Week 2: Limit of Functions

July 30 – August 3, 2012

1. Limits of functions at a point

2. Limits at infinity

Lecture 4

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2. Limits at infinity

Meaning of limits

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$f(x)$ is not defined at $x = 1$, but as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2. We say that *the limit of $f(x)$ as x approaches to* 1*, is* 2.

Definition of limits

Definition

Let a be a real number. Suppose the function $y = f(x)$ is defined for all x near a on both sides. That is, for some real numbers b, c with $b < a < c$, the intervals (b, a) and (a, c) are in the domain of $y = f(x)$.

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Example

$\lim_{x \to 1} x^2 = 1.$

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1. Calculate $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.
   Solution: The domain is $(-\infty, 1) \cup (1, \infty)$. When $x \neq 1$,
   $\dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x - 1} = x + 1$.

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2. Calculate $\lim\limits_{x\to -1} \dfrac{2x+2}{x+1}$.
   Solution: The domain is $(-\infty, -1) \cup (-1, \infty)$. When $x \neq -1$, $\dfrac{2x+2}{x+1} = 2$. Limit is 2.

Limits of functions at a point

Examples.

3. if $f(x) = \begin{cases} sin(x) & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}$ then $\lim_{x \to 0} f(x) = 0.$

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3. if $f(x) = \begin{cases} sin(x) & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}$ then $\lim_{x\to 0} f(x) = 0$.
4. Sometimes, the limit does not exist.
   $\lim_{x\to 0} \log_2 |x|$ does not exist.
   Notation: When the limit $\lim_{x\to a} f(x)$ doesn't exist because $f(x)$ becomes arbitrarily large and positive (resp. negative) as x approaches a, we write $\lim_{x\to a} f(x) = \infty$ (resp. $\lim_{x\to a} f(x) = -\infty$). Thus,

$$\lim_{x\to 0} \log_2 |x| = -\infty.$$

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   $$\lim_{x \to 0} \log_2 |x| = -\infty.$$
5. $\lim_{x \to 0} \dfrac{1}{x^2} = \infty.$

Definition of limits

Remark

One-sided limits:

(i) The limit from the left: $\lim_{x \to a^-} f(x) = L$.

(ii) The limit from the right: $\lim_{x \to a^+} f(x) = L$.

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Example

$\lim_{x \to 0} \frac{|x|}{x}$ Solution: The domain is $(-\infty, 0) \cup (0, \infty)$.

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$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution: The domain is $(-\infty, 0) \cup (0, \infty)$.

$$\frac{|x|}{x} = \begin{cases} +1 & x > 0 \\ -1 & x < 0. \end{cases}$$

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$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$

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Solution: The domain is $(-\infty, 0) \cup (0, \infty)$.

$$\frac{|x|}{x} = \begin{cases} +1 & x > 0 \\ -1 & x < 0. \end{cases}$$

$\lim_{x \to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

Existence of limits

Remark

The limit $\lim_{x \to a} f(x) = L$ exists if and only if

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$$\lim_{x \to 0} \frac{|x|}{x}$$

As $\lim_{x \to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x \to 0^-} \frac{|x|}{x} = -1$, $\lim_{x \to 0} \frac{|x|}{x}$ doesn't exist.

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Example

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As $\lim\limits_{x \to 0^+} \dfrac{|x|}{x} = 1$ and $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} = -1$, $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ doesn't exist.

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$L \neq \pm\infty$, otherwise, the limit doesn't exist.

Existence of limits

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$$\lim_{x\to 0} \frac{1}{x}$$

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<u>Solution:</u> The domain is $(-\infty, 0) \cup (0, \infty)$. As $x \to 0^+$, $\frac{1}{x}$ gets larger and larger, indicated by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

As ∞ is not a real number, we say $\lim_{x \to 0} \frac{1}{x}$ does not exist.

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Remark

If $x = k$ is a vertical asymptote of $y = f(x)$, then $\lim_{x \to k} f(x)$ doesn't exist.

Rules for calculating limits

Basic limit rules. Suppose that $a \in \mathbb{R}$ and that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and are finite real numbers. Then

(i) $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(iii) $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.

A similar set of rules hold for left- and right-hand limits.

Rules for calculating limits

Limit rule for compositions.

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to L} g(x) = g(L)$ then

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

An easier way of remembering this rule will be given later.

Recall from last class

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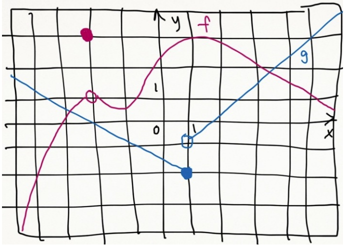
Rules for calculating limits

Example.



Use the limit laws and the graphs of f and g in the above figure to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$

Rules for calculating limits

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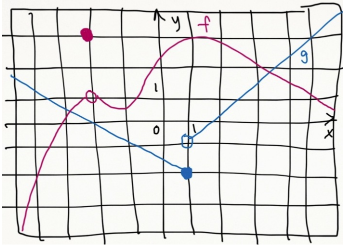

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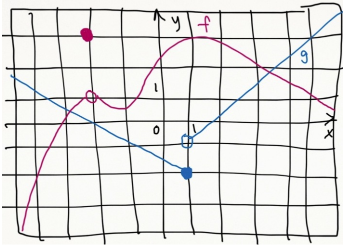

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(a) $\lim\limits_{x\to -2}[f(x)+5g(x)]$

(b) $\lim\limits_{x\to 1}[f(x)g(x)]$

(c) $\lim\limits_{x\to 2}\dfrac{f(x)}{g(x)}$

Limits at infinity

There are many situations where it is desirable to know what the eventual state of a system will be.

Example: Suppose that an initially unpolluted lake containing 1 gigalitre (10^9 litres) of water has a river flowing through it at a rate of 1 megalitre (10^6 litres) per day. A factory is built next to the lake and discharges ten thousand (10^4) litres of pollution per day. Environmental authorities want to know what the eventual impact of the pollution will be. By making some simple assumptions, it can be shown that the amount $P(t)$ of pollution (in litres) in the lake after t days of the factory's operation is given by

$$P(t) = \frac{10^9}{101}\left(1 - e^{-101t/10^5}\right). \tag{1}$$

To see whether the pollution in the lake eventually stabilizes, and what the level of pollution will be, one studies the behavior of $P(t)$ as $t \to \infty$.

Calculating limits at infinity

Definition (imprecise).
if f is a function defined on an interval (a, ∞), or $(-\infty, \infty)$, then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.
Sometimes we also write

$$f(x) \longrightarrow L \qquad \text{as} \qquad x \longrightarrow \infty$$

Examples. True or False?

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(e) $\lim\limits_{x \to \infty} sin(x) = 0$

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(f) $\lim_{x\to\infty} e^x = \infty$

(g) $\lim_{x\to\infty} \sqrt{x} = 2$

(h) $\lim_{x\to\infty} ln(x) = 100$

We have a similar definition for the limit as x goes to $-\infty$:
Let f be a function defined on some interval $(-\infty, a)$ or $(-\infty, \infty)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large and negative.

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Question. What is the value of $\lim_{x \to -\infty} e^x$?

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Question. What is the value of $\lim\limits_{x \to -\infty} e^x$?

The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L.$$

Basic rules. Suppose that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist and are finite real numbers. Then

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(v) if $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} \frac{1}{f(x)} = 0$.

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(v) if $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} \dfrac{1}{f(x)} = 0$.

Examples.

Basic rules. Suppose that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist and are finite real numbers. Then

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(a) $\lim_{x\to\infty}(\frac{1}{x} + 1 - e^{-2x})$

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(a) $\lim_{x\to\infty}(\frac{1}{x} + 1 - e^{-2x})$

(b) $\lim_{x\to-\infty}(\frac{3}{x} + 1 - cos(x))$

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Example. Find $\lim_{x \to 0} x^2 \sin(1/x)$.

The squeeze theorem for limits at infinity. Suppose that f, g and h are all defined on the interval (b, ∞), where $b \in \mathbb{R}$. If

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Example. Use the squeeze theorem to find $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x}$.

Indeterminate forms.

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$$\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

where $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to\infty$ are said to be limits of the form $\frac{\infty}{\infty}$.

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- $$\lim_{x\to\infty} \frac{e^x}{x}$$ (deal with this more later)