Week 10: The logarithmic and exponential functions, exponential growth

October 9 – October 12, 2012

1. General exponential and logarithmic functions
   - Exponential growth and decay

2. Inverse functions
   - The inverse trigonometric functions
   - The hyperbolic functions

Lecture 18
Lecture 19

Recall from last class

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- $(e^x)^r = e^{rx}$.

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(ii) $$\lim_{x\to 2^+} e^{3/(2-x)}$$

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(Important) derivative. $\frac{d}{dx} e^x = e^x$. Moreover, up to multiplication by a constant, this is the *only* function which is its own derivative!!

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(iii) On what interval is the curve $y = xe^{3x}$ concave upward?

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Example. Evaluate the integral

$$\int \frac{e^{1/x}}{x^2} \, dx.$$

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Example. Solve for x:

1. $2^{x-5} = 3$.

Armed with this definition of an arbitrary exponential function, we can improve one of the properties of $\ln x$. We showed earlier that $\ln(a^r) = r \ln a$ when r is *rational*. But now, if x is any *real* number, we notice that

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We list this, and a few other properties. The book justifies a few of these properties more. In the following, we assume that x, y are real numbers and a, b are strictly positive.

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(ii) Differentiate the function $h(t) = t^3 - 3^t$.

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Example. Calculate $\int_0^5 3^x \, dx$.

We can now state the **general power rule**, namely if n is *any* real number and f is given by $f(x) = x^n$, then

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Example. Differentiate $y = x^{e^x}$.

General Logarithmic Functions. If a is positive and not equal to one, then a^x is one-to-one. Hence it has an inverse, denoted by $\log_a$, which is called the **logarithmic function with base** a. That is,

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How does the graphs of the general logarithm look like?

It is easy to check the following important **change of base formula**, relating general logarithms to the natural logarithm: If $a > 0$, $a \neq 1$, then

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Example. Calculate the derivative of the function given by $f(x) = \log_{10}(2 + \sin x)$.

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We record this as

$$e = \lim_{x\to 0}(1+x)^{1/x} \qquad \text{or equivalently} \qquad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

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(viii) The inverse function of $y = e^{3x}$ is $y = \frac{1}{3}\ln x$.

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Theorem

The only solutions of the differential equation $\frac{dy}{dt} = ky$ *are the exponential functions*

$$y(t) = y(0)e^{kt}.$$

Example: Population growth. The CIA World Fact Book estimated that the population of Australia in July 2009 was 21,262,641 and that its 2009 growth rate was 1.195%. Estimate the population of Australia in July 2015, assuming that its growth rate remains constant.

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Remark. In population studies, the constant of proportionality k is often called the *(relative) growth rate*.

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Some Half-Lives.

Yield	Fission Product	Half-life
6.8%	Cesium-134	2 years
6.3%	Iodine-135	7 hours
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(a) Suppose that $m(t)$ denotes the mass of Carbon-14 present in a fossil at time t. Find a formula for $m(t)$.

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(a) Suppose that $m(t)$ denotes the mass of Carbon-14 present in a fossil at time t. Find a formula for $m(t)$.

(b) The proportion of Carbon-14 in an animal fossil is 0.6%. Assume that when the animal died the proportion was 1%. Use this information to estimate when the animal died.

Compound interest. Suppose that an amount A_0 is invested at an interest rate of r per annum and is compounded n times a year. Then the investment $A(t)$ in t years is given by

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Notice how close this is to the amount for daily compounding. However, the formula using continuous compounding is much easier to manipulate mathematically.

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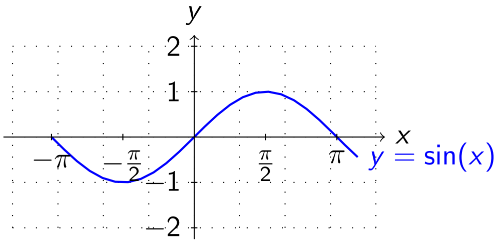

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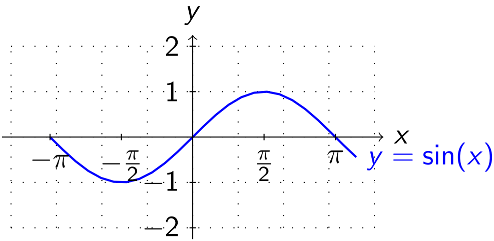

Notice that this is not one-to-one, but if we restrict x to lie in $[\frac{-\pi}{2}, \frac{\pi}{2}]$, then $\sin x$ *is* one-to-one, hence it has an inverse. We denote this inverse by $\sin^{-1} x$.

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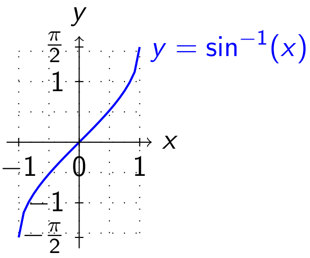

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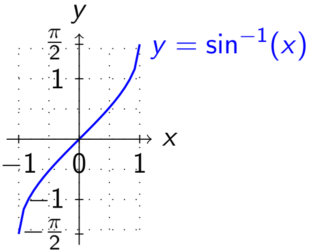

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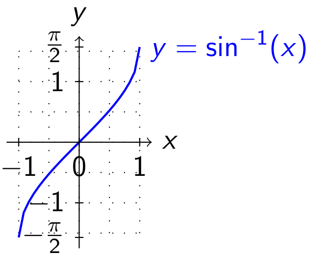

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$$\sin^{-1} x = y \iff \sin y = x \qquad \text{and} \qquad \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}.$$

That is, if $x \in [-1, 1]$, then $\sin^{-1} x$ is the number between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x. The domain of $\sin^{-1} x$ is $[-1, 1]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

The cancellation formulas for $\sin^{-1}$ are:

$$\begin{aligned} \sin^{-1}(\sin x) &= x \qquad \text{for } \frac{-\pi}{2} \leq x \leq \frac{\pi}{2} \\ \sin(\sin^{-1} x) &= x \qquad \text{for } -1 \leq x \leq 1. \end{aligned}$$

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Example. If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) the derivative $f'(x)$, and (c) the domain of f'.

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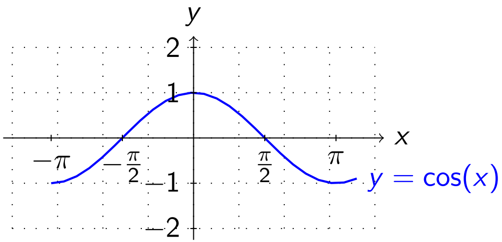

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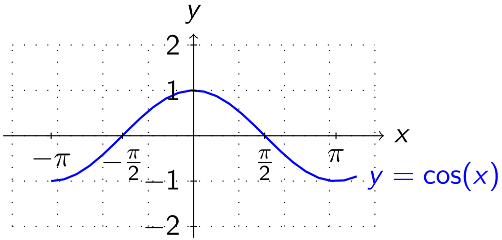

We must again restrict the domain of $\cos x$ so that it is one-to-one. We restrict x to lie in $[0, \pi]$, then define the inverse function, denoted by $\cos^{-1} x$.

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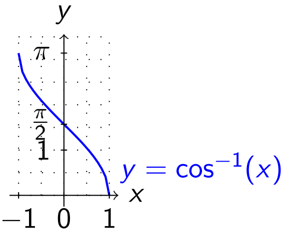

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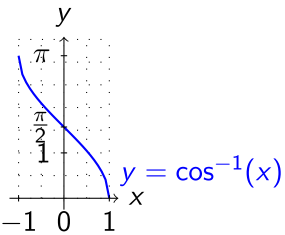

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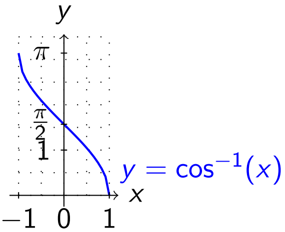

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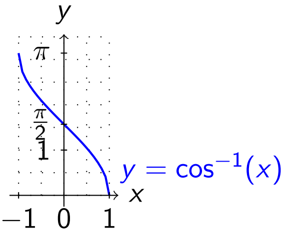

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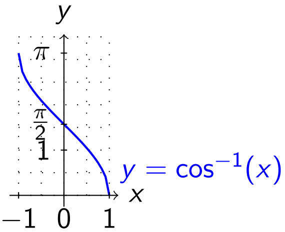

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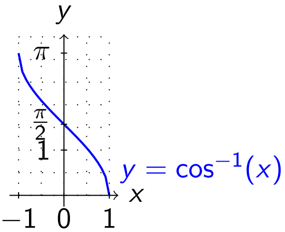

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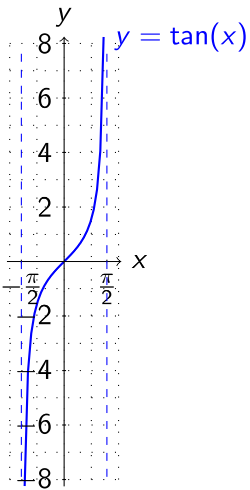

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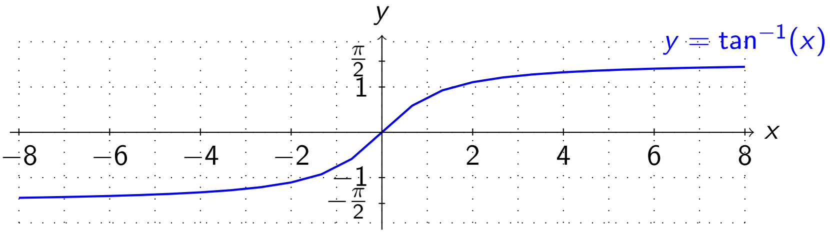

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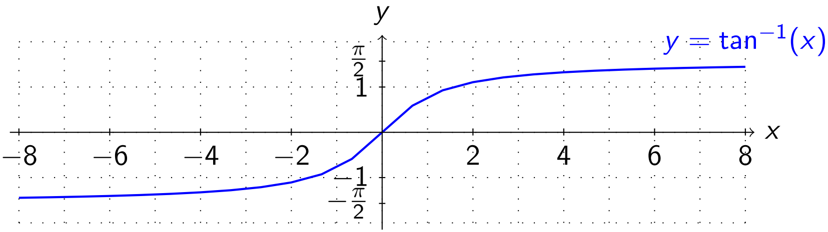

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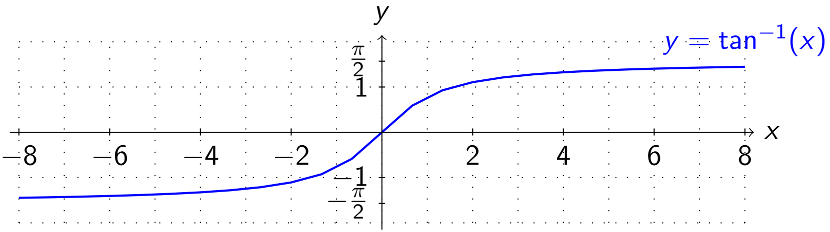

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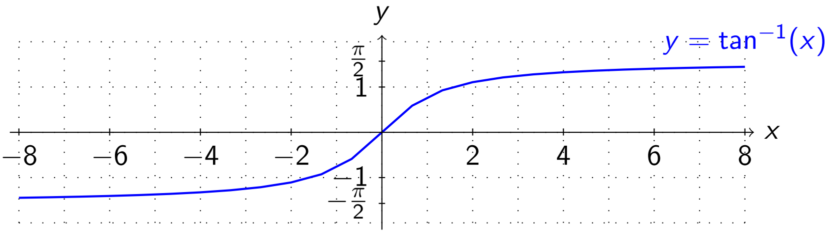

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Remarks. Note that $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc are often also called arcsin, arccos, arctan, etc. Also, note that one can discuss the definition and basic properties (such as derivatives) of the inverse functions to sec, csc, and cot.

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Example. Simplify the expression $\cos(\tan^{-1} x)$.

Example. Differentiate the function given by $f(x) = x \tan^{-1} \sqrt{x}$.

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We draw the graphs of sinh, cosh, and tanh:

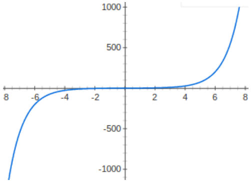

Figure 1: $y = \sinh x$

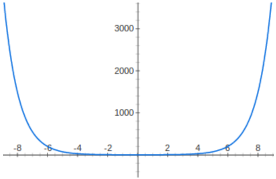

Figure 2: $y = \cosh x$

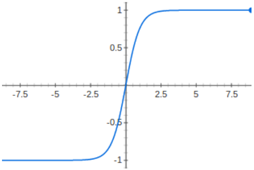

Figure 3: $y = \tanh x$

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We now can get some insight into why these functions are called "hyperbolic". Notice that if t is a real number, then the point $(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$. It lies on the right branch because $\cosh t \geq 1$:

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Example. Find the derivative: $y = e^{\cosh 3x}$.

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Remark. In the textbook there are similar formulas for the inverse hyperbolic functions and their derivatives. Although you will have webassign questions and a tutorial quiz on the hyperbolic functions and their inverses, I will not emphasize them on the final exam to the same extent as the trigonometric functions and their inverses. **That is, you are expected to be familiar with hyperbolic functions and their inverses. I may ask you questions involving these functions on the final exam, but if I do, they will be very basic questions.**

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Because one cannot say in advance exactly what a limit of the form $\frac{\infty}{\infty}$ will be, we call it an *indeterminate form*. Other indeterminate forms include

$$0\cdot\infty, \qquad \tfrac{0}{0}, \qquad \infty-\infty, \qquad 0^0, \qquad \infty^0 \qquad \text{and} \qquad 0^\infty.$$

Each of the above limits (a), (b) and (c) can be calculated using the standard trick of dividing the denominator through by the fastest growing term. It is easy to tell which terms grow fastest because we are working with polynomials. What if this is not the case?

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In each case, does the denominator win out, or the numerator, or is there some kind of compromise?

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In fact, this kind of intuitive reasoning is generally true!

L'Hôpital's rule

Suppose that f and g are differentiable functions, $a \in \mathbb{R}$, and $g'(x) \neq 0$, except possibly at a. Suppose also that either one of the two following conditions hold:

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If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists or is $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$