Week 1: Functions

July 23 – July 27, 2012

Lecture 2

Who am I?

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1. Introduction
   - Sets
   - Inequalities

2. Functions
   - Representation of functions
   - A catalogue of functions

Sets

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Example. True or false?

$$\frac{1}{2} \in \mathbb{Z}; \qquad \sqrt{2} \in \mathbb{R}; \qquad 3 \in \mathbb{N}; \qquad \pi \notin \mathbb{Q}.$$

Intervals. Suppose that $a, b \in \mathbb{R}$ and $a < b$.

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- $[a, b]$ is a *closed* interval;
  (a, b) is an *open* interval;
  $(a, b]$ is neither open nor closed.

Definition

Suppose that A and B are two sets. We say that A is a *subset* of B if $x \in A$ implies that $x \in B$, and we will denote this by $A \subseteq B$. If A is a subset of B then we also say that B *contains* the set A.

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4. Solve $\dfrac{3}{1-x} < \dfrac{2}{x}$.

Absolute values.

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4. *Geometric Interpretation.*

Example. Solve

(a) $|x-4| < 3$

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(a) $|x-4| < 3$ (b) $|3x+5| \geq 2$ (c) $\left|\dfrac{x-1}{x-5}\right| < 1.$

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In this course, we begin a mathematical study of functions whose inputs and outputs are real numbers.

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Example and terminology. A function f with domain $[0, \infty)$ is given by the rule

$$f(x) = x^2 \qquad \forall x \in [0, \infty).$$

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The range of a function. Suppose that f is a function. The *range* of f, denoted by $\mathrm{Ran}(f)$, is defined by

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- If the domain of f is $[2, 4]$, the range of f is $[4, 16]$.

The graph of a function. If f is a function then its graph is the set of ordered pairs

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Note that although the concepts of even and odd functions and increasing and decreasing functions will not be covered in lecture, you are still expected to be familiar with them (refer to the textbook).

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where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$. The a_j are called the *coefficients* of the polynomial and n is the *degree* of the polynomial (as long as a_n is nonzero). If $n = 1$ the polynomial is called linear, $n = 2$ quadratic, $n = 3$ cubic, etc.

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Rational functions. Suppose that p and q are polynomials. A function f is called a *rational function* if

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : q(x) \neq 0\}$$

and

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- How to compute sin, cos, tan, sec, cosec, and cot of the values $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, etc.

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- the sum and difference formulae

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

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- double-angle formulae

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x\\ \cos(2x) &= \cos^2 x - \sin^2 x\\ \tan(2x) &= \frac{2\tan x}{1-\tan^2 x}.\end{aligned}$$

The exponential and logarithm functions.

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Roots.

Functions of the form $x^{\frac{1}{n}}$. e.g. $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Constructing new functions from known ones

Combining functions. If two functions f and g have the same domain A, we can construct new functions $f+g$, $f-g$ and $f \cdot g$ each with domain A. These are defined *pointwise* by the following formulae:

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The domain of $\dfrac{f}{g}$ is $\{x \in A : g(x) \neq 0\}$.

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If f has domain A and $\mathrm{Ran}(f)$ is a subset of $\mathrm{Dom}(g)$ then we can define a new function $g \circ f$ on A by the rule

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Example

$$\cos(x^2) = f \circ g(x),$$

where $f(x) = \cos(x)$ and $g(x) = x^2$.

Examples

- Let $f(x) = x^2$ with domain $[0, \infty)$ and $g(x) = sin(x)$ with domain $\mathbb{R}$. Then $(g \circ f)(x) = sin(x^2)$ and $(f \circ g)(x) = (sin(x))^2$. Note the parentheses! What are the domains and ranges of $g \circ f$ and $f \circ g$?

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- Let $f(x) = x^3$ with domain $[-2, 2]$ and $g(x) = ln(x)$ with maximal domain $(0, \infty)$. Does $g \circ f$ make sense on the domain of f? Note that if we write an expression like $ln(x^3)$ where we don't specify the domain of x^3, we implicitly mean that the domain of $ln(x^3)$ is the maximal one.

Piecewise defined functions

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$$f(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1. \end{cases}$$

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Can you think of another piecewise defined function that we encountered in the first lecture?

Function transformations. Once the shapes of the graphs of basic functions are known, we can alter the shape of the graph by altering the function in simple ways.

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New graph	Obtained from $y = f(x)$ by
$y = f(x) + a$	translating the graph upwards by a units
$y = f(x) - a$	translating the graph downwards by a units
$y = f(x + a)$	translating the graph to the left by a units
$y = f(x - a)$	translating the graph to the right by a units
$y = cf(x)$	stretching the graph vertically by factor c
$y = (1/c)f(x)$	compressing the graph vertically by factor c
$y = f(cx)$	compressing the graph horizontally by factor c
$y = f(x/c)$	stretching the graph horizontally by factor c
$y = -f(x)$	reflecting the graph about the x-axis
$y = f(-x)$	reflecting the graph about the y-axis

Examples. Sketch the graphs given by

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(b) $y = 3\sin(2x + \pi)$