Fourier Transform Theorems

- Addition Theorem
- Shift Theorem
- Convolution Theorem
- Similarity Theorem
- Rayleigh's Theorem
- Differentiation Theorem

Addition Theorem

$$\mathcal{F}\{f+g\} = F + G$$

Proof:

$$\mathcal{F}\{f+g\}(s) = \int_{-\infty}^{\infty} [f(t)+g(t)]\, e^{-j2\pi st} dt$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt + \int_{-\infty}^{\infty} g(t) e^{-j2\pi st} dt \\ &= F(s) + G(s) \end{aligned}$$

Shift Theorem

$$\mathcal{F}\,\{f(t-t_0)\}(s) = e^{-j2\pi s t_0}F(s)$$

Proof:

$$\mathcal{F}\,\{f(t-t_0)\}(s) = \int_{-\infty}^{\infty} f(t-t_0)e^{-j2\pi st}dt$$

Multiplying the r.h.s. by $e^{j2\pi st_0}e^{-j2\pi st_0} = 1$ yields:

$$\begin{aligned}
&\mathcal{F}\,\{f(t-t_0)\}(s)\\
&= \int_{-\infty}^{\infty} f(t-t_0)e^{-j2\pi st}e^{j2\pi st_0}e^{-j2\pi st_0}dt\\
&= e^{-j2\pi st_0}\int_{-\infty}^{\infty} f(t-t_0)e^{-j2\pi s(t-t_0)}dt.
\end{aligned}$$

Substituting $u = t - t_0$ and $du = dt$ yields:

$$\begin{aligned}
\mathcal{F}\,\{f(t-t_0)\}(s) &= e^{-j2\pi st_0}\int_{-\infty}^{\infty} f(u)e^{-j2\pi su}du\\
&= e^{-j2\pi st_0}F(s).
\end{aligned}$$

Shift Theorem Example

$$
\begin{aligned}
&\mathcal{F}\left\{\sin\left(2\pi\left(t+1/4\right)\right)\right\}(s)\\
&= e^{\frac{j2\pi s}{4}}\mathcal{F}\left\{\sin(2\pi t)\right\}\\
&= e^{\frac{j\pi s}{2}}\cdot\frac{j}{2}\left[\delta(s+1)-\delta(s-1)\right]\\
&= \frac{j}{2}\left[e^{\frac{j\pi s}{2}}\delta(s+1)-e^{\frac{j\pi s}{2}}\delta(s-1)\right]\\
&= \frac{j}{2}\left[e^{\frac{j\pi(-1)}{2}}\delta(s+1)-e^{\frac{j\pi(+1)}{2}}\delta(s-1)\right]\\
&= \frac{j}{2}\left[-j\delta(s+1)-j\delta(s-1)\right]\\
&= \frac{1}{2}\left[\delta(s+1)+\delta(s-1)\right]\\
&= \mathcal{F}\left\{\cos(2\pi s)\right\}
\end{aligned}
$$

Shift Theorem (variation)

$$\mathcal{F}^{-1}\{F(s-s_0)\}(t) = e^{j2\pi s_0 t} f(t)$$

Proof:

$$\mathcal{F}^{-1}\{F(s-s_0)\}(t) = \int_{-\infty}^{\infty} F(s-s_0) e^{j2\pi st} ds$$

Multiplying the r.h.s. by $e^{j2\pi s_0 t} e^{-j2\pi s_0 t} = 1$ yields:

$$\begin{aligned}
&\mathcal{F}^{-1}\{F(s-s_0)\}(t) \\
&= \int_{-\infty}^{\infty} F(s-s_0) e^{j2\pi st} e^{j2\pi s_0 t} e^{-j2\pi s_0 t} ds \\
&= e^{j2\pi s_0 t} \int_{-\infty}^{\infty} F(s-s_0) e^{j2\pi (s-s_0) t} ds.
\end{aligned}$$

Substituting $u = s - s_0$ and $du = ds$ yields:

$$\begin{aligned}
\mathcal{F}^{-1}\{F(s-s_0)\}(t) &= e^{j2\pi s_0 t} \int_{-\infty}^{\infty} F(u) e^{j2\pi ut} du \\
&= e^{j2\pi s_0 t} f(t).
\end{aligned}$$

Convolution Theorem

$$\mathcal{F}\{f * g\} = F \cdot G$$

Proof:

$$\mathcal{F}\{f * g\}(s)$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u) g(t-u) du \right] e^{-j2\pi st} dt$$

Changing the order of integration:

$$\mathcal{F}\{f * g\}(s)$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(t-u) e^{-j2\pi st} dt \right] du$$

By the Shift Theorem, we recognize that

$$\left[\int_{-\infty}^{\infty} g(t-u) e^{-j2\pi st} dt \right] = e^{-j2\pi su} G(s)$$

so that

$$\begin{aligned} \mathcal{F}\{f * g\}(s) &= \int_{-\infty}^{\infty} f(u) e^{-j2\pi su} G(s) du \\ &= G(s) \int_{-\infty}^{\infty} f(u) e^{-j2\pi su} du \\ &= G(s) \cdot F(s) \end{aligned}$$

Convolution Theorem Example

The pulse, Π, is defined as:

$$\Pi(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The triangular pulse, Λ, is defined as:

$$\Lambda(t) = \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\Lambda = \Pi * \Pi$. Using this fact, we can compute $\mathcal{F}\{\Lambda\}$:

$$\begin{aligned} \mathcal{F}\{\Lambda\}(s) &= \mathcal{F}\{\Pi * \Pi\}(s) \\ &= \mathcal{F}\{\Pi\}(s) \cdot \mathcal{F}\{\Pi\}(s) \\ &= \frac{\sin(\pi s)}{\pi s} \cdot \frac{\sin(\pi s)}{\pi s} \\ &= \frac{\sin^2(\pi s)}{\pi^2 s^2}. \end{aligned}$$

Convolution Theorem (variation)

$$\mathcal{F}^{-1}\{F * G\} = f \cdot g$$

Proof:

$$\mathcal{F}^{-1}\{F * G\}(t)$$
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(u) G(s-u) du \right] e^{j2\pi st} ds$$

Changing the order of integration:

$$\mathcal{F}^{-1}\{F * G\}(t)$$
$$= \int_{-\infty}^{\infty} F(u) \left[\int_{-\infty}^{\infty} G(s-u) e^{j2\pi st} ds \right] du$$

By the Shift Theorem, we recognize that

$$\left[\int_{-\infty}^{\infty} G(s-u) e^{j2\pi st} ds \right] = e^{j2\pi tu} g(t)$$

so that

$$\begin{aligned}
\mathcal{F}^{-1}\{F * G\}(t) &= \int_{-\infty}^{\infty} F(u) e^{j2\pi tu} g(t) du \\
&= g(t) \int_{-\infty}^{\infty} F(u) e^{j2\pi tu} du \\
&= g(t) \cdot f(t)
\end{aligned}$$

Similarity Theorem

$$\mathcal{F}\ \{f(at)\}(s) = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Proof:

$$\mathcal{F}\ \{f(at)\}(s) = \int_{-\infty}^{\infty} f(at) e^{-j2\pi st} dt$$

There are two cases.

- $a > 0$. Multiplying the integral by $a/|a| = 1$ and the exponent by $a/a = 1$ yields:

  $$\mathcal{F}\ \{f(at)\}(s) = \frac{1}{|a|} \int_{-\infty}^{\infty} f(at) e^{-j2\pi(s/a)at} a dt$$

  We now make the substitution $u = at$ and $du = adt$:

  $$\begin{aligned} \mathcal{F}\ \{f(at)\}(s) &= \frac{1}{|a|} \int_{-\infty}^{\infty} f(u) e^{-j2\pi(s/a)u} du \\ &= \frac{1}{|a|} F\left(\frac{s}{a}\right) \end{aligned}$$

Similarity Theorem (contd.)

- $a < 0$. Multiplying the integral by $|a|/|a| = 1$ and the exponent by $-|a|/a = 1$ and using the fact that $a = -|a|$ yields:

$$\mathcal{F}\{f(at)\}(s) =$$
$$\frac{1}{|a|}\int_{t=-\infty}^{t=\infty} f(-|a|t)e^{-j2\pi(s/a)(-|a|t)}|a|dt$$

We now make the substitution $u = -|a|t$ and $du = -|a|dt$:

$$\begin{aligned}
\mathcal{F}\{f(at)\}(s) &= -\frac{1}{|a|}\int_{u=\infty}^{u=-\infty} f(u)e^{-j2\pi(s/a)u}du \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)e^{-j2\pi(s/a)u}du \\
&= \frac{1}{|a|}F\left(\frac{s}{a}\right)
\end{aligned}$$

Similarity Theorem Example

Let's compute, $G(s)$, the Fourier transform of:

$$g(t) = e^{-t^2/9}.$$

We know that the Fourier transform of a Gaussian:

$$f(t) = e^{-\pi t^2}$$

is a Gaussian:

$$F(s) = e^{-\pi s^2}.$$

We also know that :

$$\mathcal{F}\,\{f(at)\}(s) = \frac{1}{|a|}F\left(\frac{s}{a}\right).$$

We need to write $g(t)$ in the form $f(at)$:

$$g(t) = f(at) = e^{-\pi(at)^2}.$$

Let $a = \frac{1}{3\sqrt{\pi}}$:

$$g(t) = e^{-t^2/9} = e^{-\pi\left(\frac{1}{3\sqrt{\pi}}t\right)^2} = f\left(\frac{1}{3\sqrt{\pi}}t\right).$$

It follows that:

$$G(s) = 3\sqrt{\pi}\,e^{-\pi(3\sqrt{\pi}s)^2}.$$

Rayleigh's Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Proof:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f^*(t) f(t) dt$$

Substituting $\{\mathcal{F}^{-1}\{F\}\}^*(t)$ for $f^*(t)$:

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds \right]^* f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F^*(s) e^{-j2\pi st} ds \right] f(t)\, dt$$

Rayleigh's Theorem (contd.)

Changing the order of integration:

$$
\begin{aligned}
&= \int_{t=-\infty}^{t=\infty} \left[\int_{s=-\infty}^{s=\infty} F^*(s) e^{-j2\pi st} ds \right] f(t)\, dt \\
&= \int_{s=-\infty}^{s=\infty} F^*(s) \left[\int_{t=-\infty}^{t=\infty} e^{-j2\pi st} f(t)\, dt \right] ds \\
&= \int_{-\infty}^{\infty} F^*(s) F(s)\, ds \\
&= \int_{-\infty}^{\infty} |F(s)|^2\, ds
\end{aligned}
$$

Differentiation Theorem

$$\mathcal{F}\{f'\}(s) = j2\pi s F(s)$$

Proof:

$$f = \mathcal{F}^{-1}\{F\}$$

Therefore

$$f(t) = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds$$

Differentiating both sides with respect to t:

$$f'(t) = \int_{-\infty}^{\infty} j2\pi s F(s) e^{j2\pi st} ds$$

or

$$f' = \mathcal{F}^{-1}\{j2\pi s F(s)\}$$

Taking the Fourier transform of both sides:

$$\begin{aligned} \mathcal{F}\{f'\}(s) &= \mathcal{F}\{\mathcal{F}^{-1}\{j2\pi s F(s)\}\} \\ &= j2\pi s F(s) \end{aligned}$$

Differentiation Theorem Example

$$
\begin{aligned}
& \mathcal{F}\left\{\frac{d\sin(2\pi t)}{dt}\right\}(s) \\
&= j2\pi s\mathcal{F}\left\{\sin(2\pi t)\right\}(s) \\
&= j2\pi s\cdot\frac{j}{2}\left[\delta(s+1)-\delta(s-1)\right] \\
&= -\pi s\left[\delta(s+1)-\delta(s-1)\right] \\
&= \pi\left[s\delta(s-1)-s\delta(s+1)\right] \\
&= \pi\left[(+1)\delta(s-1)-(-1)\delta(s+1)\right] \\
&= \pi\left[\delta(s-1)+\delta(s+1)\right] \\
&= \pi\mathcal{F}\left\{2\cos(2\pi t)\right\}(s) \\
&= \mathcal{F}\left\{2\pi\cos(2\pi t)\right\}(s)
\end{aligned}
$$