Week 13: Techniques of integration

October 29 – November 2, 2012

1. Techniques of integration
   - Trigonometric integrals
   - The method of partial fractions

Lecture 23

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$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

    and progressively lower the powers until the integral can be evaluated. Also use

$$\sin x \cos x = \tfrac{1}{2} \sin 2x.$$

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Recall from last class

Example. Integrate

$$\int \tan^5 x \, dx.$$

Trigonometric substitutions

We now look at integrals of the form $\int \sqrt{a^2 - x^2}\, dx$, where $a > 0$. A good strategy to evaluate this integral is to change variables from x to θ by using the substitution $x = a \sin\theta$.

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These kind of substitutions are called *inverse* substitutions, and we can make an inverse substitution of the form, for instance $x = a \sin \theta$ as long as it defines a one-to-one function. We do this by restricting θ to lie in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Table of substitutions.

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Now we do lots of examples!

Example. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\, dx$.

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Example. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example. Evaluate $\int \frac{x}{\sqrt{x^2+4}}\,dx$.

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Example. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

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Example. Evaluate $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $a > 0$.

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- f is *improper* if the degree of the denominator q is less than or equal to the degree of the numerator p;
- a quadratic polynomial is *irreducible* if it has no real linear factors. (Equivalently, a quadratic $ax^2 + bx + c$ is irreducible if its discriminant $b^2 - 4ac$ is negative, so it has no real roots.)

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Example. Evaluate $\displaystyle\int \frac{x}{x^2+2x+10}\,dx$.

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2. It can be shown using algebra that every proper rational function f can be written as a unique sum of functions of the form
$$\frac{A}{(x-a)^k} \quad \text{and} \quad \frac{Bx+C}{(x^2+bx+c)^k}, \tag{1}$$
where the quadratic x^2+bx+c is *irreducible*. We call this sum the *partial fractions decomposition* of f.

The overall strategy (contd.).

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$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

Example. Use long division to write the integrand as a polynomial plus a *proper* rational function:

$$\int \frac{x^4 - 5x^3 + 12x^2 - 21x + 35}{x^3 - 3x^2 + 4x - 12}\, dx.$$

Partial fractions decompositions. To find the partial fractions decomposition of a proper rational function $\frac{p}{q}$, we express q as a product of real linear factors and real irreducible quadratic factors. The form of the partial fractions decomposition is determined by this factorization. There are four cases, depending on the type of factorization.

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$$\frac{x-3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$
$$\frac{x^2-x+7}{x(2x+1)(x-3)} = \frac{A}{x} + \frac{B}{2x+1} + \frac{C}{x-3}.$$

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Finding the coefficients: Method 2

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Note carefully how the repeated factors appear on the right-hand side. The constants A, B and C in each case can be determined as in the previous example.

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Example.

Find the partial fractions decomposition of $\dfrac{x^2-3x+8}{x(x-2)^2}$.

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$$\frac{x^3-2x+4}{(x^2+5)(x^2+x+1)} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{x^2+x+1}.$$

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Example.

Find the partial fractions decomposition of $\dfrac{4x^2+2x+1}{(x+1)(x^2+x+1)}$.

Case 4: The denominator has a repeated irreducible quadratic factor.

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$$\frac{x^2+x}{(x^2+9)^3} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2} + \frac{Ex+F}{(x^2+9)^3}$$

$$\frac{x^3-2x+4}{(x-2)(x^2+x+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}.$$

As before, the constants appearing in each example can be determined by algebra.

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Example. Write down the *form* of partial fractions decomposition for the rational function given by

$$\frac{4x^4 - 3x^2 + x - 9}{x^3(x-7)(x^2+3)^2(x^2+x+2)}.$$

(We won't find the constant coefficients.)

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Example. Find

(a) $\displaystyle\int \frac{8x^3 - 12x^2 - 13x - 5}{2x^2 - 3x - 2}\,dx$

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(a) $$\int \frac{8x^3 - 12x^2 - 13x - 5}{2x^2 - 3x - 2}\,dx$$

(b) $$\int \frac{4x^2 - 15x + 29}{(x-5)(x^2 - 4x + 13)}\,dx.$$