# A Tutorial on Deep Latent Variable Models of Natural Language

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This manuscript was written to complement a tutorial at EMNLP 2018 and is intended to serve both as an introduction to deep latent variable models as well as a review of recent advances. Its current state is somewhere between a tutorial and notes. The accompanying slides and code can be found at http://nlp.seas.harvard.edu/latent-nlp-tutorial.html

# 1 Introduction

Latent variable models, and probabilistic graphical models more generally, provide a declarative language for specifying prior knowledge and structural relationships in complex datasets. They have a long and rich history in natural language processing, having contributed to fundamental advances such as statistical alignment for translation (Brown et al., 1993), topic modeling (Blei et al., 2003), unsupervised part-of-speech tagging (Brown et al., 1992), and grammar induction (Klein and Manning, 2004), among others. *Deep learning*, broadly construed, is a toolbox for learning rich representations (i.e., features) of data through numerical optimization. Deep learning is the current dominant paradigm in natural language processing, and some of the major successes include language modeling (Bengio et al., 2003; Mikolov et al., 2010; Zaremba et al., 2014), machine translation (Sutskever et al., 2014; Cho et al., 2014; Bahdanau et al., 2015; Vaswani et al., 2017), and natural language understanding tasks such as question answering and natural language inference.

There has been much recent, exciting work on combining the complementary strengths of latent variable models and deep learning. Latent variable modeling makes it easy to explicitly specify model constraints through conditional independence properties, while deep learning makes it possible to parameterize these conditional likelihoods with powerful function approximators. While these "deep latent variable" models provide a rich, flexible framework for modeling many real-world phenomena, difficulties exist: deep parameterizations of conditional likelihoods usually make posterior inference intractable, and latent variable objectives often complicate backpropagation by introducing points of non-differentiability. This tutorial explores these issues in depth through the lens of *variational inference* (Jordan et al., 1999; Wainwright and Jordan, 2008), a key technique for performing approximate posterior inference.

The term "deep latent variable" models can also refer to the use of neural networks to perform latent variable *inference* ("deep inference"). In the context of variational inference, this means that we train an inference network to output the parameters of an approximate posterior distribution given the set of variables to be conditioned upon (Kingma and Welling, 2014; Rezende et al., 2014; Mnih and Gregor, 2014). We will devote a significant portion of the tutorial to this setting.

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**Tutorial Outline** The tutorial will be organized as follows: section 2 introduces notation and briefly describes the basics of neural networks; section 3 presents several archetypical latent variable models of text and their "deep" variants; section 4 surveys applications of latent variable models introduced in the previous section; section 5 deals with learning and performing posterior inference in latent variable models, both in the case where exact inference over the latent variables is tractable and when it is not; section 6 focuses on amortized variational inference and variational autoencoders, a central framework for learning deep generative models; section 7 briefly touches on other methods for learning latent variable models; and finally, section 8 concludes.

While our target audience is the community of natural language processing (NLP) researchers/practioners, we hope that this tutorial will also be of use to researchers in other areas. We have therefore organized the tutorial to be modular: sections 3 and 4, which are more specific to NLP, are largely independent from sections 5 and 6, which mostly deal with the general problem of learning deep generative models.

**Scope** This tutorial will focus on learning latent variable models whose joint distribution can be expressed as a *directed graphical model* (DGM),<sup>1</sup> and we will mostly do this through variational inference. Specifically, we will not cover (or only briefly touch on) undirected graphical models such as restricted Boltzmann Machines (and more broadly, Markov random fields), posterior inference based on Markov chain Monte Carlo sampling, spectral learning of latent variable models, and non-likelihood-based approaches such as generative adversarial networks (Goodfellow et al., 2014). While each of these topics is a rich area of active research on its own, we have chosen to limit scope of this tutorial to directed graphical models and variational inference in order to provide a more thorough background on fundamental ideas and key techniques, as well as a detailed survey of recent advances.

# 2 Preliminaries

**Notation** Throughout, we will assume the following notation:

- x: an observation, usually a sequence of tokens,  $x = x_1, \dots, x_T$ , sometimes denoted with  $x_{1:T}$
- $x^{(1:N)}$ : a dataset of N i.i.d observations,  $x^{(1:N)} = x^{(1)}, \dots, x^{(N)}$
- z: an unobserved latent variable, which may be a sequence or other structured object
- **z**: a latent vector (we will use z to denote a general latent variable, and **z** in the specific case where the latent variable is a continuous vector, e.g.  $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ )
- $\theta$ : generative model parameters
- $\lambda$ : variational parameters
- $p(x, z; \theta)$ , or  $p_{x,z}(x, z; \theta)$ : generative model parameterized by  $\theta$
- $p(x \mid z; \theta)$ , or  $p_{x\mid z}(x \mid z; \theta)$ : likelihood model parameterized by  $\theta$
- $q(z; \lambda)$ : variational distribution parameterized by  $\lambda$
- $\Delta^{V-1}$ : the standard V-simplex, i.e. the set  $\{\pi \in \mathbb{R}^V \mid \pi_v \geq 0 \text{ for all } v, \text{ and } \sum_{v=1}^V \pi_v = 1\}$

We will use  $p(x; \theta)$  to denote the probability mass (or density) function of a random variable evaluated at x, where the density is parameterized by  $\theta$ . While the random variable over which the distribution is induced will be often clear from context, when it is not clear we will either use subscripts to identify the random

<sup>&</sup>lt;sup>1</sup>Directed graphical models are also sometimes referred to as *generative models*, since their directed nature makes it easy to generate data through ancestral sampling.

variable, e.g.  $p_z(z;\theta)$  and  $p_x(x;\theta)$ , or a different letter, e.g.  $q(z;\lambda)$ . We will sometimes overload notation and use  $p(x=x_0;\theta)$  with the same p to refer to the probability of the event that the random variable x takes on the value  $x_0$  where the probability distribution is again parameterized by  $\theta$ , i.e.  $p(x=x_0;\theta)=p_x(x_0;\theta)$ . For brevity of notation, and since the distinction will encumber rather than elucidate at the level of abstraction this tutorial is aiming for, we will also overload variables (e.g. x,z) to both refer to a random variable (a measurable function from a sample space  $\Omega$  to  $\mathbb{R}^d$ ) and its realization (an element of  $\mathbb{R}^d$ ).

**Neural Networks** We now briefly introduce the neural network machinery to be used in this tutorial. Deep networks are parameterized non-linear functions, which transform an input z into features h using parameters  $\pi$ . We will in particular make use of the multilayer perceptron (MLP), which computes features as follows:

$$h = \text{MLP}(\mathbf{z}; \pi) = V\sigma(W\mathbf{z} + \mathbf{b}) + a$$

where  $\sigma$  is an element-wise nonlinearity, such as  $\tanh$ , ReLU, or the logistic sigmoid, and the set of neural network parameters is  $\pi = \{V, W, a, b\}$ .

We will also make use of the recurrent neural network (RNN), which maps a sequence of inputs  $\mathbf{z}_{1:T}$  into a sequence of features  $h_{1:T}$ , as follows:

$$h_t = \text{RNN}(h_{t-1}, \mathbf{z}_t; \pi) = \sigma(\mathbf{U}\mathbf{z}_t + \mathbf{V}h_{t-1} + \mathbf{b}),$$

where  $\pi = \{V, U, b\}$ , and where for concreteness we have parameterized the above RNN as an Elman RNN (Elman, 1990). While modern architectures typically use the slightly more complicated long short-term memory (LSTM) (Hochreiter and Schmidhuber, 1997) or gated recurrent units (GRU) (Cho et al., 2014) RNNs instead, we will remain agnostic with regard to the specific RNN parameterization, and simply use RNN( $h_{t-1}, \mathbf{z}_t; \pi$ ) to encompass the LSTM/GRU parameterizations as well.

**RNN Language Models** We will shortly introduce several archetypal latent variable models of a sentence  $x = x_1, \dots, x_T$  consisting of T words. Before we do so, however, we briefly introduce the RNN language model, an incredibly effective model of a sentence that uses no latent variables. Indeed, because RNN language models are such good sentence models, we will need a very good reason to make use of latent variables, which as we will see complicate learning and inference. Sections 3 and 4 attempt to motivate the need for introducing latent variables into our sentence models, and the remaining sections discuss how to learn and do inference with them.

Using the notation  $x_{< t}$  as shorthand for the sequence  $x_1, \ldots, x_{t-1}$ , RNN language models define a distribution over a sentence x as follows

$$p(x_{1:T}) = \prod_{t=1}^{T} p(x_t \mid x_{< t}) = \prod_{t=1}^{T} \text{softmax}(Wh_t)_{x_t},$$
(1)

where

$$h_t = \text{RNN}(h_{t-1}, \mathbf{x}_{t-1}; \theta), \tag{2}$$

and  $\mathbf{x}_{t-1}$  is the word embedding corresponding to the (t-1)-th word in the sequence. The softmax function over a vector  $\mathbf{s} \in \mathbb{R}^V$  applies an element-wise exponentiation to  $\mathbf{s}$  and renormalizes to obtain a distribution, i.e. softmax :  $\mathbb{R}^V \to \Delta^{V-1}$ ,  $\mathbf{y} = \operatorname{softmax}(\mathbf{s})$  means that each element of  $\mathbf{y}$  is given by

$$y_k = \frac{\exp(s_k)}{\sum_{v=1}^V \exp(s_v)}.$$

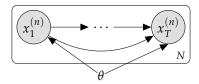


Figure 1: Graphical model corresponding to an RNN language model.

Because RNN language models are a central tool in deep NLP, for the remainder of the tutorial we will use the notation

$$x_{1:T} \sim p(x_{1:T}) = \text{RNNLM}(x_{1:T}; \theta)$$

to mean that x is distributed according to an RNN language model (as above) with parameters  $\theta$ , where  $\theta$  includes the parameters of the RNN itself as well as the matrix W used in defining the per-word distributions at each step t.

Finally, we note that the RNNLM model above makes no independence assumptions, and simply decomposes the probability of the sequence x using the chain rule of probability. We show a graphical model corresponding to an RNN language model in Figure 1.

# 3 Archetypal Latent Variable Models of Sentences

We now present several archetypal latent variable models of sentences  $x = x_1, ..., x_T$ , which will serve to ground the discussion. All of the models we introduce will provide us with a joint distribution  $p(x, z; \theta)$  over the observed words x and unobserved latent variables z. These models will differ in terms of whether the latent variables z are discrete, continuous, or structured, and we will provide both a shallow and deep variant of each model. We will largely defer the motivating applications for these models to the next section, but we note that on an intuitive level, the latent variables in these three types of models have slightly different interpretations. In particular, discrete latent variables can be interpreted as inducing a clustering over data points, continuous latent variables can be interpreted as providing a dimensionality reduction of data points, and structured latent variables can be interpreted as unannotated structured objects (i.e., objects with interdependent pieces or parts) in the data; these interpretations will be expanded upon below.

In addition to describing models in each of the three categories listed above, we will briefly sketch out what inference – that is, calculating the posterior over latent variables  $p(z \mid x; \theta)$  – in these models looks like, as a way of motivating the later sections of the tutorial.

#### 3.1 A Discrete Latent Variable Model

We begin with a simple discrete latent variable model, namely, a latent-variable Naive Bayes model (i.e., a mixture of categoricals model). This model assumes a sentence  $x = x_1, \dots, x_T$  is generated according to the following process:

1. Draw latent variable  $z \in \{1, ..., K\}$  from a Categorical prior  $p(z; \mu)$  with parameter  $\mu \in \Delta^{K-1}$ . That is,  $p(z = k; \mu) = \mu_k$ .

<sup>&</sup>lt;sup>2</sup>Because discrete structured latent variables consist of interdependent discrete latent variables, we can also think of them as inducing interdependent clusterings of the parts of each data point.

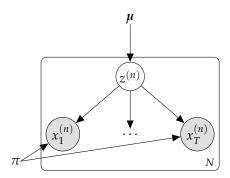


Figure 2: Naive Bayes graphical model. For simplicity, all sequences are depicted as having T tokens. All distributions are categorical, and the parameters are  $\mu \in \Delta^{K-1}$  and  $\pi = \{\pi_k \in \Delta^{V-1}\}_{k=1}^K$ .

2. Given z, draw each token  $x_t \in \{1, ..., V\}$  in x independently from a Categorical distribution with parameter  $\pi_z \in \Delta^{V-1}$ . That is,  $p(x_t = v; \pi_z) = \pi_{z,v}$ , where  $\pi_{z,v}$  is the probability of drawing word index v given the latent variable z. Thus, the probability of the sequence  $x = x_1, ..., x_T$  given z is

$$p(x \mid z; \; \boldsymbol{\pi}_z) = \prod_{t=1}^T \pi_{z,x_t} \; .$$

Letting  $\theta = [\mu, \pi_1, \dots, \pi_K]$  be all the parameters of our model, the full joint distribution is

$$p(x,z;\theta) = p(z;\mu) \times p(x|z;\pi_z) = \mu_z \times \prod_{t=1}^{T} \pi_{z,x_t}.$$
 (3)

This model assumes that each token in x is generated independently, conditioned on z. This assumption is clearly naive (hence the name) but greatly reduces the number of parameters that need to be estimated.<sup>3</sup> The total number of parameters in this generative model is K + KV, where we have K parameters for  $\mu$  and V parameters in  $\pi_z$  for each of the K values of z.<sup>4</sup>

Despite the Naive Bayes assumption, the above model becomes interesting when we have N sentences  $\{x^{(n)}\}_{n=1}^N$  that we assume to be generated according to the above process. Indeed, since each sentence  $x^{(n)}$  comes with a corresponding latent variable  $z^{(n)}$  governing its generation, we can see the  $z^{(n)}$  values as inducing a clustering over the sentences  $\{x^{(n)}\}_{n=1}^N$ ; sentences generated by the same value of  $z^{(n)}$  belong to the same cluster. We show a graphical model depicting this scenario in Figure 2.

**Making the Model "Deep"** One of the reasons we are interested in deep latent variable models is that neural networks make it simple to define flexible distributions without using too many parameters. As an example, we can formulate a sentence model similar to the Naive Bayes model, but which avoids the Naive Bayes assumption above (whereby each token is generated independently given z) using an RNN. An RNN will allow the probability of  $x_t$  to depend on the entire history  $x_{<t} = x_1, \dots, x_{t-1}$  of tokens preceding  $x_t$ . In this deep variant, we might then define the probability of x given latent variable z as

$$p(x \mid z) = \text{RNNLM}(x; \, \pi_z). \tag{4}$$

<sup>&</sup>lt;sup>3</sup>In our formulation we model text as *bag-of-words* and thus ignore position information. It is also possible to model position-specific probabilities within Naive Bayes with additional parameters  $\pi_{z,v,t}$  for  $t=1,\ldots T$ . This would result in KVT parameters.

<sup>&</sup>lt;sup>4</sup>The model is overparameterized since we only need V-1 parameters for a Categorical distribution over a set of size V. This is rarely an issue in practice.

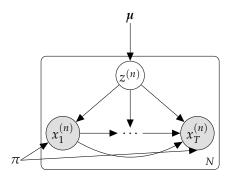


Figure 3: Graphical model representation of a categorical latent variable model with tokens generated by an RNN. For simplicity, all sequences are depicted as having T tokens. The  $z^{(n)}s$  are drawn from a Categorical distribution with parameter  $\mu$ , while  $x^{(n)}$  is drawn from an RNNLM $(x; \pi_{z^{(n)}})$ . These RNNLMs have parameters  $\pi = \{\pi_k\}_{k=1}^K$ .

That is, the probability of x given z is given by an RNNLM with parameters that are specific to the z that is drawn. We then obtain the joint distribution  $p(x, z; \theta)$  by substituting (4) into the term for  $p(x \mid z)$  in (3). We show the corresponding graphical model in Figure 3.

Note that this deep model allows for avoiding the Naive Bayes assumption while still using only  $O(Kd^2 + Vd)$  parameters, assuming  $h_t \in \mathbb{R}^d$ . The Naive Bayes model, on the other hand, requires O(KV) parameters, and O(KVT) parameters if we use position-specific distributions. Thus, as long as  $d^2$  is not too large, we can save parameters under the deep model as K and V get large.

**Inference** For discrete latent variable models, inference – that is, calculating the posterior  $p(z \mid x)$  – can typically be performed by enumeration. In particular, using Bayes's rule we have

$$p(z \mid x; \theta) = \frac{p(x, z; \theta)}{p(x; \theta)} = \frac{p(z; \theta)p(x \mid z; \theta)}{\sum_{k=1}^{K} p(z = k; \theta)p(x \mid z; \theta)},$$

where the latent variable z is assumed to take on one of K values. Calculating the denominator  $\sum_{k=1}^{K} p(z=k;\theta)p(x\,|\,z;\theta)$  is clearly the most computationally expensive part of inference, but is generally considered to be tractable as long as K is not too big. Note, however, that the model's parameterization can affect how quickly we can calculate this denominator: under the Naive Bayes model we can accumulate x's word counts once, and evaluate their probability under the K categorical distributions, whereas for the RNN based model we need to run K different RNNs over x.

#### 3.2 A Continuous Latent Variable Model

We now consider models in which the latent variables are vectors in  $\mathbb{R}^d$ , rather than integer-valued. We begin with a continuous analog of the Naive Bayes model in the previous subsection. In particular, we will assume a sequence is generated according to the following process:

- 1. Draw latent variable **z** from  $\mathcal{N}(\mu, I)$ —that is, from a Normal distribution with mean  $\mu$  and identity covariance matrix I.
- 2. Given  $\mathbf{z}$ , draw each token  $x_t \in \{1, ..., V\}$  in x independently from a Categorical distribution with softmax( $W\mathbf{z}$ ) as its parameters.

Note that the above model closely resembles the shallow model of Section 3.1, except that instead of using the latent variable *z* to index into a set of Categorical distribution parameters, we let the parameters of a

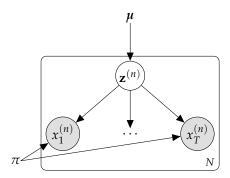


Figure 4: Continuous Naive Bayes model. The  $\mathbf{z}^{(n)}$  have a normal distribution  $\mathcal{N}(\boldsymbol{\mu},\mathbf{I})$ , and each token  $x_t^{(n)}$  has a Categorical distribution with parameter softmax $(\mathbf{W}\mathbf{z}^{(n)})$ . (For consistency with the previous models, we let  $\pi = \{\mathbf{W}\}$ ). Note that the dependence structure is identical to that in Figure 2; the only difference is the type of latent variable and the parameterizations.

Categorical distribution be a function of the latent **z**. We can thus view the latent variable **z** as a lower dimensional representation of our original sentence x. Letting  $\pi = \{W\}$  and  $\theta = \{\mu\} \cup \pi$ , we have the joint density

$$p(x, \mathbf{z}; \theta) = p(\mathbf{z}; \boldsymbol{\mu}) \times p(x \mid \mathbf{z}; \boldsymbol{W}) = \mathcal{N}(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{I}) \times \prod_{t=1}^{T} p(x_t \mid \mathbf{z}; \boldsymbol{W})$$
$$= \mathcal{N}(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{I}) \times \prod_{t=1}^{T} \operatorname{softmax}(\boldsymbol{W}\mathbf{z})_{x_t}. \tag{5}$$

We show a corresponding graphical model in Figure 4. Note that the dependence structure in Figure 4 is identical to that depicted in Figure 2; we have, however, changed the type of latent variable (from discrete to continuous) and the parameterizations of the corresponding distributions.

**Making the Model "Deep"** As in Section 3.1, we may replace the Naive Bayes distribution over tokens with one parameterized by an RNN. We thus have the generative process:

- 1. Draw latent variable **z** from  $\mathcal{N}(\mu, I)$ .
- 2. Given **z**, draw each token  $x_t \in \{1, ..., V\}$  in x from a *conditional* RNN, CRNNLM( $x_{1:T}; \pi, \mathbf{z}$ ).

We use the notation CRNNLM( $x_{1:T}$ ;  $\pi$ ,  $\mathbf{z}$ ) to refer to the distribution over sentences induced by conditioning an ordinary RNNLM on some vector  $\mathbf{z}$ , by concatenating  $\mathbf{z}$  onto the RNN's input at each time-step. In particular, we define

$$CRNNLM(x_{1:T}; \pi, \mathbf{z}) = \prod_{t=1}^{T} softmax(\mathbf{W}\mathbf{h}_{t})_{x_{t}},$$
 (6)

where

$$h_t = \text{RNN}(h_{t-1}, [\mathbf{x}_{t-1}; \mathbf{z}]; \pi). \tag{7}$$

Compare Equations (6) and (7) with Equations (1) and (2) in Section 2.

Then, letting  $\pi$  contain the parameters of the CRNNLM, we may write the joint density as

$$p(\mathbf{x}, \mathbf{z}; \theta) = p(\mathbf{z}; \boldsymbol{\mu}) \times p(\mathbf{x} | \mathbf{z}; \boldsymbol{W}) = \mathcal{N}(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{I}) \times \text{CRNNLM}(x_{1:T}; \pi, \mathbf{z}),$$
 (8)

and we show the corresponding graphical model in Figure 5. As before, the dependence structure mirrors that in Figure 3 exactly.

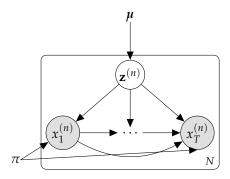


Figure 5: Continuous latent variable model with tokens generated by an RNN. The  $\mathbf{z}^{(n)}$  have a normal distribution  $\mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$ , and the  $x^{(n)}$  have a CRNNLM $(x_{1:T}^{(n)}; \pi, \mathbf{z}^{(n)})$  distribution, where  $\pi$  contains the parameters of the CRNNLM. Note that the dependence structure is identical to that in Figure 3; the only difference is the type of latent variable and the parameterizations.

Inference Again using Bayes's rule, the posterior under continuous latent variable models is given by

$$p(\mathbf{z} \mid x; \theta) = \frac{p(x, \mathbf{z}; \theta)}{p(x; \theta)} = \frac{p(\mathbf{z}; \theta)p(x \mid \mathbf{z}; \theta)}{\int p(\mathbf{z}; \theta)p(x \mid \mathbf{z}; \theta) d\mathbf{z}}.$$

Unlike with discrete latent variables, however, calculating the denominator above will in general be intractable. Indeed, using deep models will generally prevent us from exactly calculating the denominator above even in relatively simple cases. This concern motivates many of the methods we discuss in later sections.

### 3.3 A Structured, Discrete Latent Variable Model

Finally, we will consider a model with multiple interrelated discrete latent variables per data-point. In particular, we will consider the Hidden Markov Model (HMM) (Rabiner, 1989), which models a sentence by assuming there is a discrete latent variable responsible for generating each word in the sentence, and that each of these latent variables depends only on the latent variable responsible for generating the previous word. Concretely, an HMM assumes that a sentence is generated according to the following process:

First, begin with variable  $z_0$ , which is always equal to the special "start" state 0 (i.e.  $z_0 = 0$ ). Then, for t = 1, ..., T

- 1. Draw latent variable  $z_t \in \{1, ..., K\}$  from a Categorical distribution with parameter  $\mu_{z_{t-1}} \in \Delta^{K-1}$ .
- 2. Draw observed token  $x_t \in \{1, ..., V\}$  from a Categorical distribution with parameter  $\pi_{z_t} \in \Delta^{V-1}$ .

The above generative process gives rise to the following joint distribution over the sentence  $x = x_1, ..., x_T$  and latent variables  $z = z_1, ..., z_T$ :

$$p(x,z;\theta) = p(z_{1},...,z_{T};\mu) \times p(x_{1},...,x_{T} | z_{1},...,z_{T};\pi)$$

$$= \prod_{t=1}^{T} p(z_{t} | z_{t-1}; \mu_{z_{t-1}}) \times \prod_{t=1}^{T} p(x_{t} | z_{t}; \pi_{z_{t}})$$

$$= \prod_{t=1}^{T} \mu_{z_{t-1},z_{t}} \times \prod_{t=1}^{T} \pi_{z_{t},x_{t}}$$
(9)

We show the corresponding graphical model in Figure 6.

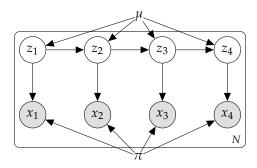


Figure 6: HMM graphical model for a sequence of length T=4.

It is important to note that the second equality above makes some significant independence assumptions: it assumes that the probability of  $z_t$  depends only on  $z_{t-1}$  (and not on  $z_{< t-1}$  or  $x_{< t}$ ), and it assumes that the probability of  $x_t$  depends only on  $z_t$  (and not on  $z_{< t}$  or  $x_{< t}$ ). These "Markov" (i.e., independence) assumptions are what give the Hidden Markov Model its name. We also note that we have referred to an HMM as a "structured" latent variable model because the latent sequence  $z = z_1, \ldots, z_T$  is structured in the sense that it contains multiple components that are interdependent, as governed by  $\mu$ .

**Making the Model "Deep"** We can create a "deep" HMM (c.f., Tran et al. (2016b); Johnson et al. (2016)) by viewing the  $\mu_z$  and  $\pi_z$  categorical parameters as being parameterizable in their own right, and parameterizing them with neural network components. For example, we might parameterize an HMM's categorical distributions as follows:

$$p(z_t | z_{t-1}) = \operatorname{softmax}(\operatorname{MLP}(z_{t-1}; \mu))$$
  $p(x_t | z_t) = \operatorname{softmax}(\operatorname{MLP}(z_{t-1}; \pi)),$ 

where  $\mu$  and  $\pi$  now refer to the parameters of the corresponding MLPs. Note that the graphical model in Figure 6 remains correct for the deep HMM; indeed, graphical models only show the dependency structure of the model (which has not been changed), and not the particular parameterization chosen. We also note that a deep parameterization may allow us to use fewer parameters as K and V get large. In particular, a standard HMM requires  $O(K^2 + KV)$  parameters, whereas if the MLPs above have d hidden units we require only  $O(Kd + Vd + d^2)$  parameters.

**Inference** For structured, discrete latent variable models we have

$$p(z \mid x; \theta) = \frac{p(x, z; \theta)}{p(x; \theta)} = \frac{p(z; \theta)p(x \mid z; \theta)}{\sum_{z'} p(z = z'; \theta)p(x \mid z'; \theta)},$$

where we index all possible latent structures (e.g., all sequence of discrete latent variables of length T) with z'. It is possible to compute this sum over z' structures with a dynamic program (e.g., the forward or backward algorithms (Rabiner, 1989)), and for certain models, like HMMs, it will be tractable to do so. For other, more complicated structured latent variable models the dynamic program will be intractable to compute, necessitating the use of approximate inference methods. For instance, for the Factorial HMM (FHMM) (Ghahramani and Jordan, 1996) depicted in Figure 7, which generates each word  $x_t$  by conditioning on the current state of *several* independent first-order Markov chains, calculation of the denominator above will be exponential in depth and therefore intractable even with a dynamic program.

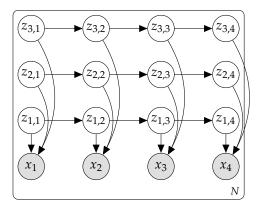


Figure 7: Factorial HMM graphical model for a sequence of length T=4.

# 4 Motivation and Examples

We now motivate the archetypal models we have introduced in the previous section by providing some examples of where similar models have been used in the literature. In general, we tend to be interested in latent variable models for any of the following interrelated reasons:

- (a) we have no or only partial supervision;
- (b) we want to make our model more interpretable through *z*;
- (c) we wish to model a multimodal distribution over x;
- (d) we would like to use latent variables to control our predictions and generations;
- (e) we are interested in learning some underlying structure or representation;
- (f) we want to better model the observed data;

and we will see that many of these apply to the models we will be discussing.

## 4.1 Examples of Discrete Latent Variable Models

As noted in Section 3, it is common to interpret discrete latent variable models as inducing a clustering over data. We now discuss several prominent NLP examples where discrete latent variable models can be interpreted in this way.

**Document Clustering** The paradigmatic example application of categorical latent variable models to text is document clustering (i.e., unsupervised document classification). In this setup we are given a set of N documents  $\{x^{(n)}\}_{n=1}^N$ , which we would like to partition into K clusters such that intra-cluster documents are maximally similar. This clustering can be useful for retrieving or recommending unlabeled documents similar to some query document. Seen from the generative modeling perspective described in Section 3.1, we view each document  $x^{(n)}$  as being generated by its corresponding latent cluster index,  $z^{(n)} \in \{1, \ldots, K\}$ , which gives us a model of  $p(x^{(n)}, z^{(n)}; \theta)$ . Since we are ultimately interested in obtaining the label (i.e., cluster index) of a document, however, we would then form the posterior over labels  $p(z^{(n)} | x^{(n)}; \theta) =$ 

$$\frac{p(x^{(n)},z^{(n)};\theta)}{\sum_{k=1}^K p(x^{(n)},z^{(n)}=k;\theta)}$$
 in order to determine the likely label for  $x^{(n)}.^5$ 

Work on document clustering goes back decades (see Willett (1988) and Aggarwal and Zhai (2012) for surveys), and many authors take the latent variable perspective described above. In terms of incorporating deep models into document clustering, it is possible to make use of neural components (such as RNNs) in modeling the generation of words in each document, as described in Section 3.1. It has been more common recently, however, to attempt to cluster real-valued vector-representations (i.e., embeddings) of documents. For instance, it is common to pre-compute document or paragraph embeddings with an unsupervised objective and then cluster these embeddings with K-Means (MacQueen et al., 1967); see Le and Mikolov (2014) and Xu et al. (2015a) for different approaches to obtaining these embeddings before clustering. Xie et al. (2016) continue to update document embeddings as they cluster, using an auxiliary loss derived from confidently clustered examples. Whereas none of these document-embedding based approaches are presented explicitly within the generative modeling framework above, they can all be viewed as identifying parameterized distributions over real vectors (e.g., Gaussians with particular means and covariances) with each cluster index, which in turn generate the *embeddings* of each document (rather than the words of each document). This sort of generative model is then (at least in principle) amenable to forming posteriors over cluster assignments, as above.

**Mixtures of Experts** Consider a supervised image captioning scenario, where we wish to produce a textual caption x in response to an image c, and we are given a training set of N aligned image-caption pairs  $\{(c^{(n)}, x^{(n)})\}_{n=1}^N$ . When training a model to generate captions similar to those in the training set, it may be useful to posit that there are in fact several "expert" captioning models represented in the training data, each of which is capable of generating a slightly different caption for the same image c. For instance, experts might differ in terms of the kind of language used in the caption (e.g., active vs. passive) or even in terms of what part of the image they focus on (e.g., foreground vs. whatever the human in the image is doing). Of course, we generally don't know beforehand which expert is responsible for generating which caption in the training data, but we can aim to capture the variance in captioning induced by these posited experts by identifying each value of a categorical latent variable z with a different posited expert, and assuming that a caption  $x^{(n)}$  is generated by first choosing one of these K experts, which in turn generates  $x^{(n)}$  conditioned on  $c^{(n)}$ . This type of model is known as a Mixture of Experts (MoE) model (Jacobs et al., 1991), and it gives rise to a joint distribution over  $x^{(n)}$  and  $z^{(n)}$ , only this time we will also condition on  $c^{(n)}$ :  $p(x^{(n)}, z^{(n)} | c^{(n)}; \theta)$ . As in the previous example, the discrete latents  $z^{(n)}$  induce a clustering, where examples are clustered by which expert generated them.

MoE models are widely used, and they are particularly suited to ensembling scenarios, where we wish to ensemble several different prediction models, which may have different areas of expertise. Garmash and Monz (2016) use an MoE to ensemble several expert neural machine translation models, and Lee et al. (2016) use a related approach, called "diverse ensembling," in training a neural image captioning system, showing that their model is able to generate more diverse captions as a result; He et al. (2018b) find that MoE also lead to more diverse responses in machine translation, and Gehrmann et al. (2018) use a similar technique for generating descriptions of restaurant databases. There has also been work in text-generation that uses an MoE model per *token*, rather than per sentence (Yin et al., 2016; Le et al., 2016; Yang et al., 2018b).<sup>6</sup> See also Eigen et al. (2013) and Shazeer et al. (2017), who use MoE *layers* in constructing neural network architectures, and in the latter case for the purpose of language modeling.

<sup>&</sup>lt;sup>5</sup>Note that if we are just interested in finding the most likely label for  $x^{(n)}$  we can simply evaluate  $\arg\max_k p(x^{(n)}, z^{(n)} = k; \theta)$ , since  $\arg\max_k p(x^{(n)}, z^{(n)} = k; \theta) = \arg\max_k \frac{p(x^{(n)}, z^{(n)} = k; \theta)}{\sum_{k'=1}^K p(x^{(n)}, z^{(n)} = k'; \theta)}$ .

<sup>&</sup>lt;sup>6</sup>Although these models have multiple discrete latents per data point, which suggests that we should perhaps consider them to be structured latent variable models, we will consider them unstructured since the interdependence between token-level latents is not made explicit in the probability model; correlations between these latents, however, are undoubtedly modeled to some extent by the associated RNNs.

Note that in the case of MoE models, we are interested in using a latent variable model not just because the training examples are not labeled with which experts generated them (reason (a) above), but also for reasons (c) and (d). That is, we attempt to capture different modes in the caption distribution (perhaps corresponding to different styles of caption, for example) by identifying a latent variable with each of these experts or modes. Similarly, we might hope to control the style of the caption by restricting our captioning system to one expert. When it comes to document clustering, on the other hand, we are interested in latent variables primarily because we don't have labeled clusters for our documents (reason (a)). Indeed, since we are interested primarily in the *posterior* over  $z^{(n)}$ , rather than the joint distribution, reasons (c) and (d) are not applicable.

## 4.2 Examples of Continuous Latent Variable Models

Continuous latent variable models can often be interpreted as performing a dimensionality reduction of data, by associating a low-dimensional vector in  $\mathbb{R}^d$  with a data point. One of the advantages of this sort of dimensionality reduction is that it becomes easier to have a finer-grained notion of similarity between data points, by calculating their distance, for example, in the dimensionally reduced space. We now discuss two prominent examples.

**Topic Models** Topic Models (Blei et al., 2003) are an enormously influential family of latent variable models, which posit a set of K latent "topics," realized as distributions over words, which in turn generate each document  $x^{(n)}$  in a collection. Each document  $x^{(n)}$  in the collection moreover has a latent distribution  $\mathbf{z}^{(n)} \in \Delta^{K-1}$  over these K topics, governing how often it chooses to discuss a particular topic. Note that the per-document distribution over topics  $\mathbf{z}^{(n)}$  in a topic model plays a similar role to the per-sentence vector  $\mathbf{z}^{(n)}$  in the shallow model of Section 3.2. Thus, in both cases we might, for instance, determine how similar two data points are by measuring the distance between their corresponding  $\mathbf{z}^{(n)}$ 's.

**Sentence Generation** Whereas, with the exception of Topic Models, NLP has historically preferred discrete latent variables, the tendency of deep models to deal with continuous *embeddings* of objects has spurred interest in continuous latent variable models for NLP. For example, it has recently become quite popular to view a sentence as being generated by a latent vector in  $\mathbb{R}^d$ , rather than by a latent label in  $\{1,\ldots,K\}$  as in the previous subsection. Thus, Bowman et al. (2016) develop a latent vector model of sentence generation, where sentences are generated with a CRNNLM, in a very similar way to that presented in Section 3.2. This model and its extensions (Yang et al., 2017; Hu et al., 2017; Kim et al., 2018), represent the dominant approach to neural latent variable modeling of sentence generation at this time.

As suggested above, viewing sentences as being generated by real vectors gives us a fine-grained notion of similarity between sentences, namely, the distance between the corresponding latent representations. The desire to compute similarities between sentences has also motivated extensive work on obtaining sentence embeddings (Le and Mikolov, 2014; Kiros et al., 2015; Joulin et al., 2016; Conneau et al., 2017; Peters et al., 2018; Pagliardini et al., 2018; Rücklé et al., 2018), which can often be interpreted as continuous latent variables, though they need not be.

# 4.3 Example of Structured, Discrete Latent Variable Models

A major motivation for using structured, discrete latent variables in NLP is the desire to infer structured discrete objects, such as parse trees or sequences or segmentations, which we believe to be represented in the data, even when they are unannotated.

Unsupervised Tagging and Parsing The simplest example application of structured, discrete latent variable modeling in NLP involves inducing the part-of-speech (POS) tags for a sentence. More formally, we are given a sentence  $x = x_1, ..., x_T$ , and we wish to arrive at a sequence  $z = z_1, ..., z_T$  of POS tags, one for each word in the sentence. HMMs, as described in Section 3.3, and their variants have historically been the dominant approach to arriving at a joint distribution  $p(x_1, ..., x_T, z_1, ..., z_T; \theta)$  over words and tags (Brown et al., 1992; Merialdo, 1994; Smith and Eisner, 2005; Haghighi and Klein, 2006; Johnson, 2007; Toutanova and Johnson, 2008; Berg-Kirkpatrick et al., 2010; Christodoulopoulos et al., 2010; Blunsom and Cohn, 2011; Stratos et al., 2016). We may then predict the POS tags for a new sentence x by calculating

$$\arg \max_{z_1,...,z_T} p(z_1,...,z_T \,|\, x_1,...,x_T;\, \theta) = \arg \max_{z_1,...,z_T} p(x_1,...,x_T,z_1,...,z_T;\, \theta).$$

There is now a growing body of work involving deep parameterizations of structured discrete latent variable models for unsupervised parsing and tagging. For instance, Tran et al. (2016b) obtain good results on unsupervised POS tagging by parameterizing the transition and emission probabilities of an HMM with neural components, as described in Section 3.3. In addition, just as recent approaches to neural document clustering have defined models that generate document *embeddings* rather than the documents themselves, there has been recent work in neural, unsupervised POS tagging based on defining neural HMM-style models that emit word *embeddings*, rather than words themselves (Lin et al., 2015; He et al., 2018a).

Unsupervised dependency parsing represents an additional, fairly simple example application of neural models with structured latent variables. In unsupervised dependency parsing we attempt to induce a sentence's dependency tree without any labeled training data. Much recent unsupervised dependency parsing is based on the DMV model of Klein and Manning (2004) and its variants (Headden III et al., 2009; Spitkovsky et al., 2010, 2011), where there are multiple discrete latent variables per word, rather than one, as in POS tagging. In particular, the DMV model can be viewed as providing a joint distribution over the words  $x_1, \ldots, x_T$  in a sentence and discrete latent variables  $z_1, \ldots, z_{3T}$  representing each left and right dependent of each word, as well as a final empty left and right dependent for each word. As in the case of unsupervised tagging, we are primarily interested in predicting the most likely dependency tree given a sentence:

$$\arg \max_{z_1,...,z_{3T}} p(z_1,...,z_{3T} \mid x_1,...,x_T; \theta) = \arg \max_{z_1,...,z_{3T}} p(x_1,...,x_T,z_1,...,z_{3T}; \theta).$$

Again, neural approaches divide between those that parameterize DMV-like models, which jointly generate the dependency tree and its words, with neural components (Jiang et al., 2016; Cai et al., 2017; Han et al., 2017), and those which define DMV-models that generate embeddings (He et al., 2018a).

**Text Generation** Structured latent variables are also widely used in text generation models and applications. For instance, Miao and Blunsom (2016) learn a summarization model by positing the following generative process for a document: we first sample a condensed, summary version of a document  $z_{1:T_1}$  from an RNNLM( $z_{1:T_1}$ ;  $\theta$ ). Then, conditioned on  $z_{1:T_1}$ , we generate the full version of the document:  $x_1, \ldots, x_{T_2}$  by sampling from a CRNNLM( $x_{1:T_2}$ ;  $\theta$ ,  $z_{1:T_1}$ ). This model gives us a joint distribution over documents  $x_{1:T_2}^{(n)}$  and their corresponding summaries  $z_{1:T_1}^{(n)}$ . Moreover, a summarization system is then merely a system that infers the posterior over summaries  $z_{1:T_1}^{(n)}$  given a document  $x_{1:T_2}^{(n)}$ .

Whereas in the model of Miao and Blunsom (2016) a document x is generated by conditioning solely on latent variables z, it is also common to posit generative models in which text x is generated by conditioning both on some latent variables z as well as on some additional contextual information c. In particular, c might represent an image for an image captioning model, a sentence in a different language for a machine translation model, or a database for data-to-document generation model. In this setting, the latent variables z would then represent some additional unobserved structure that, together with c, accounts for the

observed text x. Some notable recent examples in this vein include viewing text x as being generated by both c as well as a shorter sequence of latent variables z (Kaiser et al., 2018; Roy et al., 2018), a sequence of fertility latent variables z (Gu et al., 2018a), a sequence of iteratively refined sequences of latent variables z (Lee et al., 2018), or by a latent template or plan z (Wiseman et al., 2018).

# 5 Learning and Inference

After defining a latent-variable model, we are typically interested in being able to do two related tasks: (1) we would like to be able to learn the parameters  $\theta$  of the model, and (2) once trained, we would like to be able to perform *inference* over the model. That is, we'd like to be able to compute the posterior distribution  $p(z \mid x; \theta)$  (or approximations thereof) over the latent variables, given some data x. As we will see, these two tasks are intimately connected because learning often uses inference as a subroutine. On an intuitive level, this is because if we *knew*, for instance, the value of z given x, learning  $\theta$  would be simple: we would simply maximize  $p(x \mid z_{known}; \theta)$ . Thus, as we will see, learning often involves alternately inferring likely z values, and optimizing the model assuming these inferred z's.

The dominant approach to learning latent variable models in a probabilistic setting is to maximize the log marginal likelihood. This is equivalent to minimizing  $KL[p_{\star}(x)||p(x;\theta)]$ , the KL-divergence between the true data distribution  $p_{\star}(x)$  and the model distribution  $p(x;\theta)$ , where the latent variable z has been marginalized out. It is also possible to approximately minimize other divergences between  $p_{\star}(x)$  and  $p(x;\theta)$ , e.g. the Jensen-Shannon divergence or the Wasserstein distance. In the context of deep latent variable models, such methods often utilize a separate model (discriminator/critic) which learns to distinguish between samples from  $p_{\star}(x)$  from samples from  $p(x;\theta)$ . The generative model  $\theta$  is trained "adversarially" to fool the discriminator. This gives rise to a family of models known as Generative Adversarial Networks (GANs) (Goodfellow et al., 2014). While not the main focus of the this tutorial, we review GANs and their applications to text modeling in section 7.3.

## 5.1 Directly Maximizing the Log Marginal Likelihood

We begin with cases where the log marginal likelihood, i.e.

$$\log p(x; \theta) = \log \sum_{z} p(x, z; \theta)$$

is tractable to evaluate. (The sum should be replaced with an integral if *z* is continuous). This is equivalent to assuming posterior inference is tractable, since

$$p(z \mid x; \theta) = \frac{p(x, z; \theta)}{p(x; \theta)}.$$

Calculating the log marginal likelihood is indeed tractable in some of the models that we have seen so far, such as categorical latent variable models where K is not too big, or certain structured latent variable models (like HMMs) where dynamic programs allow us to efficiently sum over all the z assignments. In such cases, maximum likelihood training of our parameters  $\theta$  then corresponds to solving the following maximization problem:

$$\underset{\theta}{\arg\max} \sum_{n=1}^{N} \log p(x^{(n)}; \theta),$$

where we have assumed *N* examples in our training set.

In cases where  $p(x, z; \theta)$  is parameterized by a deep model, the above maximization problem is not tractable to solve exactly. We will assume, however, that  $p(x, z; \theta)$  is differentiable with respect to  $\theta$ . The main tool for optimizing such models, then, is gradient-based optimization. In particular, define the log marginal likelihood over the training set  $x^{(1:N)} = [x_1, \dots, x_N]$  as

$$L(\theta) = \log p(x^{(1:N)}; \theta) = \sum_{n=1}^{N} \log p(x^{(n)}; \theta) = \sum_{n=1}^{N} \log \sum_{z} p(x^{(n)}, z; \theta).$$

The gradient is given by

$$\begin{split} \nabla_{\theta}L(\theta) &= \sum_{n=1}^{N} \frac{\nabla_{\theta} \sum_{z} p(x^{(n)}, z; \theta)}{p(x^{(n)}; \theta)} \qquad \text{(chain rule)} \\ &= \sum_{n=1}^{N} \sum_{z} \frac{p(x^{(n)}, z; \theta)}{p(x^{(n)}; \theta))} \nabla_{\theta} \log p(x^{(n)}, z; \theta) \qquad \text{(since } \nabla p(x, z) = p(x, z) \nabla \log p(x, z)) \\ &= \sum_{n=1}^{N} \mathbb{E}_{p(z \mid x^{(n)}; \theta)} [\nabla_{\theta} \log p(x^{(n)}, z; \theta)] \end{split}$$

Note that the above gradient expression involves an expectation over the posterior  $p(z \mid x^{(n)}; \theta)$ , and is therefore an example of how inference is used as a subroutine in learning. With this expression for the gradient in hand, we may then learn by updating the parameters as

$$\theta^{(i+1)} = \theta^{(i)} + \eta \nabla_{\theta} L(\theta^{(i)}),$$

where  $\eta$  is the learning rate and  $\theta^{(0)}$  is initialized randomly. In practice the gradient is calculated over *minibatches* (i.e. random subsamples of the training set), and adaptive algorithms (Duchi et al., 2011; Zeiler, 2012; Kingma and Ba, 2015) are often used.

# 5.2 Expectation Maximization (EM) Algorithm

The Expectation Maximization (EM) algorithm (Dempster et al., 1977) is an iterative method for learning latent variable models with tractable posterior inference. It maximizes a lower bound on the log marginal likelihood at each iteration. Given randomly-initialized starting parameters  $\theta^{(0)}$ , the algorithm updates the parameters via the following alternating procedure:

- 1. E-step: Derive the posterior under current parameters  $\theta^{(i)}$ , i.e.,  $p(z \mid x^{(n)}; \theta^{(i)})$  for all n = 1, ..., N.
- 2. M-step: Define the expected complete data likelihood as

$$Q(\theta, \theta^{(i)}) = \sum_{n=1}^{N} \mathbb{E}_{p(z \mid x^{(n)}; \, \theta^{(i)})} [\log p(x^{(n)}, z; \, \theta)]$$

Maximize this with respect to  $\theta$ , holding  $\theta^{(i)}$  fixed

$$\theta^{(i+1)} = \operatorname*{arg\,max}_{\theta} Q(\theta, \theta^{(i)})$$

It can be shown that EM improves the log marginal likelihood at each iteration, i.e.

$$\sum_{n=1}^{N} \log p(x^{(n)}; \theta^{(i+1)}) \ge \sum_{n=1}^{N} \log p(x^{(n)}; \theta^{(i)}).$$

As a simple example, let us apply the above recipe to the Naive Bayes model in section 3.1, with K = 2:

1. E-step: for each n = 1, ..., N, calculate

$$p(z \mid x^{(n)}; \, \theta^{(i)}) = \frac{p(x^{(n)}, z; \, \theta^{(i)})}{\sum_{z' \in \{1,2\}} p(x^{(n)}, z'; \, \theta^{(i)})} = \frac{(\mu_1^{(i)})^{\mathbb{I}[z=1]} (1 - \mu_1^{(i)})^{\mathbb{I}[z=2]} \prod_{t=1}^{T} \pi_{z, x_t^{(n)}}^{(i)}}{(\mu_1^{(i)}) \prod_{t=1}^{T} \pi_{1, x_t^{(n)}}^{(i)} + (1 - \mu_1^{(i)}) \prod_{t=1}^{T} \pi_{2, x_t^{(n)}}^{(i)}}.$$

Note that above we have written the prior over z in terms of a single parameter  $\mu_1$ , since we must have  $\mu_2 = 1 - \mu_1$ .

2. M-step: The expected complete data likelihood is given by

$$Q(\theta, \theta^{(i)}) = \sum_{n=1}^{N} \sum_{z} p(z \mid x^{(n)}; \ \theta^{(i)}) (\log p(x^{(n)} \mid z; \ \pi) + \log p(z; \ \mu_1))$$

To maximize the above with respect to  $\mu_1$ , we can differentiate and set the resulting expression to zero. Using the indicator notation  $\mathbb{1}[\cdot]$  to refer to a function that returns 1 if the condition in the bracket holds and 0 otherwise, we have

$$\begin{split} \frac{\partial Q(\theta,\theta^{(i)})}{\partial \mu_{1}} &= \frac{\partial \sum_{n} \sum_{z} p(z \mid x^{(n)}; \, \theta^{i}) \log(\mu_{1}^{\mathbb{I}[z=1]} (1-\mu_{1})^{\mathbb{I}[z=2]})}{\partial \mu_{1}} = 0 \\ & \Longrightarrow \frac{\sum_{n=1}^{N} \sum_{z} p(z \mid x^{(n)}; \, \theta^{(i)}) \mathbb{I}[z=1]}{\mu_{1}} = \frac{\sum_{n=1}^{N} \sum_{z} p(z \mid x^{(n)}; \, \theta^{(i)}) \mathbb{I}[z=2]}{1-\mu_{1}} \\ & \Longrightarrow \mu_{1}(\sum_{n=1}^{N} \sum_{z} p(z \mid x^{(n)}; \, \theta^{(i)})) = \sum_{n=1}^{N} \sum_{z} p(z \mid x^{(n)}; \, \theta^{(i)}) \mathbb{I}[z=1] \\ & \Longrightarrow \mu_{1}^{(i+1)} = \frac{\sum_{n=1}^{N} \sum_{z} p(z \mid x^{(n)}; \, \theta^{(i)}) \mathbb{I}[z=1]}{N} = \frac{\sum_{n=1}^{N} q_{1}^{(n)}}{N}, \end{split}$$

where  $q_1^{(n)} = \sum_z p(z \mid x^{(n)}; \theta^{(i)}) \mathbb{1}[z=1] = \mathbb{E}_{p(z \mid x^{(n)}; \theta^{(i)})} \mathbb{1}[z=1]$ . (We can verify that the above is indeed the maximum since  $\frac{\partial^2 Q(\theta, \theta^{(i)})}{\partial \mu_1^2} < 0$ ). A similar derivation for  $\pi_{z,v}$  yields

$$\pi_{z,v}^{(i+1)} = \frac{\sum_{n=1}^{N} q_z^{(n)} \sum_{t=1}^{T} \mathbb{1}[x_t^{(n)} = v] / T}{\sum_{t=1}^{N} q_z^{(n)}}.$$

Note that these updates are analogous to the maximum likelihood parameters of a Naive Bayes model in the supervised case, except that the empirical counts  $\sum_{i=1}^N \mathbb{1}[z^{(n)}=z]$  have been replaced with the expected counts  $\sum_{i=1}^N q_z^{(n)}$  under the posterior distribution.

Let us now consider using EM to learn the parameters of the RNN model introduced at the end of Section 3.1. Here, the E-step is similar to the Naive Bayes model and follows straightforwardly from Bayes' rule:

$$p(z \mid x^{(n)}; \, \theta^{(i)}) = \frac{p(x^{(n)}, z; \, \theta^{(i)})}{\sum_{z' \in \{1,2\}} p(x^{(n)}, z'; \, \theta^{(i)})} = \frac{(\mu_1^{(i)})^{\mathbb{I}[z=1]} (1 - \mu_1^{(i)})^{\mathbb{I}[z=2]} \prod_{t=1}^T p(x_t^{(n)} \mid x_{< t}, z; \, \theta^{(i)})}{\sum_{z' \in \{1,2\}} (\mu_1^{(i)})^{\mathbb{I}[z'=1]} (1 - \mu_1^{(i)})^{\mathbb{I}[z'=2]} \prod_{t=1}^T p(x_t^{(n)} \mid x_{< t}, z'; \, \theta^{(i)})}.$$

Unlike with Naive Bayes, however, there is no closed-form update for the M step.<sup>7</sup> In this case, we can perform gradient-based optimization,

$$\theta^{(i+1)} = \theta^{(i)} + \eta \nabla_{\theta} Q(\theta, \theta^{(i)}),$$

 $<sup>^{7}</sup>$ A closed-form update exists for  $\mu$  but not for the RNN parameters.

where  $\eta$  is the learning rate and the gradient is given by

$$\nabla_{\theta} Q(\theta, \theta^{(i)}) = \sum_{i=1}^{N} \mathbb{E}_{p(z \mid x^{(n)}; \theta^{(i)})} [\nabla_{\theta} \log p(x^{(n)}, z; \theta)].$$

This variant of EM is sometimes referred to as *generalized expectation maximization* (Dempster et al., 1977; Neal and Hinton, 1998; Murphy, 2012). Note that the above expression is in fact the same as the gradient of the log marginal likelihood from the previous section, i.e.  $\nabla_{\theta}Q(\theta,\theta^{(i)}) = \nabla_{\theta}L(\theta)$ . Therefore, generalized EM is equivalent to directly performing gradient ascent on the log marginal likelihood. This connection between EM and gradient ascent on the log marginal likelihood has been noted in the literature before (Salakhutdinov et al., 2003; Berg-Kirkpatrick et al., 2010; Sutton et al., 2012), and is perhaps unsurprising given that backpropagation on the log marginal likelihood implicitly performs posterior inference (Eisner, 2016).

The connection between generalized EM and gradient ascent on the log marginal likelihood is particularly relevant to deep generative models, which will generally not admit an exact M-step. Practically speaking, we may avoid manually calculating the posteriors in the E-step and then taking gradient steps in  $Q(\theta, \theta^{(i)})$ , and instead take gradient steps directly on  $\log p(x;\theta)$ ; this is made especially convenient with automatic differentiation tools. For example, when training a deep Hidden Markov Model (HMM) as described in Section 3.3, we can use the forward (or backward) algorithm to calculate  $\log p(x;\theta)$ , and call backpropagation on the resulting value, instead of manually implementing the backward (or forward) algorithm to obtain the posteriors. Similarly, to train Probabilistic Context-Free Grammars (PCFG) with neural parameterizations of rule probabilities, we can run the inside algorithm to calculate the log marginal likelihood and call backpropagation on the resulting value instead of manually implementing the outside algorithm.

Finally, we note that the EM algorithm can in general be been seen as performing coordinate ascent on a lower bound on the log marginal likelihood (Bishop, 2006). This view (elaborated further in the next section) will become useful when considering cases where posterior inference is *intractable*, and will motivate *variational inference*—a class of methods which uses approximate but tractable posteriors in place of the true posterior.

#### 5.3 Variational Inference

So far we have considered models in which posterior inference (or equivalently, calculation of the log marginal likelihood) is tractable either via enumeration or dynamic programming. Now we consider cases in which posterior inference is intractable. Variational inference (Hinton and van Camp, 1993; Jordan et al., 1999) is a technique for approximating an intractable posterior distribution  $p(z \mid x; \theta)$  with a tractable surrogate. In the context of learning the parameters of a latent variable model, variational inference can be used in optimizing a lower bound on the log marginal likelihood that involves only an *approximate* posterior over latent variables, rather than the exact posteriors we have been considering until now.

We begin by defining a set of distributions  $\mathcal{Q}$ , known as the *variational family*, whose elements are distributions  $q(z;\lambda)$  parameterized by  $\lambda$ . That is,  $\mathcal{Q}$  contains distributions over our latent variables z. We will use  $\mathcal{Q}$  to denote the entire variational family, and  $q(z;\lambda) \in \mathcal{Q}$  to refer to a particular variational distribution within the variational family, which is picked out by  $\lambda$ . Let us assume that z is continuous. We now derive a lower bound on the marginal log-likelihood  $\log p(x;\theta) = \log \int_z p(x,z;\theta) dz$  that makes use of a  $q(z;\lambda)$  distribution.

<sup>&</sup>lt;sup>8</sup>Note, though, that taking gradient steps to maximize  $\log p(x;\theta)$  is only equivalent to doing a *single* gradient step during the M-step of generalized EM, and we are not guaranteed that the log marginal will increase monotonically as in standard EM.

<sup>&</sup>lt;sup>9</sup>See Kong et al. (2016), Yu et al. (2016, 2017b), Wiseman et al. (2018), and Kawakami et al. (2018) for concrete examples of performing direct marginalization over latent variables in deep generative models.

We can lower bound the log marginal likelihood as follows:

$$\log p(x;\theta) = \int q(z;\lambda) \log p(x;\theta) \, dz \qquad \qquad \text{(expectation of non-random quantity)}$$

$$= \int q(z;\lambda) \log \frac{p(x,z;\theta)}{p(z\,|\,x;\theta)} \, dz \qquad \qquad \text{(rewriting } p(x;\theta), \text{ but see below)}$$

$$= \int q(z;\lambda) \log \left(\frac{p(x,z;\theta)}{q(z;\lambda)} \frac{q(z;\lambda)}{p(z\,|\,x;\theta)}\right) \, dz \qquad \qquad \text{(multiplying by 1)}$$

$$= \int q(z;\lambda) \log \frac{p(x,z;\theta)}{q(z;\lambda)} \, dz + \int q(z;\lambda) \log \frac{q(z;\lambda)}{p(z\,|\,x;\theta)} \, dz \qquad \text{(distribute log)}$$

$$= \int q(z;\lambda) \log \frac{p(x,z;\theta)}{q(z;\lambda)} \, dz + \text{KL}[q(z;\lambda) \parallel p(z\,|\,x;\theta)] \qquad \text{(definition of KL divergence)}$$

$$= \mathbb{E}_{q(z;\lambda)} \log \frac{p(x,z;\theta)}{q(z;\lambda)} \quad + \text{KL}[q(z;\lambda) \parallel p(z\,|\,x;\theta)] \qquad \text{(definition of expectation)}$$

$$= \text{ELBO}(\theta,\lambda;x) \qquad + \text{KL}[q(z;\lambda) \parallel p(z\,|\,x;\theta)] \qquad \text{(definition of ELBO)}$$

$$\geq \text{ELBO}(\theta,\lambda;x) \qquad \text{(KL always non-negative)}$$

The above derivation shows that  $\log p(x;\theta)$  is equal to a quantity called the *evidence lower bound*, or ELBO, plus the KL divergence between  $q(z;\lambda)$  and the posterior distribution  $p(z\,|\,x;\theta)$ . Since the KL divergence is always non-negative, the ELBO is a lower-bound on  $\log p(x;\theta)$ , and it is this quantity that we attempt to maximize with variational inference. Before discussing the ELBO in more depth, we note that the above derivation requires that the support of the variational distribution lie within the support of the true posterior, i.e.,  $p(z\,|\,x;\theta)=0 \implies q(z;\lambda)=0$  for all  $z.^{10}$ 

The form of the ELBO is worth looking at more closely. First, note that it is a function of  $\theta$ ,  $\lambda$  (the data x is fixed), and lower bounds the log marginal likelihood  $\log p(x;\theta)$  for any  $\lambda$ . The bound is tight if the variational distribution equals the true posterior, i.e.  $q(z;\lambda) = p(z \mid x;\theta)$  for all  $z \implies \log p(x;\theta) = \text{ELBO}(\theta,\lambda;x)$ . It is also immediately evident that

$$ELBO(\theta, \lambda; x) = \log p(x; \theta) - KL[q(z; \lambda) \parallel p(z \mid x; \theta)].$$

In some scenarios the model parameters  $\theta$  are given (and thus fixed), and the researcher is tasked with finding the best variational approximation to the true posterior. Under this setup,  $\log p(x;\theta)$  is a constant and therefore maximizing the ELBO is equivalent to minimizing  $\mathrm{KL}[q(z;\lambda) \parallel p(z \mid x;\theta)]$ . However for our purposes we are also interested in learning the generative model parameters  $\theta$ .

Letting  $x^{(1:N)} = [x_1, ..., x_N]$  be the training set, the ELBO over the entire dataset is given by the sum of individual ELBOs,

$$\text{ELBO}(\theta, \lambda; \, \boldsymbol{x}^{(1:N)}) = \sum_{n=1}^{N} \text{ELBO}(\theta, \lambda^{(n)}; \, \boldsymbol{x}^{(n)}) = \sum_{n=1}^{N} \mathbb{E}_{q(z; \lambda^{(n)})} \Big[ \log \frac{p(\boldsymbol{x}^{(n)}, z; \, \theta)}{q(z; \, \lambda^{(n)})} \Big],$$

where the variational parameters are given by  $\lambda = [\lambda^{(1)}, \dots, \lambda^{(n)}]$  (i.e. we have  $\lambda^{(n)}$  for each data point  $x^{(n)}$ ). Since  $x^{(n)}$  are assumed to be drawn i.i.d, it is clear that the aggregate ELBO lower bounds the log likelihood of the training corpus

ELBO
$$(\theta, \lambda; x^{(1:N)}) \le \log p(x^{(1:N)}; \theta).$$

<sup>&</sup>lt;sup>10</sup>Otherwise, the second equality would have a division by zero. In contrast, we can have  $q(z; \lambda) = 0$  and  $p(z \mid x; \theta) > 0$  for some z, since the integral remains unchanged if we just integrate over the set  $E = \{z : q(z; \lambda) > 0\}$ .

<sup>&</sup>lt;sup>11</sup>There are many other possible parameterizations. For example, instead of nonparametric-style inference where  $\lambda$  is the union of local variational parameters  $\lambda^{(n)}$ , one could have a parameteric model with global parameters  $\phi$  that is run over  $x^{(n)}$  to produce the local variational parameters  $\lambda^{(n)}$ . This style of inference is called *amortized variational inference* (see section 6.1).

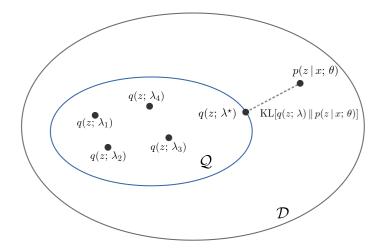


Figure 8: Illustration of variational inference.  $\mathcal{D}$  represents all possible distributions over z, and  $\mathcal{Q}$  represents the variational family with free parameters  $\lambda$ . Variational inference finds the distribution within  $\mathcal{Q}$  that is closest to  $p(z \mid x; \theta)$  in the reverse KL sense, i.e.  $\mathrm{KL}[q(z; \theta) \parallel p(z \mid x; \theta)]$ . In the above this distribution is denoted as  $q(z; \lambda^*)$ .

It is this aggregate ELBO that we wish to maximize with respect to  $\theta$  and  $\lambda$  to train our model.

### 5.3.1 Maximizing the ELBO

One possible strategy for maximizing the aggregate ELBO is coordinate ascent, where we maximize the objective with respect to the  $\lambda^{(n)}$ 's keeping  $\theta$  fixed, then maximize with respect to  $\theta$  keeping the  $\lambda^{(n)}$ 's fixed. In particular, we arrive at the following:

1. Variational E-step: For each n = 1, ..., N, maximize the ELBO for each  $x^{(n)}$  holding  $\theta^{(i)}$  fixed

$$\lambda^{(n)} = \arg\max_{\lambda} \text{ELBO}(\theta^{(i)}, \lambda; x^{(n)}) = \arg\min_{\lambda} \text{KL}[q(z; \lambda^{(n)}) \parallel p(z \mid x^{(n)}; \theta^{(i)})],$$

where the second equality holds since  $\log p(x; \theta^{(i)})$  is a constant with respect to the  $\lambda^{(n)}$ 's.

2. Variational M-step: Maximize the aggregated ELBO with respect to  $\theta$  holding the  $\lambda^{(n)}$ 's fixed

$$\theta^{(i+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \text{ELBO}(\theta, \lambda^{(n)}; \, \boldsymbol{x}^{(n)}) = \arg\max_{\theta} \sum_{n=1}^{N} \mathbb{E}_{q(\boldsymbol{z}; \lambda^{(n)})}[\log p(\boldsymbol{x}^{(n)}, \boldsymbol{z}; \, \theta)],$$

where the second equality holds since the  $\mathbb{E}_{q(z;\lambda^{(n)})}[-\log q(z;\lambda^{(n)})]$  portion of the ELBO is constant with respect to  $\theta$ .

This style of training is also known as *variational expectation maximization* (Neal and Hinton, 1998). In variational EM, the E-step, which usually performs exact posterior inference, is instead replaced with variational inference which finds the best<sup>12</sup> variational approximation to the true posterior. The E-step is illustrated in Figure 8. The M-step maximizes the expected complete data likelihood where the expectation is taken with respect to the variational posterior.<sup>13</sup>

<sup>&</sup>lt;sup>12</sup>In the "reverse KL" sense, i.e.  $KL[q(z; \lambda) \parallel p(z \mid x; \theta)]$ .

<sup>&</sup>lt;sup>13</sup>Yet another variant of EM is *Monte Carlo Expectation Maximization* (Wei and Tanner, 1990), which obtains samples from the posterior in the E-step and maximizes the Monte Carlo estimate of the complete data likelihood in the M-step. In deep generative models sampling from the true posterior requires expensive procedures such as MCMC, although there has been some work on combining variational inference with Hamiltonian Monte Carlo (Salimans et al., 2015; Hoffman, 2017).

If we consider the case where the variational family is flexible enough to include the true posterior, <sup>14</sup> then it is clear that the above reduces to the classic EM algorithm, since in the first step  $\mathrm{KL}[q(z;\lambda^{(n)}) \parallel p(z \mid x^{(n)};\theta^{(i)})]$  is minimized when  $q(z;\lambda^{(n)})$  equals the true posterior. Therefore, we can view EM as performing coordinate ascent on the ELBO where the variational family is arbitrarily flexible. Of course, this case is uninteresting since we have assumed that exact posterior inference is intractable. We are therefore interested in choosing a variational family that is flexible enough and at the same time allows for tractable optimization.

In practice, performing coordinate ascent on the entire dataset is usually too expensive. The variational Estep can instead be performed over mini-batches. As with generalized EM, the M-step can also be modified to perform gradient-based optimization. It is also possible to perform the E-step only approximately, again using gradient-based optimization. This style of approach leads to a class of methods called *stochastic variational inference (SVI)* (Hoffman et al., 2013). Concretely, for each  $x^{(n)}$  in the mini-batch (of size B) we can randomly initialize  $\lambda_0^{(n)}$  and perform gradient ascent on the ELBO with respect to  $\lambda$  for K steps,

$$\lambda_k^{(n)} = \lambda_{k-1}^{(n)} + \eta \nabla_{\lambda} \text{ ELBO}(\theta, \lambda_k^{(n)}; x^{(n)}), \quad k = 1, \dots, K$$

Then the M-step, which updates  $\theta$ , proceeds with the variational parameters  $\lambda_K^{(1)},\dots,\lambda_K^{(B)}$  held fixed

$$\theta^{(i+1)} = \theta^{(i)} + \eta \nabla_{\theta} \sum_{n=1}^{B} \mathbb{E}_{q(z \mid \Lambda_{K}^{(n)})}[\log p(x^{(n)}, z; \theta^{(i)})]$$

In general, variational inference is a rich field of active research, and we have only covered a small portion of it in this section. For example, we have not covered *coordinate ascent variational inference*, which allows for closed-form updates in the E-step for conditionally conjugate models. We refer the reader to Wainwright and Jordan (2008), Blei et al. (2017), and Zhang et al. (2017a) for further reading.

# 6 Deep Inference

In the previous sections we have discussed two ways of performing inference—that is, of calculating posterior distributions. We have either calculated the exact posterior distribution  $p(z \mid x; \theta)$  from its definition (i.e.,  $\frac{p(x,z;\theta)}{p(x;\theta)}$ ) in the case where doing so is tractable, or we have formed approximate posterior distributions  $q(z;\lambda)$  by optimizing variational parameters  $\lambda$  so as to make  $q(z;\lambda)$  as close to the true posterior as possible. In this section we discuss a third alternative, whereby we simply train a neural network to *predict* variational parameters  $\lambda$ , rather than arriving at  $\lambda$  by optimizing the ELBO with respect to them. We refer to this latter strategy as "deep inference."

### 6.1 Amortized Variational Inference and Variational Autoencoders

Let us recall the variational expectation maximization algorithm from section 5.3. The variational E-step requires that we find the best variational parameters  $\lambda^{(n)}$  for each  $x^{(n)}$ . Even in mini-batch settings, this optimization procedure can be expensive, especially if a closed-form update is not available, which is typical in deep generative models. In such cases, one could rely on iterative methods to find approximately optimal variational parameters, as in SVI (see the previous section), but this may still be prohibitively expensive; indeed, each gradient calculation  $\nabla_{\lambda}$  ELBO( $\theta, \lambda; x^{(n)}$ ) requires backpropagating gradients through the generative model.

<sup>&</sup>lt;sup>14</sup>That is, for all x there exists  $\lambda_x$  such that  $q(z; \lambda_x) = p(z \mid x; \theta)$  for all z.

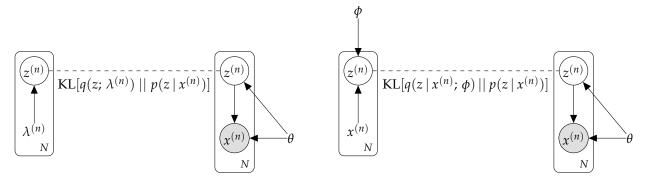


Figure 9: (Left) Traditional variational inference uses variational parameters  $\lambda^{(n)}$  for each data point  $x^{(n)}$ . (Right) Amortized variational inference employs a global inference network  $\phi$  that is run over the input  $x^{(n)}$  to produce the local variational distributions.

As an alternative, one could *predict* the variational parameters by applying a trained neural network, called an *inference network*,  $^{15}$  to the input  $x^{(n)}$  for which we would like to calculate an approximate posterior:

$$\lambda^{(n)} = \operatorname{enc}(x^{(n)}; \phi).$$

The inference network is trained (via gradient ascent) to perform variational inference for all the data points, i.e.

$$\max_{\phi} \sum_{n=1}^{N} \text{ELBO}(\theta, \text{enc}(x^{(n)}; \phi); x^{(n)}).$$

Importantly, the same encoder network (with parameters  $\phi$ ) can be used for all  $x^{(n)}$  we are interested in, and it is therefore unnecessary to optimize separate  $\lambda^{(n)}$  for each  $x^{(n)}$  we encounter. This style of inference is also known as *amortized variational inference* (AVI), as the task of performing approximate posterior inference is *amortized* across the entire dataset through the shared encoder. This is illustrated in is Figure 9. AVI is usually much faster than both SVI and traditional VI, as one can simply run the inference network over  $x^{(n)}$  to obtain the variational parameters, which should approximate the true posterior well if the inference network is sufficiently expressive and well-trained.

*Variational autoencoders (VAEs)* (Kingma and Welling, 2014; Rezende et al., 2014; Mnih and Gregor, 2014) are a family of deep generative models where the variational parameters are predicted from a deep inference network over the input, in the way just described. The *autoencoder* part of the name stems from the fact that for generative models that factorize as  $p(x, z; \theta) = p(x \mid z; \theta)p(z; \theta)$ , we can rearrange the ELBO as follows:

$$ELBO(\theta, \phi; x) = \mathbb{E}_{q(z|x;\phi)} \left[ \log \frac{p(x,z;\theta)}{q(z;\phi)} \right] \\
= \mathbb{E}_{q(z|x;\phi)} \left[ \log \frac{p(x|z;\theta)p(z;\theta)}{q(z;\phi)} \right] \\
= \mathbb{E}_{q(z|x;\phi)} [\log p(x|z;\theta)] - \text{KL}[q(z|x;\phi)||p(z;\theta)]. \tag{10}$$

Above, for brevity we have written  $\text{ELBO}(\theta, \phi; x)$  in place of  $\text{ELBO}(\theta, \text{enc}(x; \phi); x)$ , and  $q(z \mid x; \phi)$  in place of  $q(z; \text{enc}(x; \phi))$ , and we will use this notation going forward. Note that the first term in the ELBO is the expected reconstruction likelihood of x given the latent variables z, which is roughly equivalent to an autoencoding objective, and the second term can be viewed as regularization term that pushes the variational distribution to be similar to the prior.

<sup>&</sup>lt;sup>15</sup>Also referred to a recognition network or an encoder.

Method	E-step	M-step
Expectation Maximization	Exact Posterior: $q(z) = p(z \mid x; \theta)$	Exact
Log Marginal Likelihood	Exact Posterior: $q(z) = p(z \mid x; \theta)$	Gradient
Variational EM	VI: $q(z; \lambda)$ , $\lambda = \arg \max_{\lambda} ELBO(\theta, \lambda; x)$	Exact/Gradient
Stochastic Variational EM	Stochastic VI: $q(z; \lambda)$ , $\lambda = \lambda + \eta \nabla_{\lambda} \text{ELBO}(\theta, \lambda; x)$	Gradient
Variational Autoencoder	Amortized VI: $q(z; \lambda)$ , $\lambda = \text{enc}(x; \phi)$	Gradient

Table 1: Overview of the different optimization methods for training generative models. The "Expectation" or "Inference" step (E-step) corresponds to performing posterior inference, i.e. minimizing  $\mathrm{KL}[q(z)\|p(z\,|\,x;\,\theta)]$ . The "Maximization" or "Learning" step (M-step) corresponds to maximizing the complete data likelihood under the inferred posterior, i.e.  $\mathbb{E}_{q(z)}[\log p(x,z;\,\theta)]$ .

In the standard VAE setup, the inference network and the generative model are jointly trained by maximizing the ELBO with gradient ascent:

$$\theta^{(i+1)} = \theta^{(i)} + \eta \nabla_{\theta} \, \text{ELBO}(\theta^{(i)}, \phi^{(i)}; \, x^{(n)})$$
$$\phi^{(i+1)} = \phi^{(i)} + \eta \nabla_{\phi} \, \text{ELBO}(\theta^{(i)}, \phi^{(i)}; \, x^{(n)}).$$

The above updates are for a single data point, but in practice mini-batches are used. Note that unlike the coordinate ascent-style training from previous section,  $\theta$  and  $\phi$  are trained together end-to-end. In the next subsection, we will illustrate VAE training in more depth using a simple text example. Table 1 summarizes the different optimization methods we have encountered so far.

## 6.2 Training a Text VAE

Let us revisit the model described in section 3.2 and see how it may be trained with a VAE. For simplicity, we will fix the mean and variance of the prior distribution over **z**, and thereby arrive at the following generative process, as originally proposed by Bowman et al. (2016):

- 1. Sample  $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$  with  $\mathbf{z} \in \mathbb{R}^d$ .
- 2. Sample  $x \sim \text{CRNNLM}(x; \theta, \mathbf{z})$ .

We now define the variational family  $\mathcal{Q}$  to contain Gaussian distributions with diagonal covariance matrices. That is, we define  $q(\mathbf{z} \mid \mathbf{x}; \phi) = \mathcal{N}(\mathbf{z}; \mu, \operatorname{diag}(\sigma^2))$ , where  $[\mu, \sigma^2] = \operatorname{enc}(\mathbf{x}; \phi)$ , and where  $\mu \in \mathbb{R}^d$  and  $\sigma^2 \in \mathbb{R}^d_{>0}$ . A popular parameterization for  $\operatorname{enc}(x; \phi)$  is

$$h_{1:T} = \text{RNN}(x), \qquad h = \text{MLP}(h_T), \qquad \mu = \mathbf{W}_1 h + b_1, \qquad \sigma^2 = \exp(\mathbf{W}_2 h + b_2).$$

### 6.2.1 Optimizing the ELBO and the Reparameterization Trick

Given the above choice of variational family and approach to predicting  $\lambda$ , we may now attempt to maximize the ELBO with respect to  $\theta$  and  $\phi$ . The overarching strategy is to rearrange some terms to express gradient of the expectation as an expectation of the gradient, which can be estimated with Monte Carlo samples.

The gradient of the ELBO with respect to  $\theta$  is given by

$$\nabla_{\theta} \operatorname{ELBO}(\theta, \phi; x) = \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\nabla_{\theta} \log p(x, \mathbf{z}; \theta)]$$
$$= \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\nabla_{\theta} \log p(x \mid \mathbf{z}; \theta)].$$

The first equality holds because the distribution with which we are taking the expectation does not depend on  $\theta$ , so we can push the gradient inside the expectation. The second equality holds because the prior (spherical Gaussian) does not depend on  $\theta$ . The expectation in the gradient above is typically estimated with Monte Carlo samples, and one sample is often sufficient.

Let us now derive the gradient of the ELBO with respect to  $\phi$  in the general case (i.e. for any generative model/variational distribution),

$$\nabla_{\phi} \operatorname{ELBO}(\theta, \phi; x) = \nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} \left[ \log \frac{p(x, \mathbf{z}; \theta)}{q(\mathbf{z} \mid x; \phi)} \right]$$
$$= \nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\log p(x, \mathbf{z}; \theta)] - \nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\log q(\mathbf{z} \mid x; \phi)].$$

Unlike the case with  $\theta$ , we cannot simply push the gradient sign inside the expectation since the distribution with which we are taking the expectation depends on  $\phi$ .

We derive the gradients of the two terms in the above expression separately. The first term involves the standard score function gradient estimator, (Glynn, 1987; Williams, 1992; Fu, 2006)

$$\begin{split} \nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)}[\log p(x, \mathbf{z}; \theta)] &= \nabla_{\phi} \int \log p(x, \mathbf{z}; \theta) q(\mathbf{z} \mid x; \phi) \, d\mathbf{z} \\ &= \int \log p(x, \mathbf{z}; \theta) \nabla_{\phi} q(\mathbf{z} \mid x; \phi) \, d\mathbf{z} \qquad \qquad \text{(differentiate under the integral sign)} \\ &= \int \log p(x, \mathbf{z}; \theta) q(\mathbf{z} \mid x; \phi) \nabla_{\phi} \log q(\mathbf{z} \mid x; \phi) \, d\mathbf{z} \qquad \text{(since } \nabla q = q \nabla \log q) \\ &= \mathbb{E}_{q(\mathbf{z} \mid x; \phi)}[\log p(x, \mathbf{z}; \theta) \nabla_{\phi} \log q(\mathbf{z} \mid x; \phi)]. \end{split}$$

The second term is given by,

$$\nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\log q(\mathbf{z} \mid x; \phi)] = \nabla_{\phi} \int \log q(\mathbf{z} \mid x; \phi) q(\mathbf{z} \mid x; \phi) d\mathbf{z}$$

$$= \int \nabla_{\phi} \Big( \log q(\mathbf{z} \mid x; \phi) q(\mathbf{z} \mid x; \phi) \Big) d\mathbf{z} \qquad \text{(differentiate under the integral sign)}$$

$$= \int q(\mathbf{z} \mid x; \phi) \nabla_{\phi} \log q(\mathbf{z} \mid x; \phi) + \log q(\mathbf{z} \mid x; \phi) \nabla_{\phi} q(\mathbf{z} \mid x; \phi) d\mathbf{z} \qquad \text{(product rule)}$$

$$= \int \nabla_{\phi} q(\mathbf{z} \mid x; \phi) d\mathbf{z} + \int \log q(\mathbf{z} \mid x; \phi) \nabla_{\phi} q(\mathbf{z} \mid x; \phi) d\mathbf{z} \qquad \text{(apply } \nabla q = q \nabla \log q \text{ to first term)}$$

$$= 0 + \int \log q(\mathbf{z} \mid x; \phi) \nabla_{\phi} q(\mathbf{z} \mid x; \phi) d\mathbf{z} \qquad \text{(since } \int \nabla q = \nabla \int q = \nabla 1 = 0)$$

$$= \int \log q(\mathbf{z} \mid x; \phi) q(\mathbf{z} \mid x; \phi) \nabla_{\phi} \log q(\mathbf{z} \mid x; \phi) \qquad \text{(apply } \nabla q = q \nabla \log q \text{ again)}$$

$$= \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\log q(\mathbf{z} \mid x; \phi) \nabla_{\phi} \log q(\mathbf{z} \mid x; \phi)]$$

Putting it all together, we have

$$\nabla_{\phi} \operatorname{ELBO}(\theta, \phi; x) = \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} \Big[ \log \frac{p(x, \mathbf{z}; \theta)}{q(\mathbf{z} \mid x; \phi)} \nabla_{\phi} \log q(\mathbf{z} \mid x; \phi) \Big],$$

which is just policy gradient-style reinforcement learning with reward given by  $\log \frac{p(x,\mathbf{z};\theta)}{q(\mathbf{z}\,|\,x;\phi)}$ . The expectation can again be estimated with Monte Carlo samples. While the Monte Carlo gradient estimator is unbiased it will suffer from high variance, and in this case a single sample is often *not* sufficient.

<sup>&</sup>lt;sup>16</sup>Throughout this tutorial we will always assume that we can differentiate under the integral sign (i.e. swap the gradient/integral signs). This is valid under mild conditions, e.g. conditions which satisfy the hypotheses of the dominated convergence theorem.

However, we can exploit our choice of the variational family (i.e.  $q(\mathbf{z} \mid x; \phi) = \mathcal{N}(\mathbf{z}; \mu, \text{diag}(\sigma^2))$ ) to derive another estimator. First, using the KL decomposition of the ELBO, we observe that we can also express the gradient of the ELBO with respect to  $\phi$  as

$$\nabla_{\phi} \operatorname{ELBO}(\theta, \phi; x) = \nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\log p(x \mid \mathbf{z}; \theta)] - \nabla_{\phi} \operatorname{KL}[q(\mathbf{z} \mid x; \phi) || p(\mathbf{z})].$$

Beginning with the second term, the KL divergence between a diagonal Gaussian and the standard Gaussian has an analytic solution given by

$$\mathrm{KL}[q(\mathbf{z} \mid x; \phi) \| p(\mathbf{z})] = -\frac{1}{2} \sum_{i=1}^{d} (\log \sigma_{j}^{2} - \sigma_{j}^{2} - \mu_{j}^{2} + 1),$$

and therefore  $\nabla_{\phi} \text{KL}[q(\mathbf{z} \mid x; \phi) || p(\mathbf{z})]$  is easy to calculate. For the first term, notice that our variational family of Gaussian distributions is *reparameterizable* (Kingma and Welling, 2014; Rezende et al., 2014; Glasserman, 2013) in the sense that we can obtain a sample from the variational posterior by sampling from a base noise distribution and applying a deterministic transformation,

$$\epsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$
  $\mathbf{z} = \mu + \sigma \epsilon$ ,

where  $\mu$  and  $\sigma^2$  are as usual given by our encoder network. Observe that  $\mathbf{z}$  remains distributed according to  $\mathcal{N}(\mathbf{z}; \mu, \operatorname{diag}(\sigma^2))$ , but we may now express the gradient with respect to  $\phi$  as

$$\begin{split} \nabla_{\phi} \mathbb{E}_{q(\mathbf{z} \mid x; \phi)} [\log p(x \mid \mathbf{z}; \theta)] &= \nabla_{\phi} \mathbb{E}_{\epsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [\log p(x \mid \boldsymbol{\mu} + \sigma \epsilon; \theta)] \\ &= \mathbb{E}_{\epsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [\nabla_{\phi} \log p(x \mid \boldsymbol{\mu} + \sigma \epsilon; \theta)], \end{split}$$

because the expectation no longer depends on  $\phi$ . We can again approximate the expectation in the gradient above with a single sample.

This reparameterization trick just discussed empirically yields much lower-variance gradient estimators and has been instrumental in training deep generative models with VAEs. Intuitively, the reparameterized gradient estimator "differentiates through" the generative model and therefore has more information than the score function gradient estimator, which treats the generative model as a black-box reward function. Unfortunately, in many cases the variational posterior is not directly reparameterizable (e.g. if z is discrete), and extending the reparameterization trick to other families of distributions is an active research area (see section 6.4.1). <sup>17</sup>

#### 6.2.2 Posterior Collapse

We now discuss an important issue that affects training Text VAEs in practice. Recall that in the model introduced in Section 3.2 the likelihood model is allowed to fully condition on the entire history, through the RNN's state  $h_t$ :

$$h_t = \text{RNN}(h_{t-1}, [\mathbf{x}_{t-1}; \mathbf{z}])$$
$$p(x_t | x_{< t}, \mathbf{z}; \theta) = \text{softmax}(Wh_t)_{x_t},$$

where the above equations are copied from Equations (7) and (6). We might hope that in such a model  $\mathbf{z}$  would capture global aspects of the sentence/document (e.g., the topic), while the RNN learns to model local variations (e.g. placement of function words to ensure grammatically).<sup>18</sup>

 $<sup>^{17}</sup>$ The term *reparameterization trick* is a slight misnomer, since we can reparameterize (for example) a discrete distribution by applying a deterministic transformation to base Gumbel noise (section 6.4.1). However we will not be able to train  $\phi$  with gradient-based optimization in this case, since gradients will be zero almost everywhere. In VAE literature the term reparameterizable has come to colloquially mean "reparameterizable through differentiable transformations that provide nonzero gradients."

<sup>&</sup>lt;sup>18</sup>There has also been prior work on separating topics and syntax in the context of non-neural generative models (Griffiths et al., 2004; Boyd-Graber and Blei, 2008).

However, Bowman et al. (2016) observe that these types of models experience *posterior collapse*, whereby the likelihood (i.e., reconstruction) model ignores the latent variable and simply becomes a language model. That is, x and z become independent. Indeed, looking at the ELBO in Equation (10), we see that if x can be reconstructed *without* z, the model is incentivized to make the variational posterior approximately equal to the prior—that is, to have

$$\text{KL}[q(\mathbf{z} \mid x; \phi) || p(\mathbf{z})] \approx 0$$

regardless of how expressively one parameterizes  $q(\mathbf{z} \mid x; \phi)$ . More formally, Chen et al. (2017) show that this phenomenon may be justified under the "bits-back" argument: if the likelihood model is rich enough to model the true data distribution  $p_{\star}(x)$  without using any information from  $\mathbf{z}$ , then the global optimum is obtained by setting

$$p(x \mid \mathbf{z}; \theta) = p(x; \theta) = p_{\star}(x)$$
  
 $p(\mathbf{z} \mid x; \theta) = q(\mathbf{z} \mid x; \phi) = p(\mathbf{z}).$ 

Since any distribution p(x) can be factorized as  $p(x) = p(x_1) \prod_{t=2}^{T} p(x_t \mid x_{< t})$ , it is possible that a large enough RNN can model  $p_{\star}(x)$  without relying on **z**. As such, to avoid posterior collapse, past work has made conditional independence assumptions and instead used multilayer perceptrons (Miao et al., 2016, 2017) or convolutional networks (Yang et al., 2017; Semeniuta et al., 2017; Shen et al., 2018) to parameterize the likelihood model.

One way to mitigate posterior collapse from an optimization perspective is to "warm-up" the KL portion of the objective (Bowman et al., 2016; Sønderby et al., 2016). In particular, one maximizes

$$\mathbb{E}_{q(\mathbf{z} \mid x; \phi)}[\log p(x \mid \mathbf{z}; \phi)] - \beta \operatorname{KL}[q(\mathbf{z} \mid x; \phi) || p(\mathbf{z})],$$

where  $\beta$  is gradually increased from 0 to 1 over training.<sup>19</sup> Other strategies include adding auxiliary objectives which ensure that **z** is used (Dieng et al., 2017; Goyal et al., 2017a; Wang et al., 2018), randomly dropping out words during decoding (Bowman et al., 2016), thresholding the KL function so that some bits are "free" (Kingma et al., 2016), combining amortized and stochastic variational inference (Kim et al., 2018), using skip connections (Dieng et al., 2018), or working with distributions where the KL can be essentially a fixed hyperparameter (e.g. the von Mises–Fisher distribution with a fixed concentration parameter (Guu et al., 2017; Xu and Durrett, 2018)).

#### 6.2.3 Evaluation

As the ELBO always lower bounds the log marginal likelihood  $\log p(x; \theta)$ , we can evaluate the learned generative model by evaluating the ELBO across a held-out test set. We can also estimate the marginal likelihood with K importance samples,

$$p(x;\theta) = \mathbb{E}_{q(z \mid x;\phi)} \left[ \frac{p(x,z;\theta)}{q(z \mid x;\phi)} \right] \approx \frac{1}{K} \sum_{k=1}^{K} \frac{p(x,z^{(k)};\theta)}{q(z^{(k)} \mid x;\phi)},$$

where the samples are from the approximate variational posterior.

Therefore the log marginal likelihood can be estimated by

$$\log p(x;\theta) = \log \mathbb{E}_{q(z \mid x;\phi)} \left[ \frac{p(x,z;\theta)}{q(z \mid x;\phi)} \right] \approx \log \frac{1}{K} \sum_{k=1}^{K} \frac{p(x,z^{(k)};\theta)}{q(z^{(k)} \mid x;\phi)}.$$

 $<sup>^{19}\</sup>beta$ -VAEs are a class of models where the multiplier  $\beta$  on the KL portion of the ELBO is not necessarily 1 (Higgins et al., 2017). These models can be interpreted as the Langragian of a constrained optimization problem with respect to the original ELBO. Models trained with  $\beta > 1$  have been shown to learn more *disentangled* latent representations, and using VAE-like models to learn more disentangled/factorized representations is an active research area (Burgess et al., 2017; Kim and Mnih, 2018; Chen et al., 2018b).

The above estimator is not unbiased but under mild conditions converges almost surely to  $\log p(x; \theta)$  as  $K \to \infty$ .<sup>20</sup> It is also possible to maximize the above quantity directly, leading to *importance weighted autoencoders* (Burda et al., 2015) (see section 6.3.2). Wu et al. (2017b) further refine the estimate of the log marginal likelihood with annealed importance sampling (Neal, 2001).

The ELBO and the log marginal likelihood provide a quantitative estimate of the learned generative model, but from a representation learning perspective these metrics may not be useful. Indeed, Alemi et al. (2018) find that in many cases there is a family of models that achieve the same ELBO but have different reconstruction/KL terms.<sup>21</sup> It is therefore often useful to report the reconstruction term ( $\mathbb{E}_{q(z\mid x;\phi)}[\log p(x\mid z;\theta)]$ ) and the KL term ( $\mathbb{KL}[q(z\mid x;\phi)||p(z)]$ ) along with the ELBO (Bowman et al., 2016; Gulrajani et al., 2017b; Yang et al., 2017). The mutual information between x and z is also estimable though it may be computationally expensive (Hoffman and Johnson, 2016).<sup>22</sup>

Qualitative evaluation of the model includes inspecting samples from prior/variational posterior, and linearly interpolating in the latent space and evaluating samples from the interpolated latent vector.

# 6.3 Tightening the ELBO

Ideally we would like the gap between  $\log p(x; \theta)$  and  $\text{ELBO}(\theta, \phi; x)$  to be small. We have seen that the gap is equal to zero if  $q(z \mid x; \phi) = p(z \mid x; \theta)$ , since

$$KL[q(z \mid x; \phi) || p(z \mid x; \theta)] = \log p(x \mid \theta) - ELBO(\theta, \phi; x).$$

Therefore if we are able to better approximate the true posterior, we can hope to train better generative models. In this section we briefly review some recent advances that attempt to tighten the gap.

#### **6.3.1** Flows

By working with richer variational families we can better approximate the true posterior. One way to do so is through *flows*, in which a sample from a simple base density (typically Gaussian) is converted into a sample from a more complex density via a series of invertible transformations,

$$\mathbf{z}_0 \sim q(\mathbf{z}_0 \mid x; \phi) = \mathcal{N}(\mu(x), \sigma^2(x))$$
  
$$\mathbf{z}_K = f_K \circ f_{K-1} \circ \cdots \circ f_1(\mathbf{z}_0).$$

Here  $z_0$  is a sample from the variational posterior as before. Since each  $f_k$  is invertible, the log density of  $z_K$  is given by the change-of-variables formula

$$\log q_K(\mathbf{z}_K \mid x; \phi) = \log q(\mathbf{z}_0 \mid x; \phi) + \sum_{k=1}^K \log \left| \frac{\partial f_k^{-1}}{\partial \mathbf{z}_k} \right|$$
$$= \log q(\mathbf{z}_0 \mid x; \phi) - \sum_{k=1}^K \log \left| \frac{\partial f_k}{\partial \mathbf{z}_{k-1}} \right|$$

<sup>&</sup>lt;sup>20</sup>If  $\frac{p(x,z;\theta)}{q(z|x;\phi)}$  is bounded then by the strong law of large numbers  $\frac{1}{K}\sum_{k=1}^{K}\frac{p(x,z^{(k)};\theta)}{q(z^{(k)}|x;\phi)}\to p(x;\theta)$  almost surely as  $K\to\infty$ . Then the continuous mapping theorem implies  $\log\frac{1}{K}\sum_{k=1}^{K}\frac{p(x,z^{(k)};\theta)}{q(z^{(k)}|x;\phi)}\to \log p(x;\theta)$  almost surely as  $K\to\infty$ .

<sup>&</sup>lt;sup>21</sup>To be precise, Alemi et al. (2018) use the term *rate* and *distortion*. Distortion corresponds to the negative reconstruction likelihood portion of the ELBO, and rate corresponds to the KL[ $q(z|x;\phi)|m(z)$ ], where m(z) is a *variational marginal* distribution. Hence the rate is not necessarily equal to the KL portion of the ELBO, i.e. KL[ $q(z|x;\phi)|p(z)$ ]. See the paper for more discussion around m(z).

<sup>&</sup>lt;sup>22</sup>Some works directly optimize an approximation to the mutual information (Zhao et al., 2018b; Belghazi et al., 2018; Gao et al., 2018). These works typically require adversarial training.

where  $\left|\frac{\partial f_k}{\partial z_{k-1}}\right|$  is the absolute value of the determinant of the Jacobian of  $f_k$ . (Here the parameters of  $f_k$  have been subsumed into  $\phi$ ). Then the ELBO using  $q_K$  as the variational posterior is

$$\begin{split} \text{ELBO}(\theta, \phi; \, x) &= \mathbb{E}_{q_K(\mathbf{z}_K \,|\, x; \phi)}[\log p(x, \mathbf{z}_K; \, \theta) - \log q_K(\mathbf{z}_K \,|\, x; \, \phi)] \\ &= \mathbb{E}_{q(\mathbf{z}_0 \,|\, x; \phi)}\Big[\log p(x, \mathbf{z}_K; \, \theta) - \log q(\mathbf{z}_0 \,|\, x; \, \phi) + \sum_{k=1}^K \log \Big| \frac{\partial f_k}{\partial \mathbf{z}_{k-1}} \Big| \Big] \end{split}$$

The key to making flows efficient is making the determinant of the Jacobian easy to calculate. For example, Rezende and Mohamed (2015) use *normalizing flows* where each  $f_k$  is parameterized as

$$f_k(\mathbf{z}_{k-1}) = \mathbf{z}_{k-1} + \boldsymbol{u}_k h(\boldsymbol{w}_k^{\top} \mathbf{z}_{k-1} + \boldsymbol{b}_k)$$

where  $u_k$ ,  $w_k$ ,  $b_k$  are learnable parameters (part of  $\phi$ ) and  $h(\cdot)$  is a differentiable non-linear function with derivative  $h'(\cdot)$ . Then the matrix determinant lemma gives

$$\left|\frac{\partial f_k}{\partial \mathbf{z}_{k-1}}\right| = |1 + \mathbf{u}_k^\top (h'(\mathbf{w}_k^\top \mathbf{z}_{k-1} + b_k) \mathbf{w}_k)|,$$

which is very efficient to calculate. Another popular type of flow is the inverse autoregressive flow (IAF) (Kingma and Ba, 2015). Letting  $\mathbf{z}_{k,< d}$  refer to the first d-1 dimensions of the latent vector at the k-th flow step, IAF defines an invertible transformation via

$$f_k(\mathbf{z}_{k-1}) = \boldsymbol{\mu}_k + \boldsymbol{\sigma}_k \odot \mathbf{z}_{k-1},$$

where  $\mu_{k,d} \in \mathbb{R}$  and  $\sigma_{k,d} > 0$  (the d-th parameters of the above transformation) are the output from a neural network over the previous dimensions  $\mathbf{z}_{k-1,< d}$ ,

$$\mu_{k,d}$$
,  $\sigma_{k,d} = \text{MLP}_{k,d}(\mathbf{z}_{k-1, < d})$ 

In this case the Jacobian is given by a triangular matrix, and hence we can efficiently calculate the log determinant,

$$\log \left| \frac{\partial f_k}{\partial \mathbf{z}_{k-1}} \right| = \sum_{d=1}^D \log \sigma_{k,d}.$$

Note that the MLP's above do not have to be invertible, as long as they only depend on the previous dimensions, and further, we can efficiently calculate  $\mu_k$ ,  $\sigma_k$  in a single forward pass using masked autoregressive networks (Germain et al., 2015). Hence training remains efficient.

Other types of flows include householder flows (Tomczak and Welling, 2016; van den Berg et al., 2018), neural autoregressive flows (Huang et al., 2018), and *continuous* normalizing flows (Chen et al., 2018a; Grathwohl et al., 2018a). Continuous flows are particularly interesting in that they instead use neural networks to parameterize a differential equation that represents how the layer changes in continuous time.

#### 6.3.2 Importance Sampling

Another way to tighten the gap between the ELBO and the log marginal likelihood is through importance sampling (Burda et al., 2015). To illustrate this, first note that, using Jensen's inequality, we have the following:

$$p(x;\theta) = \mathbb{E}_{q(z \mid x;\phi)} \left[ \frac{p(x,z;\theta)}{q(z \mid x;\phi)} \right] \implies \log p(x;\theta) \ge \mathbb{E}_{q(z \mid x;\phi)} \left[ \log \frac{p(x,z;\theta)}{q(z \mid x;\phi)} \right].$$

It is moreover clear that Jensen's inequality allows us to form an analogous inequality for *any* unbiased estimator of  $p(x; \theta)$ , not just  $\frac{p(x,z;\theta)}{q(z\,|\,x;\phi)}$ . In particular, consider the following unbiased estimator, which uses multiple independent samples  $z^{(1:K)} = [z^{(1)}, \ldots, z^{(K)}]$  from  $q(z\,|\,x;\phi)$ :

$$I_K = \frac{1}{K} \sum_{k=1}^{K} \frac{p(x, z^{(k)}; \theta)}{q(z^{(k)} | x; \phi)}.$$

Then, using an argument analogous to the one above, and letting  $q(z^{(1:K)} | x; \phi) = \prod_{k=1}^K q(z^{(k)} | x; \phi)$ , we have

$$\log p(x; \theta) \ge \mathbb{E}_{q(z^{(1:K)} | x; \phi)} \Big[ \log \frac{1}{K} \sum_{k=1}^{K} \frac{p(x, z^{(k)}; \theta)}{q(z^{(k)} | x; \phi)} \Big] = \mathbb{E}[\log I_K].$$

Burda et al. (2015) prove that under mild conditions, we have

$$\log p(x; \theta) \ge \mathbb{E}[\log I_K] \ge \mathbb{E}[\log I_{K-1}],$$

and further, that  $\lim_{K\to\infty} \mathbb{E}[\log I_K] = \log p(x;\theta)$ , where the convergence is in the strongest (i.e. almost sure) sense. If K=1, we recover the vanilla VAE, and therefore optimizing the above objective for K>1 maximizes a tighter bound. Under this setup,  $q(z\mid x;\phi)$  is viewed as an importance sampling distribution (and  $\frac{p(x,z;\theta)}{q(z\mid x;\phi)}$  the importance weights), and so this type of approach is known as an *Importance Weighted Autoencoder (IWAE)* (Burda et al., 2015).

In the general case the gradients with respect to  $\theta$  and  $\phi$  are given by

$$\begin{split} & \nabla_{\theta} \mathbb{E}[\log I_K] = \mathbb{E}_{q(z^{(1:K)} \mid x; \phi)} \Big[ \sum_{k=1}^K w^{(k)} \nabla_{\theta} \log p(x, z^{(k)}; \theta) \Big], \\ & \nabla_{\phi} \mathbb{E}[\log I_K] = \mathbb{E}_{q(z^{(1:K)} \mid x; \phi)} \Big[ \sum_{k=1}^K (\log I_K - w^{(k)}) \nabla_{\phi} \log q(z^{(k)} \mid x; \phi) \Big], \end{split}$$

where

$$w^{(k)} = \frac{p(x, z^{(k)}) / q(z^{(k)} \mid x; \phi)}{\sum_{i=1}^{K} p(x, z^{(j)}) / q(z^{(j)} \mid x; \phi)}$$

are the normalized importance weights (see Mnih and Rezende (2016) for the derivation). The expectations here can again be estimated with a single sample from  $q(z^{(1:K)} \mid x; \phi)$ , which is equivalent to K samples from  $q(z \mid x; \phi)$ . In the Gaussian case it is still possible to apply the reparameterization trick to the IWAE objective to derive low-variance gradient estimator for  $\phi$  (see Burda et al. (2015) for the expression of reparameterized gradient estimator, which is different from score function gradient estimator shown above), and this reparameterized estimator should be used instead. Empirically IWAEs typically outperform VAEs in terms of log marginal likelihood even with a few samples (Burda et al., 2015; Mnih and Rezende, 2016). However, the IWAE objective may hinder the training of the inference network by reducing the signal-to-noise ratio of the gradient estimator (Rainforth et al., 2018), though recent work targets this issue by applying reparameterization twice (Tucker et al., 2018).

These ideas have also been extended to sequential latent variable models with particle filtering (Maddison et al., 2017a; Nasseth et al., 2018; Le et al., 2018a), in which we no longer simply take  $q(z^{(1:K)} | x; \phi)$  to be the product of  $q(z | x; \phi)$ 's. Cremer et al. (2017) and Domke and Sheldon (2018) provide an alternative interpretation of IWAE as optimizing the standard ELBO but with a different variational distribution.

#### 6.3.3 Amortization Gap

In VAEs we restrict the variational family to be the class of distributions whose parameters are obtainable by running a parameteric model (i.e. inference network) over the input. This choice allows for fast training/inference but may be too strict of a restriction. (Note that this is usually not an issue with traditional VI which typically works with variational families that allow for closed-from expressions for the best  $\lambda$  for a given x).

Letting  $\lambda^*$  be the best variational parameter for a given data point, i.e.

$$\lambda^{\star} = \underset{\lambda}{\arg\min} \operatorname{KL}[q(z; \lambda) \parallel p(z \mid x; \theta)]$$

we can break down the *inference gap* (the gap between the variational posterior from the inference network and the true posterior) as follows (Cremer et al., 2018)

$$\underbrace{\mathrm{KL}[q(z\,|\,x;\,\phi)\parallel p(z\,|\,x;\,\theta)]}_{\text{inference gap}} = \underbrace{\mathrm{KL}[q(z;\,\lambda^{\star})\parallel p(z\,|\,x;\,\theta)]}_{\text{approximation gap}} + \underbrace{\mathrm{KL}[q(z\,|\,x;\,\phi)\parallel p(z\,|\,x;\,\theta)] - \mathrm{KL}[q(z;\,\lambda^{\star})\parallel p(z\,|\,x;\,\theta)]}_{\text{amortization gap}}$$

Therefore the inference gap consists of two parts: the *approximation gap*, which is the gap between the true posterior and the best possible variational posterior within Q, and the *amortization gap*, which quantifies the gap between inference network posterior and the best possible variational posterior.

To reduce the approximation gap we can work with richer variational families (e.g. by applying flows), and to reduce the amortization gap we can better optimize  $\lambda$  for each data point. Cremer et al. (2018) find that both approximation/amortization gaps contribute significantly to the final inference gap, and there has been some recent work on actively reducing the amortization gap. For example one could use an inference network to initialize the variational parameters and subsequently run iterative refinement (e.g. gradient ascent) to refine them (Hjelm et al., 2016; Krishnan et al., 2018; Kim et al., 2018). Alternatively, Marino et al. (2018) utilize meta-learning (Andrychowicz et al., 2016) to learn to perform better inference. Similar ideas have also been explored in Salakhutdinov and Larochelle (2010), Cho et al. (2013), Salimans et al. (2015), and Pu et al. (2017).

### 6.4 Working with Other (e.g. Discrete) Distributions

#### 6.4.1 Extending the Reparameterization Trick

The reparameterization trick is often crucial in training VAEs as it allows us to efficiently obtain low-variance estimators of the gradient. Ruiz et al. (2016), Naesseth et al. (2017), and Figurnov et al. (2018) generalize the reparameterization trick to work with other distributions (e.g. Gamma, Dirichlet). Of particular interest are methods that extend reparameterization to discrete distributions utilizing the *Gumbel-Max trick* (Papandreou and Yuille, 2011; Hazan and Jaakkola, 2012; Maddison et al., 2014). Concretely, suppose z is the one hot representation of a categorical random variable with K categories and unnormalized scores  $\alpha$ , i.e.

$$p(z_k = 1; \alpha) = \frac{\alpha_k}{\sum_{i=1}^K \alpha_i}$$

Then we can draw a sample from  $p(z; \alpha)$  by adding independent Gumbel noise to the logits and finding the arg max,

$$k = \arg\max_{i} [g_i + \log \alpha_i]$$

$$z_k = 1, \quad (z_i = 0 \text{ for all } i \neq k)$$

where  $g_i$ 's are independent Gumbel noise.<sup>23</sup> More succinctly, we have

$$z = \underset{u \in \Delta^{K-1}}{\arg\max} (\log \alpha + g)^{\top} u,$$

where  $\Delta^{K-1}$  is the *K*-simplex,  $g = [g_1, \dots, g_K]$  is the vector of Gumbel noise, and  $\alpha = [\alpha_1, \dots, \alpha_K]$ . In this case we have  $z \sim p(z; \alpha)$ , and while this shows that we can reparameterize a discrete distribution, we cannot perform gradient-based optimization because the arg max function has zero gradients almost everywhere.

If we replace the arg max above with a softmax and a temperature term  $\tau > 0$ , i.e.

$$s = \operatorname{softmax}\left(\frac{\log \alpha + g}{\tau}\right)$$

$$s_k = \frac{\exp((\log \alpha_k + g_k)/\tau)}{\sum_{i=1}^K \exp((\log \alpha_i + g_i)/\tau)},$$

we say that  $s = [s_1, ..., s_K]$  is drawn from a *Gumbel-Softmax* (Jang et al., 2017) or a *Concrete* (Maddison et al., 2017b) distribution with parameters  $\alpha$ ,  $\tau$ . The Gumbel-Softmax defines a distribution on the simplex  $\Delta^{K-1}$ , and the density is given by

$$p(s; \alpha, \tau) = (K - 1)! \tau^{K - 1} \prod_{k = 1}^{K} \left( \frac{\alpha_k s_k^{-\tau - 1}}{\sum_{j = 1}^{K} \alpha_j s_j^{-\tau}} \right).$$

(See Jang et al. (2017) and Maddison et al. (2017b) for the derivation). Notably this distribution is reparameterizable by construction since we can draw a sample by (1) drawing *i.i.d* Gumbel noise  $g = [g_1, \ldots, g_K]$ , (2) transforming the noise as  $(g_i + \log \alpha_i)/\tau$ , and (3) applying a softmax to the transformed logits. Importantly, unlike the arg max case, the softmax relaxation has nonzero gradients.<sup>24</sup> While *s* is no longer discrete, we can anneal the temperature  $\tau \to 0$  as training progresses and hope that this will approximate a sample from a discrete distribution.

Going back to VAEs, suppose now  $\alpha = \text{enc}(x; \phi)$  and let  $q_s(s \mid x; \phi, \tau)$  be the Gumbel-Softmax distribution (note that  $\tau$  could in theory also be a function of x). If  $q_z(z \mid x; \phi)$  is the original categorical distribution, we

$$s = \operatorname{softmax}(\beta) = \underset{u \in \Delta^{K-1}}{\operatorname{arg max}} \beta^{\top} u + H(u),$$

where  $H(u) = -\sum_{k=1}^{K} u_k \log u_k$  is the entropy term. There has been recent work on differentiable relaxations of arg max with different regularizers, e.g. the *sparsemax* adds a squared penalty (Martins and Astudillo, 2016; Niculae et al., 2018),

$$\operatorname{sparsemax}(\beta) = \underset{u \in \Delta^{K-1}}{\arg\max} \, \beta^\top u - \frac{1}{2} \|u\|_2^2.$$

This is equivalent to a Euclidean projection of  $\beta$  onto the simplex, which encourages sparsity (since the projection is likely to hit an edge of the simplex).

<sup>&</sup>lt;sup>23</sup>We can sample from a Gumbel distribution by sampling a uniform random variable  $u \sim \mathcal{U}[0,1]$  and applying the transformation  $g = -\log(-\log u)$ .

<sup>&</sup>lt;sup>24</sup>Note that replacing the arg max with the softmax is equivalent to adding an entropy regularizer to the arg max problem, i.e.

might hope that the gradient of the ELBO obtained from the Gumbel-Softmax distribution will approximate the true gradient, i.e.

$$\begin{split} \nabla_{\phi} \mathbb{E}_{q_{z}(z \mid x; \phi, \tau)} \Big[ \log \frac{p_{x, z}(x, z; \theta)}{q_{z}(z \mid x; \phi)} \Big] &\approx \nabla_{\phi} \mathbb{E}_{q_{s}(s \mid x; \phi)} \Big[ \log \frac{p_{x, s}(x, s; \theta, \tau)}{q_{s}(s \mid x; \phi)} \Big] \\ &= \mathbb{E}_{g \sim \text{Gumbel}} \Big[ \nabla_{\phi} \log \frac{p_{x, s}(x, f(g, \alpha, \tau); \theta, \tau)}{q_{s}(f(g, \alpha, \tau) \mid x; \phi)} \Big], \end{split}$$

where f applies the required softmax transformation to the Gumbel noise g to obtain s. Here we have employed subscripts in the density functions to emphasize the random variables over which the distributions are induced:  $p_{x,z}(x,z;\theta) = p_{x|z}(x\,|\,z;\theta)p_z(z;\theta)$  is the joint density under the original model while  $p_{x,s}(x,s;\theta,\tau) = p_{x|z}(x\,|\,z=s;\theta)p_s(s;\theta,\tau)$  is the density under the new, relaxed model (note that  $p_{x,s}$  still makes use of the original likelihood model  $p_{x|z}$ ). Unlike the score function gradient estimator, the above estimator is biased and the variance will diverge to infinity as  $\tau \to 0$ . Recent work (Tucker et al., 2017; Grathwohl et al., 2018b) combines the above estimator with the standard score function estimator to obtain even lower-variance gradient estimators. Finally, since we are still using the original likelihood model  $p_{x|z}$  to define the relaxed joint distribution  $p_{x,s}$ , the above relaxation is only applicable if the likelihood model  $p_{x|z}$ , which was originally defined over a discrete latent space  $z \in \{0,1\}^K$  (and the observed data), is well-defined for a continuous latent space  $z \in \Delta^{K-1}$ . For example if z is a parse tree and  $p_{x|z}(x\,|\,z;\theta)$  uses the parse tree to define a complex computational graph (e.g. as in Recurrent Neural Network Grammars (Dyer et al., 2016)), then the techniques described above are not applicable because  $p_{x|z}(x\,|\,z;\theta)$  would not make sense for non-discrete z.

#### 6.4.2 Non-Gaussian Prior/Posterior Distributions

Here we briefly review some other distributions that have been considered in the context of VAEs. Johnson et al. (2016) introduce structured variational autoencoders, which encode structure in the latent space through probabilistic graphical models. Structured VAEs have been used to model latent sequences (Chung et al., 2015; Fraccaro et al., 2016; Serban et al., 2017; Zaheer et al., 2017; Krishnan et al., 2017; Liu et al., 2018), graphs (Kipf and Welling, 2016; Jin et al., 2018), trees (Yin et al., 2018; Corro and Titov, 2018; Li et al., 2019) and other structured objects (Kusner et al., 2017; Dai et al., 2018). There has also been some work on extending VAEs to the nonparametric Bayesian setting (Tran et al., 2016a; Goyal et al., 2017b; Naslinick and Smyth, 2017; Miao et al., 2017; Singh et al., 2017; Bodin et al., 2017), and the von Mises–Fisher distribution is becoming an interesting alternative to the Gaussian (Guu et al., 2017; Davidson et al., 2018; Xu and Durrett, 2018).

Other classes of distributions that have been considered include mixture of Gaussians (Dilokthanakul et al., 2016), learned priors (Tomczak and Welling, 2018; Huang et al., 2017), hierarchical (Sønderby et al., 2016; Zhao et al., 2017; Park et al., 2018), and discrete (Rolfe, 2017; van den Oord et al., 2017) distributions.

<sup>&</sup>lt;sup>25</sup>See the appendix of Maddison et al. (2017b) for further discussion on which components can/should be relaxed. In particular, under the KL formulation of the ELBO, there is some debate as to whether the KL should be calculated between the original categorical distributions (which can be readily calculated in closed-form) or between the relaxed Gumbel-Softmax distributions (which requires sampling).

<sup>&</sup>lt;sup>26</sup>Some other strategies for reducing the variance of the estimator include Rao-Blackwellization (Ranganath et al., 2014) and sophisticated control variates (Mnih and Gregor, 2014; Miller et al., 2017; Deng et al., 2018).

<sup>&</sup>lt;sup>27</sup>However see Choi et al. (2018) for an example of relaxing a TreeLSTM to work with "soft" parse trees in the context of text classification tasks.

## 7 Other Methods

In this section we briefly review some other methods that have been used to train latent variable models.

## 7.1 Wake-Sleep Algorithm

The wake-sleep algorithm (Hinton et al., 1995) is a method for training deep directed graphical models, and can be thought of as a precursor to variational autoencoders. The wake-sleep algorithm also makes use of a *recognition network* which produces an approximate posterior distribution over the latent variables given the data, i.e.  $q(z \mid x; \phi)$ . Training proceeds as follows:

1. Wake phase: sample a data point  $x \sim p_{\star}(x)$  and latent variable  $\hat{z} \sim q(z \mid x; \phi)$ . Take a gradient step with respect to  $\theta$  to maximize the joint likelihood,

$$\theta^{(i+1)} = \theta^{(i)} + \eta \nabla_{\theta} \log p(x, \hat{z}; \theta^{(i)})$$

2. Sleep phase: sample "phantom" data from the generative model, i.e.  $\hat{z} \sim p(z)$ ,  $\hat{x} \sim p(x \mid \hat{z}; \theta)$ . Take a gradient step with respect to  $\phi$  to maximize  $\hat{z}$ ,

$$\phi^{(i+1)} = \phi^{(i)} + \eta \nabla_{\phi} \log q(\hat{z} \,|\, \hat{x}; \, \phi^{(i)})$$

It is clear that the wake phase is training the generative model to maximize a Monte Carlo estimate of the expected complete data likelihood under the variational posterior, i.e.  $\mathbb{E}_{q(z\,|\,x;\,\phi)}[\log p(x,z;\,\theta)]$ . Hence the wake phase is equivalent to the variational M-step. In the sleep-phase the recognition network is trained to maximize a Monte Carlo estimate of  $\mathbb{E}_{p(x,z;\theta)}[\log q(z\,|\,x;\,\phi)]$ , which is equivalent to minimizing  $\mathrm{KL}[p(z\,|\,x;\,\theta)\|q(z\,|\,x;\,\phi)]$ . Recall that the variational E-step trains the recognition network to minimize  $\mathrm{KL}[q(z\,|\,x;\,\phi)\|p(z\,|\,x;\,\theta)]$ . Therefore the sleep-phase almost corresponds to the variational E-step, but the KL direction is reversed. We have seen before that in the variational E-step, minimizing the KL is equivalent to maximizing the ELBO. But this involves an expectation over  $q(z\,|\,x;\,\phi)$ , which is difficult to optimize, especially if the latent variable is not reparameterizable. In contrast, the sleep phase minimizes the other direction  $\mathrm{KL}[p(z\,|\,x;\,\theta)\|q(z\,|\,x;\,\phi)]$  and therefore does not have this issue.

While the wake-sleep algorithm does not maximize a lower bound on the log marginal likelihood, in practice it performs well (Hinton et al., 2006; Ba et al., 2015; Mnih and Rezende, 2016; Le et al., 2018b). Bornschein and Bengio (2015) introduce the reweighted wake-sleep algorithm, which combines importance sampling with wake-sleep to further improve performance.

## 7.2 Reversible Neural Generative Models

Suppose z and x are both continuous, and consider the following generative model

$$z \sim p_z(z),$$
  $x = f(z; \theta),$ 

where  $p_z(z)$  is a simple prior (e.g. Gaussian) and x is obtained by applying a deterministic function f to z. If f is invertible, it is possible to evaluate  $\log p(x; \theta)$  using the change-of-variables formula,

$$\log p(x;\theta) = \log p_z(f^{-1}(x)) + \log \left| \frac{\partial f^{-1}(x)}{\partial x} \right|,$$

<sup>&</sup>lt;sup>28</sup>This is in contrast to generative models considered so far, in which the *parameters* of the distribution  $p(x | z; \theta)$  are given by applying f to the latent variable z.

where  $\left|\frac{\partial f^{-1}(x)}{\partial x}\right|$  is the absolute value of the determinant of the Jacobian. There has been recent work (Dinh et al., 2015, 2017; Papamakarios et al., 2017; Kingma and Dhariwal, 2018) on applying such methods with f parameterized in a way to allow for fast evaluation of the determinant (e.g. using ideas from autoregressive density estimation with neural networks (Larochelle and Murray, 2011; Germain et al., 2015) to make the Jacobian a triangular matrix, in which case the log determinant is simply the summation of the log diagonals). Similar transformations have also been applied to variational posteriors to make them more flexible in the context of VAEs (Rezende and Mohamed, 2015; Kingma et al., 2016) (see section 6.3.1). Chen et al. (2018a) and Grathwohl et al. (2018a) introduce *continuous normalizing flows*, where the discrete flow steps are mapped to continuous time and a neural network parameterizes the continuous time-dynamics via an ordinary differential equation.

While deep generative models parameterized with reversible neural networks have achieved impressive results in image generation (Dinh et al., 2017; Kingma and Dhariwal, 2018), they are unfortunately difficult to apply to language since text is typically modeled as a discrete variable. He et al. (2018a) circumvent this issue by modeling pretrained word embeddings instead for unsupervised POS tagging and dependency parsing.

### 7.3 Generative Adversarial Networks

Let us consider a similar generative model as above,  $z \sim p(z)$ ,  $x = f(z; \theta)$ , where  $f : \mathbb{R}^d \to \mathbb{R}^n$  is differentiable but not necessarily invertible (hence we no longer require that d = n). In this case we cannot easily evaluate  $p(x; \theta)$  using the change-of-variables formula, although the density can be defined as the partial derivative of the cumulative density function,

$$p(x; \theta) = \frac{\partial^n}{\partial x_1 \dots \partial x_n} \int_{\{z: f(z; \theta) < x\}} p(z).$$

Implicit probabilistic models are generative models like the above which specify a stochastic procedure for generating samples from the model (Mohamed and Lakshminarayanan, 2016). This is in contrast to prescribed or explicit probabilistic models in which the modeler specifies an explicit parameterization of the density  $p(x;\theta)$ .<sup>29</sup> In this tutorial we have mostly considered explicit probabilistic models, whose training involved maximizing the log likelihood (or a lower bound on it) directly. Implicit models are instead trained with likelihood-free inference, which (roughly) are a class of methods that estimate the density ratio  $\frac{p_{\star}(x)}{p(x;\theta)}$  or the density difference  $p_{\star}(x) - p(x;\theta)$  instead of directly working with the likelihood  $p(x;\theta)$  (Marin et al., 2012; Gutmann and Hyvärinen, 2012; Gutmann et al., 2014; Bernton et al., 2017).

Generative Adversarial Networks (GAN) (Goodfellow et al., 2014) are deep, implicit generative models trained with adversarial training where a *discriminator* (or a *critic*) is trained to distinguish between samples from the generative model  $p(x; \theta)$  versus the true data distribution  $p_{\star}(x)$ . The generative model is trained "adversarially" to fool the discriminator.

Formally, let  $D: \mathcal{X} \to (0,1)$  be the discriminator (typically parameterized as a neural network) that maps the data  $x \in \mathcal{X}$  to the interval (0,1). The discriminator parameters  $\psi$  and the generative model parameters  $\theta$  are trained based on the following minimax objective:

$$\epsilon \sim \mathcal{N}(0,1), \quad x = \mu + \sigma \epsilon,$$

or we can explicitly define it by specifying the density, i.e.

$$p(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

<sup>&</sup>lt;sup>29</sup>The distinction between implicit/explicit generative models is therefore based on how the generative model is defined, rather than the distribution itself. For example, if  $x \sim \mathcal{N}(\mu, \sigma^2)$ , we can implicitly define the model by giving a stochastic procedure for generating the data, i.e.

$$\min_{\theta} \max_{\psi} \mathbb{E}_{p_{\star}(x)}[\log D(x; \psi)] + \mathbb{E}_{p(x; \theta)}[\log(1 - D(x; \psi))],$$

where the expectations are approximated with samples from  $p_{\star}(x)$  and  $p(x;\theta)$ . Interpreting  $D(x;\psi)$  as the probability the discriminator assigns to the event that x is sampled from the true data distribution, we can see that the discriminator training objective is the standard log likelihood objective from logistic regression. The generative model is trained to minimize the probability that the discriminator will correctly classify the generated sample as fake.<sup>30</sup> If the observed domain  $\mathcal{X}$  is continuous (e.g. dequantized images) then we can backpropagate the gradients from the discriminator to the generative model. For text this is generally not the case and thus we must resort to other techniques.

Goodfellow et al. (2014) show that the above objective approximately minimizes the Jensen-Shannon divergence between the data/model distributions, which is given by

$$JS[p_{\star}(x) \| p(x;\theta)] = \frac{1}{2} \left( KL \left[ p_{\star}(x) \left\| \frac{p_{\star}(x) + p(x;\theta)}{2} \right] + KL \left[ p(x;\theta) \left\| \frac{p_{\star}(x) + p(x;\theta)}{2} \right] \right).$$

This is notably different from the usual maximum likelihood training which minimizes  $KL[p_*(x)||p(x;\theta)]$ . The GAN objective has been generalized to (approximately) minimize other measures, such as f-divergences (Nowozin et al., 2016) and the Wasserstein distance (Arjovsky et al., 2017; Tolstikhin et al., 2018), which requires different parameterizations of the discriminator  $\psi$ .

Empirically GANs perform remarkably well and are able to generate impressive-looking images (Radford et al., 2016; Salimans et al., 2016; Zhang et al., 2017b; Miyato et al., 2018; Karras et al., 2018; Brock et al., 2018). While a rich research program has formed around GANs (and implicit models in general) over the past few years, <sup>31</sup> their applications to text modeling has been somewhat limited by the difficulty associated with optimizing the GAN objective when the output space is discrete.

#### 7.3.1 Generative Adversarial Networks for Text

Training the discriminator is straightforward with gradient-based methods. For the generative model if the domain  $\mathcal{X}$  is continuous and  $D(x; \psi)$  is differentiable with respect to x, then we can use the reparameterization trick to obtain low variance gradient estimators

$$\begin{split} \nabla_{\theta} \mathbb{E}_{p(x;\theta)}[\log(1 - D(x;\psi)] &= \nabla_{\theta} \mathbb{E}_{p(z)}[\log(1 - D(f(z;\theta);\psi)] \\ &= \mathbb{E}_{p(z)}[\nabla_{\theta} \log(1 - D(f(z;\theta);\psi))] \\ &= \mathbb{E}_{p(z)}\Big[ -\frac{1}{1 - D(x;\psi)} \nabla_{x} D(x;\psi) \frac{\partial x}{\partial \theta} \Big], \end{split}$$

where the expectation is approximated with Monte Carlo samples. Notice that this requires the gradient of  $D(x; \psi)$  with respect to x, which is multiplied with the Jacobian  $\frac{\partial x}{\partial \theta}$  to obtain the gradient with respect to  $\theta$ .

If  $\mathcal{X}$  is discrete however, we run into several issues. First, we cannot even have  $x = f(z; \theta)$  since a deterministic function that maps from a continuous z to a discrete x will either be uninteresting (e.g. a constant function) or badly behaved (e.g. an argmax, which is differentiable almost everywhere but has zero gradients). A tempting strategy in this case is to define the likelihood  $p(x | z; \theta)$  whose *parameters* are the output

<sup>&</sup>lt;sup>30</sup>In practice the generator is trained to maximize  $\mathbb{E}_{p(x;\theta)}[-\log D(x;\psi)]$  as this provides stronger gradient signals early in training.

<sup>&</sup>lt;sup>31</sup>For example see https://github.com/hindupuravinash/the-gan-zoo for a list of GAN variants.

<sup>&</sup>lt;sup>32</sup>Note that automatic differentiation packages never actually instantiate the full Jacobian, but we have written it explicitly here for illustrative purposes.

from f, e.g. for a single token x

$$p(x | z; \theta) = \pi_x$$
  $\pi = f(z; \theta) = \text{softmax}(\text{MLP}(z; \theta)).$ 

Note that this is now an *explicit generative model*, much like the models we have considered in previous sections. Then the score function gradient estimator is

$$\mathbb{E}_{p(x;\theta)}[\log(1 - D(x;\psi))\nabla_{\theta}\log p(x;\theta)].$$

Unfortunately we run into another issue here because  $\log p(x; \theta)$  was assumed to be intractable in the first place! Researchers have proposed to various ways to mitigate this problem.

**Relaxing the Discrete Space** Several works (Gulrajani et al., 2017a; Rajeswar et al., 2017; Press et al., 2017) relax the discrete space into a continuous space, for example by using the softmax function instead of the one-hot representation of x,

$$x = f(z; \theta) = \text{softmax}(\text{MLP}(z; \theta)).$$

If x is a sequence of discrete symbols, then the output would be a softmax-ed vector for each time step, i.e. a  $T \times V$  dense matrix where T is the number of tokens and V is the vocabulary size. For example, we may have separate MLPs for each position,

$$x = [x_1, \dots x_T]$$
  $x_i = f_i(z; \theta) = \operatorname{softmax}(\operatorname{MLP}_i(z; \theta)).$ 

(Other parameterizations are possible, for example one might use an RNN or a transposed convolutional network to share parameters.)

In this relaxed setup, as was the case with images it is straightforward to apply the reparameterization trick to obtain gradients for the generative model. At first this strategy might seem hopeless since (1) the discriminator can easily discriminate between a discrete sample  $x \sim p_*(x)$  and a generated (continuous) sample  $x \sim p(x;\theta)$ , and (2) the Jensen-Shannon divergence, which is implicitly minimized by the GAN objective, is infinite for two distributions with disjoint support (as is the case here). However in practice this strategy does work to an extent, especially if one instead implicitly minimizes the Wasserstein distance (Arjovsky et al., 2017). Gulrajani et al. (2017a) note that training might be feasible in this regime since under mild conditions, the Wasserstein distance is finite and differentiable almost everywhere (i.e. even if the supports between the two distributions do not overlap). For actual sample generation it is common practice to replace the softmax with an arg max in these models.

Another way to "relax" the discrete space to learn an auxiliary continuous space in which to generate / discriminate (Zhao et al., 2018a; Subramanian et al., 2018). For example, Zhao et al. (2018a) use the adversarial autoencoder framework (Makhzani et al., 2015) to jointly learn (1) a sentence-level autoencoder which encodes into and decodes from a continuous space, and (2) a GAN which learns to generate/discriminate in the same continuous space. Tolstikhin et al. (2018) show that this style of training approximately minimizes the Wasserstein distance between the model/data distributions, and Gu et al. (2018b) apply similar techniques to model dialogue responses.

**Non-Latent Variable Models** A simple alternative is to model  $p(x; \theta)$  without an explicit latent variable, e.g. as an RNN language model (Yu et al., 2017a; Lin et al., 2017; Guo et al., 2018). Then the score function gradient estimator is straightforward to calculate, and this setup resembles sequence-level training of deep models (Ranzato et al., 2016) where the reward comes from a learned discriminator. Such methods usually require initializing with a pretrained language model (trained with maximum likelihood), in addition to other techniques from reinforcement learning. Li et al. (2017), Wu et al. (2017a), and Yang et al. (2018a) extend this setup to the conditional case for neural machine translation and dialogue modeling.

Some works instead perform adversarial training on the hidden states of the generative model to avoid optimization challenges associated with score function gradient estimators (Lamb et al., 2016; Xu et al., 2017). We emphasize that these models are not latent variable models in the sense we have been employing in this tutorial. Instead they should be viewed as language models optimized with adversarial training. The motivation for this setting is avoiding *exposure bias* (Bengio et al., 2015; Ranzato et al., 2016; Wiseman and Rush, 2016), which describes the mismatch between training (which uses ground-truth history) and generation (which uses model-generated history). However, whether adversarially trained models are able to meaningfully mitigate exposure bias is an open question (Tevet et al., 2018; Caccia et al., 2018).

We conclude this section by noting that training implicit models for text is very much an open problem. Evaluation of implicit models itself is difficult, as unlike explicit generative models it is not possible to tractably estimate the log likelihood. Even if log likelihood is estimable, Theis et al. (2016) show that models that achieve good log likelihoods do not necessarily generate good samples. Cífka et al. (2018) and Semeniuta et al. (2018) explore a variety of deep generative models of language across various metrics and find that different models do well on different metrics.

# 8 Discussion: Role of Latent Variables in Deep Learning

Deep latent variable models, which combine the composability and interpretability of graphical models with the flexible modeling capabilities of deep networks, are an exciting area of research. It is nonetheless worth discussing again *why* we would want to use latent variable models in the first place, especially given that from a pure performance standpoint, models that do not formally make use of latent variables such as LSTMs and transformer networks are incredibly effective (Vaswani et al., 2017; Melis et al., 2018b; Merity et al., 2018). Even from the perspective of learning interesting/meaningful structures (e.g. topic modeling, word alignment, unsupervised tagging), one may argue that these meaningful representations can be implicitly captured within the hidden layers of a deep network. Indeed, the recent, astonishing success of pretrained (non-latent variable) language models as generic feature extractors (Peters et al., 2018; Radford et al., 2018; Devlin et al., 2018) suggests that deep networks can capture significant amount of linguistic knowledge without the explicit use of latent variables.

These are valid points, and it is entirely possible that deterministic deep networks are "all you need". However we think that this is unlikely to be the case. For one, latent variable models *can* surpass the performance of deterministic deep networks if properly optimized. For example, deterministic (soft) attention (Bahdanau et al., 2015) was generally thought to outperform latent variable (hard) attention (Xu et al., 2015b), until recent work showed that when properly optimized (albeit with a more expensive training procedure), latent variable attention can outperform deterministic attention (Deng et al., 2018; Shankar et al., 2018; Wu et al., 2018). Similarly, language models equipped with a latent variable per token (Yang et al., 2018b) are currently the state-of-the-art in language modeling. Framing certain aspects of a network as essentially approximating a latent variable objective often yields valuable insights and new avenues for further work. Notably, the interpretation of dropout (Hinton et al., 2012; Srivastava et al., 2014) as optimizing a latent variable objective has led to rich extensions and improvements (Kingma et al., 2015; Gal and Ghahramani, 2016a,b; Ma et al., 2017; Melis et al., 2018a).

Finally, latent variable modeling gives us a declarative language with which to explicitly inject both *inductive bias* and domain-specific *constraints* into models. Inductive bias, or the inherent "preferences" of a model (or learning algorithm), is crucial for learning and generalization. It can help mitigate against model misspecification, allow for data-efficient learning, and through a carefully crafted generative model, enable interesting structures to emerge. Furthermore, if we have constraints on the representations learned by a model, such as that they represent a valid parse tree, or be interpretable, or allow for controlling the model's predictions, we can enforce these constraints in a principled way through latent variables.

## 9 Conclusion

This tutorial has focused on learning deep latent variable models of text, in particular, models that can be expressed as directed graphical models. We have reviewed some archetypical models of text including their applications and explored learning such models through maximizing the log marginal likelihood or a lower bound on it. We have devoted a significant portion of the tutorial to amortized variational inference, a key technique for learning deep latent variable models in which a separate inference network is trained to perform approximate posterior inference.

We again emphasize that many important areas were not covered in the tutorial. Notably we have not covered undirected graphical models, posterior inference through Markov chain Monte Carlo, spectral learning of latent variable models, and non-likelihood-based approaches. These areas are all important on their own, and the intersection of latent variable models and deep learning remains an exciting avenue for much future work.

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## References

- Charu C. Aggarwal and ChengXiang Zhai. 2012. A Survey of Text Clustering Algorithms. In *Mining Text Data*, pages 77–128. Springer.
- Alexander A. Alemi, Ben Poole, Ian Fischer, Joshua V. Dillon, Rif A. Saurous, and Kevin Murphy. 2018. Fixing a Broken ELBO. In *Proceedings of ICML*.
- Marcin Andrychowicz, Misha Denil, Sergio Gomez, Matthew Hoffman, David Pfau, Tom Schaul, and Nando de Freitas. 2016. Learning to Learn by Gradient Descent by Gradient Descent. In *Proceedings of NIPS*.
- Martín Arjovsky, Soumith Chintala, and Léon Bottou. 2017. Wasserstein Generative Adversarial Networks. In *Proceedings of ICML*.
- Jimmy Ba, Ruslan R Salakhutdinov, Roger B Grosse, and Brendan J Frey. 2015. Learning Wake-Sleep Recurrent Attention Models. In *Proceedings of NIPS*.
- Dzmitry Bahdanau, Kyunghyun Cho, and Yoshua Bengio. 2015. Neural Machine Translation by Jointly Learning to Align and Translate. In *Proceedings of ICLR*.
- Mohamed Ishmael Belghazi, Aristide Baratin, Sai Rajeswar, Sherjil Ozair, Yoshua Bengio, Aaron Courville, and R Devon Hjelm. 2018. MINE: Mutual Information Neural Estimation. In *Proceedings of ICML*.
- Samy Bengio, Oriol Vinyals, Navdeep Jaitly, and Noam Shazeer. 2015. Scheduled Sampling for Sequence Prediction with Recurrent Neural Networks. In *Proceedings of NIPS*.
- Yoshua Bengio, Rejean Ducharme, and Pascal Vincent. 2003. A Neural Probabilistic Language Model. *Journal of Machine Learning Research*, 3:1137–1155.
- Rianne van den Berg, Leonard Hasenclever, Jakub M. Tomczak, and Max Welling. 2018. Sylvester Normalizing Flows for Variational Inference. In *Proceedings of UAI*.
- Taylor Berg-Kirkpatrick, Alexandre Bouchard-Cote, John DeNero, and Dan Klein. 2010. Painless Unsupervised Learning with Features. In *Proceedings of NAACL*.

- Espen Bernton, Pierre E. Jacob, Mathieu Gerber, and Christian P. Robert. 2017. Inference in Generative Models Using the Wasserstein Distance. *arXiv:1701.05146*.
- Eli Bingham, Jonathan P. Chen, Martin Jankowiak, Fritz Obermeyer, Neeraj Pradhan, Theofanis Karaletsos, Rohit Singh, Paul Szerlip, Paul Horsfall, and Noah D. Goodman. 2018. Pyro: Deep Universal Probabilistic Programming. arXiv:1810.09538.
- Christopher M. Bishop. 2006. Pattern Recognition and Machine Learning. Springer.
- David M. Blei, Alp Kucukelbir, and Jon D. McAuliffe. 2017. Variational Inference: A Review for Statisticians. *Journal of the American Statistical Association*, 112(518):859–877.
- David M Blei, Andrew Y Ng, and Michael I Jordan. 2003. Latent dirichlet allocation. *Journal of machine Learning research*, 3(Jan):993–1022.
- Phil Blunsom and Trevor Cohn. 2011. A Hierarchical Pitman-Yor process HMM for Unsupervised Part of Speech Induction. In *Proceedings of ACL*. Association for Computational Linguistics.
- Erik Bodin, Iman Malik, Carl Henrik Ek, and Neill D. F. Campbell. 2017. Nonparametric Inference for Auto-Encoding Variational Bayes. In *Proceedings of NIPS Workshop on Advances in Approximate Bayesian Inference*.
- Jorg Bornschein and Yoshua Bengio. 2015. Reweighted Wake-Sleep. In Proceedings of ICLR.
- Samuel R. Bowman, Luke Vilnis, Oriol Vinyal, Andrew M. Dai, Rafal Jozefowicz, and Samy Bengio. 2016. Generating Sentences from a Continuous Space. In *Proceedings of CoNLL*.
- Jordan L. Boyd-Graber and David M. Blei. 2008. Syntactic Topic Models. In Proceedings of NIPS.
- Andrew Brock, Jeff Donahue, and Karen Simonyan. 2018. Large Scale GAN Training for High Fidelity Natural Image Synthesis. *arXiv*:1809.11096.
- Peter F Brown, Peter V Desouza, Robert L Mercer, Vincent J Della Pietra, and Jenifer C Lai. 1992. Class-based N-gram Models of Natural Language. *Computational Linguistics*, 18(4):467–479.
- Peter F. Brown, Stephen A. Della Pietra, Vincent J. Della Pietra, and Robert L. Mercer. 1993. The Mathematics of Statistical Machine Translation: Parameter Estimation. *Computational linguistics*, 19(2):263–311.
- Yuri Burda, Roger Grosse, and Ruslan Salakhutdinov. 2015. Importance Weighted Autoencoders. In Proceedings of ICLR.
- Christopher P. Burgess, Irina Higgins, Arka Pal, Loic Matthey, Nick Watters, Guillaume Desjardins, and Alexander Lerchner. 2017. Understanding Disentangling in β-VAE. In *Proceedings of NIPS Workshop on Learning Disentangled Representations*.
- Massimo Caccia, Lucas Caccia, William Fedus, Hugo Larochelle, Joelle Pineau, and Laurent Charlin. 2018. Language GANs Falling Short. *arXiv:1811.02549*.
- Jiong Cai, Yong Jiang, and Kewei Tu. 2017. CRF Autoencoder for Unsupervised Dependency Parsing. In *Proceedings of EMNLP*.
- Ricky T. Q. Chen, Yulia Rubanova, Jesse Bettencourt, and David Duvenaud. 2018a. Neural Ordinary Differential Equations. In *Proceedings of NIPS*.
- Tian Qi Chen, Xuechen Li, Roger Grosse, and David Duvenaud. 2018b. Isolating Sources of Disentanglement in Variational Autoencoders. In *Proceedings of NIPS*.
- Xi Chen, Diederik P. Kingma, Tim Salimans, Yan Duan, Prafulla Dhariwal, John Schulman, Ilya Sutskever, and Pieter Abbeel. 2017. Variational Lossy Autoencoder. In *Proceedings of ICLR*.
- Kyunghyun Cho, Bart van Merrienboer, Caglar Gulcehre, Dzmitry Bahdanau, Fethi Bougares, Holger Schwenk, and Yoshua Bengio. 2014. Learning Phrase Representations using RNN Encoder-Decoder for Statistical Machine Translation. In *Proceedings of EMNLP*.
- Kyunghyun Cho, Tapani Raiko, Alexander Ilin, and Juha Karhunen. 2013. A Two-Stage Pretraining Algorithm for Deep Boltzmann Machines. In *Proceedings of ICANN*.

- Jihun Choi, Kang Min Yoo, and Sang goo Lee. 2018. Learning to Compose Task-Specific Tree Structures. In *Proceedings* of AAAI.
- Christos Christodoulopoulos, Sharon Goldwater, and Mark Steedman. 2010. Two Decades of Unsupervised POS Induction: How far have we come? In *Proceedings of EMNLP*.
- Junyoung Chung, Kyle Kastner, Laurent Dinh, Kratarth Goel, Aaron Courville, and Yoshua Bengio. 2015. A Recurrent Latent Variable Model for Sequential Data. In *Proceedings of NIPS*.
- Ondřej Cífka, Aliaksei Severyn, Enrique Alfonseca, and Katja Filippova. 2018. Eval All, Trust a Few, Do Wrong to None: Comparing Sentence Generation Models. *arXiv:1804.07972*.
- Alexis Conneau, Douwe Kiela, Holger Schwenk, Loïc Barrault, and Antoine Bordes. 2017. Supervised Learning of Universal Sentence Representations from Natural Language Inference Data. In *Proceedings of EMNLP*.
- Caio Corro and Ivan Titov. 2018. Differentiable Perturb-and-Parse: Semi-Supervised Parsing with a Structured Variational Autoencoder. arXiv:1807.09875.
- Chris Cremer, Xuechen Li, and David Duvenaud. 2018. Inference Suboptimality in Variational Autoencoders. In *Proceedings of ICML*.
- Chris Cremer, Quaid Morris, and David Duvenaud. 2017. Reinterpreting Importance-Weighted Autoencoders. In *Proceedings of ICLR Workshop*.
- Hanjun Dai, Yingtao Tian, Bo Dai, Steven Skiena, and Le Song. 2018. Syntax-Directed Variational Autoencoder for Structured Data. In *Proceedings of ICLR*.
- Tim R. Davidson, Luca Falorsi, Nicola De Cao, Thomas Kipf, and Jakub M. Tomczak. 2018. Hyperspherical Variational Auto-Encoders. In *Proceedings of UAI*.
- Arthur P. Dempster, Nan M. Laird, and Donald B. Rubin. 1977. Maximum Likelihood from Incomplete Data via the EM Algorithm. *Journal of the Royal Statistical Society, Series B*, 39(1):1–38.
- Yuntian Deng, Yoon Kim, Justin Chiu, Demi Guo, and Alexander M. Rush. 2018. Latent Alignment and Variational Attention. In *Proceedings of NIPS*.
- Jacob Devlin, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. 2018. BERT: Pre-training of Deep Bidirectional Transformers for Language Understanding. *arXiv:1810.04805*.
- Adji B. Dieng, Yoon Kim, Alexander M. Rush, and David M. Blei. 2018. Avoiding Latent Variable Collapse with Generative Skip Models. In *Proceedings of the ICML Workshop on Theoretical Foundations and Applications of Deep Generative Models*.
- Adji B. Dieng, Chong Wang, Jianfeng Gao, and John Paisley. 2017. TopicRNN: A Recurrent Neural Network With Long-Range Semantic Dependency. In *Proceedings of ICLR*.
- Nat Dilokthanakul, Pedro A.M. Mediano, Marta Garnelo, Matthew C.H. Lee, Hugh Salimbeni, Kai Arulkumaran, and Murray Shanahan. 2016. Deep Unsupervised Clustering with Gaussian Mixture Variational Autoencoders. arXiv:1611.02648.
- Laurent Dinh, David Krueger, and Yoshua Bengio. 2015. NICE: Non-linear Independent Components Estimation. In *Proceedings of ICLR 2015 Workshop*.
- Laurent Dinh, Jascha Sohl-Dickstein, and Samy Bengio. 2017. Density Estimation using Real NVP. In *Proceedings of ICLR*.
- Justin Domke and Daniel Sheldon. 2018. Importance Weighting and Variational Inference. In Proceedings of NIPS.
- John Duchi, Elad Hazan, and Yoram Singer. 2011. Adaptive Subgradient Methods for Online Learning and Stochastic Optimization. *Journal of Machine Learning Research*, 12.
- Chris Dyer, Adhiguna Kuncoro, Miguel Ballesteros, and Noah A. Smith. 2016. Recurrent Neural Network Grammars. In *Proceedings of NAACL*.
- David Eigen, Marc'Aurelio Ranzato, and Ilya Sutskever. 2013. Learning Factored Representations in a Deep Mixture of Experts. *arXiv*:1312.4314.

- Jason Eisner. 2016. Inside-Outside and Forward-Backward Algorithms Are Just Backprop (Tutorial Paper). In *Proceedings of the Workshop on Structured Prediction for NLP*.
- Jeffrey L. Elman. 1990. Finding Structure in Time. Cognitive Science, 14(2):179–211.
- Michael Figurnov, Shakir Mohamed, and Andriy Mnih. 2018. Implicit Reparameterization Gradients. In *Proceedings of NIPS*.
- Marco Fraccaro, Soren Kaae Sonderby, Ulrich Paquet, and Ole Winther. 2016. Sequential Neural Models with Stochastic Layers. In *Proceedings of NIPS*.
- Michael C. Fu. 2006. Gradient Estimation. Handbooks in operations research and management science, 13:575-616.
- Yarin Gal and Zoubin Ghahramani. 2016a. A Theoretically Grounded Application of Dropout in Recurrent Neural Networks. In *Proceedings of NIPS*.
- Yarin Gal and Zoubin Ghahramani. 2016b. Dropout as a Bayesian Approximation: Representing Model Uncertainty in Deep Learning. In *Proceedings of ICML*.
- Shuyang Gao, Rob Brekelmans, Greg Ver Steeg, and Aram Galstyan. 2018. Auto-Encoding Total Correlation Explanation. In *Proceedings of ICML*.
- Ekaterina Garmash and Christof Monz. 2016. Ensemble Learning for Multi-source Neural Machine Translation. In *Proceedings of COLING*.
- Sebastian Gehrmann, Falcon Z. Dai, Henry Elder, and Alexander M. Rush. 2018. End-to-end content and plan selection for natural language generation.
- Mathieu Germain, Karol Gregor, Iain Murray, and Hugo Larochelle. 2015. MADE: Masked Autoencoder for Distribution Estimation. In *Proceedings of ICML*.
- Zoubin Ghahramani and Michael I. Jordan. 1996. Factorial Hidden Markov Models. In Proceedings of NIPS.
- Paul Glasserman. 2013. Monte Carlo Methods in Financial Engineering, volume 53. Springer Science & Business Media.
- Peter Glynn. 1987. Likelihood Ratio Gradient Estimation: An Overview. In Proceedings of Winter Simulation Conference.
- Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. 2014. Generative Adversarial Nets. In *Proceedings of NIPS*.
- Anirudh Goyal, Alessandro Sordoni, Marc-Alexandre Cote, Nan Rosemary Ke, and Yoshua Bengio. 2017a. Z-Forcing: Training Stochastic Recurrent Networks. In *Proceedings of NIPS*.
- Prasoon Goyal, Zhiting Hu, Xiaodan Liang, Chenyu Wang, and Eric Xing. 2017b. Nonparametric Variational Autoencoders for Hierarchical Representation Learning. In *Proceedings of ICCV*.
- Will Grathwohl, Ricky T. Q. Chen, Jesse Bettencourt, Ilya Sutskever, and David Duvenaud. 2018a. FFJORD: Free-form Continuous Dynamics for Scalable Reversible Generative Models. *arXiv*:1810.01367.
- Will Grathwohl, Dami Choi, Yuhuai Wu, Geoffrey Roeder, and David Duvenaud. 2018b. Backpropagation through the Void: Optimizing Control Variates for Black-box Gradient Estimation. In *Proceedings of ICLR*.
- Thomas L. Griffiths, Mark Steyvers, David M. Blei, and Joshua B. Tenenbaum. 2004. Integrating Topics and Syntax. In *Proceedings of NIPS*.
- Jiatao Gu, James Bradbury, Caiming Xiong, Victor O.K. Li, and Richard Socher. 2018a. Non-Autoregressive Neural Machine Translation. In *Proceedings of ICLR*.
- Xiaodong Gu, Kyunghyun Cho, Jungwoo Ha, and Sunghun Kim. 2018b. DialogWAE: Multimodal Response Generation with Conditional Wasserstein Auto-Encoder. *arXiv*:1805.12352.
- Ishaan Gulrajani, Faruk Ahmed, Martin Arjovsky, and Aaron Courville Vincent Dumoulin. 2017a. Improved Training of Wasserstein GANs. In *Proceedings of NIPS*.
- Ishaan Gulrajani, Kundan Kumar, Faruk Ahmed, Adrien Ali Taiga, Francesco Visin, David Vazquez, and Aaron Courville. 2017b. PixelVAE: A Latent Variable Model for Natural Images. In *Proceedings of ICLR*.

- Jiaxian Guo, Sidi Lu, Han Cai, Weinan Zhang, Yong Yu, and Jun Wang. 2018. Long Text Generation via Adversarial Training with Leaked Information. In *Proceedings of AAAI*.
- Michael U. Gutmann, Ritabrata Dutta, Samuel Kaski, and Jukka Corander. 2014. Likelihood-free Inference via Classification. *arXiv:1407.4981*.
- Michael U. Gutmann and Aapo Hyvärinen. 2012. Noise-Contrastive Estimation of Unnormalized Statistical Models, with Applications to Natural Image Statistics. *Journal of Machine Learning Research*, 3:307–361.
- Kelvin Guu, Tatsunori B. Hashimoto, Yonatan Oren, and Percy Liang. 2017. Generating Sentences by Editing Prototypes. arXiv:1709.08878.
- Aria Haghighi and Dan Klein. 2006. Prototype-driven Learning for Sequence Models. In Proceedings of ACL.
- Wenjuan Han, Yong Jiang, and Kewei Tu. 2017. Dependency Grammar Induction with Neural Lexicalization and Big Training Data. In *Proceedings of EMNLP*.
- Tamir Hazan and Tommi Jaakkola. 2012. On the Partition Function and Random Maximum A-Posteriori Perturbation. In *Proceedings of ICML*.
- Junxian He, Graham Neubig, and Taylor Berg-Kirkpatrick. 2018a. Unsupervised Learning of Syntactic Structure with Invertible Neural Projections. In *Proceedings of EMNLP*.
- Xuanli He, Gholamreza Haffari, and Mohammad Norouzi. 2018b. Sequence to Sequence Mixture Model for Diverse Machine Translation. In *Proceedings of CoNLL*.
- William P Headden III, Mark Johnson, and David McClosky. 2009. Improving Unsupervised Dependency Parsing with Richer Contexts and Smoothing. In *Proceedings of NAACL*.
- Irina Higgins, Loic Matthey, Arka Pal, Christopher Burgess, Xavier Glorot, Matthew Botvinick, Shakir Mohamed, and Alexander Lerchner. 2017.  $\beta$ -VAE: Learning Basic Visual Concepts with a Constrained Variational Framework. In *Proceedings of ICLR*.
- Geoffrey Hinton, Nitish Srivastava, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov. 2012. Improving Neural Networks by Preventing Co-Adaptation of Feature Detectors. arXiv:1207.0580.
- Geoffrey E. Hinton and Drew van Camp. 1993. Keeping the Neural Networks Simple by Minimizing the Description Length of the Weights. In *Proceedings of COLT*.
- Geoffrey E. Hinton, Peter Dayan, Brendan J. Frey, and Radford M. Neal. 1995. The Wake-Sleep Algorithm for Unsupervised Neural Networks. *Science*, 268:1158–1161.
- Geoffrey E. Hinton, Simon Osindero, and Yee-Whye Teh. 2006. A Fast Learning Algorithm for Deep Belief Nets. *Neural Computation*, 18:1527–1554.
- R Devon Hjelm, Kyunghyun Cho, Junyoung Chung, Russ Salakhutdinov, Vince Calhoun, and Nebojsa Jojic. 2016. Iterative Refinement of the Approximate Posterior for Directed Belief Networks. In *Proceedings of NIPS*.
- Sepp Hochreiter and Jurgen Schmidhuber. 1997. Long Short-Term Memory. Neural Computation, 9(8):1735–1780.
- Matthew D. Hoffman. 2017. Learning Deep Latent Gaussian models with Markov chain Monte Carlo. In *Proceedings of ICML*.
- Matthew D. Hoffman, David M. Blei, Chong Wang, and John Paisley. 2013. Stochastic Variational Inference. *Journal of Machine Learning Research*, 13:1303–1347.
- Matthew D. Hoffman and Matthew J. Johnson. 2016. ELBO Surgery: Yet Another Way to Carve Up the Variational Evidence Lower Bound. In *Proceedings of NIPS Symposium on Advances in Approximate Bayesian Inference*.
- Zhiting Hu, Zichao Yang, Xiaodan Liang, Ruslan Salakhutdinov, and Eric P Xing. 2017. Toward Controlled Generation of Text. In *Proceedings of ICML*.
- Chin-Wei Huang, David Krueger, Alexandre Lacoste, and Aaron Courville. 2018. Neural Autoregressive Flows. In *Proceedings of ICML*.

- Chin-Wei Huang, Ahmed Touati, Laurent Dinh, Michal Drozdzal, Mohammad Havaei, Laurent Charlin, and Aaron Courville. 2017. Learnable Explicit Density for Continuous Latent Space and Variational Inference. In *Proceedings of ICML Workshop on Principled Approaches to Deep Learning*.
- Robert A Jacobs, Michael I Jordan, Steven J Nowlan, and Geoffrey E Hinton. 1991. Adaptive Mixtures of Local Experts. *Neural Computation*, 3(1):79–87.
- Eric Jang, Shixiang Gu, and Ben Poole. 2017. Categorical Reparameterization with Gumbel-Softmax. In *Proceedings of ICLR*.
- Yong Jiang, Wenjuan Han, and Kewei Tu. 2016. Unsupervised Neural Dependency Parsing. In Proceedings of EMNLP.
- Wengong Jin, Regina Barzilay, and Tommi Jaakkola. 2018. Junction Tree Variational Autoencoder for Molecular Graph Generation. In *Proceedings of ICML*.
- Mark Johnson. 2007. Why doesnt EM find good HMM POS-taggers? In Proceedings of EMNLP-CoNLL.
- Matthew Johnson, David K. Duvenaud, Alex Wiltschko, Ryan P. Adams, and Sandeep R. Datta. 2016. Composing Graphical Models with Neural Networks for Structured Representations and Fast Inference. In *Proceedings of NIPS*.
- Michael Jordan, Zoubin Ghahramani, Tommi Jaakkola, and Lawrence Saul. 1999. Introduction to Variational Methods for Graphical Models. *Machine Learning*, 37:183–233.
- Armand Joulin, Edouard Grave, Piotr Bojanowski, and Tomas Mikolov. 2016. Bag of Tricks for Efficient Text Classification. *arXiv:1607.01759*.
- Lukasz Kaiser, Samy Bengio, Aurko Roy, Ashish Vaswani, Niki Parmar, Jakob Uszkoreit, and Noam Shazeer. 2018. Fast Decoding in Sequence Models Using Discrete Latent Variables. In *Proceedings of ICML*.
- Tero Karras, Timo Aila, Samuli Laine, and Jaakko Lehtinen. 2018. Progressive Growing of GANs for Improved Quality, Stability, and Variation. In *Proceedings of ICLR*.
- Kazuya Kawakami, Chris Dyer, and Phil Blunsom. 2018. Unsupervised Word Discovery with Segmental Neural Language Models. *arXiv:1811.09353*.
- Hyunjik Kim and Andriy Mnih. 2018. Disentangling by Factorising. In *Proceedings of ICML*.
- Yoon Kim, Sam Wiseman, Andrew C. Miller, David Sontag, and Alexander M. Rush. 2018. Semi-Amortized Variational Autoencoders. In *Proceedings of ICML*.
- Diederik P. Kingma and Jimmy Ba. 2015. Adam: A Method for Stochastic Optimization. In Proceedings of ICLR.
- Diederik P. Kingma and Prafulla Dhariwal. 2018. Glow: Generative Flow with Invertible 1x1 Convolutions. In *Proceedings of NIPS*.
- Diederik P. Kingma, Tim Salimans, and Max Welling. 2016. Improving Variational Inference with Autoregressive Flow. *arXiv:1606.04934*.
- Diederik P. Kingma and Max Welling. 2014. Auto-Encoding Variational Bayes. In Proceedings of ICLR.
- Durk P. Kingma, Tim Salimans, and Max Welling. 2015. Variational Dropout and the Local Reparameterization Trick. In *Proceedings of NIPS*.
- Thomas N. Kipf and Max Welling. 2016. Variational Graph Auto-Encoders. In *Proceedings of NIPS Bayesian Deep Learning Workshop*.
- Ryan Kiros, Yukun Zhu, Ruslan R Salakhutdinov, Richard Zemel, Raquel Urtasun, Antonio Torralba, and Sanja Fidler. 2015. Skip-thought Vectors. In *Proceedings of NIPS*.
- Dan Klein and Christopher D Manning. 2004. Corpus-based Induction of Syntactic Structure: Models of Dependency and Constituency. In *Proceedings of ACL*.
- Lingpeng Kong, Chris Dyer, and Noah A. Smith. 2016. Segmental Recurrent Neural Networks. In Proceedings of ICLR.
- Rahul G. Krishnan, Dawen Liang, and Matthew Hoffman. 2018. On the Challenges of Learning with Inference Networks on Sparse, High-dimensional Data. In *Proceedings of AISTATS*.

- Rahul G. Krishnan, Uri Shalit, and David Sontag. 2017. Structured Inference Networks for Nonlinear State Space Models. In *Proceedings of AAAI*.
- Matt J. Kusner, Brooks Paige, and Jose Miguel Hernandez-Lobato. 2017. Grammar Variational Autoencoder. In *Proceedings of ICML*.
- Alex Lamb, Anirudh Goyal, Ying Zhang, Saizheng Zhang, Aaron Courville, and Yoshua Bengio. 2016. Professor Forcing: A New Algorithm for Training Recurrent Networks. In *Proceedings of NIPS*.
- Hugo Larochelle and Iain Murray. 2011. The Neural Autoregressive Distribution Estimator. In Proceedings of AISTATS.
- Phong Le, Marc Dymetman, and Jean-Michel Renders. 2016. LSTM-Based Mixture-of-Experts for Knowledge-Aware Dialogues. In *Proceedings of the 1st Workshop on Representation Learning for NLP*.
- Quoc Le and Tomas Mikolov. 2014. Distributed Representations of Sentences and Documents. In Proceedings of ICML.
- Tuan Anh Le, Maximilian Igl, Tom Rainforth, Tom Jin, and Frank Wood. 2018a. Auto-Encoding Sequential Monte Carlo. In *Proceedings of ICLR*.
- Tuan Anh Le, Adam R. Kosiorek, N. Siddharth, Yee Whye Teh, and Frank Wood. 2018b. Revisiting Reweighted Wake-Sleep. *arXiv:1805.10469*.
- Jason Lee, Elman Mansimov, and Kyunghyun Cho. 2018. Deterministic Non-Autoregressive Neural Sequence Modeling by Iterative Refinement. In *Proceedings of EMNLP*.
- Stefan Lee, Senthil Purushwalkam Shiva Prakash, Michael Cogswell, Viresh Ranjan, David Crandall, and Dhruv Batra. 2016. Stochastic Multiple Choice Learning for Training Diverse Deep Ensembles. In *Proceedings of NIPS*.
- Bowen Li, Jianpeng Cheng, Yang Liu, and Frank Keller. 2019. Dependency Grammar Induction with a Neural Variational Transition-based Parser. In *Proceedings of AAAI*.
- Jiwei Li, Will Monroe, Tianlin Shi, Sebastien Jean, Alan Ritter, and Dan Jurafsky. 2017. Adversarial Learning for Neural Dialogue Generation. In *Proceedings of EMNLP*.
- Chu-Cheng Lin, Waleed Ammar, Chris Dyer, and Lori Levin. 2015. Unsupervised POS Induction with Word Embeddings. arXiv:1503.06760.
- Kevin Lin, Dianqi Li, Xiaodong He, Ming-ting Sun, and Zhengyou Zhang. 2017. Adversarial Ranking for Language Generation. In *Proceedings of NIPS*.
- Hao Liu, Lirong He, HAoli Ba, Bo Dai, Kun Bai, and Zenglin Xu. 2018. Structured Inference for Recurrent Hidden Semi-Markov Model. In *Proceedings of IJCAI*.
- Xuezhe Ma, Yingkai Gao, Zhiting Hu, Yaoliang Yu, Yuntian Deng, and Eduard Hovy. 2017. Dropout with Expectation-linear Regularization. In *Proceedings of ICLR*.
- James MacQueen et al. 1967. Some Methods for Classification and Analysis of Multivariate Observations. In *Proceedings* of the fifth Berkeley symposium on mathematical statistics and probability, volume 1, pages 281–297. Oakland, CA, USA.
- Chris J. Maddison, Dieterich Lawson, George Tucker, Nicolas Heess, Mohammad Norouzi, Andriy Mnih, Arnaud Doucet, and Yee Whye Teh. 2017a. Filtering Variational Objectives. In *Proceedings of NIPS*.
- Chris J. Maddison, Andriy Mnih, and Yee Whye Teh. 2017b. The Concrete Distribution: A Continuous Relaxation of Discrete Random Variables. In *Proceedings of ICLR*.
- Chris J. Maddison, Daniel Tarlow, and Tom Minka. 2014. A\* Sampling. In Proceedings of NIPS.
- Alireza Makhzani, Jonathon Shlens, Navdeep Jaitly, Ian Goodfellow, and Brendan Frey. 2015. Adversarial Autoencoders. arXiv:1511.05644.
- Jean-Michel Marin, Pierre Pudlo, Christian P. Robert, and Robin Ryder. 2012. Approximate Bayesian Computational Methods. *Statistics and Computing*, 22:1167–1180.
- Joseph Marino, Yisong Yue, and Stephan Mandt. 2018. Iterative Amortized Inference. In Proceedings of ICML.

- André F. T. Martins and Ramón Fernandez Astudillo. 2016. From Softmax to Sparsemax: A Sparse Model of Attention and Multi-Label Classification. In *Proceedings of ICML*.
- Gábor Melis, Charles Blundell, Tomáš Kočiskỳ, Karl Moritz Hermann, Chris Dyer, and Phil Blunsom. 2018a. Pushing the Bounds of Dropout. *arXiv:1805.09208*.
- Gábor Melis, Chris Dyer, and Phil Blunsom. 2018b. On the State of the Art of Evaluation in Neural Language Models. In *Proceedings of ICLR*.
- Bernard Merialdo. 1994. Tagging English Text with a Probabilistic Model. Computational Linguistics, 20(2):155-171.
- Stephen Merity, Nitish Shirish Keskar, and Richard Socher. 2018. Regularizing and optimizing LSTM language models. In *International Conference on Learning Representations*.
- Yishu Miao and Phil Blunsom. 2016. Language as a Latent Variable: Discrete Generative Models for Sentence Compression. In *Proceedings of EMNLP*.
- Yishu Miao, Edward Grefenstette, and Phil Blunsom. 2017. Discovering Discrete Latent Topics with Neural Variational Inference. In *Proceedings of ICML*.
- Yishu Miao, Lei Yu, and Phil Blunsom. 2016. Neural Variational Inference for Text Processing. In Proceedings of ICML.
- Tomas Mikolov, Martin Karafiat, Lukas Burget, Jan Cernocky, and Sanjeev Khudanpur. 2010. Recurrent Neural Network Based Language Model. In *Proceedings of INTERSPEECH*.
- Andrew C. Miller, Nicholas J. Foti, Alexander D'Amour, and Ryan P. Adams. 2017. Reducing Reparameterization Gradient Variance. In *Proceedings of NIPS*.
- Takeru Miyato, Toshiki Kataoka, Masanori Koyama, and Yuichi Yoshida. 2018. Spectral Normalization for Generative Adversarial Networks. In *Proceedings of ICLR*.
- Andriy Mnih and Danilo J. Rezende. 2016. Variational Inference for Monte Carlo Objectives. In Proceedings of ICML.
- Andryi Mnih and Karol Gregor. 2014. Neural Variational Inference and Learning in Belief Networks. In *Proceedings of ICML*.
- Shakir Mohamed and Balaji Lakshminarayanan. 2016. Learning in Implicit Generative Models. arXiv:1610.03483.
- Kevin P. Murphy. 2012. Machine Learning: A Probabilistic Perspective. The MIT Press.
- Christian A. Naesseth, Francisco J. R. Ruiz, Scott W. Linderman, and David M. Blei. 2017. Reparameterization Gradients through Acceptance-Rejection Sampling Algorithms. In *Proceedings of AISTATS*.
- Eric Naslinick and Padhraic Smyth. 2017. Stick-Breaking Variational Autoencoders. In Proceedings of ICLR.
- Christian A. Nasseth, Scott W. Linderman, Rajesh Ranganath, and David M. Blei. 2018. Variational Sequential Monte Carlo. In *Proceedings of AISTATS*.
- Radford M. Neal. 2001. Annealed Importance Sampling. Statistics and Computing, 11:125-139.
- Radford. M. Neal and Geoffrey E. Hinton. 1998. A New View of the EM Algorithm that Justifies Incremental, Sparse and Other Variants. *Learning in Graphical Models*.
- Vlad Niculae, André F. T. Martins, Mathieu Blondel, and Claire Cardie. 2018. SparseMAP: Differentiable Sparse Structured Inference. In *Proceedings of ICML*.
- Sebastian Nowozin, Botond Cseke, and Ryota Tomioka. 2016. f-GAN: Training Generative Neural Samplers using Variational Divergence Minimization. In *Proceedings of NIPS*.
- Aaron van den Oord, Oriol Vinyals, and Koray Kavukcuoglu. 2017. Neural Discrete Representation Learning. In *Proceedings of NIPS*.
- Matteo Pagliardini, Prakhar Gupta, and Martin Jaggi. 2018. Unsupervised Learning of Sentence Embeddings Using Compositional N-gram Features. In *Proceedings of NAACL*.
- George Papamakarios, Theo Pavlakou, and Iain Murray. 2017. Masked Autoregressive Flow for Density Estimation. In *Proceedings of NIPS*.

- George Papandreou and Alan L. Yuille. 2011. Perturb-and-Map Random Fields: Using Discrete Optimization to Learn and Sample from Energy Models. In *Proceedings of ICCV*.
- Yookoon Park, Jaemin Cho, and Gunhee Kim. 2018. A Hierarchical Latent Structure for Variational Conversation Modeling. In *Proceedings of NAACL*.
- Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. 2017. Automatic Differentiation in PyTorch.
- Matthew Peters, Mark Neumann, Mohit Iyyer, Matt Gardner, Christopher Clark, Kenton Lee, and Luke Zettlemoyer. 2018. Deep Contextualized Word Representations. In *Proceedings of NAACL*.
- Ofir Press, Amir Bar, Ben Bogin, Jonathan Berant, and Lior Wolf. 2017. Language Generation with Recurrent Generative Adversarial Networks without Pre-training. *arXiv*:1706.01399.
- Yunchen Pu, Zhe Gan, Ricardo Henao, Chunyuan Li, Shaobo Han, and Lawrence Carin. 2017. VAE Learning via Stein Variational Gradient Descent. In *Proceedings of NIPS*.
- Lawrence R. Rabiner. 1989. A Tutorial on Hidden markov Models and Selected Applications in Speech Recognition. *Proceedings of the IEEE*, 77(2):257–286.
- Alec Radford, Luke Metz, and Soumith Chintala. 2016. Unsupervised Representation Learning with Deep Convolutional Generative Adversarial Networks. In *Proceedings of ICLR*.
- Alec Radford, Karthik Narasimhan, Tim Salimans, and Ilya Sutskever. 2018. Improving Language Understanding by Generative Pre-Training. OpenAI blog:https://blog.openai.com/language-unsupervised.
- Tom Rainforth, Adam R. Kosiorek, Tuan Anh Le, Chris J. Maddison, Maximilian Igl, Frank Wood, and Yee Whye Teh. 2018. Tighter Variational Bounds are Not Necessarily Better. In *Proceedings of ICML*.
- Sai Rajeswar, Sandeep Subramanian, Francis Dutil, Christopher Pal, and Aaron Courville. 2017. Adversarial Generation of Natural Language. *arXiv*:1705.10929.
- Rajesh Ranganath, Sean Gerrish, and David M. Blei. 2014. Black Box Variational Inference. In Proceedings of AISTATS.
- Marc'Aurelio Ranzato, Sumit Chopra, Michael Auli, and Wojciech Zaremba. 2016. Sequence Level Training with Recurrent Neural Networks. In *Proceedings of ICLR*.
- Danilo J. Rezende and Shakir Mohamed. 2015. Variational Inference with Normalizing Flows. In *Proceedings of ICML*.
- Danilo J. Rezende, Shakir Mohamed, and Daan Wierstra. 2014. Stochastic Backpropagation and Approximate Inference in Deep Generative Models. In *Proceedings of ICML*.
- Jason Tyler Rolfe. 2017. Discrete Variational Autoencoders. In Proceedings of ICLR.
- Aurko Roy, Ashish Vaswani, Arvind Neelakantan, and Niki Parmar. 2018. Theory and Experiments on Vector Quantized Autoencoders. *arXiv:1805.11063*.
- Andreas Rücklé, Steffen Eger, Maxime Peyrard, and Iryna Gurevych. 2018. Concatenated *p*-mean Word Embeddings as Universal Cross-Lingual Sentence Representations. *arXiv:1803.01400*.
- Francisco J. R. Ruiz, Michalis K. Titsias, and David M. Blei. 2016. The Generalized Reparameterization Gradient. In *Proceedings of NIPS*.
- Ruslan Salakhutdinov and Hugo Larochelle. 2010. Efficient Learning of Deep Boltzmann Machines. In *Proceedings of AISTATS*.
- Ruslan Salakhutdinov, Sam Roweis, and Zoubin Ghahramani. 2003. Optimization with EM and Expectation-Conjugate-Gradient. In *Proceedings of ICML*.
- Tim Salimans, Ian Goodfellow, Wojciech Zaremba, Vicki Cheung, Alec Radford, and Xi Chen. 2016. Improved Techniques for Training GANs. In *Proceedings of NIPS*.
- Tim Salimans, Diederik Kingma, and Max Welling. 2015. Markov Chain Monte Carlo and Variational Inference: Bridging the Gap. In *Proceedings of ICML*.

- Stanislau Semeniuta, Aliaksei Severyn, and Erhardt Barth. 2017. A Hybrid Convolutional Variational Autoencoder for Text Generation. *arXiv:1702.02390*.
- Stanislau Semeniuta, Aliaksei Severyn, and Sylvain Gellyi. 2018. On Accurate Evaluation of GANs for Language Generation. In *Proceedings of the ICML Workshop on Theoretical Foundations and Applications of Deep Generative Models*.
- Iulian Vlad Serban, Alessandro Sordoni, Ryan Lowe, Laurent Charlin, Joelle Pineau, Aaron Courville, and Yoshua Bengio. 2017. A Hierarchical Latent Variable Encoder-Decoder Model for Generating Dialogues. In *Proceedings of AAAI*.
- Shiv Shankar, Siddhant Garg, and Sunita Sarawagi. 2018. Surprisingly Easy Hard-Attention for Sequence to Sequence Learning. In *Proceedings of EMNLP*.
- Noam Shazeer, Azalia Mirhoseini, Krzysztof Maziarz, Andy Davis, Quoc Le, Geoffrey Hinton, and Jeff Dean. 2017. Outrageously Large Neural Networks: The Sparsely-Gated Mixture-of-Experts Layer. In *Proceedings of ICLR*.
- Dinghan Shen, Yizhe Zhang, Ricardo Henao, Qinliang Su, and Lawrence Carin. 2018. Deconvolutional Latent-Variable Model for Text Sequence Matching. In *Proceedings of AAAI*.
- Rachit Singh, Jeffrey Ling, and Finale Doshi-Velez. 2017. Structured Variational Autoencoders for the Beta-Bernoulli Process. In *Proceedings of NIPS Workshop on Advances in Approximate Bayesian Inference*.
- Noah A. Smith and Jason Eisner. 2005. Contrastive Estimation: Training Log-Linear Models on Unlabeled Data. In *Proceedings of ACL*.
- Casper Kaae Sønderby, Tapani Raiko, Lars Maaløe, Søren Kaae Sønderby, and Ole Winther. 2016. Ladder Variational Autoencoders. In *Proceedings of NIPS*.
- Valentin I Spitkovsky, Hiyan Alshawi, Angel X Chang, and Daniel Jurafsky. 2011. Unsupervised dependency parsing without gold part-of-speech tags. In *Proceedings of the conference on empirical methods in natural language processing*, pages 1281–1290. Association for Computational Linguistics.
- Valentin I Spitkovsky, Hiyan Alshawi, Daniel Jurafsky, and Christopher D Manning. 2010. Viterbi Training Improves Unsupervised Dependency Parsing. In *Proceedings of CoNLL*.
- Nitish Srivastava, Geoffrey Hinton, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov. 2014. Dropout: A Simple Way to Prevent Neural Networks from Overfitting. *Journal of Machine Learning Research*.
- Karl Stratos, Michael Collins, and Daniel Hsu. 2016. Unsupervised Part-of-Speech Tagging with Anchor Hidden Markov Models. *Transactions of the Association for Computational Linguistics*, 4:245–257.
- Sandeep Subramanian, Sai Rajeswar, Alessandro Sordoni, Adam Trischler, Aaron Courville, and Christopher Pal. 2018. Towards Text Generation with Adversarially Learned Neural Outlines. In *Proceedings of NIPS*.
- Ilya Sutskever, Oriol Vinyals, and Quoc Le. 2014. Sequence to Sequence Learning with Neural Networks. In *Proceedings* of NIPS.
- Charles Sutton, Andrew McCallum, et al. 2012. An Introduction to Conditional Random Fields. *Foundations and Trends® in Machine Learning*, 4(4):267–373.
- Guy Tevet, Gavriel Habib, Vered Shwartz, and Jonathan Berant. 2018. Evaluating Text GANs as Language Models. arXiv:1810.12686.
- Lucas Theis, Aaron van den Oord, and Matthias Bethge. 2016. A Note on the Evaluation of Generative Models. In *Proceedings of ICLR*.
- Ilya Tolstikhin, Olivier Bousquet, Sylvain Gelly, and Bernhard Schoelkopf. 2018. Wasserstein Auto-Encoders. In *Proceedings of ICLR*.
- Jakub M. Tomczak and Max Welling. 2016. Improving Variational Auto-Encoders using Householder Flow. In Proceedings of NIPS Bayesian Deep Learning Workshop.
- Jakub M. Tomczak and Max Welling. 2018. VAE with a VampPrior. In Proceedings of AISTATS.
- Kristina Toutanova and Mark Johnson. 2008. A Bayesian LDA-based Model for Semi-Supervised Part-of-Speech Tagging. In *Proceedings of NIPS*.

- Dustin Tran, Rajesh Ranganath, and David M. Blei. 2016a. The Variational Gaussian Process. In Proceedings of ICLR.
- Ke Tran, Yonatan Bisk, Ashish Vaswani, Daniel Marcu, and Kevin Knight. 2016b. Unsupervised Neural Hidden Markov Models. In *Proceedings of the Workshop on Structured Prediction for NLP*.
- George Tucker, Dieterich Lawson, Shixiang Gu, and Chris J. Maddison. 2018. Doubly Reparameterized Gradient Estimators for Monte Carlo Objectives. *arXiv:1810.04152*.
- George Tucker, Andriy Mnih, Chris J. Maddison, Dieterich Lawson, and Jascha Sohl-Dickstein. 2017. REBAR: Low-variance, Unbiased Gradient Estimates for Discrete Latent Variable Models. In *Proceedings of NIPS*.
- Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Łukasz Kaiser, and Illia Polosukhin. 2017. Attention is All You Need. In *Proceedings of NIPS*.
- Martin J. Wainwright and Michael I. Jordan. 2008. Introduction to Variational Methods for Graphical Models. *Foundations and Trends in Machine Learning*, 1:1–103.
- Wenlin Wang, Zhe Gan, Wenqi Wang, Dinghan Shen, Jiaji Huang, Wei Ping, Sanjeev Satheesh, and Lawrence Carin. 2018. Topic Compositional Neural Language Model. In *Proceedings of AISTATS*.
- Greg C. G. Wei and Martin A. Tanner. 1990. A Monte Carlo Implementation of the EM Algorithm and the Poor Mans Data Augmentation Algorithms. *Journal of the American Statistical Association*, 85:699–704.
- Peter Willett. 1988. Recent Trends in Hierarchic Document Clustering: A Critical Review. *Information Processing & Management*, 24(5):577–597.
- Ronald J. Williams. 1992. Simple Statistical Gradient-following Algorithms for Connectionist Reinforcement Learning. *Machine Learning*, 8.
- Sam Wiseman and Alexander M. Rush. 2016. Sequence-to-Sequence Learning as Beam-Search Optimization. In *Proceedings of EMNLP*.
- Sam Wiseman, Stuart M. Shieber, and Alexander M. Rush. 2018. Learning Neural Templates for Text Generation. In *Proceedings of EMNLP*.
- Lijun Wu, Yingce Xia, Li Zhao, Fei Tian, Tao Qin, Jianhuang Lai, and Tie-Yan Liu. 2017a. Adversarial Neural Machine Translation. In *Proceedings of ACML*.
- Shijie Wu, Pamela Shapiro, and Ryan Cotterell. 2018. Hard Non-Monotonic Attention for Character-Level Transduction. In *Proceedings of EMNLP*.
- Yuhuai Wu, Yuri Burda, Ruslan Salakhutdinov, and Roger Grosse. 2017b. On the Quantitative Analysis of Decoder-Based Generative Models. In *Proceedings of ICLR*.
- Junyuan Xie, Ross Girshick, and Ali Farhadi. 2016. Unsupervised Deep Embedding for Clustering Analysis. In *Proceedings of ICML*.
- Jiacheng Xu and Greg Durrett. 2018. Spherical Latent Spaces for Stable Variational Autoencoders. In *Proceedings of EMNLP*.
- Jiaming Xu, Peng Wang, Guanhua Tian, Bo Xu, Jun Zhao, Fangyuan Wang, and Hongwei Hao. 2015a. Short Text Clustering via Convolutional Neural Networks. In *Proceedings of NAACL*.
- Kelvin Xu, Jimmy Ba, Ryan Kiros, Kyunghyun Cho, Aaron Courville, Ruslan Salakhutdinov, Richard Zemel, and Yoshua Bengio. 2015b. Show, Attend and Tell: Neural Image Caption Generation with Visual Attention. In *Proceedings of ICML*.
- Zhen Xu, Bingquan Liu, Baoxun Wang, Chengjie Sun, Xiaolong Wang, Zhuoran Wang, and Chao Qi. 2017. Neural Response Generation via GAN with an Approximate Embedding Layer. In *Proceedings of EMNLP*.
- Zhen Yang, Wei Chen, Feng Wang, and Bo Xu. 2018a. Improving Neural Machine Translation with Conditional Sequence Generative Adversarial Nets. In *Proceedings of NAACL*.
- Zhilin Yang, Zihang Dai, Ruslan Salakhutdinov, and William W. Cohen. 2018b. Breaking the Softmax Bottleneck: A High-Rank RNN Language Model. In *Proceedings of ICLR*.

- Zichao Yang, Zhiting Hu, Ruslan Salakhutdinov, and Taylor Berg-Kirkpatrick. 2017. Improved Variational Autoencoders for Text Modeling using Dilated Convolutions. In *Proceedings of ICML*.
- Jun Yin, Xin Jiang, Zhengdong Lu, Lifeng Shang, Hang Li, and Xiaoming Li. 2016. Neural Generative Question Answering. In *Proceedings of AAAI*.
- Pengcheng Yin, Chunting Zhou, Junxian He, and Graham Neubig. 2018. StructVae: Tree-structured Latent Variable Models for Semi-supervised Semantic Parsing. In *Proceedings of ACL*.
- Lantao Yu, Weinan Zhang, Jun Wang, and Yong Yu. 2017a. SeqGAN: Sequence Generative Adversarial Nets with Policy Gradient. In *Proceedings of AAAI*.
- Lei Yu, Phil Blunsom, Chris Dyer, Edward Grefenstette, and Tomas Kocisky. 2017b. The Neural Noisy Channel. In *Proceedings of ICLR*.
- Lei Yu, Jan Buys, and Phil Blunsom. 2016. Online Segment to Segment Neural Transduction. In Proceedings of EMNLP.
- Manzil Zaheer, Amr Ahmed, and Alexander J. Smola. 2017. Latent LSTM Allocation: Joint Clustering and Non-Linear Dynamic Modeling of Sequence Data. In *Proceedings of ICML*.
- Wojciech Zaremba, Ilya Sutskever, and Oriol Vinyals. 2014. Recurrent Neural Network Regularization. *arXiv:1409.2329*. Matthew D. Zeiler. 2012. Adadelta: An Adaptive Learning Rate Method. *arXiv:1212.5701*.
- Cheng Zhang, Judith Butepage, Hedvig Kjellstrom, and Stephan Mandt. 2017a. Advances in Variational Inference. arXiv:1711.05597.
- Han Zhang, Tao Xu, Hongsheng Li, Shaoting Zhang, Xiaogang Wang, Xiaolei Huang, and Dimitris Metaxas. 2017b. StackGAN: Text to Photo-realistic Image Synthesis with Stacked Generative Adversarial Networks. In *Proceedings of ICCV*.
- Junbo Zhao, Yoon Kim, Kelly Zhang, Alexander M Rush, and Yann LeCun. 2018a. Adversarially Regularized Autoencoders. In *Proceedings of ICML*.
- Shengjia Zhao, Jiaming Song, and Stefano Ermon. 2017. Learning Hierarchical Features from Generative Models. In *Proceedings of ICML*.
- Shengjia Zhao, Jiaming Song, and Stefano Ermon. 2018b. InfoVAE: Information Maximizing Variational Autoencoders. In *Proceedings of the ICML Workshop on Theoretical Foundations and Applications of Deep Generative Models*.