Week 9: The natural logarithm and exponential functions

October 2 – October 5, 2012

1. The natural logarithm & exponential
   - The natural logarithm
   - The exponential function

Lecture 17
Lecture 18

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(ii) $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$ for all positive real numbers x and y.

(iii) $\ln(x^r) = r\ln(x)$ whenever r is a rational number and x is a positive real number.

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Also, if $y = \ln x$ and $x > 0$,

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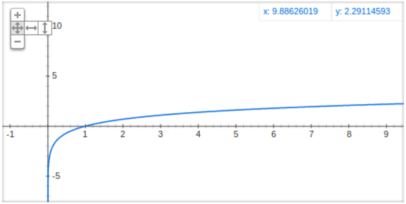
Graph for ln(x)



Important definition. By the intermediate value theorem, there is a number where $\ln x$ equals 1. We denote this number by e (for "Euler"). It is (in my opinion) perhaps the most important number in mathematics. It is irrational, and **approximately** equal to 2.718.

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Example. Differentiate the function given by $y = \ln(\sin x)$.

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Example. Suppose f is defined by $f(x) = \ln|x|$. What is $f'(x)$?

We record this formula:

$$\frac{d}{dx} \ln|x| = \frac{1}{x}.$$

Hence, we obtain the formula

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hence,

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provided that f is differentiable and restricted to an interval I where it is nonzero.

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Logarithmic differentiation

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Example. Find $\dfrac{dy}{dx}$ if $y = \left(\dfrac{(3x^2+4)(x+2)}{x^3+5x}\right)^{3/5}$ and $x > 0$.

The exponential function

Since $\frac{d}{dx}\ln x = \frac{1}{x} > 0$ on $(0, \infty)$, $\ln x$ is an increasing differentiable function, hence it has a differentiable inverse, which we denote by exp. The domain of exp is $\mathbb{R}$ and the range of exp is $(0, \infty)$.

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- $\exp(\ln(x)) = x$ and $\ln(\exp(x)) = x$.

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Also, the cancellation equations $\exp(\ln(x)) = x$ and $\ln(\exp(x)) = x$ become

$$e^{\ln x} = x \qquad \text{and} \qquad \ln(e^x) = x.$$

The first cancellation formula requires $x > 0$, but the second holds for all x.

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(e) $\ln x + \ln(x-1) = 1$

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3. $(e^x)^r = e^{rx}$.