the wavefront aberration over the telescope aperture. This spatial filtering prior to sensing results in higher accuracy measurement of the wavefront than traditional unfiltered wavefront sensing with the same superficial density of sensing elements on the telescope's aperture. SFWFS AO can produce correspondingly better AO correction, and thus better image quality (all other factor being equal) when Strehl ratios are sufficiently high (approximately 60% or higher) in the sensing wavelength bandpass. Poyneer & Macintosh (2004) showed that, under such circumstances, residual phase aberrations after SFWFS typically show an almost flat power spectrum within the AO control area, followed by a sharp rise in power (per unit spatial frequency). In contrast, wavefront sensing that does not guard against high frequency noise being aliased into the measurement of the aberration produces an uncorrected aberration power spectrum that increases as the square or other low power of the spatial frequency — k^2 for Palomar AO (Sivaramakrishnan et al. 2001) — until a spatial frequency that is more or less the Nyquist frequency defined by the sensor spacing on the aperture.

Spatially-filtered wavefront sensing (SFWFS) results in a non-monotonically decreasing expected value of Φ with spatial frequency: typically there is a bump in the power spectrum of the AO-corrected phase just outside the control radius. With perfect sensing and correction the residual wavefront phase aberration has a power spectrum as shown in Figure 1. This power spectrum has no residual phase aberrations in a spatial frequency band $|k_1| < k_{AO}, |k_2| < k_{AO}$, where k_{AO} is the spatial Nyquist frequency of a square grid of wavefront sensors on the pupil, also loosely known as the 'AO control radius'.

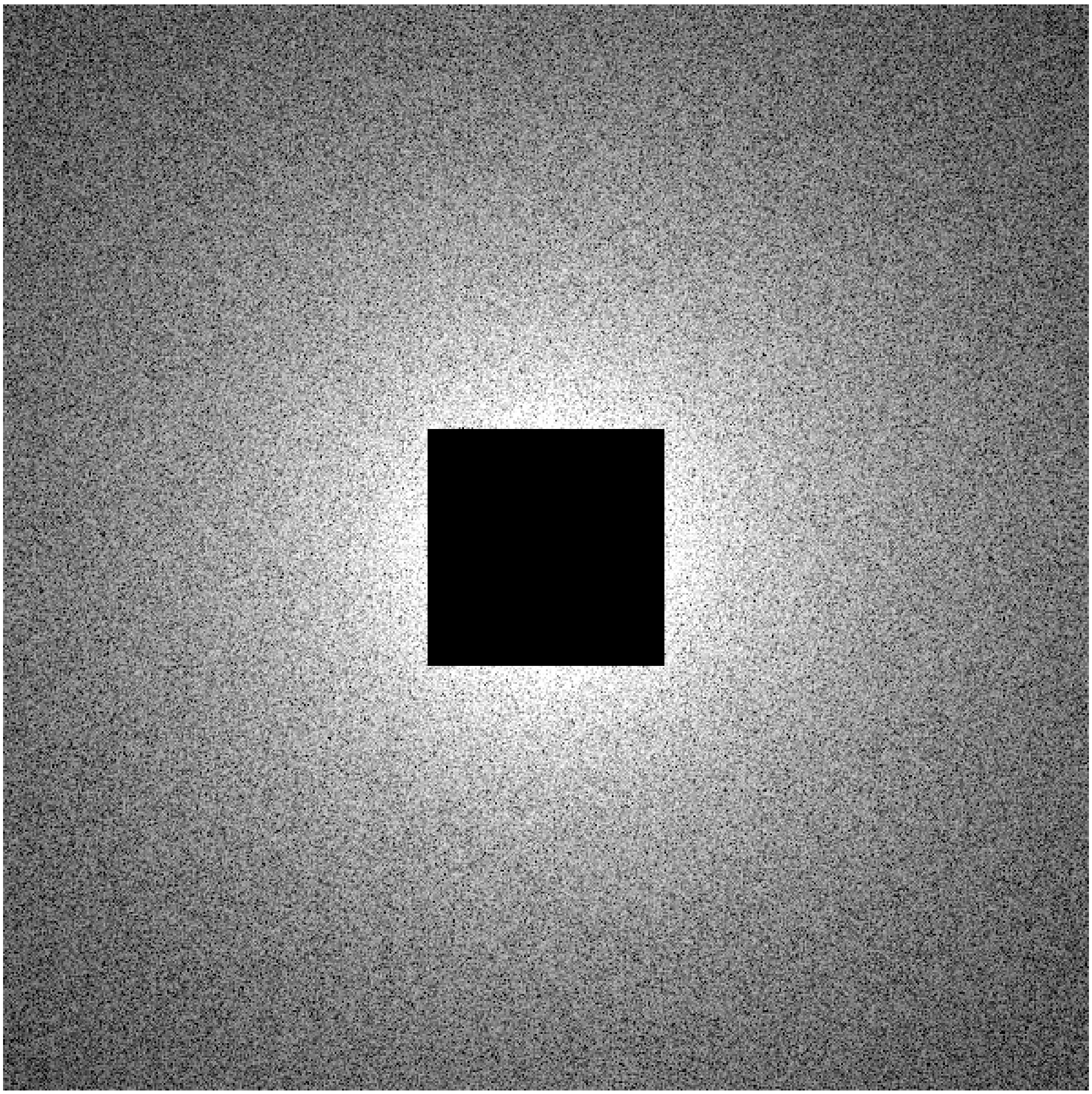
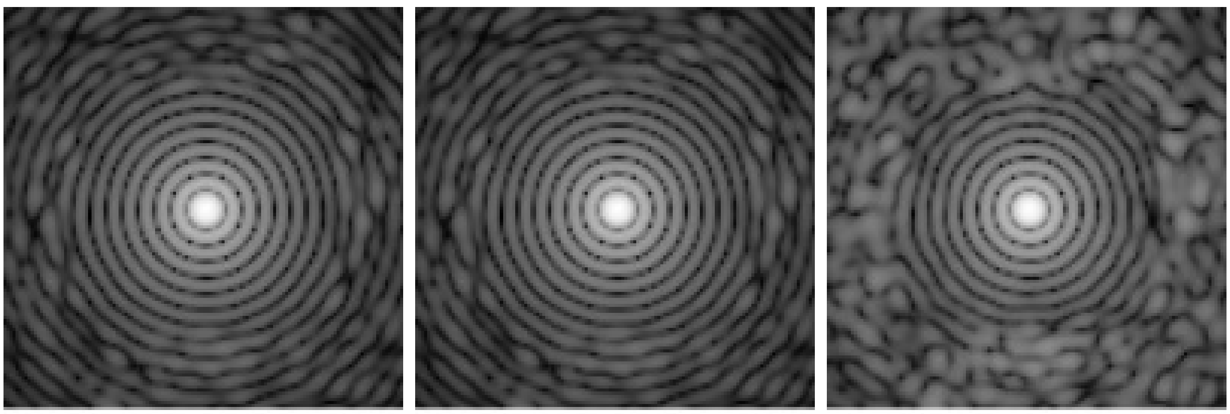


Fig. 1.— The power spectrum of residual phase aberrations. Perfect wavefront correction follows perfect spatially-filtered wavefront sensing within the sensing spatial frequency passband (dark central square) of a square grid of 64 wavefront sensors across the aperture diameter (after Poyneer & Macintosh (2004)).



$SR\,0.99$ $SR\,0.97$ $SR\,0.90$

Fig. 2.— The full numerical PSF of an unobscured pupil with the aberration whose power spectrum is shown in Figure 1. The phase aberrations are scaled to produce Strehl ratios shown below each PSF. The leftmost 99% Strehl ratio PSF is qualitatively similar to a space-based high dynamic range PSF, the 97% Strehl ratio PSF in the middle to a next-generation Extreme Adaptive Optics system's predicted PSF in the K-band, and the 90% Strehl PSF to the best H-band near-IR adaptive optics images on AEOS. Real AO system PSFs have other features not reproduced in this simple, unobstructed aperture simulation. The square spatial frequency control band of the wavefront sensing and correction system is visible as a square pattern imprinted in the PSF.

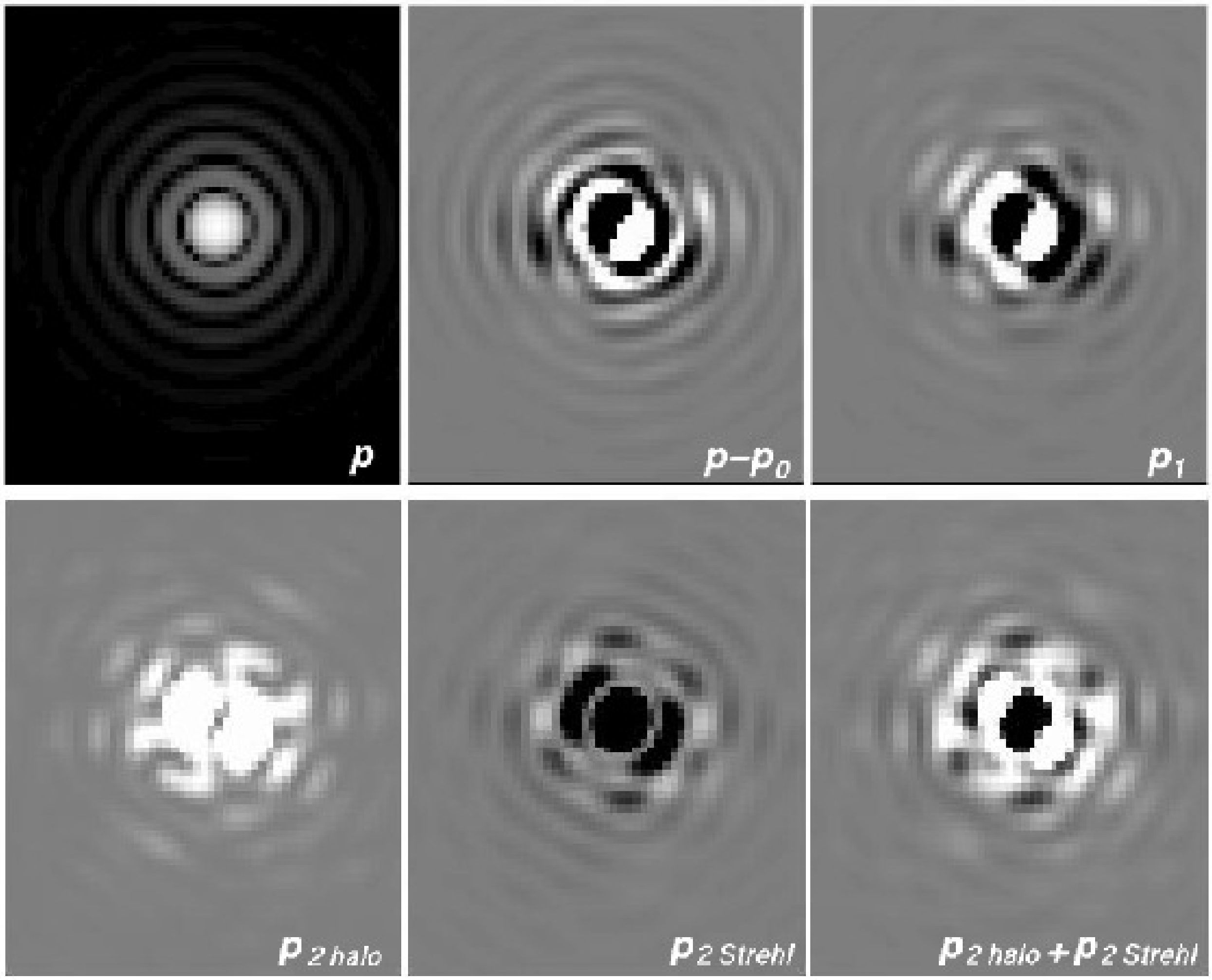


Fig. 3.— A 94% Strehl ratio PSF, p, formed by a circular aperture, can be decomposed into individual terms in the power series in the Fourier transform of the phase over the aperture. Here the terms up to second order are examined. In the top left panel (p) black represents zero. In the other panels middle grey represents zero, since these terms can be negative in places. The antisymmetry of the first order term p_1 and the symmetry of the second order terms are apparent. The positive halo term, $p_{2,\ halo}$, is zero at the image center, as is the pinned first order term p_1. At these Strehl ratios the image degradation can be dominated near the core of the PSF by the pinned second order term $p_{2,\ Strehl}$. This is the first term that depresses the peak value of the perfect PSF is $p_{2,\ Strehl}$. It's value at the origin is the variance of the phase aberrations over the aperture (as measured in square radians), a fact known as the Maréchal approximation. The sum of the first order term (taken over the entire image plane) is zero. So is the sum of both the second order terms. Each order (except the zero-order term p_0) does not contribute any net power to the PSF — power is merely redistributed over the image plane by each term.

3.1. A numerical example

We generate a single realization of a Kolmogorov-spectrum power spectrum, using a Markov-based method described by Lane et al. (1992), with $D/r_o = 15$ (r_o being the Fried length; roughly speaking this the distance over which the phase might be expected to vary by a radian). After Fourier transforming this phase screen we set all power within the control band to zero, and re-transform the Fourier-filtered array back to a perfectly-sensed and corrected phase aberration.

Our control band is 32 cycles across the pupil in the x_1 and x_2 directions. The dark square in the center of this figure shows the effect of our simulated correction in spatial frequency space. Our phase screen is 1040 pixels on a side, which is more than twice the size of the 512-sample diameter pupils over which we place this phase screen. The variance of the phase after this Fourier-filtered simulation of wavefront sensing and correction is about 0.07 square radians, which correspnds to a Strehl ratio of about 93%. We rescale the values in this phase array to provide Strehl ratios of 0.99, 0.97, and 0.90 for this study (Figure 2).

This 'residual wavefront aberration' (or it's transform, rather), along with the bump near the AO control radius, is first convolved by the square root of the PSF, *i.e.,* by a, in the first term p_1, of the PSF expansion, and the power spectrum of the phase errors suffers the same convolution in the second order terms. The convolution simply erases all information in Φ or $\Phi\Phi^*$ on scales smaller than λ/D for conventional (un-apodized) pupils.

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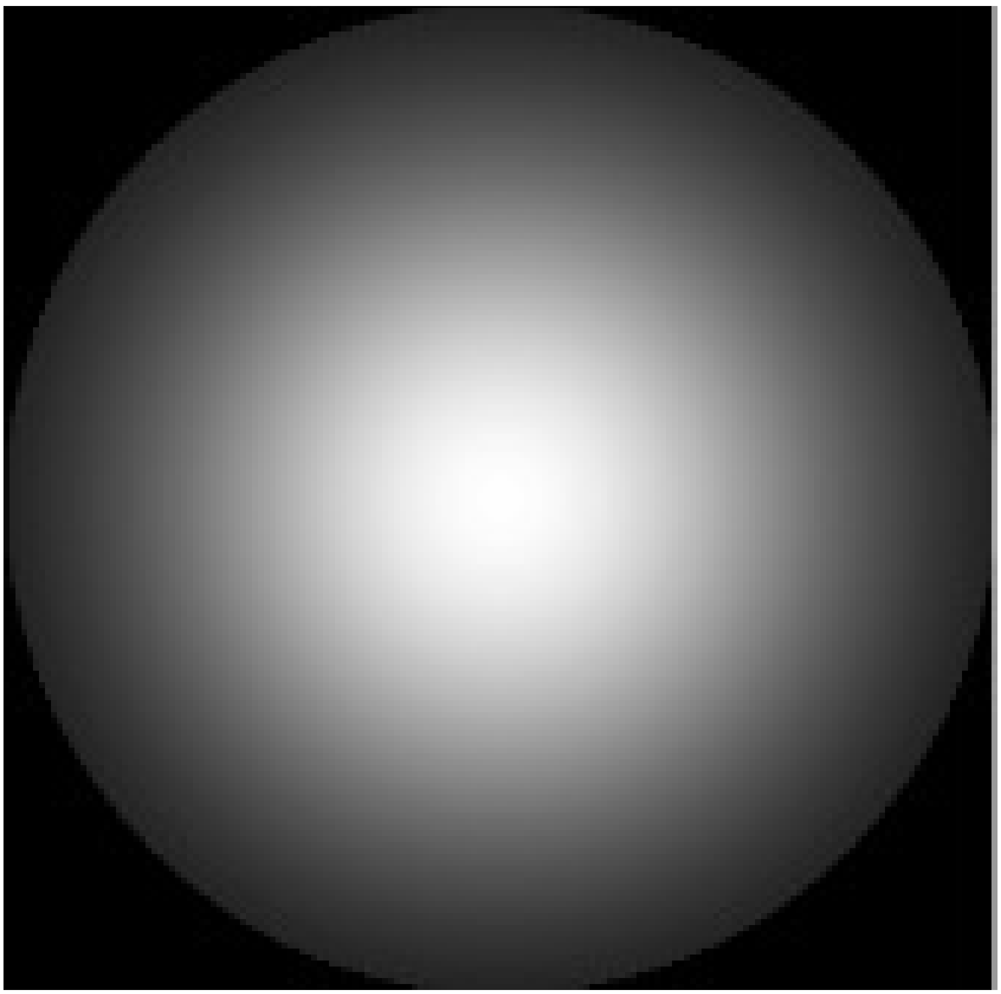

Fig. 4.— An apodized pupil with a Gaussian radial profile. The standard deviation of the Gausssian is half the radius of the parent aperture. The transmission at the edge of the pupil is 14% of the transmission at the center. This pupil is used to calculate the PSF described in Figure 6.

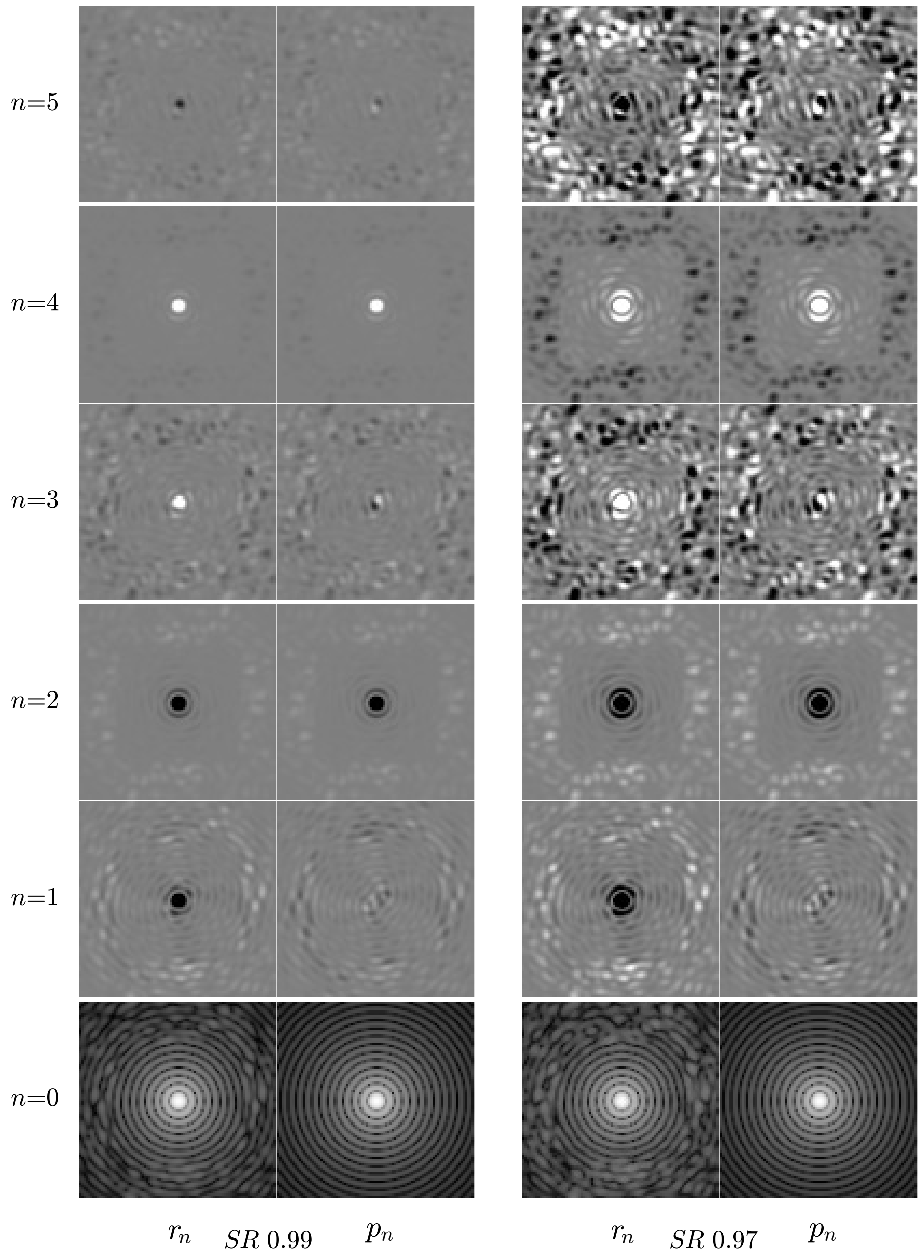

Fig. 5.— The first six terms in the power series expansion for an unobstructed circular aperture at 99% (left) and 97% (right) Strehl ratios. The n-th term of the power series is p_n, and the residual after the first n terms of the series are subtracted from the full numerical PSF are r_n. Thus the full numerical PSF is r_0, and the perfect PSF is p_0. The bottom row is stretched logarithmically. Greyscales between left right images of any particular order are the same.

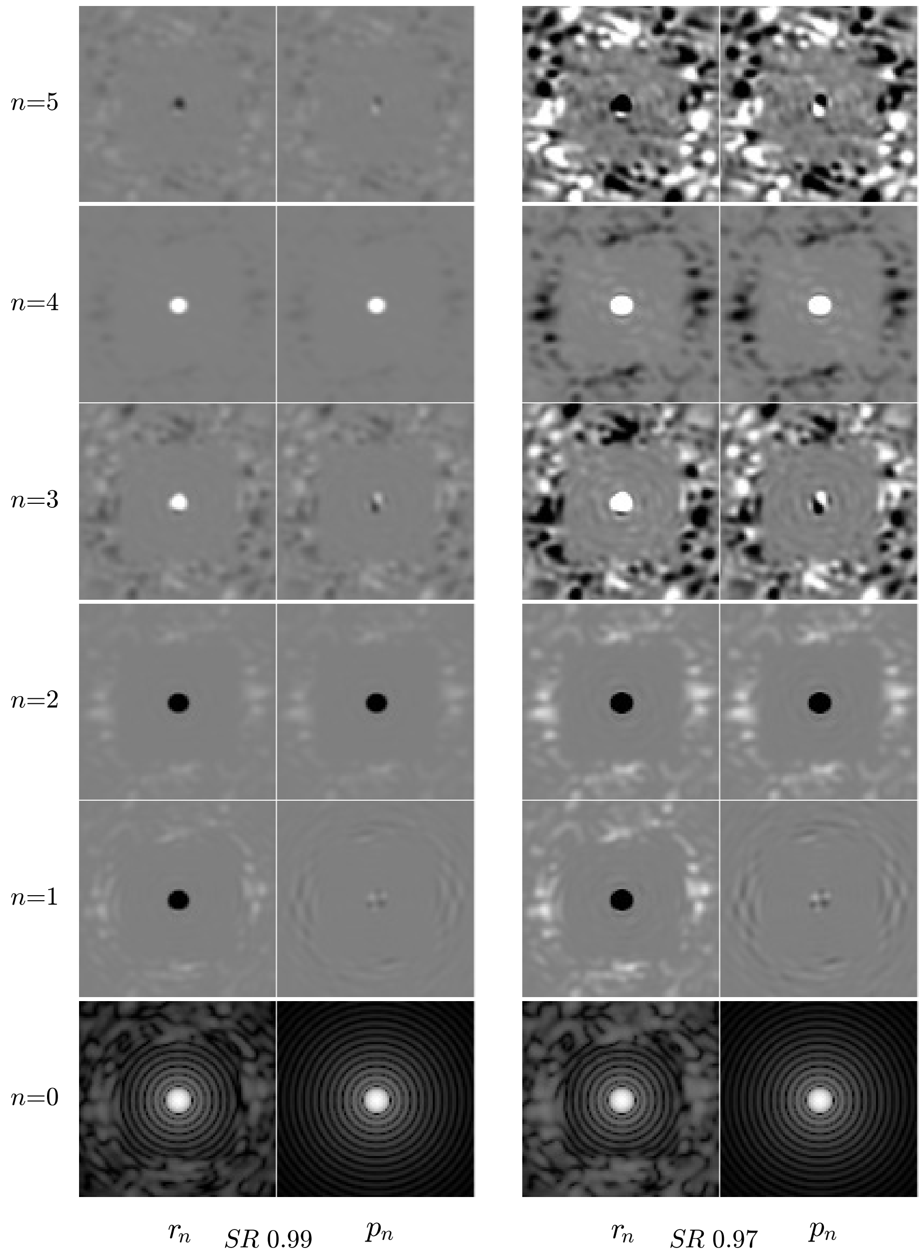

Fig. 6.— The first six terms in the power series expansion for an unobstructed *apodized* circular aperture at 99% (left) and 97% (right) Strehl ratios. The n-th term of the power series is p_n, and the residual after the first n terms of the series are subtracted from the full numerical PSF are r_n. Thus the full numerical PSF is r_0, and the perfect PSF is p_0. The bottom row is stretched logarithmically. Greyscales between left right images of any particular order are the same. Compare the 'ringiness' of the PSF near the core, within the AO-controlled area, with that of the unapodized pupil in the previous figure.