Week 6: Optimization, anti-derivatives, definite integral

August 27 – August 31, 2012

1. Antiderivatives

2. Area
   - Riemann Sums
   - Definite integral

Lecture 12

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

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1. what is the *domain* of f
2. *Intercepts:*
   (i) the y-intercept is $f(0)$
   (ii) the x-intercepts are found by solving $f(x) = 0$

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1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
   (i) y-axis symmetry (reflection) if $f(-x) = f(x)$
   (ii) symmetry by rotation of 180^o about the origin if $f(-x) = -f(x)$
   (iii) periodicity if $f(x + T) = f(x)$ for some number T

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1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
4. *Asymptotes:*
   (i) $y = L$ is an horizontal asymptote if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$
   (ii) $x = a$ is a vertical asymptote if $\lim_{x\to a^\pm} f(x) = \pm\infty$ for one (or more) combination of $\pm$

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6. *Local maxima/minima:* determined by locating critical numbers and using the first (or second) derivative test

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2. *Intercepts:*
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4. *Asymptotes:*
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6. *Local maxima/minima:*
7. *Concavity and points of inflection:* determined by the sign of $f''(x)$

Optimization Problems

We have already seen that an optimization problem may be reduced to finding the absolute maximum or minimum points of an appropriate function f over some interval I. We already have techniques for doing this if f is continuous and I is closed. If either of these conditions fail, then one can try locating maxima and minima by graphing the function.

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Example. A cylindrical can is to hold 1 L of oil. Find the dimensions that will minimize the the cost of the metal to manufacture the can.

Example. Find the area of the largest square that can be inscribed in the circle given by the equation

$$x^2 + y^2 = 4.$$

1 Antiderivatives

2 Area
- Riemann Sums
- Definite integral

Antiderivatives

Thus far we have concentrated on finding the derivative f' of a given function f. For example, given a function f which describes the position of a particle at time t, or g which measures the volume of water in a tank at time t, we have investigated how to find functions f' and g' which measure the instantaneous rate of change of f and g.

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Now we look at the reverse question. That is, if we can measure the velocity of a particle at each time t, can we find the position of the particle? If we can measure the rate at which water leaks from a tank, can we determine the amount of water that leaked out in a given time period? That is, given *the derivative* of a function, f', can we determine f?

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Definition

A function F is said to be an *antiderivative* (or *primitive*) of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives

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Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Antiderivatives

A table of antiderivatives.

In the following table, C is an arbitrary constant.

Function	Antiderivative
$cf(x)$	$cF(x)+C$
$f(x)+g(x)$	$F(x)+G(x)+C$
x^n (where $n \neq -1$)	$\dfrac{x^{n+1}}{n+1}+C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Differential equations. A *differential equation* is an equation involving the derivatives of a function. We can solve some very basic differential equations by looking for antiderivatives. Although antiderivatives usually involve arbitrary constants, we may be able to uniquely determine these constants by some extra conditions.

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Example (physics!). A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/sec and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

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What is area?

We are now going to turn our focus to the other main branch of calculus, which deals with *integrals*. We will see how the notion of integral naturally arises when we consider the area under a curve, much as the notion of derivative naturally arose when we considered the tangent line to a curve. The concepts of differential and integral calculus are related by the so-called Fundamental Theorem of Calculus, which is one of humanity's great accomplishments.

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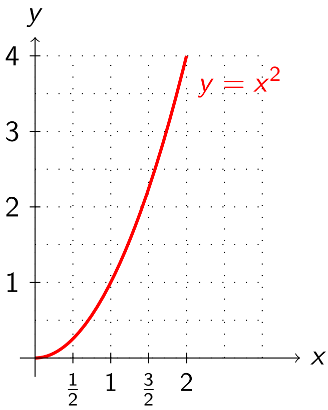
We are motivated by the following problem: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

What is area?

This question is easy to answer if the curve is a horizontal line, and only slightly more difficult if the curve is any non-vertical line. But what if the curve is not a line?

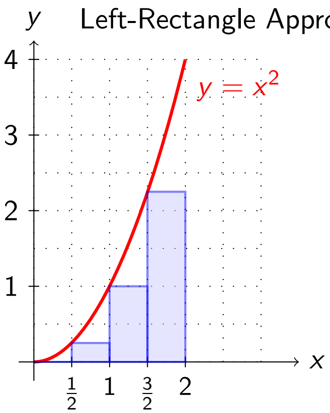
Example. Consider the curve $y = x^2$ between 0 and 2. Let's try to estimate the area under the curve:

Riemann Sums



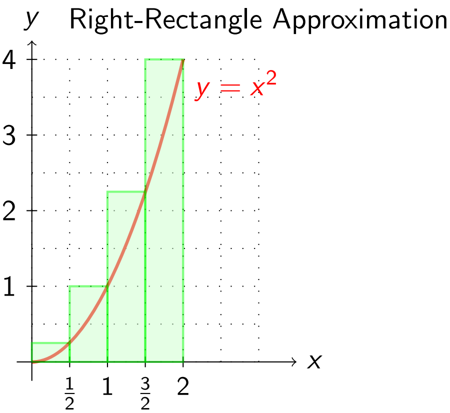
We want to find the area under the graph of $y = x^2$ in between $x = 0$ and $x = 2$.

Riemann Sums



Left Rectangle Approximation $=$
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

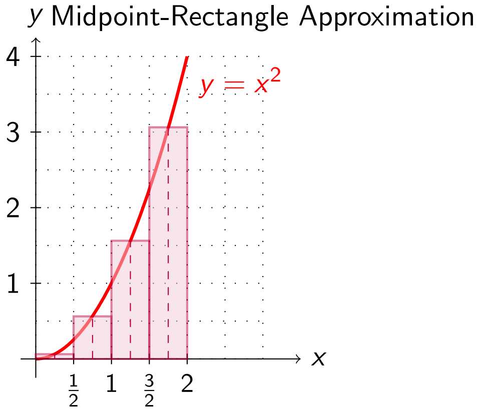
Riemann Sums



Left Rectangle Approximation $=$
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

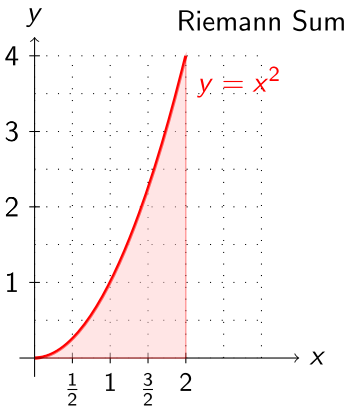
Right Rectangle Approximation $=$
$= \frac{1}{2} \cdot [(\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2 + 2^2] = 3.75.$

Riemann Sums



Midpoint Rectangle Approximation $=$
$= \frac{1}{2} \cdot [(\frac{1}{4})^2 + (\frac{3}{4})^2 + (\frac{5}{4})^2 + (\frac{7}{4})^2] = 2.625.$

Riemann Sums



Actual area is $2\frac{2}{3}$.

Area under a curve

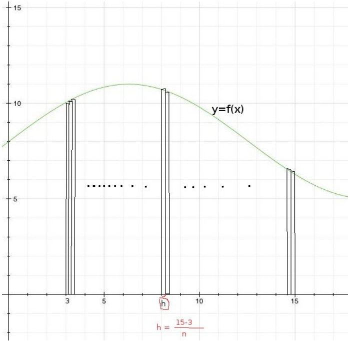


Figure 1: Computing area under $y = 3\sin(x/4) + 8$ in between $x = 3$ and $x = 15$

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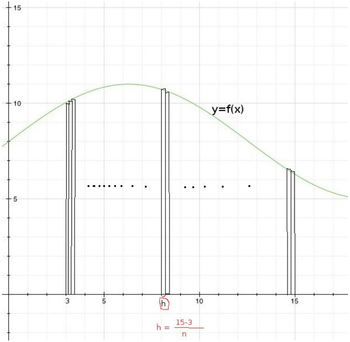

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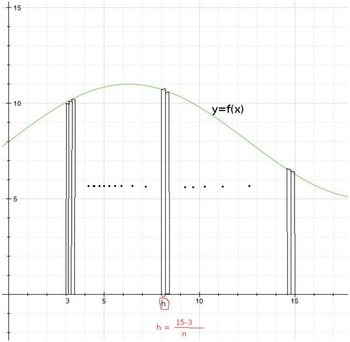

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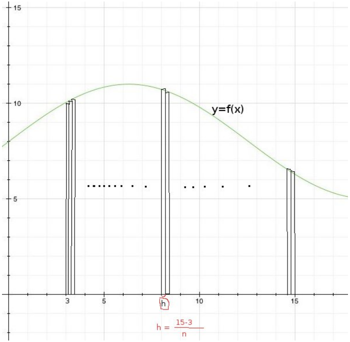

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Let $x_0 = 3$, $x_1 = 3 + h$, $\ldots x_{n-1} = 15 - h$, $x_n = 15$. Then, the approximated area is

$$h \cdot \left[f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right]$$

where $h = (15 - 3)/n$.

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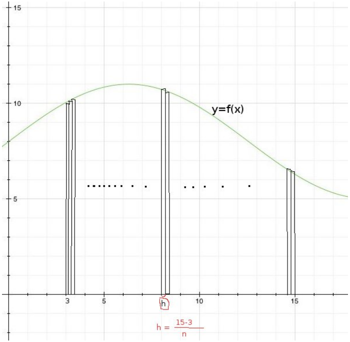

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where $h = (15 - 3)/n$.

Taking the limit as $n \to \infty$, we compute the area.

We more generally define the **area** of the region S that lies under the graph of the continuous function f to be the limit of the sum of areas of approximating rectangles. That is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\triangle x + f(x_2)\triangle x + \ldots + f(x_n)\triangle x].$$

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Note: It can be shown that we get the same value if we use left endpoints rather than right endpoints. In fact, we can take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. These numbers $x_1^*, \ldots, x_n^*$ are called *sample points*.

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Over a small amount of time, velocity doesn't change too much! So we can approximate our total distance traveled by multiplying the velocity on an interval by the time difference of 5 seconds. It is plausible that as we take more and more frequent measurements and repeat this procedure, we would get the *exact* distance traveled. We can interpret this as another problem of finding the area under the curve of a given function!

Riemann Sums

A limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x = \lim_{n\to\infty} [f(x_1^*)\triangle x + \ldots + f(x_n^*)\triangle x]$$

arises when we compute the area under the curve $y = f(x)$. Limits like this occur in a variety of situations, sometimes when f is not a positive function. We now define the notion of a Riemann sum and of a definite integral.

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Riemann sums.

Start with a function f defined on an interval $[a, b]$. Divide $[a, b]$ into n smaller subintervals by choosing points $x_1, \ldots, x_{n-1}$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

We call the resulting set of subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ a **partition** P of $[a, b]$.

Riemann Sums

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Then we choose **sample points** $x_1^*, \ldots, x_n^*$ in the subintervals so that x_i^* is in the ith subinterval $[x_{i-1}, x_i]$. The sample points could be left or right endpoints or any number in between. Here is an example of what a partition looks like:

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A **Riemann sum** associated to a partition P and a function f is made by evaluating f at the sample points, multiplying by the lengths of the corresponding subintervals, and then adding:

$$\sum_{i=1}^{n} f(x_i^*)\triangle x_i = f(x_1^*)\triangle x_1 + \ldots + f(x_n^*)\triangle x_n.$$

Riemann Sums

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If we take *all* possible partitions of $[a, b]$ and *all* possible choices of sample points, we can try to take the limit of *all* possible Riemann sums as n becomes large. However, since our subintervals have different lengths, we need to make sure that all these lengths $\triangle x_i$ approach 0. We do this by requiring that the largest subinterval length, $\max \triangle x_i$, approaches zero.

Definite integral

Definition

If f is a function defined on $[a, b]$, the **definite integral of** f **from** a **to** b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \triangle x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \triangle x_i$$

provided this limit exists. If it does exist, we say that f is *integrable* on $[a, b]$.

$y = x^2$

Definite integral

Remarks.

(i) The definition is saying that the definite integral is a **number** that can be approximated to within any degree of accuracy by a Riemann sum. The number does not depend on x. In fact, we can use any letter other than x without changing the value of the integral.

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(ii) The symbol $\int$ is called an **integral sign**. In the above notation, $f(x)$ is called the **integrand**, a is the **lower limit of integration** and b is the **upper limit of integration**. Together they are **the limits of integration**. The symbol dx has no meaning by itself. The procedure of calculating an integral is called **integration**.

Theorem.

If f is continuous on $[a, b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$. That is, the definite integral $\int_a^b f(x)\,dx$ exists.

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(i) Although in the definition of partition we allowed subintervals to have different widths, for most intents and purposes it suffices to consider partitions where all subintervals have the same width. If our function f is integrable, this will not affect the value of the integral. Such partitions are called *regular partitions*.

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(ii) If f takes on both positive and negative values, then the definite integral gives us the *net area* or *signed area* between the graph of f and the x-axis.

Examples.

Let's evaluate the following integrals by interpreting each in terms of areas:

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Note: The book goes into more detail (pg. 209-211) on how to evaluate Riemann sums using the definition, and on the "Midpoint Rule." Both topics are recommended reading, although you will not be tested on the midpoint rule or on more complicated examples of Riemann sums.

Properties of the Definite Integral

Basic properties. Suppose that a, b are real numbers and f is integrable on $[a, b]$.

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$$\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

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$$\int_a^b (f \pm g)(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$
3. $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$.

Properties of the Definite Integral

Example.

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\, dx$.

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Another property. Let a, b, c be real numbers, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Properties of the Definite Integral

Example.
Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\, dx$.

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Example. Suppose that $\int_0^{10} f(x)\, dx = 17$ and $\int_0^8 f(x)\, dx = 12$, find $\int_8^{10} f(x)\, dx$.

Comparison properties of the integral.

Suppose that $a \leq b$.

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4. If $|f|$ is integrable on $[a, b]$ then
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$