Week 4: Differentiation

August 13 – August 17, 2012

▸ Lecture 8

Rules for differentiation

So far we have calculated derivatives working directly from the definition, that is, by taking the limit as $h \to 0$ of the difference quotient. However, in practice this is often both difficult and tedious. Luckily, there exist some nice rules that make it quite easy to compute the derivatives of most functions we are familiar with.

Rules for differentiation

So far we have calculated derivatives working directly from the definition, that is, by taking the limit as $h \to 0$ of the difference quotient. However, in practice this is often both difficult and tedious. Luckily, there exist some nice rules that make it quite easy to compute the derivatives of most functions we are familiar with.

Some basic derivatives.

$f(x)$	$f'(x)$
C, where C is a constant	0

Rules for differentiation

So far we have calculated derivatives working directly from the definition, that is, by taking the limit as $h \to 0$ of the difference quotient. However, in practice this is often both difficult and tedious. Luckily, there exist some nice rules that make it quite easy to compute the derivatives of most functions we are familiar with.

Some basic derivatives.

$f(x)$	$f'(x)$
C, where C is a constant	0
x^n, where n is a real number	nx^{n-1}

Rules for differentiation

So far we have calculated derivatives working directly from the definition, that is, by taking the limit as $h \to 0$ of the difference quotient. However, in practice this is often both difficult and tedious. Luckily, there exist some nice rules that make it quite easy to compute the derivatives of most functions we are familiar with.

Some basic derivatives.

$f(x)$	$f'(x)$
C, where C is a constant	0
x^n, where n is a real number	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

Rules for differentiation

So far we have calculated derivatives working directly from the definition, that is, by taking the limit as $h \to 0$ of the difference quotient. However, in practice this is often both difficult and tedious. Luckily, there exist some nice rules that make it quite easy to compute the derivatives of most functions we are familiar with.

Some basic derivatives.

$f(x)$	$f'(x)$
C, where C is a constant	0
x^n, where n is a real number	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\dfrac{1}{x}$

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

(ii) $(f-g)'(x) = f'(x) - g'(x)$,

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

(ii) $(f-g)'(x) = f'(x) - g'(x)$,

(iii) $(C \cdot f)'(x) = C \cdot f'(x)$.

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

(ii) $(f-g)'(x) = f'(x) - g'(x)$,

(iii) $(C \cdot f)'(x) = C \cdot f'(x)$.

True or false?.

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

(ii) $(f-g)'(x) = f'(x) - g'(x)$,

(iii) $(C \cdot f)'(x) = C \cdot f'(x)$.

True or false?.

(i) If $f(x) = \sin(x)$ and $g(x) = e^x$, then $(f+g)'(x) = \cos(x) + e^x$.

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

(ii) $(f-g)'(x) = f'(x) - g'(x)$,

(iii) $(C \cdot f)'(x) = C \cdot f'(x)$.

True or false?.

(i) If $f(x) = \sin(x)$ and $g(x) = e^x$, then $(f+g)'(x) = \cos(x) + e^x$.

(ii) If $f(x) = x^2$, $g(x) = x^{-3}$, and $h(x) = x^4$, then $(f+g+h)'(x) = 3x^3$.

Sum, difference, and constant multiple rules

Suppose that f and g are differentiable at x and C is a real number. Then $f+g$, $f-g$ and Cf are differentiable at x. Moreover,

(i) $(f+g)'(x) = f'(x) + g'(x)$,

(ii) $(f-g)'(x) = f'(x) - g'(x)$,

(iii) $(C \cdot f)'(x) = C \cdot f'(x)$.

True or false?.

(i) If $f(x) = \sin(x)$ and $g(x) = e^x$, then $(f+g)'(x) = \cos(x) + e^x$.

(ii) If $f(x) = x^2$, $g(x) = x^{-3}$, and $h(x) = x^4$, then $(f+g+h)'(x) = 3x^3$.

(iii) If $C = \pi$, then $\frac{d}{dx}(C^2) = 2\pi$.

The product and quotient rules

If f and g are both differentiable, then

(i) $(fg)' = fg' + gf'$ (the *product rule*)

The product and quotient rules

If f and g are both differentiable, then

(i) $(fg)' = fg' + gf'$ (the *product rule*)

(ii) $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$ (the *quotient rule*).

The product and quotient rules

If f and g are both differentiable, then

(i) $(fg)' = fg' + gf'$ (the *product rule*)

(ii) $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ (the *quotient rule*).

That is, fg is differentiable with derivative given by formula (i), and if $\frac{f}{g}$ is defined, then it is differentiable, with derivative given by formula (ii). *Note that these formulas are not particularly intuitive!*

Examples.

The product and quotient rules

If f and g are both differentiable, then

(i) $(fg)' = fg' + gf'$ (the *product rule*)

(ii) $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ (the *quotient rule*).

That is, fg is differentiable with derivative given by formula (i), and if $\frac{f}{g}$ is defined, then it is differentiable, with derivative given by formula (ii). *Note that these formulas are not particularly intuitive!*

Examples.

(i) Compute the derivative of $y = x^2 \sin x$

The product and quotient rules

If f and g are both differentiable, then

(i) $(fg)' = fg' + gf'$ (the *product rule*)

(ii) $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ (the *quotient rule*).

That is, fg is differentiable with derivative given by formula (i), and if $\frac{f}{g}$ is defined, then it is differentiable, with derivative given by formula (ii). *Note that these formulas are not particularly intuitive!*

Examples.

(i) Compute the derivative of $y = x^2 \sin x$

(ii) Compute the derivative of $F(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$.

The chain rule

Theorem

Suppose that g is differentiable at the point x and f is differentiable at the point $g(x)$. Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

The chain rule

Theorem

Suppose that g is differentiable at the point x and f is differentiable at the point $g(x)$. Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Remark: Sometimes the chain rule is written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

While this is easier to remember, it omits some important information, namely the fact that f' is evaluated at $g(x)$.

The chain rule

Theorem

Suppose that g is differentiable at the point x and f is differentiable at the point $g(x)$. Then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Remark: Sometimes the chain rule is written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

While this is easier to remember, it omits some important information, namely the fact that f' is evaluated at $g(x)$.

Example. Suppose that $y = \sin(x^3 + 2x)$. Find $\dfrac{dy}{dx}$.

Examples.

(i) Differentiate $y = (x^3 - 1)^{100}$.

Examples.

(i) Differentiate $y = (x^3 - 1)^{100}$.

(ii) Differentiate the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

Examples.

(i) Differentiate $y = (x^3 - 1)^{100}$.

(ii) Differentiate the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

(iii) If $f(x) = \sin(\cos(\tan x))$, what is $f'(x)$?

Examples.

(i) Differentiate $y = (x^3 - 1)^{100}$.

(ii) Differentiate the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

(iii) If $f(x) = \sin(\cos(\tan x))$, what is $f'(x)$?

(iv) Here is a table of values for f, g, f', g'.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

Examples.

(i) Differentiate $y = (x^3 - 1)^{100}$.

(ii) Differentiate the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

(iii) If $f(x) = \sin(\cos(\tan x))$, what is $f'(x)$?

(iv) Here is a table of values for f, g, f', g'.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.

Examples.

(i) Differentiate $y = (x^3 - 1)^{100}$.

(ii) Differentiate the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

(iii) If $f(x) = \sin(\cos(\tan x))$, what is $f'(x)$?

(iv) Here is a table of values for f, g, f', g'.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0.$

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.
5. **Product Rule** $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.
5. **Product Rule** $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.
6. **Quotient Rule** $\left[\dfrac{u(x)}{v(x)}\right]' = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$, when $v(x) \neq 0$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.
5. **Product Rule** $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.
6. **Quotient Rule** $\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$, when $v(x) \neq 0$.
7. **Exponential Rule** If $f(x) = e^{ax}$, $f'(x) = ae^{ax}$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.
5. **Product Rule** $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.
6. **Quotient Rule** $\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$, when $v(x) \neq 0$.
7. **Exponential Rule** If $f(x) = e^{ax}$, $f'(x) = ae^{ax}$.
8. **Logarithm Rule** if $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, for $x \neq 0$.

Recall from last class

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.
5. **Product Rule** $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.
6. **Quotient Rule** $\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$, when $v(x) \neq 0$.
7. **Exponential Rule** If $f(x) = e^{ax}$, $f'(x) = ae^{ax}$.
8. **Logarithm Rule** if $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, for $x \neq 0$.
9. **Chain Rule** If $h(x) = f \circ g(x)$ is the composition of two functions f and g, then

$$h'(x) = f'[g(x)] \cdot g'(x).$$

The following example cannot be solved using the rules for differentiation (why not?). We must work directly from the difference quotient.

Example.

The following example cannot be solved using the rules for differentiation (why not?). We must work directly from the difference quotient.

Example. Suppose that f with domain $\mathbb{R}$ is defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine whether or not f is differentiable at 0.

The following example cannot be solved using the rules for differentiation (why not?). We must work directly from the difference quotient.

Example. Suppose that f with domain $\mathbb{R}$ is defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine whether or not f is differentiable at 0.

Exercise. Is $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ differentiable at 0?

Implicit differentiation

Many functions can be described by a rule of the form $y = f(x)$, i.e., one variable expressed explicitly in terms of another. However, sometimes it convenient or even necessary to define f by an equation relating the variables x and y.

Implicit differentiation

Many functions can be described by a rule of the form $y = f(x)$, i.e., one variable expressed explicitly in terms of another. However, sometimes it convenient or even necessary to define f by an equation relating the variables x and y.

Example. The function f with domain $[0, 4]$ is defined by

$$y = f(x), \qquad y \geq 0, \qquad x^2 + y^2 = 16.$$

Implicit differentiation

Many functions can be described by a rule of the form $y = f(x)$, i.e., one variable expressed explicitly in terms of another. However, sometimes it convenient or even necessary to define f by an equation relating the variables x and y.

Example. The function f with domain $[0, 4]$ is defined by

$$y = f(x), \qquad y \geq 0, \qquad x^2 + y^2 = 16.$$

Example. The function g with domain $(-a, a)$ is defined by

$$y = g(x), \qquad y < 0, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Implicit differentiation

In other cases it may not be easy (or possible) to express y explicitly as a function of x:

Implicit differentiation

In other cases it may not be easy (or possible) to express y explicitly as a function of x:

- spiral: $$\frac{1}{2}\ln(x^2+y^2)=\tan^{-1}\left(\frac{y}{x}\right)$$

Implicit differentiation

In other cases it may not be easy (or possible) to express y explicitly as a function of x:

- spiral: $$\frac{1}{2}\ln(x^2+y^2) = \tan^{-1}\left(\frac{y}{x}\right)$$
- heart: $$(x^2+y^2-1)^3 - x^2y^3 = 0.$$

Implicit differentiation

In other cases it may not be easy (or possible) to express y explicitly as a function of x:

- spiral: $\frac{1}{2}\ln(x^2+y^2) = \tan^{-1}\left(\frac{y}{x}\right)$
- heart: $(x^2+y^2-1)^3 - x^2y^3 = 0.$

However, implicit differentiation gives a way of calculating the derivative $\frac{dy}{dx}$ *without* expressing y explicitly as a function of x. It's easiest to see how to perform implicit differentiation by considering some examples.

Implicit differentiation

In other cases it may not be easy (or possible) to express y explicitly as a function of x:

- spiral: $\frac{1}{2}\ln(x^2+y^2)=\tan^{-1}\left(\frac{y}{x}\right)$
- heart: $(x^2+y^2-1)^3-x^2y^3=0.$

However, implicit differentiation gives a way of calculating the derivative $\frac{dy}{dx}$ *without* expressing y explicitly as a function of x. It's easiest to see how to perform implicit differentiation by considering some examples.

Example. Suppose that y is a function of x and that

$$4x^2+y^2=16.$$

Calculate $\frac{dy}{dx}$.

Implicit differentiation

Example. Suppose that y is a function of x, implicitly related by the equation

$$y^4 + x^3 - x^2 \sin(3y) = 8.$$

Find the equation of the tangent to the corresponding curve at the point where $(x, y) = (2, 0)$.

Related rates

Many physical processes involve quantities (such as temperature, volume, concentration, velocity) that change with time. If Q is a quantity that varies with time, then the derivative $\frac{dQ}{dt}$ gives the rate of change of that quantity with respect to time.

Related rates

Many physical processes involve quantities (such as temperature, volume, concentration, velocity) that change with time. If Q is a quantity that varies with time, then the derivative $\frac{dQ}{dt}$ gives the rate of change of that quantity with respect to time.

Sometimes, changing one quantity will cause another quantity to change but perhaps at a different rate. For example, if the radius of a balloon changes, so too will the volume of the balloon, but their rates of change will be different. It may be easier to measure the rate of change of the volume of the balloon compared to the rate of change of the radius. Then we can use this measurement to determine the rate of change of the radius. Such a problem is commonly called a *related rates* problem.

Related rates

Many physical processes involve quantities (such as temperature, volume, concentration, velocity) that change with time. If Q is a quantity that varies with time, then the derivative $\frac{dQ}{dt}$ gives the rate of change of that quantity with respect to time.

Sometimes, changing one quantity will cause another quantity to change but perhaps at a different rate. For example, if the radius of a balloon changes, so too will the volume of the balloon, but their rates of change will be different. It may be easier to measure the rate of change of the volume of the balloon compared to the rate of change of the radius. Then we can use this measurement to determine the rate of change of the radius. Such a problem is commonly called a *related rates* problem.

The *chain rule* and *implicit differentiation* are often useful for solving problems involving two related rates of change.

Related rates

Example (Stewart, Problem 2.7.9). If a snowball melts so that its surface area decreases at a rate of $1\mathrm{cm}^2/\mathrm{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully
2. Draw a diagram if possible

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully
2. Draw a diagram if possible
3. Introduce notation. Assign symbols to all quantities that are functions of time.

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully
2. Draw a diagram if possible
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully
2. Draw a diagram if possible
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully
2. Draw a diagram if possible
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution
6. Use the chain rule to differentiate both sides of the equation with respect to t.

Related rates

Strategy for solving related rates problems (Stewart, p.129).

1. Read problem carefully
2. Draw a diagram if possible
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution
6. Use the chain rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Related rates

Example (Stewart, Problem 2.7.13). Two cars start moving from the same point. One travels south at $60\mathrm{mi}/\mathrm{h}$ and the other travels west at $25\mathrm{mi}/\mathrm{h}$. At what rate is the distance between the cars increasing two hours later?

Related rates

Example (Stewart, Problem 2.7.13). Two cars start moving from the same point. One travels south at $60\mathrm{mi}/\mathrm{h}$ and the other travels west at $25\mathrm{mi}/\mathrm{h}$. At what rate is the distance between the cars increasing two hours later?

Example (Stewart, Problem 2.7.25). Gravel is being dumped from a conveyor belt at a rate of $30\mathrm{ft}^3/\mathrm{min}$, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line.

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line. That is,

$$f'(a) \sim \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line. That is,

$$f'(a) \sim \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Thus, simplifying,

$$y = f(a+h) \sim f'(a) \cdot h + f(a).$$

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line. That is,

$$f'(a) \sim \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Thus, simplifying,

$$y = f(a+h) \sim f'(a) \cdot h + f(a).$$

For very small h, the curve almost coincides with the tangent line. Thus, the value of the tangent line at $x = a + h$ gives an approximate value for $f(a+h)$.

Linear Approximation

We could also write

$$f(x) \sim f(a) + f'(a)(x - a).$$

This value is called the *linear approximation* or *tangent line approximation* or *linearization* of f at a. Be aware that we are just introducing three new names for a concept that we have already encountered!

Linear Approximation

We could also write

$$f(x) \sim f(a) + f'(a)(x - a).$$

This value is called the *linear approximation* or *tangent line approximation* or *linearization* of f at a. Be aware that we are just introducing three new names for a concept that we have already encountered!

Example. Estimate $\sqrt{16.001}$ without using a calculator.

Linear Approximation

We could also write

$$f(x) \sim f(a) + f'(a)(x - a).$$

This value is called the *linear approximation* or *tangent line approximation* or *linearization* of f at a. Be aware that we are just introducing three new names for a concept that we have already encountered!

Example. Estimate $\sqrt{16.001}$ without using a calculator.

You can use a calculator to see that the difference between the actual value of $\sqrt{16.001}$ and the approximation we just calculated is only (almost) $-.0000000195$!!!

Differential Approximation

The differential approximation. Suppose that $y = f(x)$ for some differentiable function f and fix a point a in $\mathrm{Dom}(f)$. A small change $x = a + dx$ from a will produce a corresponding change $dy = f(a + dx) - f(a)$. We call dx the *differential* of x, it is an independent variable. We call dy the *differential* of y, it is a dependent variable.

Differential Approximation

The differential approximation. Suppose that $y = f(x)$ for some differentiable function f and fix a point a in $\mathrm{Dom}(f)$. A small change $x = a + dx$ from a will produce a corresponding change $dy = f(a + dx) - f(a)$. We call dx the *differential* of x, it is an independent variable. We call dy the *differential* of y, it is a dependent variable.

That is, since

$$f(x) - f(a) = f'(a)(x - a), \qquad \text{we have} \qquad dy = f'(a)dx$$

whenever dx is small. This is called the *differential approximation*.

Differential Approximation

The differential approximation. Suppose that $y = f(x)$ for some differentiable function f and fix a point a in $\mathrm{Dom}(f)$. A small change $x = a + dx$ from a will produce a corresponding change $dy = f(a + dx) - f(a)$. We call dx the *differential* of x, it is an independent variable. We call dy the *differential* of y, it is a dependent variable.

That is, since

$$f(x) - f(a) = f'(a)(x - a), \qquad \text{we have} \qquad dy = f'(a)dx$$

whenever dx is small. This is called the *differential approximation*.

Example. The radius of a sphere is measured as 45 mm (to the nearest millimetre) and its volume is calculated. Estimate (i) the error and (ii) the percentage error for the calculated volume.