Week 7: Fundamental Theorem of Calculus

September 3 – September 7, 2012

1. Fundamental Theorem of Calculus
   - FTC and applications
   - Evaluating integrals

Lecture 14

Comparison properties of the integral.

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3. If $m \leq f(x) \leq M$ for all x in $[a, b]$ then
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$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$
4. If $|f|$ is integrable on $[a, b]$ then
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

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For example, $g(a) = \int_a^a f(t)\,dt = 0$, and $g(b) = \int_a^b f(t)\,dt$. If f is positive, then $g(x)$ can be interpreted as the area under the graph of f from a to x, the “area so far” function.

Theorem (Fundamental Theorem of Calculus)

Suppose a function f is continuous on $[a, b]$.

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Remark.

Note that part 1 says that if f is integrated, then differentiated, we get the original function f. Part 2 says that if we take a function F, differentiate it, then integrate the result, we get the original function F, but in the form $F(b) - F(a)$. Together, the two parts say that differentiation and integration are *inverse processes*.

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(iii) $g(x) = \int_3^{17} \sqrt{1+t^2}\,dt$.

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(c) We use the notation $F(x)\Big|_a^b = F(b) - F(a)$. Another common notation is $F(x)\Big]_a^b$, which is used in the book.

(d) When we apply the FTC, part 2, we may use *any* antiderivative of f (recall that if $F' = f$, the most general form for an antiderivative of f is $F + C$, where C is an arbitrary constant).

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(e) The FTC, part 2 is quite plausible from a physical perspective. Recall that when we considered the "Distance Problem" several lectures ago, we determined that the area under the velocity curve was equal to the total distance travelled. That is, if velocity and position are given by $v(t)$ and $s(t)$ respectively at time t, then $s'(t) = v(t)$, so s is an antiderivative of v, and $\int_a^b v(t)\,dt = s(b) - s(a)$.

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Note that the definite integral is a *number*, whereas the indefinite integral is a *function*, or a *family of functions*. The Fundamental Theorem of Calculus, part 2 tells us that if f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$. Also note that an indefinite integral can represent either a particular antiderivative of f or an entire family of antiderivatives.

A table of indefinite integrals.

In the following table, c, C, and k are arbitrary constants.

f	$\int f(x)\,dx$
$cf(x)$	$c(\int f(x)\,dx) + C$
$f(x) \pm g(x)$	$(\int f(x)\,dx) \pm (\int g(x)\,dx) + C$
k	$kx + C$
x^n (where $n \neq -1$)	$\dfrac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

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(iii) True or False: $\int_0^{\pi} \sec^2 x \, dx = \tan x \Big|_0^{\pi} = 0.$

(iv) Find the indefinite integral: $\int (1-t)(2+t^2) \, dt.$

Applications.

Suppose f is continuous on $[a, b]$ and F is an antiderivative of f, that is, $F' = f$. Then the FTC, part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Applications.

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$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

We know $F'(x)$ is the rate of change of $y = F(x)$ with respect to x, and $F(b) - F(a)$ is the *net* change in y when x changes from a to b. Thus the FTC part 2 says that the integral of a rate of change over an interval is the net change over that interval.

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For instance,

1. If $V(t)$ is the volume of water in a tank at time t, then $V'(t)$ is the rate at which water flows into the tank at time t. Hence $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$ is the net change in the amount of water in the tank between times t_1 and t_2.

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2. If the rate of growth of a population is $\frac{dP}{dt}$, then $\int_{t_1}^{t_2} \frac{dP}{dt}\,dt = P(t_2) - P(t_1)$, the net change in population during time t_1 to t_2.

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3. If an object has position function s and velocity v, then $\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$, the net change of position (or *displacement*) of the particle during the period from t_1 to t_2.

Recall from last class

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2. $$g(x) = \int_1^{x^4} \sec t\,dt.$$
3. $$g(x) = \int_2^{\frac{1}{x}} \sin^4 t\,dt.$$
4. Evaluate the following integral:

$$\int_0^{\frac{\pi}{4}} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$$

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3. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

Further Examples.

(a) If $v(t) = 3t - 5$ gives the velocity of a particle moving along a line, find the displacement and the total distance traveled by the particle during the time interval $0 \le t \le 3$.

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(b) Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

The average value of a function

The average value of finitely many numbers $y_1, y_2, \ldots, y_n$ is given by

$$y_{\text{ave}} = \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

In general, how do we calculate the average value of a function $y = f(x)$, where $a \leq x \leq b$?

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Definition

Suppose that f is continuous on $[a, b]$. Then the *average value* f_{ave} of f on $[a, b]$ is defined by the formula

$$f_{\mathrm{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The average value is sometimes denoted by $\overline{f}$.

The average value of a function

Example. Find the average value of the function f over $[-2, 3]$, where

$$f(x) = 1 + x^2.$$

Suppose that $T(t)$ is the temperature (in $^\circ$C) at time t (in hours) and that T_{ave} is the average temperature on the time interval $[0, 24]$. Is there a specific time t_0 in $[0, 24]$ when the temperature $T(t_0)$ is equal to the average temperature T_{ave}? More generally, given a function f, is there a specific value c for which $f(c) = f_{\mathrm{ave}}$?

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The answer is yes! This is called the mean value theorem for integrals.

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Theorem (The Mean Value Theorem for integrals)

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Example. Find all numbers c that satisfy the conclusion of the MVT for integrals when $f(x) = 1 + x^2$ and $[a, b] = [-2, 3]$.

Integration by substitution

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However, there is a useful method called “integration by substitution” that we can apply. We illustrate it with the above example. If we let $u = 1 + x^2$, then $du = 2x\,dx$. Hence we may formally write

$$\begin{aligned}\int 2x\sqrt{1+x^2}\,dx &= \int \sqrt{1+x^2}(2xdx)\\ &= \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C\\ &= \frac{2}{3}(x^2+1)^{3/2} + C\end{aligned}$$

Integration by substitution

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

That is, *it is permissible to work with dx and du after integral signs as if they are differentials.*

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(i) Evaluate $\int \sqrt{2x+1}\,dx$.

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(i) Evaluate $\int \sqrt{2x+1}\,dx$.

(ii) Evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

(iii) Evaluate $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx$.

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2. If f is odd $(f(-x) = f(x))$, then
$$\int_{-a}^{a} f(x)\,dx = 0.$$

Midterm Review

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§2.5, #36 Find the derivative of the function

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}}.$$