Digital Communication

ASSIGNMENT - 1

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17EC10003

Prolate Spheroidal Wave Functions

Introduction

Claude E. Shannon (1916–2001) once posed the question:

> ***"To what extent are functions, which are confined to a finite bandwidth, also concentrated in the time domain?"***

This open question was answered by David Slepian (1923-2007) at Bell Laboratories in a series of seminal papers dated back to 1960s.
"We found a second-order differential equation that commuted with an integral operator that was at the heart of the problem," as commented by D. Slepian.

This statement best testifies to their findings: the prolate spheroidal wave functions of order zero (PSWFs), being the eigenfunctions of a second-order singular Sturm-Liouville equation, are coincidentally the spectrum of an integral operator related to the finite Fourier transform.

In this assignment we revisit some of the intrinsic properties of the PSWFs and try to explore them further.

Generating PSWFs

There are different ways proposed to generate the Prolate Spheroidal Wave Functions in digital domain for our calculation purposes. We followed the ways discussed in Spectral Analysis for Physical Applications by D.B. Percival and A.T.Walden.

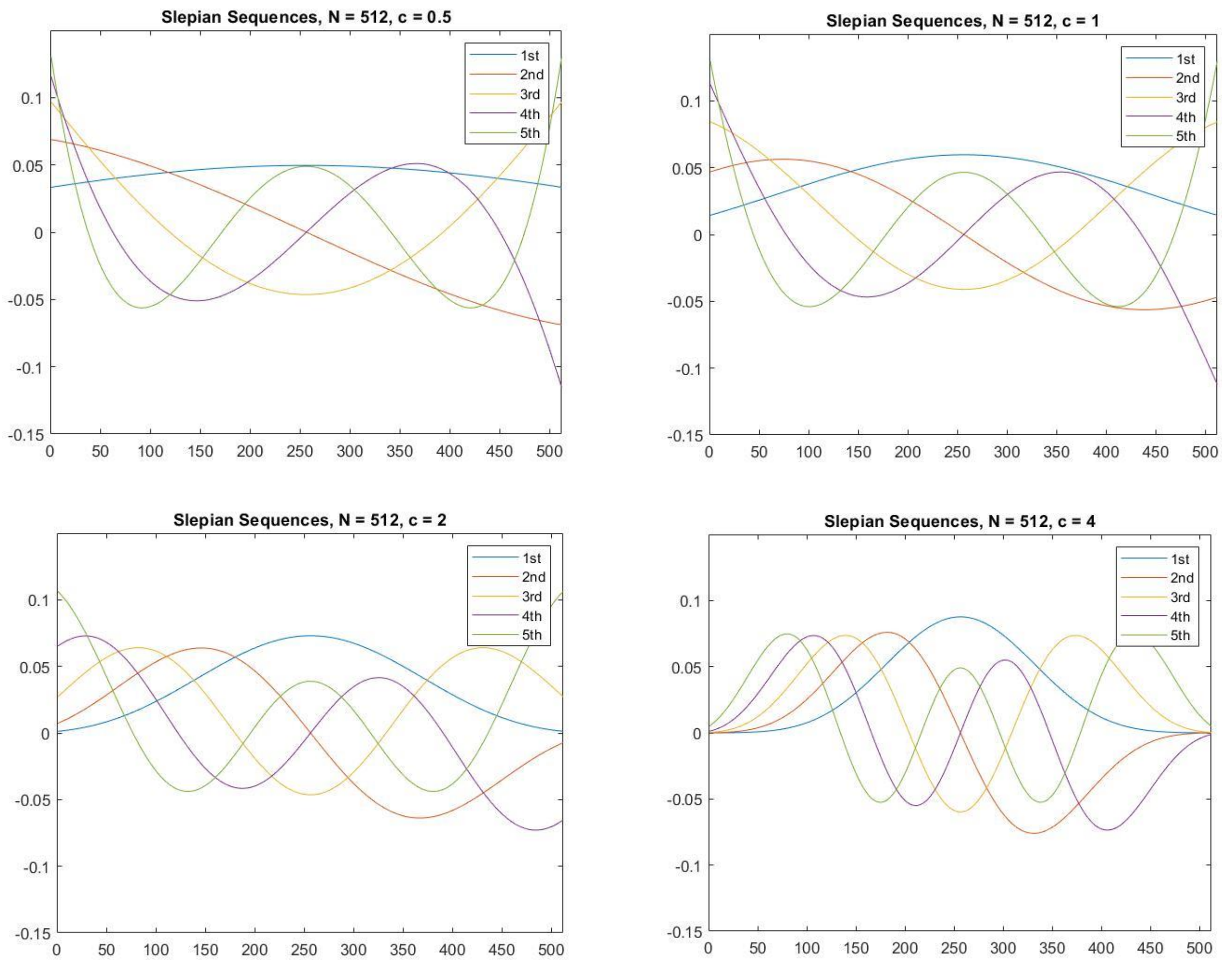
They used tridieig() to get the eigenvalues 1:k, if we want first k PSWFs, of the sparse tridiagonal matrix. It then uses inverse iteration using the exact eigenvalues on the starting vector with approximate shape, to get the eigenvectors required.

In contrast, Our method was to create the sparse matrix using spdiags() use the MATLAB eigs() function to find its eigenvalue and the eigenvectors. Then only the last k eigenvectors were used to find the exact eigenvalues using the Toeplitz sinc matrix, a method similar to the one discussed in their book.

The results obtained are shown on the next page.
Also, the Eigen Values of each PSWFs is shown below in the table.

Eigen Values for different NW values.

NW/k	**1**	**2**	**3**	**4**	**5**
0.5	0.78	0.20	0.01	0.00	0.00
1	0.98	0.74	0.24	0.02	0.00
2	0.99	0.99	0.95	0.72	0.27
4	1.00	1.00	0.99	0.99	0.99



Plots of various PSWFs for different NW.

DISCUSSION:

1) We can see from the plots that the even order (0 based ordering) PSWFs are even functions while the odd order PSWFs are odd functions.

2) We can also see that as NW increases, more PSWFs have their eigenvalues close to 1. This is because after 2NW-1 order, the energy under the wave becomes significantly low.

3) As a result, we can see that the PSWFs after that order do not converges to 0 at the extreme ends of the plot.

4) We can also see that the PSWFs are ordered with decreasing eigenvalues.

Verifying Properties.

In this part, we verify the orthogonality of the PSWFs. We consider two cases, one for infinite support, and other for the finite support.

Although the same was repeated at each value of NW, we only consider the case of NW = 4 as it gave the most accurate results due to the reason discussed above.

Infinite support

N	1	2	3	4	5
1	**1.0**	**0.0**	**0.0**	**0.0**	**0.0**
2	**0.0**	**1.0**	**0.0**	**0.0**	**0.0**
3	**0.0**	**0.0**	**1.0**	**0.0**	**0.0**
4	**0.0**	**0.0**	**0.0**	**1.0**	**0.0**
5	**0.0**	**0.0**	**0.0**	**0.0**	**1.0**

Finite support

N	1	2	3	4	5
1	**1.0**	**0.0**	**0.0**	**0.0**	**0.0**
2	**0.0**	**1.0**	**0.0**	**0.0**	**0.0**
3	**0.0**	**0.0**	**1.0**	**0.0**	**0.1**
4	**0.0**	**0.0**	**0.0**	**1.0**	**0.0**
5	**0.0**	**0.0**	**0.1**	**0.0**	**0.9**

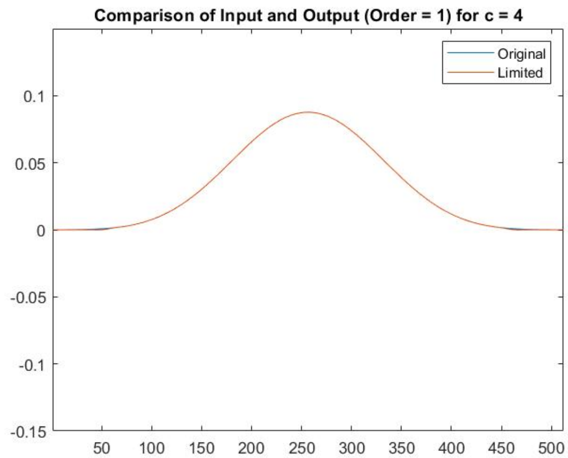
For other NW values, we had same result for the infinite support but for finite support the values were more off from the original.

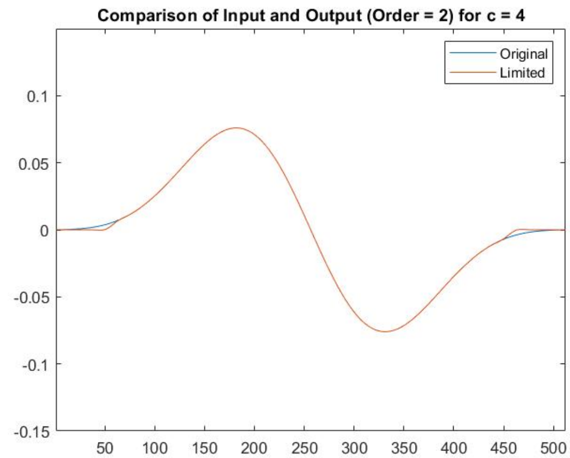
Now, we needed to show the most intrinsic property of the PSWFs, that they are indeed the eigen function of the finite Fourier transform, i.e. the Band time limiting operation.

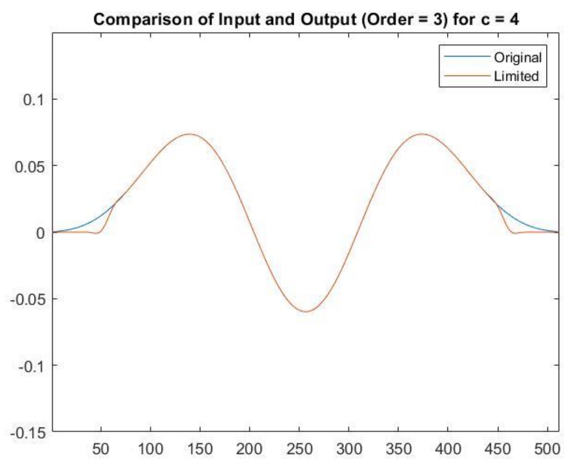
This was achieved by only taking a certain number of values in the sequence, thus time limiting it, and then passing it through a low pass filter. We then compared its distance with that of the required one, that is, **BD(F)** with **λ(F)**. For that we calculated the relative MSE between the two.

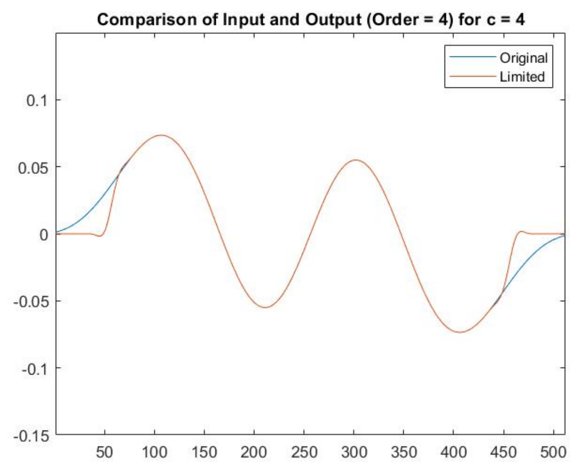
The results are shown in the next page.

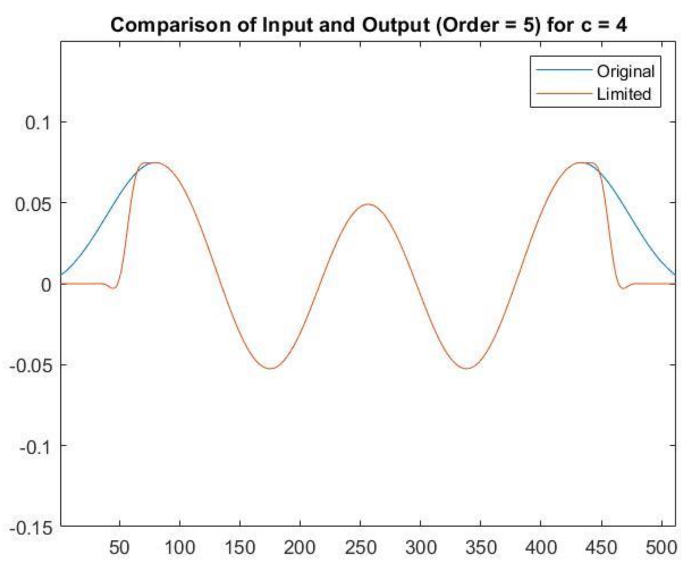
Comparison between the Calculated and Observed waves.











Relative MSE between BD(F) and λ(F) for various NW

NW/N	1	2	3	4	5
0.5	**0.02**	**0.03**	**0.03**	**0.04**	**0.04**
1	**0.01**	**0.02**	**0.03**	**0.03**	**0.04**
2	**0.00**	**0.01**	**0.02**	**0.03**	**0.03**
4	**0.00**	**0.00**	**0.00**	**0.01**	**0.02**

DISCUSSION:

1) Some of the important properties that can be directly observed is that the PSWFs are orthonormal to each other.

2) When taken in finite support, we lose some energy and thus they are no longer orthonormal but still they are orthogonal to some extent. In our case, the dot product between two PSWFs were occasionally non zero due to error in conversion from continuous to discrete domain.

3) This orthonormal property can be used to expand other function with PSWFs as bases.

4) We can see from the plots that the two functions, the Band time limited and the scaled versions are indeed similar. Also, we can verify that by looking at the MSE between these two for different order and as well as NW values.

5) Again, we can see that the error increases as we go to higher order, specifically, above 2NW-1.

EXPANDING FUNCTIONS IN PSWFs DOMAIN.

Now, we will be exploiting the orthonormal property of the PSWFs to expand any given arbitrary signal. We will use, for our purpose, the following function as the input signal.

$$y(t) = e^{-t^2}$$

This function is square integrable so we can use it.

Now, we can write any function as sum of the basis function times some coefficient. We need to find those coefficients.

$$y(t) = \sum \alpha_i PS(t)_i$$

Take dot product with $PS(t)_i$ on both sides,

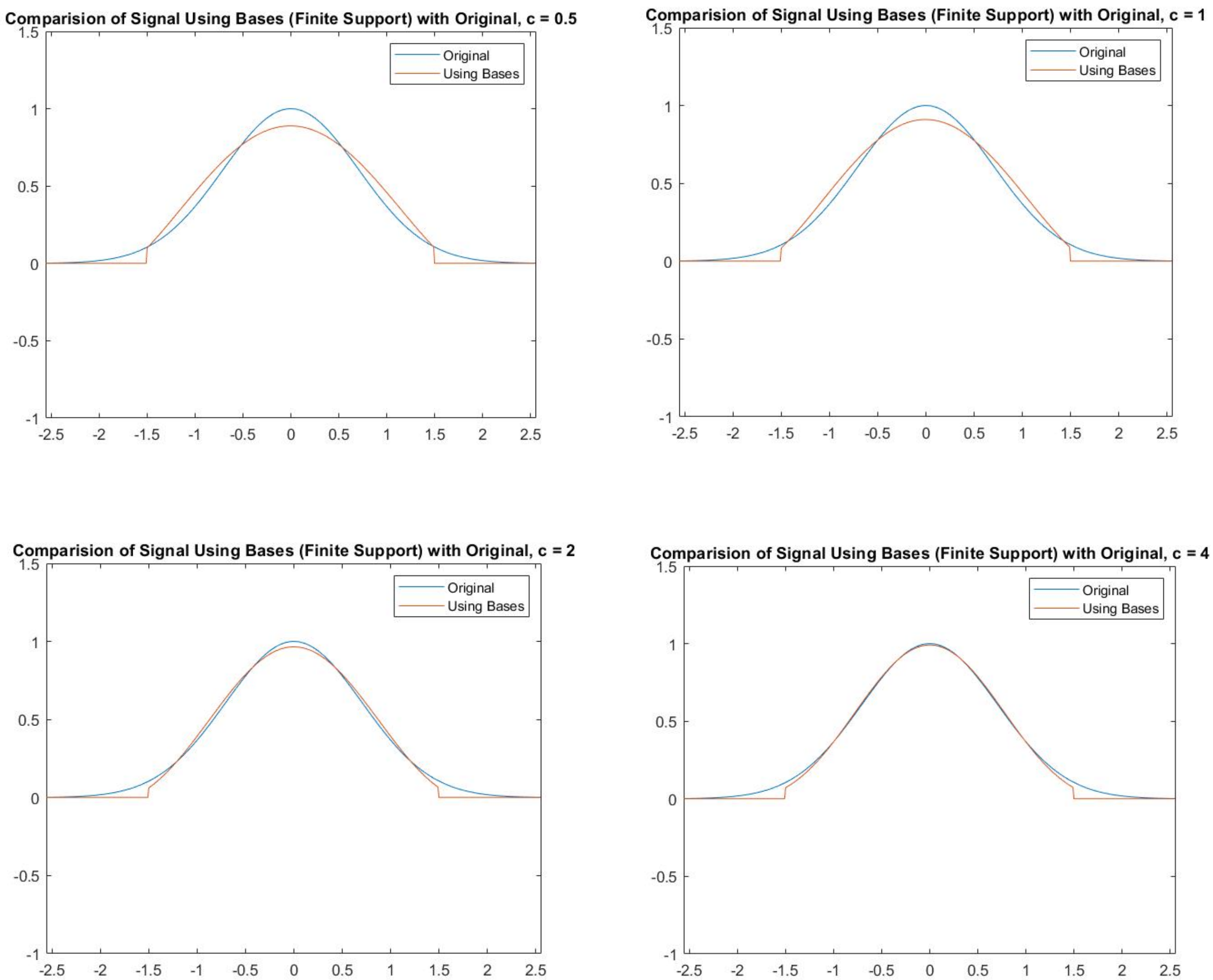
$$\alpha_i = \frac{y(t) \cdot PS(t)_i}{||PS(t)_i||^2}$$

This can be easily written using the orthogonality argument of the PSWFs.

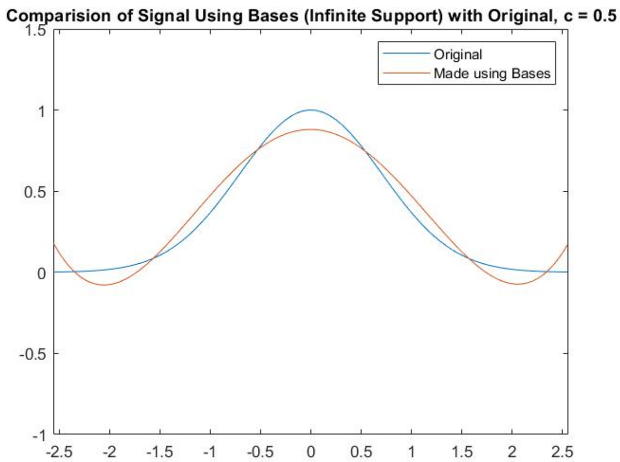
For infinite domain, we can simply write the norm as 1.

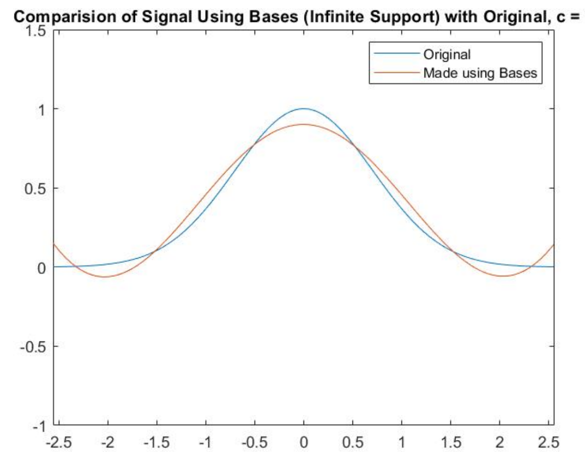
Note that in finite domain case, the vectors are not exactly orthonormal. We need to consider the norm of each wave, but we can ignore the fact that the dot products with others are non-zero.

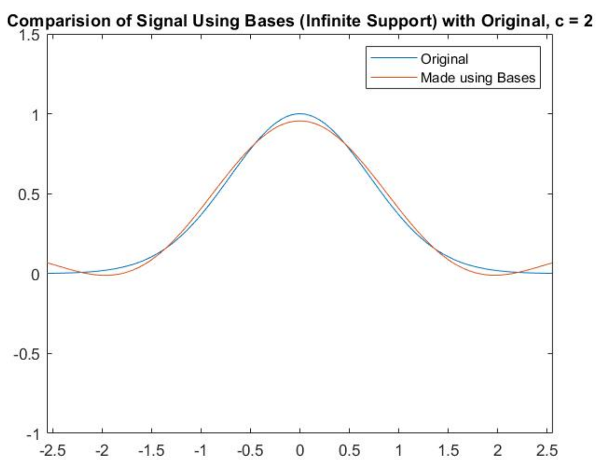
Comparison of Original Signal with Expanded Signal in Finite Support

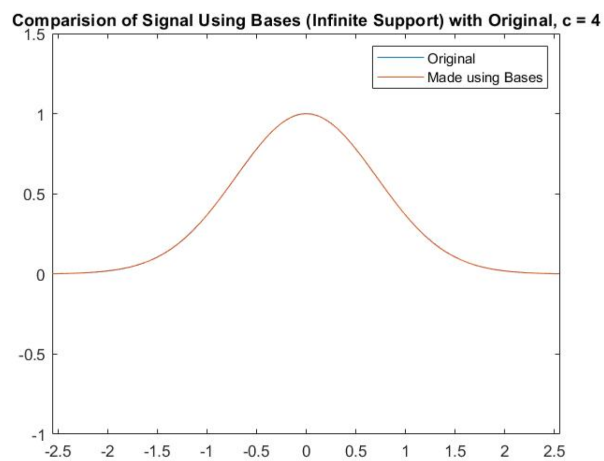


Comparison of Original Signal with Expanded Signal in Infinite Support









The following were the coefficients observed.

For Finite Support

c/Order	α1	α2	α3	α4	α5
0.5	**8.3045**	**-0.0232**	**-6.1730**	**0.0354**	**3.8948**
1	**9.5346**	**-0.0302**	**-4.6994**	**0.0333**	**3.1843**
2	**10.6929**	**-0.0453**	**-2.9753**	**0.0251**	**1.2048**
4	**11.1643**	**-0.0543**	**-0.3886**	**0.0073**	**-0.2280**

For Infinite Support

c/Order	α1	α2	α3	α4	α5
0.5	**8.5514**	**-0.0289**	**-6.0094**	**0.0369**	**3.6219**
1	**9.7339**	**-0.0360**	**-4.4507**	**0.0332**	**2.9438**
2	**10.7996**	**-0.0498**	**-2.6131**	**0.0206**	**1.1799**
4	**11.1927**	**-0.0559**	**-0.1666**	**0.0014**	**0.1552**

To check how close the two signals were, we calculated the Energy Ratio between the two signals in both scenarios.

Finite Support

c	0.5	1	2	4
Energy Ratio	**0.965**	**0.975**	**0.993**	**0.995**

Infinite Support

c	0.5	1	2	4
Energy Ratio	**0.976**	**0.983**	**0.996**	**1.000**

DISCUSSION:

1) This is an important feature to notice that we can now study the characteristics of any function using their expansion in PSFWs domain. We can now study how any given signal will behave when we try to limit it in time domain and frequency domain.

2) We can clearly see the trend that as NW increases, we get more and more energy inside the expanded signal with respect to the original signal.

3) Another important, yet obvious thing to notice is that expansion in infinite support conserves more energy than expansion in finite support.

4) As discussed above, for the calculation in finite domain, the dot product with others are not exactly zero, thus their contribution will also come while calculating the exact values. We, for our case, have neglected their effect.

REFERENCES:

1) Percival, D.B. and Walden, A.T., Spectral Analysis for Physical Applications.
2) C. Flammer, Spheroidal Wave Functions.
3) Ian C. Moore, Michael Cada, Prolate spheroidal wave functions, an introduction to the Slepian series and its properties.
4) D. Slepian, H.O. Pollak, Prolate spheroidal wave functions, Fourier Analysis and uncertainty.
5) H. Xiao, V. Roklin, N. Yarvin, Prolate spheroidal wave functions, quadrature and interpolation, Inverse Problems.
6) Li-Lian Wang, A Review of Prolate Spheroidal Wave Functions from the Perspective of Spectral Methods.