Week 8: Inverse functions, differentiating inverse functions, the natural logarithm

September 24 – September 28, 2012

1. Inverse functions
   - Inverse functions and calculus

2. The natural logarithm

Recall from last class

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

That is, *it is permissible to work with dx and du after integral signs as if they are differentials*.

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Example.

1. Evaluate $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx$.

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Inverse functions

Recall that we think of functions as machines that take inputs from the domain, do stuff with them, and give outputs. An important question is "for a 'machine' f, when does there exist a 'machine' g that undoes f?". If such a g exists, we call it the inverse of f.

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Example. Suppose f is a function such that $f(0) = 0$ and $f(1) = 0$. Is there a function g that undoes f?

One-one functions

Definition

A function f is said to be **one-to-one** if it doesn't attain the same value twice. That is,

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for any $x_1, x_2 \in \mathrm{Dom}(f)$.

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Graphically, f is one-to-one if and only if no horizontal line intersects its graph more than once (this is called the *horizontal line test*, compare this with the *vertical line test* to test if f is a function in the first place). One-to-one functions are sometimes called **injective** functions.

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(v) f given by $f(x) = \cos(x)$.

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Suppose f is a function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

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Example. Suppose f is defined by $f(x) = y = x^3$, so f maps x to y. The inverse function of f is given by $f^{-1}(x) = x^{1/3}$ because

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x.$$

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(ii) (important!) The -1 in f^{-1} is **NOT** an exponent. That is, $f^{-1}(x)$ is **NOT** equal to $\frac{1}{f(x)}$ in general. For example, if $f(x) = x^3$ then what is $f^{-1}(8)$? What is $\frac{1}{f(8)}$? (However, we *can* write $(f(x))^{-1} = \frac{1}{f(x)}$.)

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(iii) Since f and f^{-1} "undo" each other, we have the rules $f^{-1}(f(x)) = x$ for every x in A, and $f(f^{-1}(y)) = y$ for every y in B.

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So given f, how do we explicitly find the inverse function f^{-1}? We first write $y = f(x)$, then try to solve for x in terms of y, then at the end switch the variables x and y to express f^{-1} as a function of x.

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If f has an inverse, notice that $f(a) = b$ if and only if $f^{-1}(b) = a$. That is, (a, b) is in the graph of f if and only if (b, a) is in the graph of f^{-1}. Since the point (b, a) is obtained from (a, b) by reflecting about the line $y = x$, we get the graph of f^{-1} by reflecting the graph of f about the line $y = x$.

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Example. Sketch the graph of f and f^{-1} if $f(x) = \sqrt{x - 2}$.

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Theorem

If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Inverse function theorem

Intuitively, we know that if f is differentiable then the graph has no kinks or vertical tangent lines. If the graph of f^{-1} is obtained by reflecting about the line $y = x$, then if the graph of f has no kinks, then neither will the graph of f^{-1}. The only way to get vertical tangent lines to the graph of f^{-1} is if there are horizontal tangent lines to the graph of f. This motivates the following important theorem.

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Theorem (The inverse function theorem)

Suppose f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

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Example. If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.

The natural logarithm

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$$V(0) = 1, \quad V(1) = 2, \quad V(2) = 4, \quad V(3) = 8$$

and so on. In general, $V(t) = 2^t$ if t is a nonnegative integer.

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Question. Can we graph $V(t) = 2^t$ if $t \in \mathbb{R}$?
Only if we have defined 2^t when $t \in \mathbb{R}$.

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- What is $2^{\sqrt{3}}$?
- What is 2^{π}?

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There is lots of work to be done here before we can draw the graph of V. At the moment we only have the following: If b is a positive real number and r is a rational number with $r = p/q$ then b^r is defined to be the unique positive qth root of b^p. We do not yet have a definition for b^r when $r \notin \mathbb{Q}$.

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1. Define the natural logarithm function ln as a function from $(0, \infty)$ to $\mathbb{R}$ using an integral formula.

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3. Define b^r when $r \notin \mathbb{Q}$ using the exp and ln functions.

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- $\ln x < 0$ if $0 < x < 1$.

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(iii) $\ln(x^r) = r\ln(x)$ whenever r is a rational number and x is a positive real number.

The natural logarithm

Example. Expand the expression $\ln \frac{(x^2+5)^4 \sin x}{x^3+1}$.