The Capacitated Vehicle Routing Problem With Time Windows As A Mixed Integer Linear Program

Ananda Chakrabarti

Abstract

This report presents a Mixed-Integer Linear Programming (MILP) formulation for the Capacitated Vehicle Routing Problem with Time Windows (CVRPTW). Unlike formulations with hard time windows which may easily become infeasible when operational constraints conflict, this approach penalizes late deliveries in the objective function while maintaining solution feasibility. The model is implemented using Gurobi and evaluates on a network of 80 customers in an 80 mile radius centered around a distribution hub. The solver achieves an acceptable solution within 20 minutes, demonstrating strong computational performance and validating that the soft constraint formulation effectively balances route distance minimization with time window adherence. The result demonstrates the viability of this approach for mid-sized regional distribution problems.

1 Introduction

The Vehicle Routing Problem (VRP) is a fundamental combinatorial optimization problem in operations research with widespread applications in logistics, supply chain management, and transportation planning [5, 8]. Given a fleet of vehicles, a central depot, and a set of geographically dispersed customers with known demands, the VRP seeks to determine optimal delivery routes that minimize total travel distance while satisfying operational constraints. Minimizing the total travel distance also implicitly minimizes fuel consumption and carbon emissions (and maximizes profits, of course), although more sophisticated modeling explicitly takes these factors into account as well [2].

The Capacitated Vehicle Routing Problem with Time Windows (CVRPTW) extends the classical VRP by imposing two critical real-world constraints: (i) each vehicle has a limited cargo capacity that cannot be exceeded, and (ii) each customer specifies a preferred time window during which service must be delivered [6, 7]. These constraints make the problem more complex but also more realistic for practical applications such as last-mile delivery, food distribution, and e-commerce logistics.

This project presents a Mixed-Integer Linear Programming (MILP) formulation for the CVRPTW using the Gurobi optimization solver [1]. The MILP uses a soft time window constraint which allows deliveries to occur outside the preferred time window but penalizes such violations in the objective function. This relaxation greatly improves solution feasibility while maintaining strong preference for on-time delivery, providing a more flexible model that better reflects real-world operational trade-offs.

MILPs are solved using branch-and-bound algorithms combined with cutting plane methods [4]. The solver begins by relaxing integrality constraints to obtain a linear programming (LP) relaxation, providing a lower bound on the optimal objective value. At each node of the search tree, the LP relaxation is solved; if the solution satisfies integrality constraints, it becomes a candidate incumbent. Otherwise, branching occurs by partitioning the solution space based on a fractional variable. Cutting planes—valid inequalities that tighten the LP relaxation without eliminating integer-feasible solutions—are added to strengthen bounds and reduce the search space. Modern solvers like Gurobi employ sophisticated heuristics for node selection, variable branching strategies, and automatic cut generation. For large-scale instances, computational performance depends critically on formulation tightness: weak formulations with loose LP relaxations lead to extensive branching, while tight formulations yield faster convergence.

To validate this problem formulation, I generated a geographically realistic problem simulation centered around a distribution hub, solved the problem using Gurobi's MILP solver, and visualized solutions on real map backgrounds using OpenStreetMap tiles[3]. I demonstrated the effectiveness of the approach on a test case with 80 customers.

The remainder of this report is organized as follows. Section 2 presents the mathematical formulation of the CVRPTW with soft time windows. Section 3 describes the computational implementation including data generation and solver configugation, and presents numerical results on the test case. Finally, section 4 concludes with a discussion of findings and possible future work.

2 Problem Formulation

Consider a distribution network consisting of a central depot (node 0) and a set of customers $\mathcal{C} = \{1, 2, \ldots, n\}$ requiring delivery of commodities. A homogeneous fleet of V vehicles, each with capacity Q, is available at the depot. Each customer $i \in \mathcal{C}$ has demand q_i and specifies a preferred time window $[e_i, \ell_i]$ for service. The objective is to design vehicle routes minimizing total travel distance while respecting capacity limits and time window preferences.

Here, *soft time windows* are employed that permit late deliveries with a penalty in the objective function. This ensures feasibility when strict adherence is impossible while maintaining strong preference for on-time service.

Let $\mathcal{N} = \{0\} \cup \mathcal{C}$ denote all nodes. Each node i has its associated geographical coordinate (x_i, y_i). I use the geodesic distance d_{ij} and travel time t_{ij} between nodes $i, j \in \mathcal{N}$. Each customer requires a fixed service duration δ. The notation used for the CVRPTW parameters is summarized in Table 1. The decision variables are:

- $a_{ij} \in \{0, 1\}$: Equals 1 if a vehicle travels from node i to node j, $i, j \in \mathcal{N}$, $i \neq j$, 0 otherwise.
- $s_i \geq 0$: Service start time at customer $i \in \mathcal{C}$.
- $y_i \geq 0$: Cumulative vehicle load before serving customer $i \in \mathcal{C}$.
- $\sigma_i \geq 0$: Time window slack (lateness) at customer $i \in \mathcal{C}$.

Parameter	**Description**
$\mathcal{N}$	Set of all nodes: $\{0, 1, 2, \ldots, n\}$ where 0 is the depot
$\mathcal{C}$	Set of all customer nodes: $\{1, 2, \ldots, n\}$
(x_i, y_i)	Geographical coordinate of node i, for all $i \in \mathcal{N}$
V	Number of vehicles available at the depot
Q	Capacity of each vehicle (identical for all vehicles)
d_{ij}	Geodesic distance between nodes i and j, for all $i, j \in \mathcal{N}$, $i \neq j$
t_{ij}	Travel time from node i to node j, for all $i, j \in \mathcal{N}$, $i \neq j$
q_i	Demand at customer $i \in \mathcal{C}$ (units of commodity)
e_i	Earliest service (delivery window) start time at customer $i \in \mathcal{C}$
ℓ_i	Latest preferred (delivery window) service start time at customer $i \in \mathcal{C}$
δ	Fixed service duration at each customer location
W	Penalty weight for time window violations (soft constraint)
M	Large constant for "deactivating" time sequencing constraints when $a_{ij} = 0$.

Table 1: Notation for CVRPTW parameters

The MILP for the CVRPTW with soft time window constraints is formulated as follows:

$$\min_{\{a_{ij},\ s_i,\ y_i,\ \sigma_i\}_{i,j\in\mathcal{N}}} \quad \sum_{i\in\mathcal{N}}\sum_{\substack{j\in\mathcal{N}\\ j\neq i}} d_{ij}a_{ij} + W\sum_{i\in\mathcal{C}}\sigma_i \tag{1a}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{C}} x_{0j} \leq V, \tag{1b}$$

$$\sum_{j\in\mathcal{C}} x_{0j} = \sum_{j\in\mathcal{C}} x_{j0}, \tag{1c}$$

$$\sum_{\substack{i\in\mathcal{N}\\ i\neq j}} a_{ij} = 1, \quad \forall j \in \mathcal{C}, \tag{1d}$$

$$\sum_{\substack{j\in\mathcal{N}\\ j\neq i}} a_{ij} = 1, \quad \forall i \in \mathcal{C}, \tag{1e}$$

$$s_j \geq s_i + t_{ij} + \delta - M(1 - a_{ij}), \quad \forall i \in \mathcal{N}, j \in \mathcal{C}, i \neq j, \tag{1f}$$

$$y_j \geq y_i + q_i a_{ij} - M(1 - a_{ij}), \quad \forall i, j \in \mathcal{C}, i \neq j, \tag{1g}$$

$$y_i + q_i \leq Q, \quad \forall i \in \mathcal{C}, \tag{1h}$$

$$e_i \leq s_i \leq \ell_i + \sigma_i, \quad \forall i \in \mathcal{C}, \tag{1i}$$

$$s_0 = T_{\text{start}},\ y_0 = 0, \tag{1j}$$

$$a_{ij} \in \{0,1\},\ s_i \geq 0,\ y_i \geq 0,\ \sigma_i \geq 0, \quad \forall i, j \in \mathcal{N}, i \neq j. \tag{1k}$$

The objective (1a) minimizes total travel distance plus a weighted penalty for time window violations/late deliveries, where W is the weight for the penalty on late deliveries (which also controls the trade-off between route length and punctuality). Constraint (1b) limits fleet size, while (1c) ensures vehicles return to the depot. Constraints (1d) and (1e) guarantee each customer is visited exactly once. The time sequencing constraint (1f) enforces that service at j begins after completing service at i and traveling from i to j, using M (set to 10^6) to deactivate when arc (i, j) is unused (i.e., when $a_{ij} = 0$). Similarly, (1g) tracks cumulative load along routes. Capacity is enforced by (1h). Time windows are handled by (1i) with lateness slack σ_i. Finally, (1j) initializes depot conditions (at business day start), and (1k) defines the domain of the decision variables.

This soft time window formulation provides operational flexibility: when $\sigma_i = 0$, service is on-time for customer i; when $\sigma_i > 0$, service is late by σ_i hours but the solution still remains feasible. The penalty discourages lateness while allowing the solver to find feasible solutions when perfect adherence is impossible.

3 Results

The proposed CVRPTW formulation is evaluated on a test instance representative of regional distribution operations. All computations were performed using Gurobi 12.0.2 on an Apple M2 processor with 8 cores and 8 GB RAM, running under macOS. The implementation is in Python, utilizing the Gurobi Python API for model construction and optimization.

A random distribution network was modelled centered at a depot with coordinates (39.9°N, 82.7°W). This roughly corresponds to the location of an actual Amazon shipping centre near Columbus, Ohio[1]. I generated 80 customer locations uniformly distributed within an 80-mile radius centered at the depot, in an attempt to represent a reasonably sized regional delivery area. Customer locations were generated using polar coordinates with geodesic distance calculations via the GeoPy library, ensuring geographic accuracy. Each customer has unit demand ($q_i = 1$), and a fleet of 5 vehicles with capacity $Q = 16$ units each is available, yielding zero capacity margin (total demand = total capacity = 80 units). Business hours were simulated to span 8:00 AM to 6:00 PM ($T_{\text{start}} = 8$, $T_{\text{end}} = 18$). Preferred delivery time windows for each customer were randomly generated with durations between 1 and 3 hours, with earliest service times uniformly distributed across the simulation time. Service duration is fixed at $\delta = 0.25$ hours (15 minutes). Travel times were computed as $t_{ij} = 0.6 \cdot (d_{ij}/80)$, corresponding to an average speed of approximately 25 mph accounting for urban and rural road conditions (the travel factor of 0.6 can be adjusted depending on geographical setting). The soft time window penalty weight is set to $W = 1000$, strongly prioritizing on-time delivery while permitting violations when necessary for feasibility.

This yielded a model with 13,124 constraints, 6,804 variables (6,561 binary routing variables and 243 continuous variables for time, load, and slack). After Gurobi's presolve phase, the model reduced to 12,893 rows and 6,721 columns. I set a time limit of 20 minutes for the solver to allow for sufficient exploration of the solution space, and it terminated with a best objective value of 2124.71 and an optimality gap of 10.7% (indicating that the true optimal solution was within 10.7% of this value). The solver explored over 213,000 branch-and-bound nodes and performed approximately 14.8 million simplex iterations. The first feasible solution is found within seconds, and the objective value steadily improves throughout the search, demonstrating the algorithm's effectiveness at progressively refining solution quality. The final 10.7% gap still represents strong solution quality suitable for operational deployment, especially given the

[1] Of course, this is a scaled down toy problem setup compared to what real world industries solve. I understand that Amazon and other companies have extremely sophisticated solvers capable of rapidly solving way more detailed MILPs with a much larger number of customers and fleet of delivery vehicles, trucks with significantly higher delivery capacity, traffic delays, etc.

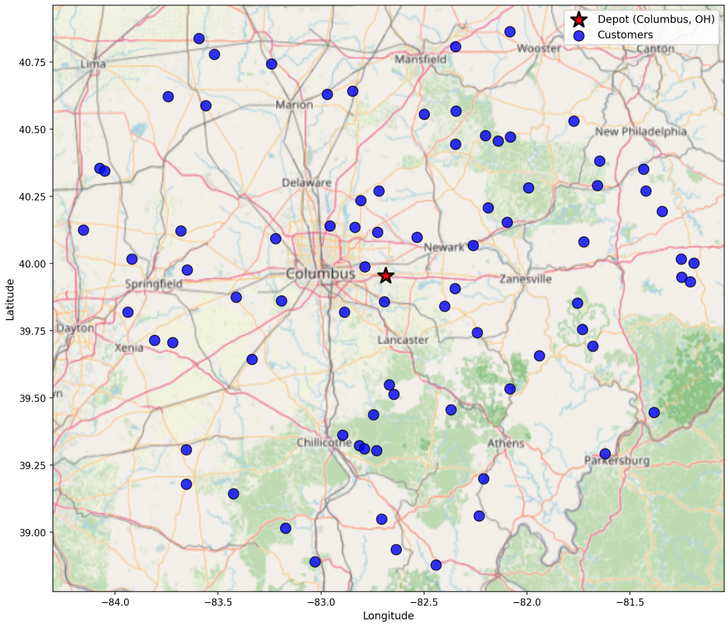

Figure 1: Geographic network showing depot and 80 customer locations within an 80-mile radius centered at the depot. Map tiles were generated using OpenStreetMap.

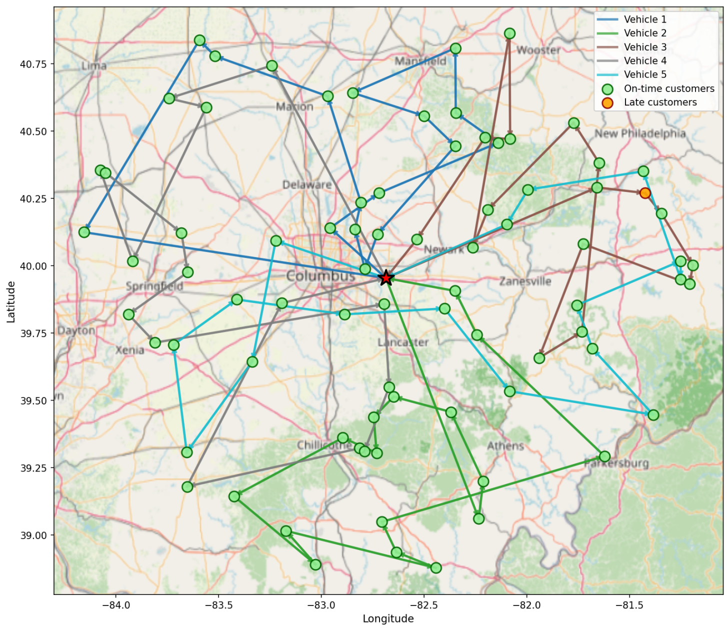

Figure 2: Optimized vehicle routes for 80 customers. Five vehicles serve all customers with minimal route overlap. Green markers indicate on-time deliveries, orange markers indicate late deliveries. Arrows show route direction.

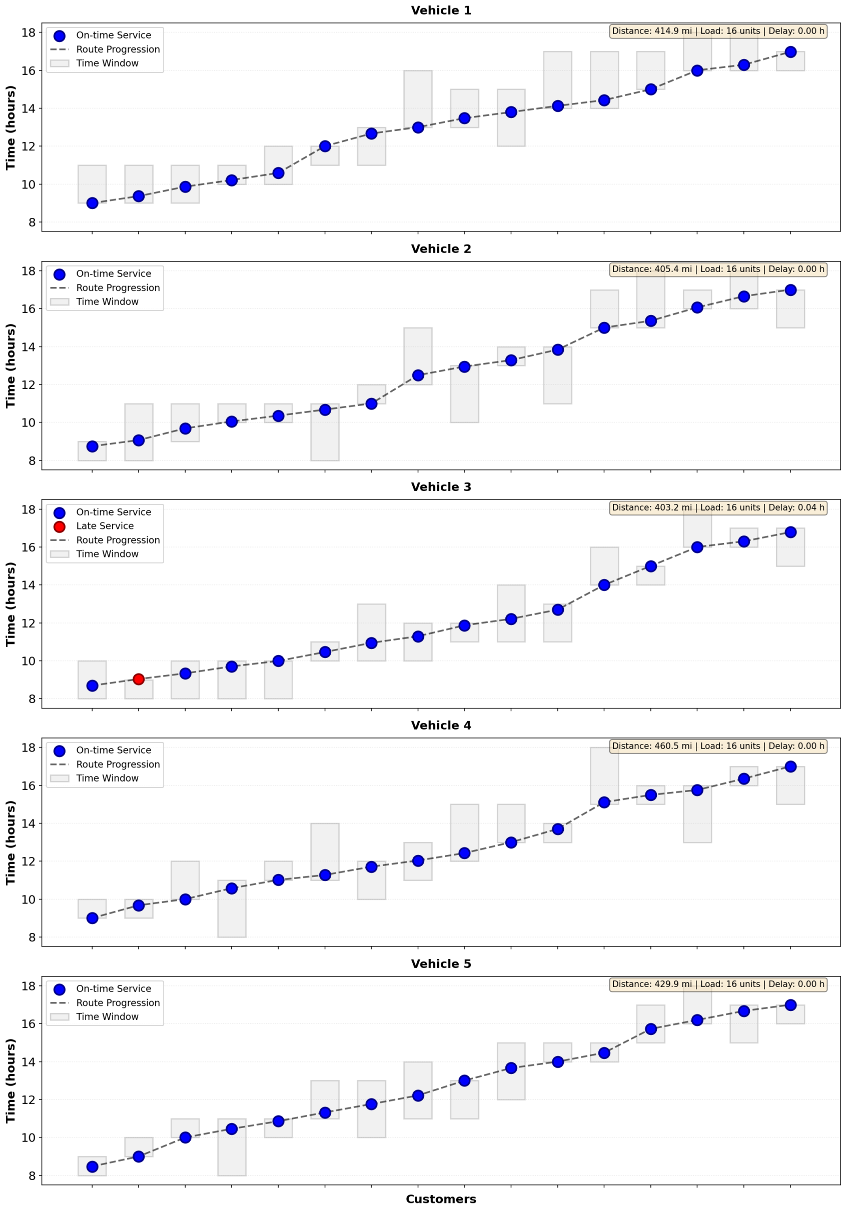

Figure 3: Time window adherence timeline for all five vehicle routes, simulated for 10 hours from 8:00 AM to 6:00 PM. Each row corresponds to one vehicle. Gray bars show the preferred delivery time windows for each customer, blue dots indicate on-time service, red dots indicate late service.

problem's combinatorial complexity with over 6,500 binary variables.

The results are presented in Figures 1, 2 and 3. It is observed that even with a soft time-window constraint, the time-window constraint is still effectively enforced for the most part, with just 1 out of the 80 customers facing a delayed delivery, which amounted to 2.4 minutes outside that customer's delivery window.

4 Conclusion

This report presented a MILP formulation for the CVRPTW with soft time windows, implemented and validated on a realistic 80-customer distribution instance. Instead of hard time window constraints, we use penalized slack variables that permit time window violations while maintaining strong preference for on-time delivery through a tunable penalty weight. The computational results demonstrate that this formulation achieves near-optimal solutions (10.7% gap) within reasonable time limits (20 minutes) while attaining solid on-time performance. The soft time window approach proves particularly valuable for operational planning where hard time windows may be overly restrictive or lead to infeasibility. Future work includes a more realistic modeling of road networks such as highways and urban grids, heterogeneous fleet extensions with vehicle-specific capacities and costs and integration of stochastic travel times to account for traffic variability.

References

[1] Rimmi Anand, Divya Aggarwal, and Vijay Kumar. A comparative analysis of optimization solvers. *Journal of Statistics and Management Systems*, 20(4):623–635, 2017.

[2] Tolga Bektaş and Gilbert Laporte. The pollution-routing problem. *Transportation research part b: methodological*, 45(8):1232–1250, 2011.

[3] Jonathan Bennett. *OpenStreetMap*. Packt Publishing Ltd, 2010.

[4] Stephen Boyd and Jacob Mattingley. Branch and bound methods. *Notes for EE364b, Stanford University*, 2006:07, 2007.

[5] George B Dantzig and John H Ramser. The truck dispatching problem. *Management science*, 6(1):80–91, 1959.

[6] Ted K Ralphs, Leonid Kopman, William R Pulleyblank, and Leslie E Trotter. On the capacitated vehicle routing problem. *Mathematical programming*, 94(2):343–359, 2003.

[7] Marius M Solomon. Algorithms for the vehicle routing and scheduling problems with time window constraints. *Operations research*, 35(2):254–265, 1987.

[8] Paolo Toth and Daniele Vigo. *The vehicle routing problem*. SIAM, 2002.