Week 3: Limits, continuity, derivatives of functions

July 6 – August 10, 2012

1. Continuity
   - Continuous functions
   - The intermediate value theorem

2. Derivatives
   - Meaning and Computation of derivatives
   - Differentiable functions

Lecture 6

Indeterminate forms.

Limits of the type

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$ are said to be limits of the form $\frac{\infty}{\infty}$.

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- $$\lim_{x\to\infty} \frac{x}{x^2}$$
- $$\lim_{x\to\infty} \frac{x^3}{x}$$
- $$\lim_{x\to\infty} \frac{e^x}{x}$$ (deal with this more later)

Since we cannot determine in advance what kind of limiting behavior something of the form $\frac{\infty}{\infty}$ has, we say that $\frac{\infty}{\infty}$ is an *indeterminate form*. Note that $\infty - \infty$ and $\frac{0}{0}$ are also indeterminate forms.

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Limits of the form $f(x)/g(x)$. To calculate a limit of the form

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where both $f(x)$ and $g(x)$ tend to infinity as $x \to \infty$, one approach is to divide both f and g by the *fastest growing term* appearing in the denominator g.

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Examples.

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Examples.

1. Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x + 1}{7 + 6x^2}$.

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Examples.

1. Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x + 1}{7 + 6x^2}$.
2. Find $\displaystyle\lim_{x\to\infty} \frac{x^2 + 3x}{\sqrt{2x^4 + 3} - 4x}$.

Limits of the form $\sqrt{f(x)} - \sqrt{g(x)}$. The trick is to multiply both numerator and denominator by the 'conjugate squared,' and then expand the numerator as a difference of squares.

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Example. Evaluate $\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$, if it exists.

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   - Continuous functions
   - The intermediate value theorem

2. Derivatives
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Definition

Suppose that f is defined on some open interval containing the point a. If

$$\lim_{x \to a} f(x) = f(a)$$

then we say that f is *continuous* at a; otherwise, we say that f is *discontinuous* at a.

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Definition. Suppose that f is a real-valued function defined on a closed interval $[a, b]$. We say that

(a) f is continuous at the endpoint a if $\lim_{x \to a^+} f(x) = f(a)$,

(b) f is continuous at the endpoint b if $\lim_{x \to b^-} f(x) = f(b)$,

(c) f is continuous on the closed interval $[a, b]$ if f is continuous on the open interval (a, b) and at each of the endpoints a and b.

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True or False?. $f(x) = x^2$ is continuous everywhere.

Discontinuities

A discontinuity can be further classified as a removable, jump or infinite discontinuity.

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(ii) $f(x) = \frac{1}{x}$ has an infinite discontinuity at $x = 0$.

(iii) $f(x) = \begin{cases} x^2 & \text{if } x \leq 3 \\ 1 & \text{if } x > 3, \end{cases}$ has a jump discontinuity at $x = 3$.

Combining continuous functions

Proposition

Suppose that the functions f and g are continuous at a point a. Then $f+g$, $f-g$ and fg are continuous at a. If $g(a) \neq 0$ then f/g is also continuous at a.

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Suppose that f is continuous at a and that g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Corollary

Suppose that f and g are continuous on their domains and that $\lim_{x \to a} f(x)$ belongs to $\mathrm{Dom}(g)$. Then

$$\lim_{x \to a} g(f(x)) = g\big(\lim_{x \to a} f(x)\big).$$

("you can move limits inside continuous functions")

Combining continuous functions

Example. Suppose that a and b are real numbers and consider the function f with domain $\mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 - a^2 & \text{if } x < 0 \\ \cos x + b & \text{if } x \geq 0. \end{cases}$$

For what values of a and b will f be continuous at 0?

The intermediate value theorem (IVT)

Theorem (The intermediate value theorem)

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one real number c in $[a, b]$ such that $f(c) = z$.

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Example. Show that the equation

$$e^x = 3\cos x - 1$$

has at least one positive solution.

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Average rate of change of $f(x)$ between $x = a$ and $x = b$

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\end{aligned}$$

Average rate of change

Let us look at an example:

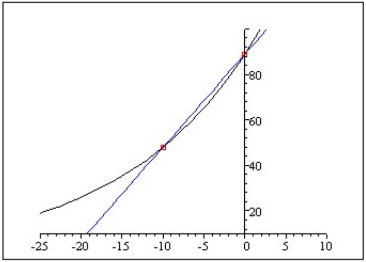

Figure 1: Graph of $f(x)$ between $x = -10$ and $x = 0$.

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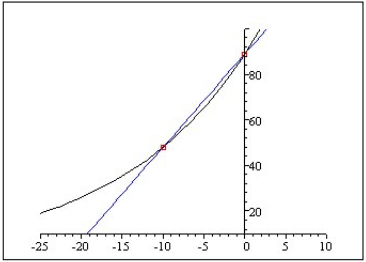

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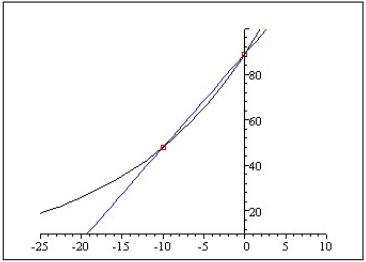

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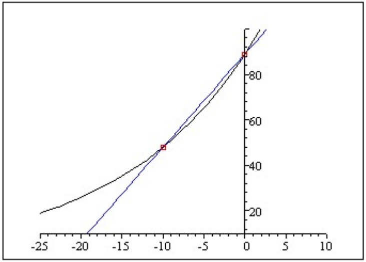
Average rate of change of $f(x)$ between $x = -10$ and $x = 0$

$$= \frac{f(0) - f(-10)}{(0 - (-10))} = \frac{88 - 48}{10} = \frac{40}{10} = 4.$$

Average rate of change



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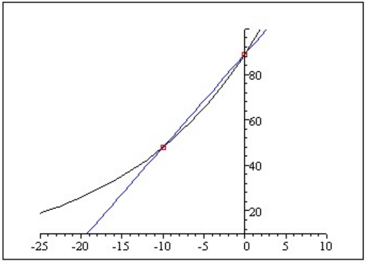

Note that,

Average rate of change of $f(x)$ between $x = -10$ and $x = 0$

$=$ Slope of the secant line joining $(-10, 48)$ and $(0, 88)$.

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Taking $x = a + h$, $h \to 0$ as $x \to a$. So, we can rewrite as

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Remark

Instantaneous rate of change of $f(x)$ at $x = a$ = Slope of the tangent line to $f(x)$ at the point $x = a$.

Slope of the tangent line

To find the slope of the tangent line to $f(x)$ at a point $P = (x_0, f(x_0))$, consider another point $Q = (x_0 + h, f(x_0 + h))$ near P.

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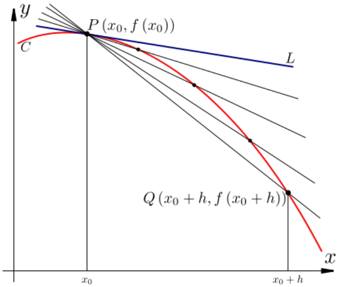

Figure 2: Secant lines between P and Q, as Q approaches P

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Solution:

$$\begin{aligned} \text{Average rate of change} &= \frac{f(x) - f(2)}{x-2} \\ &= \frac{4x - 3x^2 - (4\cdot 2 - 3\cdot 2^2)}{x-2} \end{aligned}$$

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\end{aligned}$$

$$\text{Instantaneous rate of change of } f(x) \text{ at } 2 = \lim_{x \to 2} -3x - 2 = -8.$$

Definition of derivative

Definition

The derivative of a function $f(x)$, denoted $f'(x)$, gives at each point a, the value of the instantaneous rate of change of $f(x)$ at $x = a$. That is,

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Mathematically,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Remark

$f'(a) =$ Slope of the tangent line to $f(x)$ at the point $x = a$.

Important example

Suppose an object moves along a straight line according to the equation of motion $y = f(t)$, where y is the displacement of the object from the origin at time t, then we call f the *position function* of the object. The *velocity* of the object at $t = a$ is defined to be the instantaneous rate of change of the position function when $t = a$. The *speed* of the object at $t = a$ is defined to be the absolute value of the velocity when $t = a$.
Suppose an arrow is shot upwards on the moon with a velocity of 58 meters per second, then its height in meters after t seconds is given by

$$H = 58t - 0.83t^2.$$

Find the velocity and speed of the arrow after 1 second.

Example

Let's compute the derivative f' of the function $f(x) = \sqrt{x}$ and find the domain.

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Solution: Apply the definition. The slope of the tangent line is given by

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h}$$

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\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\
&= \lim_{h \to 0} \frac{2h + h^2}{h} \\
&= \lim_{h \to 0} (2 + h) = 2
\end{aligned}
$$

Equation of the tangent line

Example

Find the equation of the tangent line to the graph $f(x) = x^2$ at $x = 1$.

Solution: Apply the definition. The slope of the tangent line is given by

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The tangent line to the graph $f(x) = x^2$ at $x = 1$ is a line through $(1, f(1)) = (1, 1)$ with the slope 2. By the point-slope form, we get the equation of the tangent line,

$$y = 2(x-1)+1 \Longrightarrow y = 2x-1.$$

Definition

Suppose that f is defined on some open interval containing the point x. We say that f is *differentiable* at x if $f'(x)$ exists.

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Remark on notation. Other notation for $f'(x)$ includes $\frac{d}{dx}f(x)$ and $\frac{df}{dx}(x)$. If $y = f(x)$ then the derivative is often denoted by y' or $\frac{dy}{dx}$. The ratio

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Example.

If $f(x) = |x|$, does $f'(0)$ exist?

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If f is differentiable at a then it is continuous at a.

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A function could fail to be differentiable for several reasons.

- The graph has a jump or a vertical tangent line, equivalently, the function is discontinuous.
- The graph has a sharp corner: the secant lines from the left and the right have different limits.

Higher order derivatives

If f is a differentiable function with derivative f', sometimes f' may also be differentiable. If this is the case, then the derivative of f' is denoted f'' and called the *second derivative of* f.

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If f'' is differentiable then its derivative f''' is called the *third derivative of* f. The prime notation f''' is only practical until the third derivative. For higher order derivatives we introduce new notation.

Higher order derivatives

Notation. Suppose that f can be differentiated n times. Then its nth derivative is denoted by $f^{(n)}$. If $y = f(x)$ then we also write

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Higher order derivatives have various interpretations depending on the context. A geometric interpretation of the second derivative will be given in the next chapter. One interpretation that features in mechanics is that of *acceleration*. As before, suppose that the position of a object that moves in a straight line is given by $y(t)$ at time t. Then $v(t) = y'(t)$ is the velocity of the particle and $a(t) = v'(t)$ is its acceleration at time t. Note that $a(t) = y''(t)$, so acceleration is the second derivative of position.

Rules for differentiation

So far we have calculated derivatives working directly from the definition, that is, by taking the limit as $h \to 0$ of the difference quotient. However, in practice this is often both difficult and tedious. Luckily, there exist some nice rules that make it quite easy to compute the derivatives of most functions we are familiar with.

Some basic derivatives.

$f(x)$	$f'(x)$
C, where C is a constant	0
x^n, where n is a real number	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$1/x$