Week 5: Maxima and Minima, Mean Value Theorem

August 20 – August 24, 2012

Lecture 10

1. Linear Approximation

2. Maxima and Minima
   - The Mean Value Theorem
   - Derivatives and shapes of graphs

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line.

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line. That is,

$$f'(a) \sim \frac{f(a+h) - f(a)}{h} \quad \text{for } h \text{ small.}$$

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line. That is,

$$f'(a) \sim \frac{f(a+h) - f(a)}{h} \quad \text{for } h \text{ small.}$$

Thus, simplifying,

$$y = f(a+h) \sim f'(a) \cdot h + f(a).$$

Linear Approximation

We have, the slope of the tangent line of $f(x)$ at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When h is very small, the slope of the secant line is very close to the slope of the tangent line. That is,

$$f'(a) \sim \frac{f(a+h) - f(a)}{h} \quad \text{for } h \text{ small.}$$

Thus, simplifying,

$$y = f(a+h) \sim f'(a) \cdot h + f(a).$$

For very small h, the curve almost coincides with the tangent line. Thus, the value of the tangent line at $x = a + h$ gives an approximate value for $f(a+h)$.

Linear Approximation

We could also write

$$f(x) \sim f(a) + f'(a)(x - a).$$

This value is called the *linear approximation* or *tangent line approximation* or *linearization* of f at a. Be aware that we are just introducing three new names for a concept that we have already encountered!

Linear Approximation

We could also write

$$f(x) \sim f(a) + f'(a)(x - a).$$

This value is called the *linear approximation* or *tangent line approximation* or *linearization* of f at a. Be aware that we are just introducing three new names for a concept that we have already encountered!

Example. Estimate $\sqrt{16.001}$ without using a calculator.

Linear Approximation

We could also write

$$f(x) \sim f(a) + f'(a)(x - a).$$

This value is called the *linear approximation* or *tangent line approximation* or *linearization* of f at a. Be aware that we are just introducing three new names for a concept that we have already encountered!

Example. Estimate $\sqrt{16.001}$ without using a calculator.

You can use a calculator to see that the difference between the actual value of $\sqrt{16.001}$ and the approximation we just calculated is only (almost) $-.0000000195$!!!

Differential Approximation

The differential approximation. Suppose that $y = f(x)$ for some differentiable function f and fix a point a in $\mathrm{Dom}(f)$. A small change $x = a + dx$ from a will produce a corresponding change $dy = f(a + dx) - f(a)$. We call dx the *differential* of x, it is an independent variable. We call dy the *differential* of y, it is a dependent variable.

Differential Approximation

The differential approximation. Suppose that $y = f(x)$ for some differentiable function f and fix a point a in $\mathrm{Dom}(f)$. A small change $x = a + dx$ from a will produce a corresponding change $dy = f(a + dx) - f(a)$. We call dx the *differential* of x, it is an independent variable. We call dy the *differential* of y, it is a dependent variable.

That is, since

$$f(x) - f(a) = f'(a)(x - a), \qquad \text{we have} \qquad dy = f'(a)dx$$

whenever dx is small. This is called the *differential approximation*.

Differential Approximation

The differential approximation. Suppose that $y = f(x)$ for some differentiable function f and fix a point a in $\mathrm{Dom}(f)$. A small change $x = a + dx$ from a will produce a corresponding change $dy = f(a + dx) - f(a)$. We call dx the *differential* of x, it is an independent variable. We call dy the *differential* of y, it is a dependent variable.

That is, since

$$f(x) - f(a) = f'(a)(x - a), \qquad \text{we have} \qquad dy = f'(a)dx$$

whenever dx is small. This is called the *differential approximation*.

Example. The radius of a sphere is measured as 45 mm (to the nearest millimetre) and its volume is calculated. Estimate (i) the error and (ii) the percentage error for the calculated volume.

1. Linear Approximation

2. Maxima and Minima
   - The Mean Value Theorem
   - Derivatives and shapes of graphs

Maximum and minimum values and critical points

Most optimization problems, such as the problem of finding the shape of a can that minimize the manufacturing cost, can be reduced to finding the maximum or minimum values of some function.

Maximum and minimum values and critical points

Most optimization problems, such as the problem of finding the shape of a can that minimize the manufacturing cost, can be reduced to finding the maximum or minimum values of some function.

Definition

Maximum and minimum values and critical points

Most optimization problems, such as the problem of finding the shape of a can that minimize the manufacturing cost, can be reduced to finding the maximum or minimum values of some function.

Definition

(a) A function f has an *absolute* or *global maximum* at c if $f(c) \geq f(x)$ for **all** $x \in \mathrm{Dom}(f)$. The number $f(c)$ is called the *maximum value* of f on $\mathrm{Dom}(f)$. f has an *absolute* or *global minimum* at c if $f(c) \leq f(x)$ for **all** $x \in \mathrm{Dom}(f)$. The number $f(c)$ is called the *minimum value* of f on $\mathrm{Dom}(f)$.

Maximum and minimum values and critical points

Most optimization problems, such as the problem of finding the shape of a can that minimize the manufacturing cost, can be reduced to finding the maximum or minimum values of some function.

Definition

(a) A function f has an *absolute* or *global maximum* at c if $f(c) \geq f(x)$ for **all** $x \in \mathrm{Dom}(f)$. The number $f(c)$ is called the *maximum value* of f on $\mathrm{Dom}(f)$. f has an *absolute* or *global minimum* at c if $f(c) \leq f(x)$ for **all** $x \in \mathrm{Dom}(f)$. The number $f(c)$ is called the *minimum value* of f on $\mathrm{Dom}(f)$.

(b) A function f has a *local maximum* at c if $f(c) \geq f(x)$ for x **near** c, that is, $f(c) \geq f(x)$ for all x in some open interval containing c. Similarly f has a *local minimum* at c if $f(c) \leq f(x)$ when x is **near** c.

True or False?.

(i) $f(x) = x^2$ has

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

(iii) $f(x) = \sin(x)$ has

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

(iii) $f(x) = \sin(x)$ has

- a local maximum at $x = \frac{\pi}{2}$

True or False?.

(i) $f(x) = x^2$ has
- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has
- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

(iii) $f(x) = \sin(x)$ has
- a local maximum at $x = \frac{\pi}{2}$
- a global minimum at $x = \frac{3\pi}{2}$

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

(iii) $f(x) = \sin(x)$ has

- a local maximum at $x = \frac{\pi}{2}$
- a global minimum at $x = \frac{3\pi}{2}$
- a unique global maximum

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

(iii) $f(x) = \sin(x)$ has

- a local maximum at $x = \frac{\pi}{2}$
- a global minimum at $x = \frac{3\pi}{2}$
- a unique global maximum
- a unique local minimum

True or False?.

(i) $f(x) = x^2$ has

- A local minimum at $x = 0$
- A global minimum at $x = 0$
- A global maximum at $x = 100$
- A local maximum at $x = 100$

(ii) $f(x) = x^3$ has

- a global minimum (anywhere)
- a local minimum(anywhere)
- a global maximum (anywhere)
- a local maximum (anywhere)

(iii) $f(x) = \sin(x)$ has

- a local maximum at $x = \frac{\pi}{2}$
- a global minimum at $x = \frac{3\pi}{2}$
- a unique global maximum
- a unique local minimum

(iv) If a function has a local minimum, it must have a global minimum

So when does a function have a global maximum or minimum???

So when does a function have a global maximum or minimum???

Theorem (extreme value theorem)

If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points c and d in $[a, b]$. (Note that c and d could very well be the endpoints of the interval!)

So when does a function have a global maximum or minimum???

Theorem (extreme value theorem)

If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points c and d in $[a, b]$. (Note that c and d could very well be the endpoints of the interval!)

The extreme value theorem tells us that a continuous function f attains its max and min values on a closed interval, but it gives no systematic approach for finding them. We now address this problem.

Theorem (Fermat's Theorem)

Suppose that f is defined on (a, b) and has a local maximum or minimum point at c for some c in (a, b). If f is differentiable at c then $f'(c) = 0$.

Theorem (Fermat's Theorem)

Suppose that f is defined on (a, b) and has a local maximum or minimum point at c for some c in (a, b). If f is differentiable at c then $f'(c) = 0$.

Limitations of Fermat's Theorem.

1. The *converse* to Fermat's theorem doesn't necessarily hold! That is, there are functions that are differentiable at c satisfying $f'(c) = 0$ but f does **not** have a local maximum or minimum at c.

Theorem (Fermat's Theorem)

Suppose that f is defined on (a, b) and has a local maximum or minimum point at c for some c in (a, b). If f is differentiable at c then $f'(c) = 0$.

Limitations of Fermat's Theorem.

1. The *converse* to Fermat's theorem doesn't necessarily hold! That is, there are functions that are differentiable at c satisfying $f'(c) = 0$ but f does **not** have a local maximum or minimum at c.
2. Fermat's theorem only applies if f is differentiable at c! Can you think of an example of a function which has a local or global minimum at c where f is **not** differentiable at c?

Although Fermat's theorem doesn't always tell us exactly where local or global extrema are located, it does suggest that we should start searching for such extrema at the points c where either $f'(c) = 0$ or where $f'(c)$ doesn't exist.

Although Fermat's theorem doesn't always tell us exactly where local or global extrema are located, it does suggest that we should start searching for such extrema at the points c where either $f'(c) = 0$ or where $f'(c)$ doesn't exist.

Definition

A *critical point* of a function f is a point c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ doesn't exist. (Note that the book uses the term "critical number" instead of "critical point.")

Although Fermat's theorem doesn't always tell us exactly where local or global extrema are located, it does suggest that we should start searching for such extrema at the points c where either $f'(c) = 0$ or where $f'(c)$ doesn't exist.

Definition

A *critical point* of a function f is a point c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ doesn't exist. (Note that the book uses the term "critical number" instead of "critical point.")

Hence we can rephrase Fermat's theorem as "**If f has a local maximum or minimum at c, then c is a critical point of f.**"

Closed Interval Method. To find the absolute maximum and minimum values of a *continuous* function f on a *closed* interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).

Closed Interval Method. To find the absolute maximum and minimum values of a *continuous* function f on a *closed* interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.

Closed Interval Method. To find the absolute maximum and minimum values of a *continuous* function f on a *closed* interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest value from Steps 1 and 2 is the absolute maximum value, while the smallest value is the absolute minimum value.

Closed Interval Method. To find the absolute maximum and minimum values of a *continuous* function f on a *closed* interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest value from Steps 1 and 2 is the absolute maximum value, while the smallest value is the absolute minimum value.

Example:. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ when $-\frac{1}{2} \leq x \leq 4$.

The Mean Value Theorem

The Mean Value Theorem (MVT) is one of the most important results for establishing the theoretical framework for calculus. Applications of the mean value theorem include

- identifying where a function is increasing or decreasing,

The Mean Value Theorem

The Mean Value Theorem (MVT) is one of the most important results for establishing the theoretical framework for calculus. Applications of the mean value theorem include

- identifying where a function is increasing or decreasing,
- identifying different types of critical points,

The Mean Value Theorem

The Mean Value Theorem (MVT) is one of the most important results for establishing the theoretical framework for calculus. Applications of the mean value theorem include

- identifying where a function is increasing or decreasing,
- identifying different types of critical points,
- determining how many zeros a polynomial has,

The Mean Value Theorem

The Mean Value Theorem (MVT) is one of the most important results for establishing the theoretical framework for calculus. Applications of the mean value theorem include

- identifying where a function is increasing or decreasing,
- identifying different types of critical points,
- determining how many zeros a polynomial has,
- proving useful inequalities.

The Mean Value Theorem

The Mean Value Theorem (MVT) is one of the most important results for establishing the theoretical framework for calculus. Applications of the mean value theorem include

- identifying where a function is increasing or decreasing,
- identifying different types of critical points,
- determining how many zeros a polynomial has,
- proving useful inequalities.

First, we state the

Theorem (Rolle's Theorem)

Suppose f is continuous on $[a, b]$, differentiable on (a, b) and such that $f(a) = f(b)$. Then, there is a $c \in (a, b)$ such that $f'(c) = 0$.

The Mean Value Theorem

Theorem (The Mean Value Theorem)

Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \textit{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

The Mean Value Theorem

Theorem (The Mean Value Theorem)

Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a,b]$.*
2. *f is differentiable on the open interval (a,b).*

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \quad \textit{equivalently} \quad f(b)-f(a) = f'(c)(b-a).$$

Important application. If $f'(x) = 0$ for all x in an interval (a,b), then f is constant on (a,b).

The Mean Value Theorem

Example. Suppose that $f(x) = x^2 - 4x + 4$. Find a number c in $(1, 4)$ that satisfies the conclusions of the mean value theorem for f on $[1, 4]$.

Recall from last class

Theorem (The Mean Value Theorem)

Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \textit{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

Applications of the MVT

In these examples, the information about a function f is obtained from information about its derivative. This is a central feature of applications of the MVT.

Applications of the MVT

In these examples, the information about a function f is obtained from information about its derivative. This is a central feature of applications of the MVT.

Example. Suppose that $f(0) = 4$ and $f'(x) \leq 3$ whenever $x \in \mathbb{R}$. How large can $f(2)$ possibly be?

Applications of the MVT

In these examples, the information about a function f is obtained from information about its derivative. This is a central feature of applications of the MVT.

Example. Suppose that $f(0) = 4$ and $f'(x) \leq 3$ whenever $x \in \mathbb{R}$. How large can $f(2)$ possibly be?

Example (Stewart 3.2.17). Show that the equation

$$1 + 2x + x^3 + 4x^5 = 0$$

has exactly one real root.

The Sign of a Derivative

We have seen that the mean value theorem gives us a way to get information about a function f from knowledge of its derivative f'. In this section we will explore this connection further and learn an easy way to tell if a function is increasing or decreasing on an interval.

The Sign of a Derivative

We have seen that the mean value theorem gives us a way to get information about a function f from knowledge of its derivative f'. In this section we will explore this connection further and learn an easy way to tell if a function is increasing or decreasing on an interval.

Definition. Suppose that a function f is defined on an interval I. We say that

The Sign of a Derivative

We have seen that the mean value theorem gives us a way to get information about a function f from knowledge of its derivative f'. In this section we will explore this connection further and learn an easy way to tell if a function is increasing or decreasing on an interval.

Definition. Suppose that a function f is defined on an interval I. We say that

a) f is (strictly) *increasing* on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) < f(x_2);$$

The Sign of a Derivative

We have seen that the mean value theorem gives us a way to get information about a function f from knowledge of its derivative f'. In this section we will explore this connection further and learn an easy way to tell if a function is increasing or decreasing on an interval.

Definition. Suppose that a function f is defined on an interval I. We say that

a) f is (strictly) *increasing* on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) < f(x_2);$$

b) f is (strictly) *decreasing* on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) > f(x_2).$$

The Sign of a Derivative

True or False?.

The Sign of a Derivative

True or False?.

(i) $f(x) = x^2$ is increasing on the interval $(4, 10)$.

The Sign of a Derivative

True or False?.

(i) $f(x) = x^2$ is increasing on the interval $(4, 10)$.

(ii) $f(x) = x^3 + 4$ is decreasing on the interval $[-2, 2]$

The Sign of a Derivative

True or False?.

(i) $f(x) = x^2$ is increasing on the interval $(4, 10)$.

(ii) $f(x) = x^3 + 4$ is decreasing on the interval $[-2, 2]$

(iii) $f(x) = \sin(x)$ is increasing on the interval $(0, \pi)$.

The Sign of a Derivative

We notice that when a function is increasing, the tangent lines to the graph have positive slope, and when a function is decreasing, the tangent lines to the graph have negative slope.

The Sign of a Derivative

We notice that when a function is increasing, the tangent lines to the graph have positive slope, and when a function is decreasing, the tangent lines to the graph have negative slope.

Increasing/decreasing test. Suppose that f is differentiable on an open interval I.

The Sign of a Derivative

We notice that when a function is increasing, the tangent lines to the graph have positive slope, and when a function is decreasing, the tangent lines to the graph have negative slope.

Increasing/decreasing test. Suppose that f is differentiable on an open interval I.

(i) If $f'(x) > 0$ for all x in I then f is increasing on I.

The Sign of a Derivative

We notice that when a function is increasing, the tangent lines to the graph have positive slope, and when a function is decreasing, the tangent lines to the graph have negative slope.

Increasing/decreasing test. Suppose that f is differentiable on an open interval I.

(i) If $f'(x) > 0$ for all x in I then f is increasing on I.

(ii) If $f'(x) < 0$ for all x in I then f is decreasing on I.

The Sign of a Derivative

We notice that when a function is increasing, the tangent lines to the graph have positive slope, and when a function is decreasing, the tangent lines to the graph have negative slope.

Increasing/decreasing test. Suppose that f is differentiable on an open interval I.

(i) If $f'(x) > 0$ for all x in I then f is increasing on I.

(ii) If $f'(x) < 0$ for all x in I then f is decreasing on I.

Example. Where is $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ increasing and where it is decreasing?

The first derivative test

Recall that **if** f has a local minimum or maximum at c, **then** c is a critical point, but not every critical point gives rise to a local maximum or minimum. However, we can now apply the preceding theorem to classify critical points to see whether f attains a local maximum, minimum, or neither.

The first derivative test

Recall that **if** f has a local minimum or maximum at c, **then** c is a critical point, but not every critical point gives rise to a local maximum or minimum. However, we can now apply the preceding theorem to classify critical points to see whether f attains a local maximum, minimum, or neither.

Corollary (The first derivative test).
Suppose that c is a critical point of a continuous function f.

The first derivative test

Recall that **if** f has a local minimum or maximum at c, **then** c is a critical point, but not every critical point gives rise to a local maximum or minimum. However, we can now apply the preceding theorem to classify critical points to see whether f attains a local maximum, minimum, or neither.

Corollary (The first derivative test).
Suppose that c is a critical point of a continuous function f.

(i) If f' is positive to the left of c and negative to the right of c, then f has a local maximum at c.

The first derivative test

Recall that **if** f has a local minimum or maximum at c, **then** c is a critical point, but not every critical point gives rise to a local maximum or minimum. However, we can now apply the preceding theorem to classify critical points to see whether f attains a local maximum, minimum, or neither.

Corollary (The first derivative test).
Suppose that c is a critical point of a continuous function f.

(i) If f' is positive to the left of c and negative to the right of c, then f has a local maximum at c.

(ii) If f' is negative to the left of c and positive to the right of c, then f has a local minimum at c.

The first derivative test

Recall that **if** f has a local minimum or maximum at c, **then** c is a critical point, but not every critical point gives rise to a local maximum or minimum. However, we can now apply the preceding theorem to classify critical points to see whether f attains a local maximum, minimum, or neither.

Corollary (The first derivative test).
Suppose that c is a critical point of a continuous function f.

(i) If f' is positive to the left of c and negative to the right of c, then f has a local maximum at c.

(ii) If f' is negative to the left of c and positive to the right of c, then f has a local minimum at c.

(iii) If f' does not change sign at c then f has neither a local maximum nor local minimum at c.

Example. Find and classify the critical points of the function f whose derivative is given by

$$f'(x) = (x-2)(x+6)(x-1)^2.$$

The Second Derivative and Concavity

The second derivative f'' measures the rate of change of f'. We can use our knowledge of f'' to give us information not just about f', but also about f itself.

The Second Derivative and Concavity

The second derivative f'' measures the rate of change of f'. We can use our knowledge of f'' to give us information not just about f', but also about f itself.

Example. Consider the graphs $y = x^2$ and $y = \sqrt{x}$ when $x > 0$. Notice that although both are increasing, one "bends up" and the other "bends down". That is, the tangent lines to x^2 on this interval lie *below* the graph of x^2 whereas the tangent lines to $\sqrt{x}$ lie *above* the graph of $\sqrt{x}$.

The Second Derivative and Concavity

The second derivative f'' measures the rate of change of f'. We can use our knowledge of f'' to give us information not just about f', but also about f itself.

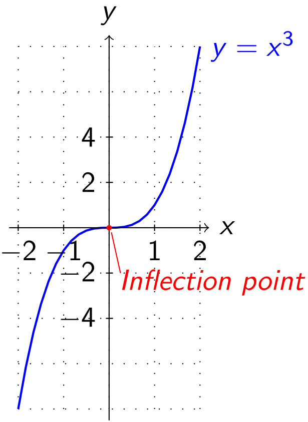
Example. Consider the graphs $y = x^2$ and $y = \sqrt{x}$ when $x > 0$. Notice that although both are increasing, one "bends up" and the other "bends down". That is, the tangent lines to x^2 on this interval lie *below* the graph of x^2 whereas the tangent lines to $\sqrt{x}$ lie *above* the graph of $\sqrt{x}$.

Definition

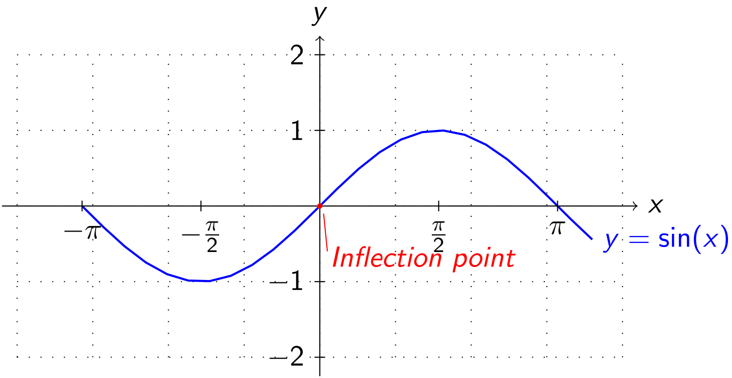
If the graph of f lies above all its tangents on an interval I then it is called *concave upward* on I. If the graph of f lies below all of its tangents on I then it is called *concave downward* on I. A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous there and the curve changes from concave upwards to concave downwards or vice versa.

The Second Derivative and Concavity

Here is an example of some inflection points on a graph:



The Second Derivative and Concavity



The Second Derivative and Concavity

Notice that when a function is concave upward, the slopes of its tangent lines are increasing, and when a function is concave downward, the slopes of its tangent lines are decreasing.

The Second Derivative and Concavity

Notice that when a function is concave upward, the slopes of its tangent lines are increasing, and when a function is concave downward, the slopes of its tangent lines are decreasing.
This suggests the following test, which can be proved with the mean value theorem.

The Second Derivative and Concavity

Notice that when a function is concave upward, the slopes of its tangent lines are increasing, and when a function is concave downward, the slopes of its tangent lines are decreasing.
This suggests the following test, which can be proved with the mean value theorem.

Concavity test. Suppose that f is twice differentiable on an interval I.

The Second Derivative and Concavity

Notice that when a function is concave upward, the slopes of its tangent lines are increasing, and when a function is concave downward, the slopes of its tangent lines are decreasing.
This suggests the following test, which can be proved with the mean value theorem.

Concavity test. Suppose that f is twice differentiable on an interval I.

(i) If $f''(x) > 0$ for all x in I then the graph of f is concave upward on I.

The Second Derivative and Concavity

Notice that when a function is concave upward, the slopes of its tangent lines are increasing, and when a function is concave downward, the slopes of its tangent lines are decreasing.
This suggests the following test, which can be proved with the mean value theorem.

Concavity test. Suppose that f is twice differentiable on an interval I.

(i) If $f''(x) > 0$ for all x in I then the graph of f is concave upward on I.

(ii) If $f''(x) < 0$ for all x in I then the graph of f is concave downward on I.

The Second Derivative and Concavity

Notice that when a function is concave upward, the slopes of its tangent lines are increasing, and when a function is concave downward, the slopes of its tangent lines are decreasing.
This suggests the following test, which can be proved with the mean value theorem.

Concavity test. Suppose that f is twice differentiable on an interval I.

(i) If $f''(x) > 0$ for all x in I then the graph of f is concave upward on I.

(ii) If $f''(x) < 0$ for all x in I then the graph of f is concave downward on I.

(iii) If $c \in I$ and f'' changes sign at c then c is a point of inflection for f.

The Second Derivative and Concavity

The concavity test allows another method for classifying the critical points of a function f.

The Second Derivative and Concavity

The concavity test allows another method for classifying the critical points of a function f.

The second derivative test. Suppose that a function f is twice differentiable at c and f'' is continuous at c.

The Second Derivative and Concavity

The concavity test allows another method for classifying the critical points of a function f.

The second derivative test. Suppose that a function f is twice differentiable at c and f'' is continuous at c.

(i) If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f;

The Second Derivative and Concavity

The concavity test allows another method for classifying the critical points of a function f.

The second derivative test. Suppose that a function f is twice differentiable at c and f'' is continuous at c.

(i) If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f;

(ii) If $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum point of f.

The Second Derivative and Concavity

The concavity test allows another method for classifying the critical points of a function f.

The second derivative test. Suppose that a function f is twice differentiable at c and f'' is continuous at c.

(i) If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f;

(ii) If $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum point of f.

(Unfortunately, notice that this test says nothing about what happens when $f''(c) = 0$ or when $f''(c)$ doesn't exist. In these cases, try using the first derivative test. In fact, the first derivative test is often easier to use.)

The Second Derivative and Concavity

Example. Sketch the graph of f, showing local maximum, local minimum and inflection points, where

$$f(x) = -2x^3 + 9x^2 + 60x - 7 \qquad \forall x \in \mathbb{R}$$

Optimization Problems

We have already seen that an optimization problem may be reduced to finding the absolute maximum or minimum points of an appropriate function f over some interval I. We already have techniques for doing this if f is continuous and I is closed. If either of these conditions fail, then one can try locating maxima and minima by graphing the function.

Optimization Problems

We have already seen that an optimization problem may be reduced to finding the absolute maximum or minimum points of an appropriate function f over some interval I. We already have techniques for doing this if f is continuous and I is closed. If either of these conditions fail, then one can try locating maxima and minima by graphing the function.

Example. A cylindrical can is to hold 1 L of oil. Find the dimensions that will minimize the the cost of the metal to manufacture the can.

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f
2. *Intercepts:*
   (i) the y-intercept is $f(0)$
   (ii) the x-intercepts are found by solving $f(x) = 0$

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
   (i) y-axis symmetry (reflection) if $f(-x) = f(x)$
   (ii) symmetry by rotation of 180^o about the origin if $f(-x) = -f(x)$
   (iii) periodicity if $f(x + T) = f(x)$ for some number T

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
4. *Asymptotes:*
   (i) $y = L$ is an horizontal asymptote if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$
   (ii) $x = a$ is a vertical asymptote if $\lim_{x\to a^\pm} f(x) = \pm\infty$ for one (or more) combination of $\pm$

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
4. *Asymptotes:*
5. *Intervals of increase/decrease:* determined by computing $f'(x)$

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
4. *Asymptotes:*
5. *Intervals of increase/decrease:*
6. *Local maxima/minima:* determined by locating critical numbers and using the first (or second) derivative test

Curve Sketching

Information gleaned from the first and second derivatives of a function can be used to sketch the function's graph. Use the following list as a guide whenever sketching the graph $y = f(x)$ by hand. (Note that, depending on the function, not every item will be relevant or easy).

1. what is the *domain* of f
2. *Intercepts:*
3. *Symmetry:*
4. *Asymptotes:*
5. *Intervals of increase/decrease:*
6. *Local maxima/minima:*
7. *Concavity and points of inflection:* determined by the sign of $f''(x)$