Week 11: Indeterminate forms, Techniques of integration

October 15 – October 19, 2012

1. Indeterminate forms

2. Integration techniques
   - Integration by parts
   - Trigonometric integrals

Lecture 21

Recall from last class

The limits of the form $\frac{\infty}{\infty}$ (also called '*indeterminate forms*') that we studied so far can be calculated using the standard trick of dividing the denominator through by the fastest growing term.

Recall from last class

The limits of the form $\frac{\infty}{\infty}$ (also called '*indeterminate forms*') that we studied so far can be calculated using the standard trick of dividing the denominator through by the fastest growing term. What about the following limits?

$$\lim_{x\to\infty} \frac{e^x}{x} \qquad\qquad \lim_{x\to\infty} \frac{\ln x}{x}$$

Recall from last class

The limits of the form $\frac{\infty}{\infty}$ (also called '*indeterminate forms*') that we studied so far can be calculated using the standard trick of dividing the denominator through by the fastest growing term. What about the following limits?

$$\lim_{x\to\infty} \frac{e^x}{x} \qquad\qquad \lim_{x\to\infty} \frac{\ln x}{x}$$

One way to solve this problem is to look at the derivatives.

Recall from last class

The limits of the form $\frac{\infty}{\infty}$ (also called '*indeterminate forms*') that we studied so far can be calculated using the standard trick of dividing the denominator through by the fastest growing term. What about the following limits?

$$\lim_{x\to\infty} \frac{e^x}{x} \qquad\qquad \lim_{x\to\infty} \frac{\ln x}{x}$$

One way to solve this problem is to look at the derivatives.
If, for example we could verify that $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0$ then this would suggest that the *rate* of increase of g is greater than that of f.

Recall from last class

The limits of the form $\frac{\infty}{\infty}$ (also called '*indeterminate forms*') that we studied so far can be calculated using the standard trick of dividing the denominator through by the fastest growing term. What about the following limits?

$$\lim_{x \to \infty} \frac{e^x}{x} \qquad \lim_{x \to \infty} \frac{\ln x}{x}$$

One way to solve this problem is to look at the derivatives.

If, for example we could verify that $\displaystyle\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0$ then this would suggest that the *rate* of increase of g is greater than that of f.

This in turn would suggest that $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$. In fact, this kind of intuitive reasoning is generally true!

L'Hôpital's rule

Suppose that f and g are differentiable functions, $a \in \mathbb{R}$, and $g'(x) \neq 0$, except possibly at a. Suppose also that either one of the two following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;

L'Hôpital's rule

Suppose that f and g are differentiable functions, $a \in \mathbb{R}$, and $g'(x) \neq 0$, except possibly at a. Suppose also that either one of the two following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
- $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$.

L'Hôpital's rule

Suppose that f and g are differentiable functions, $a \in \mathbb{R}$, and $g'(x) \neq 0$, except possibly at a. Suppose also that either one of the two following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
- $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$.

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists or is $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Remarks.

(i) The theorem also holds for limits at infinity or one-sided limits, That is, as $x \to \infty$ or $x \to -\infty$, or $x \to a^+$ or as $x \to a^-$.

Remarks.

(i) The theorem also holds for limits at infinity or one-sided limits, That is, as $x \to \infty$ or $x \to -\infty$, or $x \to a^+$ or as $x \to a^-$.

(ii) When using L'Hospital's Rule, we **do not** use the quotient rule. We differentiate the numerator and the denominator **separately**.

Remarks.

(i) The theorem also holds for limits at infinity or one-sided limits, That is, as $x \to \infty$ or $x \to -\infty$, or $x \to a^+$ or as $x \to a^-$.

(ii) When using L'Hospital's Rule, we **do not** use the quotient rule. We differentiate the numerator and the denominator **separately**.

(iii) Be sure to verify that the hypotheses in L'Hospital's rule are satisfied before applying it! We will see some examples of how things can go wrong if L'Hospital's rule is incorrectly used.

Remarks.

(i) The theorem also holds for limits at infinity or one-sided limits, That is, as $x \to \infty$ or $x \to -\infty$, or $x \to a^+$ or as $x \to a^-$.

(ii) When using L'Hospital's Rule, we **do not** use the quotient rule. We differentiate the numerator and the denominator **separately**.

(iii) Be sure to verify that the hypotheses in L'Hospital's rule are satisfied before applying it! We will see some examples of how things can go wrong if L'Hospital's rule is incorrectly used.

(iv) (non-assessable) The proof involves a more general version of the mean value theorem.

Examples. Now we will do a lot of examples! Find the following limits:

(a) $\lim\limits_{x \to \infty} \dfrac{e^x}{x}$.

Examples. Now we will do a lot of examples! Find the following limits:

(a) $\lim_{x \to \infty} \frac{e^x}{x}$.

(b) $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Examples. Now we will do a lot of examples! Find the following limits:

(a) $\lim_{x \to \infty} \frac{e^x}{x}$.

(b) $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

(c) $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Examples. Now we will do a lot of examples! Find the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$.

(b) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$.

(c) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$.

(d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

Examples. Now we will do a lot of examples! Find the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$.

(b) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$.

(c) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$.

(d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

(e) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

More Examples. Find the following limits:

1. $\displaystyle\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

More Examples. Find the following limits:

1. $$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}.$$

2. $$\lim_{x\to\infty} \frac{x+\sin x}{x}.$$

Indeterminate forms with products. Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then what is $\lim_{x\to a} f(x)g(x)$? This is called an indeterminate form of type $0 \cdot \infty$. We apply L'Hospital's rule after first writing $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$.

Indeterminate forms with products. Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then what is $\lim_{x \to a} f(x)g(x)$? This is called an indeterminate form of type $0 \cdot \infty$. We apply L'Hospital's rule after first writing $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$.

Example. Evaluate $\lim_{x \to 0^+} x \ln x$.

Indeterminate forms with differences. Now consider what happens if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, and we are looking at $\lim_{x \to a} [f(x) - g(x)]$. This is an indeterminate form of type $\infty - \infty$. We examine these by converting them into a quotient and using L'Hospital's rule.

Indeterminate forms with differences. Now consider what happens if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, and we are looking at $\lim_{x\to a}[f(x) - g(x)]$. This is an indeterminate form of type $\infty - \infty$. We examine these by converting them into a quotient and using L'Hospital's rule.

Example. Evaluate $\lim_{x\to(\frac{\pi}{2})^-} (\sec x - \tan x)$.

Indetermininate forms with powers.

Some limits involving powers are difficult to calculate because the variable is in both the base and the index. By taking the natural logarithm, the power is transformed into a product, and the problem becomes manageable.

Indetermininate forms with powers.

Some limits involving powers are difficult to calculate because the variable is in both the base and the index. By taking the natural logarithm, the power is transformed into a product, and the problem becomes manageable.

Example. Evaluate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,
- defining the natural log and exponential function.

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,
- defining the natural log and exponential function.

Some more applications of integration (some of which may be covered in Math 1014) are:

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,
- defining the natural log and exponential function.

Some more applications of integration (some of which may be covered in Math 1014) are:

- calculating the volume or surface area of a solid,

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,
- defining the natural log and exponential function.

Some more applications of integration (some of which may be covered in Math 1014) are:

- calculating the volume or surface area of a solid,
- calculating the length of a curve,

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,
- defining the natural log and exponential function.

Some more applications of integration (some of which may be covered in Math 1014) are:

- calculating the volume or surface area of a solid,
- calculating the length of a curve,
- calculating the work done by a varying force,

Applications of integration

Integration has many applications. We have seen how integration is used in

- calculating the area of a region,
- finding the average value of a function,
- defining the natural log and exponential function.

Some more applications of integration (some of which may be covered in Math 1014) are:

- calculating the volume or surface area of a solid,
- calculating the length of a curve,
- calculating the work done by a varying force,
- finding the centre of gravity of an object.

Integration by parts

To differentiate the product of two functions we use the product rule; reversing this process gives *integration by parts*:

Integration by parts

To differentiate the product of two functions we use the product rule; reversing this process gives *integration by parts*:

Integration by parts formula.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Integration by parts

To differentiate the product of two functions we use the product rule; reversing this process gives *integration by parts*:

Integration by parts formula.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If we let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we obtain an equivalent (perhaps easier to memorize) formula:

$$\int u\,dv = uv - \int v\,du.$$

The idea is to choose u and v in such a way that the integral on the right is easier to evaluate than the integral on the left.

Examples.

1. Evaluate $\int xe^x \, dx$.

Examples.

1. Evaluate $\int xe^x\,dx$.
2. Evaluate $\int \ln x\,dx$.

Examples.

1. Evaluate $\int xe^x \, dx$.
2. Evaluate $\int \ln x \, dx$.
3. Evaluate $\int t^2 \sin t \, dt$.

Recall from last class

Integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Recall from last class

Integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If we let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we obtain an equivalent (perhaps easier to memorize) formula:

$$\int u\,dv = uv - \int v\,du.$$

Recall from last class

Integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If we let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we obtain an equivalent (perhaps easier to memorize) formula:

$$\int u\,dv = uv - \int v\,du.$$

1. Evaluate $\int e^x \sin x\,dx$.

Recall from last class

Integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If we let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we obtain an equivalent (perhaps easier to memorize) formula:

$$\int u\,dv = uv - \int v\,du.$$

1. Evaluate $\int e^x \sin x\,dx$.
2. Evaluate $\int_0^{\pi^2} \sin\sqrt{x}\,dx$.

Trigonometric integrals

Now we examine integrals of the form

1. $\displaystyle\int \sin^m x \cos^n x \, dx$

Trigonometric integrals

Now we examine integrals of the form

1. $\displaystyle\int \sin^m x \cos^n x \, dx$
2. $\displaystyle\int \tan^m x \sec^n x \, dx$

Trigonometric integrals

Now we examine integrals of the form

1. $$\int \sin^m x \cos^n x \, dx$$
2. $$\int \tan^m x \sec^n x \, dx$$

There are some strategies to evaluating these integrals, but occasionally we must use some tricks and ingenuity.

Trigonometric integrals

Now we examine integrals of the form

1. $\displaystyle\int \sin^m x \cos^n x \, dx$
2. $\displaystyle\int \tan^m x \sec^n x \, dx$

There are some strategies to evaluating these integrals, but occasionally we must use some tricks and ingenuity.
We begin by looking at integrals involving powers of $\sin x$ and $\cos x$.

$$\int \sin^m x \cos^n x \, dx.$$

$$\int \sin^m x \cos^n x \, dx.$$

There are two cases to consider. The first case is if **at least one** of m or n is odd, and the second case is if **both** m and n are even.

$$\int \sin^m x \cos^n x \, dx.$$

There are two cases to consider. The first case is if **at least one** of m or n is odd, and the second case is if **both** m and n are even.

Integrals with an odd power of $\sin x$ or an odd power of $\cos x$.

$$\int \sin^m x \cos^n x \, dx.$$

There are two cases to consider. The first case is if **at least one** of m or n is odd, and the second case is if **both** m and n are even.

Integrals with an odd power of $\sin x$ or an odd power of $\cos x$.

- If there is an odd power of $\sin x$ (i.e. if m above is odd), we save one factor of $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$. Then substitute $u = \cos x$.

$$\int \sin^m x \cos^n x \, dx.$$

There are two cases to consider. The first case is if **at least one** of m or n is odd, and the second case is if **both** m and n are even.

Integrals with an odd power of $\sin x$ or an odd power of $\cos x$.

- If there is an odd power of $\sin x$ (i.e. if m above is odd), we save one factor of $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$. Then substitute $u = \cos x$.
- If there is an odd power of $\cos x$ (i.e. if n above is odd), we save one factor of $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of $\sin x$. Then substitute $u = \sin x$.

$$\int \sin^m x \cos^n x \, dx.$$

There are two cases to consider. The first case is if **at least one** of m or n is odd, and the second case is if **both** m and n are even.

Integrals with an odd power of $\sin x$ or an odd power of $\cos x$.

- If there is an odd power of $\sin x$ (i.e. if m above is odd), we save one factor of $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$. Then substitute $u = \cos x$.
- If there is an odd power of $\cos x$ (i.e. if n above is odd), we save one factor of $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of $\sin x$. Then substitute $u = \sin x$.
- (If the powers of both $\sin x$ and $\cos x$ are odd, we may use either of the above strategies).

Example. Evaluate $\int \sin^3 x \, dx$.

Example. Evaluate $\int \sin^3 x \, dx$.

Example. Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Integrals where both powers of $\sin x$ and $\cos x$ are even.

Integrals where both powers of $\sin x$ **and** $\cos x$ **are even.** We use the half-angle identities

Integrals where both powers of $\sin x$ and $\cos x$ are even. We use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

Integrals where both powers of $\sin x$ and $\cos x$ are even. We use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

and progressively lower the powers until the integral can be evaluated. Sometimes the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

is also helpful.

Integrals where both powers of $\sin x$ and $\cos x$ are even. We use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

and progressively lower the powers until the integral can be evaluated. Sometimes the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

is also helpful.

Examples.

1. Evaluate $\displaystyle\int \cos^2 x \, dx$.

Integrals where both powers of $\sin x$ and $\cos x$ are even. We use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

and progressively lower the powers until the integral can be evaluated. Sometimes the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

is also helpful.

Examples.

1. Evaluate $\displaystyle\int \cos^2 x\, dx$.
2. Evaluate $\displaystyle\int \sin^2\theta \cos^2\theta\, d\theta$.

We now look at integrals involving powers of $\tan x$ and $\sec x$, i.e.

$$\int \tan^m x \sec^n x \, dx.$$

We now look at integrals involving powers of $\tan x$ and $\sec x$, i.e.

$$\int \tan^m x \sec^n x \, dx.$$

We consider, as above, two cases. Unfortunately, they are not exhaustive (as in the case of integrals involving powers of $\sin x$ and $\cos x$).

We now look at integrals involving powers of $\tan x$ and $\sec x$, i.e.

$$\int \tan^m x \sec^n x \, dx.$$

We consider, as above, two cases. Unfortunately, they are not exhaustive (as in the case of integrals involving powers of $\sin x$ and $\cos x$).

Case 1: The integral involves an even power of $\sec x$. We save one factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$. Then substitute $u = \tan x$.

We now look at integrals involving powers of $\tan x$ and $\sec x$, i.e.

$$\int \tan^m x \sec^n x \, dx.$$

We consider, as above, two cases. Unfortunately, they are not exhaustive (as in the case of integrals involving powers of $\sin x$ and $\cos x$).

Case 1: The integral involves an even power of $\sec x$**.** We save one factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$. Then substitute $u = \tan x$.

Case 2: The integral involves an odd power of $\tan x$**.** We save one factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then substitute $u = \sec x$.

Example. Evaluate $\int \tan^8 x \sec^4 x \, dx$.

Example. Evaluate $\int \tan^8 x \sec^4 x \, dx$.

Example. Evaluate $\int \tan^5 \theta \sec^8 \theta \, d\theta$.

Other integrals involving powers of sec **and** tan**.** In other cases, the guidelines are not as clear cut. We may need to use identities, integration by parts and sometimes a little ingenuity.

Other integrals involving powers of sec **and** tan**.** In other cases, the guidelines are not as clear cut. We may need to use identities, integration by parts and sometimes a little ingenuity. The following indefinite integrals are also useful:

$$\int \tan x \, dx = \ln |\sec x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

Other integrals involving powers of sec **and** tan**.** In other cases, the guidelines are not as clear cut. We may need to use identities, integration by parts and sometimes a little ingenuity. The following indefinite integrals are also useful:

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Example. Find $\int \sec^3 x \, dx$.