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Theodore Jorgensen Jr.

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On the Dynamics of the Swing of a Golf Club

THEODORE JORGENSEN, JR.
University of Nebraska, Lincoln, Nebraska 68508
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The Lagrangian method is used to obtain two coupled differential equations describing the motion of a simple model of the swing of a golf club. These equations are simplified by a special treatment of the gravitational torques and are put in a form such that different constant torques applied by the golfer give solutions differing only in a scale factor. The equations are solved numerically for various suitable boundary conditions. It is shown that the clubhead speed achieved for a swing with a constant torque applied by the golfer increases with the hindrance to the uncocking of the wrists during the swing, and that the backswing of the club may be decreased substantially with a correspondingly small decrease in clubhead speed.

INTRODUCTION

The game of golf usually is played in beautiful parklike surroundings with lots of woodlands scattered here and there and with an occasional stream or pond to liven the view. The golfer finds, however, that the landscaping has been so placed as to penalize the unwary and reward those who can hit the ball far and straight. This study is made in the belief that an understanding of the dynamics of the swing of the club should help the golfer to increase both the distance and the precision he achieves in his shots.

There is an extensive literature on golf, most of it written by golfers who have attained the skill of virtuosi. These works are mainly books of instruction that leave the reader with little understanding of what actually happens in a golf stroke. Photographic studies shed some light on the problem by showing the common characteristics of the swings of experts. However, there seems to be no agreement on exactly what constitutes the optimum swing of a golf club.

THEORY

No attempt is made to look at the dynamics of the swing of a club by a golfer in all its complexity. The model chosen for study is shown in Fig. 1. The arms of the golfer are represented by a rod AB, which is constrained to rotate about a fixed horizontal axis at A. The golf club is represented by a rod BC, which is connected to rod AB at B, with a joint representing the wrists of the golfer. Both rods have distributed mass, indicated by M_i at a distance l_i from A for the arm and M_j at a distance l_j from B for the club. The position of the rod AB is determined by the angle θ measured counterclockwise from the horizontal, and the position of the rod BC is determined by the angle ψ, also measured counterclockwise from the horizontal. The choice of these coordinates insures that the following work is done in an inertial system. The two rods are called the golfer's *arms*, and the *club* and the hinge are called the *wrists*. It must be kept in mind, however, that our discussion and equations refer to the model just described.

The differential equations of motion are most easily obtained by the Lagrangian method. The kinetic energy of the system is a function of θ and ψ and their time derivatives $\dot{\theta}$ and $\dot{\psi}$. The potential energy is a function of θ and ψ. Elementary considerations show that the Lagrangian can be put in the form

$$L=\tfrac{1}{2}[(J+MR^2)\dot{\theta}^2+I\dot{\psi}^2]+\dot{\theta}\dot{\psi}RS\cos(\psi-\theta)-g[(G+MR)\sin\theta+S\sin\psi], \quad (1)$$

where $J=\sum M_il_i^2$ and $G=\sum M_il_i$ are the effective moment of inertia and first moment of the arms taken about an axis through A, $I=\sum M_jl_j^2$ and $S=\sum M_jl_j$ are the moment of inertia and first moment of the club taken about the axis at the golfer's wrists, M is the mass of the club, R is the distance from the axis at A to the wrists at B representing the effective length of the golfer's arms, and g is the acceleration of gravity.

When Lagrange's equations are used with this Lagrangian function, the following two differential equations are obtained:

$$\ddot{\theta}(J+MR^2)+\ddot{\psi}RS\cos(\psi-\theta)-\dot{\psi}^2RS\sin(\psi-\theta)+g(G+MR)\cos\theta=T_a, \quad (2)$$

$$\ddot{\psi}I+\ddot{\theta}RS\cos(\psi-\theta)+\dot{\theta}^2RS\sin(\psi-\theta)+gS\cos\psi=T_c. \quad (3)$$

Here T_a and T_c are the torques applied to the arms and club, respectively.

The torque T_a on the golfer's arms is the sum of the torque T_s applied by the golfer's body to the system consisting of the arms and the club and the reaction $-T_c$ to the torque exerted on the club by the wrists of the golfer. Thus T_s is obtained by adding Eqs. (2) and (3):

$$T_s=\ddot{\theta}[J+MR^2+RS\cos(\psi-\theta)] +\ddot{\psi}[I+RS\cos(\psi-\theta)]-(\dot{\psi}^2-\dot{\theta}^2)RS\sin(\psi-\theta) +g[(G+MR)\cos\theta+S\cos\psi]. \quad (4)$$

Equations (3) and (4) tell what happens in a particular golf swing only when the torques T_s and T_c and the boundary conditions are specified. The various constants concerning the arms and club are assumed to be known.

Although the golfer swings the club in the earth's gravitational field, the effect of this field on the motion of a vigorously swung club was expected to be negligible. This point was investigated starting with the study of stroboscopic photographs of golf swings, and for a full swing of a driver the average gravitational torque was estimated to be about 6% of the torque supplied by the golfer, and the average deviation from this value estimated to be about 2%. Thus the gravitational torque can be included in T_s, increasing its value by 6%, with only a 2% average deviation throughout the swing. The gravitational torque on the club, estimated to be fractionally somewhat smaller, can be treated in the same way. When the gravitation torques are

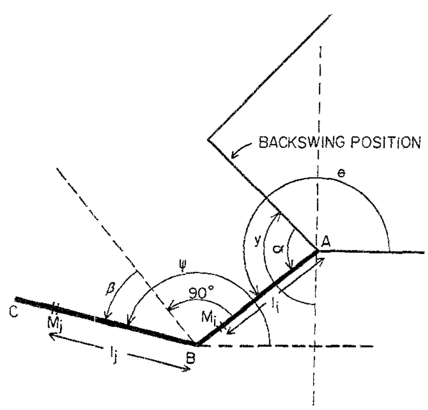

FIG. 1. Diagram showing a model of golfer's arms and golf club. The extreme position at top of backswing and an arbitrary position of the arms and club are indicated.

included in T_s and T_c, Eqs. (4) and (3) become

$$T_s=\ddot{\theta}[J+MR^2+RS\cos(\psi-\theta)] +\ddot{\psi}[I+RS\cos(\psi-\theta)]-(\dot{\psi}^2-\dot{\theta}^2)RS\sin(\psi-\theta), \quad (5)$$

and

$$T_c=\ddot{\psi}I+\ddot{\theta}RS\cos(\psi-\theta)+\dot{\theta}^2RS\sin(\psi-\theta), \quad (6)$$

and these equations describe very closely the motions involved in the swing of a golf club.

The preceding treatment of the gravitational torque produces a great simplification in the calculations without introducing significant error. First, it should be noticed that all reference to a vertical direction has been eliminated from the equations and therefore the boundary condition on the initial value of θ is not needed. Swings of the club for all possible backswings come from one and the same calculation. Second, it should be noticed that all terms on the right of each equation are of the second order in time. If both equations are divided by T_s and $(1/T_s)(d^2\theta/d^2t)$ is replaced by $d^2\theta/dz^2$, where $z=(\sqrt{T_s})t$ is a new parameter, called zeit, Eqs. (5) and (6) become

$$1=[J+MR^2+RS\cos(\psi-\theta)]\frac{d^2\theta}{dz^2} +[I+RS\cos(\psi-\theta)]\frac{d^2\psi}{dz^2} -\left[\left(\frac{d\psi}{dz}\right)^2-\left(\frac{d\theta}{dz}\right)^2\right]RS\sin(\psi-\theta) \quad (7)$$

and

$$\frac{T_c}{T_s}=I\frac{d^2\psi}{dz^2}+\frac{d^2\theta}{dz^2}RS\cos(\psi-\theta) +\left(\frac{d\theta}{dz}\right)^2RS\sin(\psi-\theta). \quad (8)$$

Thus, by the use of the parameter z, Eq. (7) is put into a form that is independent of the torque T_s on the system, and Eq. (8) has but one term, T_c/T_s, depending on the torques applied.

Equations (7) and (8) can be put into a still more useful form by using angles α and β of Fig. 1. Angle α measures the angle through which the arms have turned from the position at the top of the backswing, and angle β measures the angle by

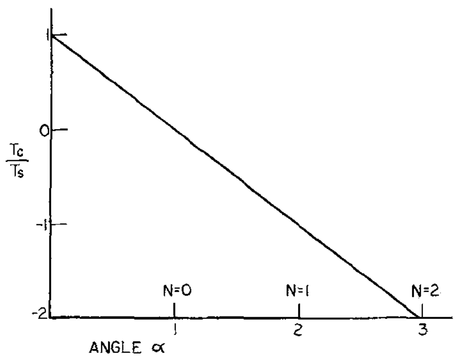

FIG. 2. The ratio Tc/Ts in units of $I/(J+I+MR^2)$, as a function of angle α, in units of $I/2RS$ and as a function of the hindrance parameter N.

which the wrists have uncocked from their position at 90° to the arms. Equation (7) becomes

$$1=(J+I+MR^2+2RS\sin\beta)\ddot{\alpha}+(I+RS\sin\beta)\ddot{\beta} +[(\dot{\beta}+\dot{\alpha})^2-\dot{\alpha}^2]RS\cos\beta, \quad (9)$$

and Eq. (8) becomes

$$T_c/T_s=I\ddot{\beta}+(I+RS\sin\beta)\ddot{\alpha}-\dot{\alpha}^2RS\cos\beta, \quad (10)$$

where here a dotted letter refers to differentiation with respect to zeit.

It is now necessary to make some assumptions concerning the manner in which the golfer swings his arms and the club. It is assumed that the torque T_s is constant for the swings under consideration. It is probably a practical impossibility for a golfer to swing a club with T_s constant, but this assumption is made as a first approximation to what actually happens. The value of T_c as it varies throughout the swing depends on the skill of the golfer.

Even when the torques are known, the differential equations cannot be solved until boundary conditions are specified. Our interest in the golf swing starts at the top of the backswing with $\alpha=\dot{\alpha}=0$. To a good approximation, a golfer starts his swing from this position having cocked his wrists so that the angle between the club and his arms is 90°, that is, $\beta=0$. Under these conditions, Eq. (10) becomes

$$T_c=T_sI(\ddot{\beta}+\ddot{\alpha}).$$

Should the golfer swing the club with perfectly flexible wrists, $T_c=0$, $\ddot{\beta}$ equals $-\ddot{\alpha}$, a negative quantity, and the clubhead starts its motion by swinging in toward the neck of the golfer. Photographs show rather that the competent golfer swings the club keeping his wrist cocked for part of the swing, maintaining $\beta=0$, $\dot{\beta}=0$, and $\ddot{\beta}=0$. Equation (10) becomes, for this condition,

$$T_c=T_s(I\ddot{\alpha}-\dot{\alpha}^2RS), \quad (11)$$

with $\ddot{\alpha}$ remaining constant and positive for constant T_s, and $\dot{\alpha}$ gradually increasing. The torque applied to the club by the wrists to maintain the condition $\beta=0$, $\dot{\beta}=0$, and $\ddot{\beta}=0$, starts then with a positive value, decreases to zero, and later becomes ever increasingly negative. For these conditions, Eq. (9) reduces to

$$\ddot{\alpha}=1/(J+I+MR^2),$$

and since $\ddot{\alpha}$ is constant, kinematics gives the relation $\dot{\alpha}^2=2\alpha\ddot{\alpha}$. Equation (11) then gives

$$T_c/T_s=(I-2\alpha RS)/(J+I+MR^2),$$

and T_c is seen to decrease linearly with angle α, starting with the positive value

$$T_c=T_sI/(J+I+MR^2)$$

and becoming zero for $\alpha=I/2RS$. The curve of T_c/T_s is shown, plotted as a function of angle α, in Fig. 2.

If the golfer makes a complete swing, keeping his wrists cocked, he will obviously not hit the ball. Somewhere in his swing he must release the torque he applies to the club by his wrists, which has been hindering the uncocking of his wrists. For purposes of calculation, it is assumed that the torque

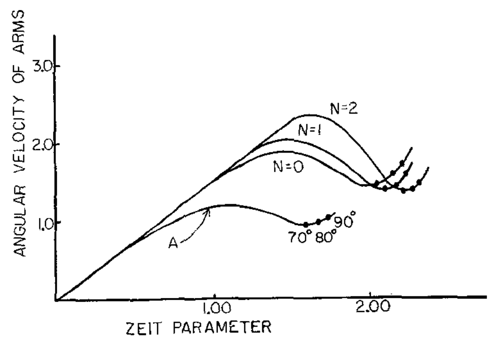

FIG. 3. The angular velocity of the golfer's arms as a function of the zeit parameter or as a function of time for various values of the hindrance parameter N. The scales are arbitrary. The curve A is for the case in which the golfer is hitting from the top. The dots indicate the points in the swing where the angle β has values of 70°, 80°, and 90°.

T_c drops quickly to zero and stays at that value after the arms have turned through some angle $\alpha=(N+1)I/2RS)$. The quantity N is called the "hindrance parameter" for the swing. If $N=0$, the uncocking of the wrists has not been hindered at all, since T_c will remain zero after it reaches that value at $\alpha=I/2RS$ (see Fig. 2). If $N=1$, the arms have been swung through an additional angle $I/2RS$, and the uncocking of the wrists has been hindered by a linearly increasing negative torque through this additional angle. The hindrance parameter is thus a measure of the delay in the uncocking of the wrists. The scale of hindrance parameters is shown in Fig. 2.

When T_c becomes zero by making the wrists perfectly flexible, Eqs. (9) and (10) take the forms

$$\ddot{\alpha}=\frac{1-\ddot{\beta}RS\sin\beta-(\dot{\alpha}+\dot{\beta})^2RS\cos\beta}{J+MR^2+RS\sin\beta}, \quad (12)$$

$$I\ddot{\beta}=\dot{\alpha}^2RS\cos\beta-\ddot{\alpha}RS\sin\beta-\ddot{\alpha}I. \quad (13)$$

These coupled equations describe the motion of the system when T_s is constant and T_c is zero. The first term on the right in Eq. (13) is the torque on the club resulting from the centrifugal force in the coordinate system rotating with angular velocity $\dot{\alpha}$. The second term on the right is a torque on the club resulting from a pseudoforce which arises from the angular acceleration of the rotating coordinate system. The third term on the right is there because $\ddot{\beta}$ is measured in a non-inertial coordinate system.

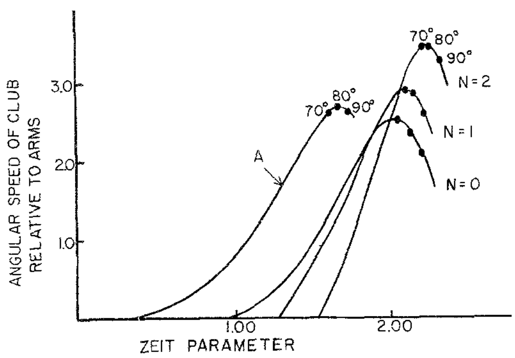

FIG. 4. The angular speed of the golf club relative to the arms as a function of the zeit parameter or as a function of time for various values of the hindrance parameter N. The scales are arbitrary. Curve A is for the case in which the golfer is hitting from the top. The dots indicate the points in the swing where the angle β has values of 70°, 80°, and 90°.

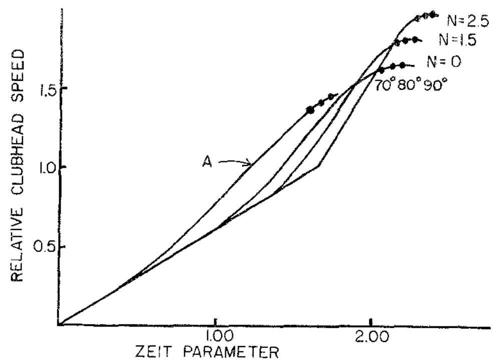

FIG. 5. Relative clubhead speed as a function of the zeit parameter or as a function of time for various values of the hindrance parameter N. The scales are arbitrary. Curve A is for the case in which the golfer is hitting from the top. The dots indicate the points in the swing where the angle β has values of 70°, 80°, and 90°.

The first term on the right in Eq. (13) is the largest of the three. This centrifugal torque makes the perfectly flexible wrists uncock, and the club swings out rapidly to hit the ball. There does not need to be any positive torque applied by the wrists to help in this motion.

Since the motion of the system up to the point where the wrists are allowed to uncock is rotational motion with constant angular acceleration, the calculation of the first part of the motion is simple and gives α and $\dot{\alpha}$ as functions of zeit:

$$\alpha(z)=\tfrac{1}{2}\ddot{\alpha}z^2, \quad (14)$$

$$\dot{\alpha}(z)=\ddot{\alpha}z. \quad (15)$$

The values of α, $\dot{\alpha}$, $\ddot{\alpha}$, β, $\dot{\beta}$, and $\ddot{\beta}$ after the point where T_c is made zero and the wrists are allowed to uncock can only be determined by solving Eqs. (12) and (13) with boundary conditions that describe what is happening at that point.

When the wrists are allowed to start uncocking, $\beta=0$, $\dot{\beta}=0$, but $\ddot{\beta}$ is determined by Eq. (13), and $\dot{\alpha}$ and $\ddot{\alpha}$ are determined by Eqs. (12), (14), and (15), for a given hindrance parameter N.

Straightforward algebraic manipulations give the following boundary conditions for this second part of the calculations.

They are

$$\ddot{\alpha}=(J+MR^2-NI)/[(J+MR^2)(J+I+MR^2)],$$

$$\dot{\alpha}^2=[(N+1)I/RS(J+I+MR^2)],$$

$$\ddot{\beta}=N/(J+MR^2),$$

$$z^2=[(N+1)I/RS](J+I+MR^2).$$

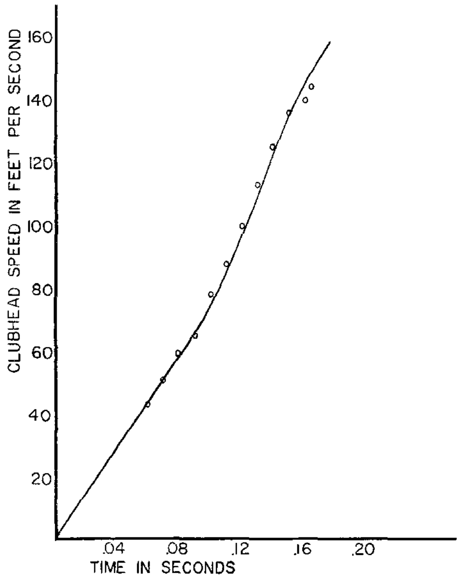

FIG. 6. Clubhead speed as a function of time for a particular swing of a No. 2 iron. The circles indicate values obtained from a stroboscopic photograph and the solid line shows the result of a solution of the differential equations for suitably chosen parameters ($R=2.0$ ft, $J=0.25$ slugs ft^2, $N=0.25$, and $T_s=103$ lb ft).

RESULTS AND DISCUSSION

Equations (12) and (13) were solved numerically for values of α, $\dot{\alpha}$, $\ddot{\alpha}$, β, $\dot{\beta}$, $\ddot{\beta}$, and V, the clubhead velocity, for equally spaced zeit intervals. These calculations were made for various golf clubs, for various choices of R and J, for various values of backswing angle γ, and for various values of the hindrance parameter. Similar calculations were made for various cases of "hitting from the top" involving early and continuous application of a positive torque by the wrists.

Samples of the results of the calculations are shown in Figs. 3–5. These samples are for a light driver swung by a golfer with fairly long arms, but results for other clubs and for a golfer with shorter arms look very much the same. The scales on these graphs are arbitrary, since both the time taken to complete the swing and the various velocities attained depend on the constant torque T_s applied by the golfer to his arms and the club. These curves then represent possible swings with many different torques applied by the golfer. The curves are plotted for various values of the hindrance parameter and for one example of hitting from the top.

The quantity $\dot{\alpha}$, the angular velocity of the golfer's arms, plotted as a function of time, is shown in Fig. 3. For the cases for a properly swung club, the values of the angular velocity $\dot{\alpha}$ increase linearly with time, up to the time the wrists are relaxed. Then the angular velocity continues to increase at a slower rate, decreases for a while, and then increases again about the time the ball is hit. This slowing up of the rate of turning of the arms and the consequent decrease in the speed of the hands of the golfer are characteristics of a proper golf swing, as has been shown by many stroboscopic photographs of experts in action. For higher hindrance parameters, the angular velocity of the arms is equal to or greater

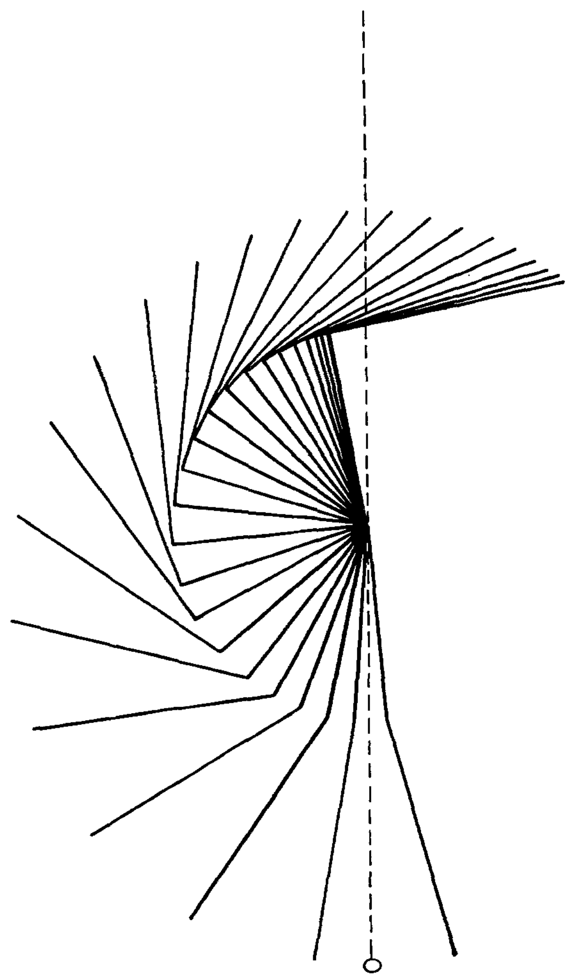

FIG. 7. A calculated stroboscopic diagram for a driver swung with a hindrance parameter $N=1$ hitting the ball when β is 90°. The backswing angle is 169°. The hands, the arms, and the club are along a straight line on impact.

than that for lower hindrance parameters over most of the swing of the club, but the fractional decrease in the speed of the hands is greater for larger hindrance parameters.

Curves showing $\dot{\beta}$, the rate of uncocking of the wrists, are given in Fig. 4. For hindrance parameter zero, $\dot{\beta}$ increases gradually, starting when the wrists are relaxed, and for larger hindrance parameters, $\dot{\beta}$ increases abruptly, showing that the club flips out under the larger centrifugal torque. The general shape of the curves for $\dot{\alpha}$ and $\dot{\beta}$ can be understood on careful reading of Eqs. (12) and (13).

The curves of most concern to the golfer are shown in Fig. 5. Here the clubhead speed is plotted as a function of time. When the wrists are relaxed, the speed increases more rapidly for a time and then approaches a maximum. The value of the clubhead speed at the maximum increases with the hindrance parameter. The curve describing the swing of a golfer hitting from the top falls considerably lower than those for swings with delayed uncocking of the wrists.

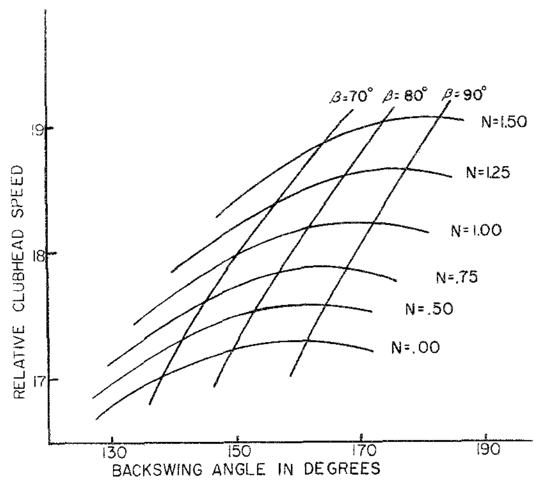

FIG. 9. Relative clubhead speeds for swings with a driver as a function of backswing angles γ for various hindrance parameters N. Curves are also shown for various values of β on impact. Example of use of these curves: For a swing with a hindrance parameter $N=1$ and the ball being hit with an angle $\beta=80°$ on impact, the backswing angle would be 161° and the relative clubhead speed would be 18.2 (on the arbitrary relative clubhead speed scale).

It should be emphasized that the different clubhead speeds achieved in these various swings result only from different wrist actions on the part of the golfer. The wrist action needed to produce a large clubhead speed is seen to be a hindrance to the uncocking of the wrists, rather than the application of a positive torque by the wrists, as advocated by some. The skill in accomplishing this hindrance to the uncocking process is probably part of the mysterious and elusive "timing" so diligently sought by many golfers.

It should be remembered that these calculations describe what happens in the simplified model of the golf swing. The question remains as to whether they also describe what a golfer does in a golf swing. In order to compare our theoretical results with experience, data of the velocity of the clubhead as a function of time are needed. Such data can be obtained from stroboscopic photographs, where the repetition rate of the light source is given, if the photographs are taken from a point

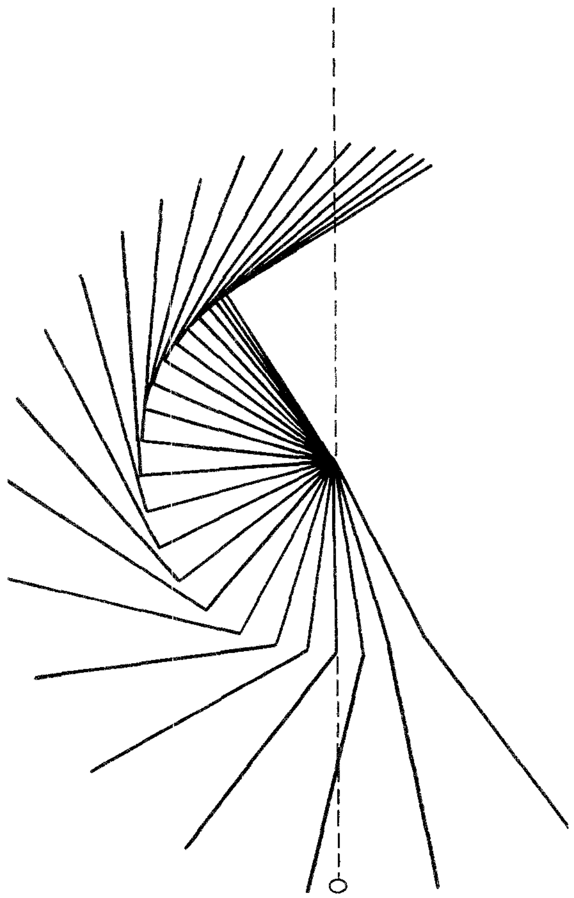

FIG. 8. The same calculated stroboscopic diagram as in Fig. 7 except that the backswing has been reduced to 147° and the ball is being hit when β is 70°, with the position of the hands somewhat ahead of the ball on impact.

on the axis of the swing. No photographs of this kind are known to exist. In *Flash*,[1] there is a photograph of the swing of Bobby Jones using a No. 2 iron, with clubhead-velocity data obtained from the photograph. These data have been corrected on the assumption that the axis of the swing was tipped 23° from the horizontal. These data are plotted in Fig. 6, using circles. The computer was asked to supply No. 2 iron calculations for various values of R, J, T_s, and hindrance parameter, and the graph of the experimental data was compared with graphs of these various calculations. The graphs were superimposed by translation along the time axis, and the best fit found is shown in Fig. 6. The graph of the experimental data shows an abrupt change in slope, as do all the calculations of the clubhead velocity for hindrance parameters of zero and greater. The agreement between theory and experiment is reassuring.

The agreement between clubhead speed for an actual swinging of a club and a calculation using the differential equations depends on the choice of various parameters. There is no way of independently determining these parameters, especially the effective moment of inertia of the golfer's arms. However, the fact that good agreement can be obtained with parameters that lie in the range of values expected from reasonable assumptions leads to considerable confidence in the results that are of interest to the golfer.

There are two general conclusions that point to what the golfer may do to increase the distance he may hit the ball with precision. The first is concerned only with the speed of the clubhead. First, it is obvious that the clubhead speed depends on the torque the golfer is able to apply to the system consisting of his arms and the club. This torque depends on his strength and on how he applies this strength. Second, however, for a given torque, assumed constant, the clubhead speed depends in a large measure on the skill with which the golfer manages the torque on the club by his wrists. When he applies this torque in such a way as to cause an early increase in angle β, called hitting from the top, he is not able to produce as large a clubhead speed as when he hinders the uncocking of the wrists by applying a negative torque throughout part of the swing. In fact, the greater the hindrance parameter, the greater is the possible clubhead speed that can be produced for a given torque on the arms and club, as may be seen from Fig. 5. These curves are drawn for hindrance parameters as large as 2.5, although it is unlikely that anyone can manage to hit, using such a large parameter.

The second general conclusion is concerned with the small change in clubhead speed resulting from a sizable decrease in backswing angle. A hint of this effect may be seen in the curves of Fig. 5. In the vicinity of the maxima of these curves are three points marked with circles. These are, from left to right, points for which angle β has values 70°, 80°, and 90°, respectively. In Fig. 7 is shown a simulation of a stroboscopic photograph of the swing shown in Fig. 5 for the hindrance parameter 1.0. In it the ball is being hit when $\beta = 90°$. In Fig. 8 is shown the same swing, but the ball is being hit with $\beta = 70°$ and the hands on impact are leading the clubhead by about 5 in. In each of these figures the angle α is the same when β becomes 90°, but the backswing angle from the vertical, angle γ in Fig. 1, is successively 169° and 147°. For this particular swing the backswing angle can be decreased from 169° to 147° with only a 1.4% decrease in clubhead speed. Many photographs of experts show swings in which the hands on impact are leading the clubhead by greater amounts than those mentioned here. We conclude, on the basis of our calculations, that they are trading decrease in backswing for increase in precision, with negligible decrease in clubhead speed.[2]

These results are presented in another way in Fig. 9. In this figure, clubhead speed is plotted against the backswing angle γ for various hindrance parameters shown for a light driver with an arm length of 2.25 ft and an effective moment of inertia of the arms of 0.2500 slugs ft^2. Curves of angles β of 70°, 80°, and 90° are also shown. These curves show that for a hindrance parameter of one, the backswing can be decreased from 172° to 150° with only a 1.4% decrease in clubhead speed.

So far the discussion has been mainly concerned with one club, the driver. Calculations were made for other clubs, and the results in general looked very much the same as for the driver. Two differences of interest were noted: both the backswing angles γ and the clubhead speeds V, for a constant torque T_s and a particular angle β on impact, decrease in going from the driver to the nine iron. The relative magnitude of these changes can be shown by one example. For swings with hindrance parameter of one and for cocking angle at impact $\beta=70°$, the driver, the two iron, the five iron, and the nine iron require backswing angles of 147°, 134°, 131°, and 126°, respectively. The corresponding clubhead speeds are in the same order: 173, 149, 143, and 135 on the arbitrary scale mentioned earlier.

[1] H. E. Edgerton and J. R. Killian, Jr., *Flash* (Charles T. Branford Co., Boston, 1954).

[2] A. Palmer, *My Swing and Yours* (Simon & Schuster, Inc., New York, 1965), Chap. 4.

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The Classical Atom Revisited

J. H. TUTSCH
Department of Mathematics, University of Wisconsin, Rock County Center, Janesville, Wisconsin 53545
(Received 21 August 1969; revision received 26 November 1969)

The existence of nonplanetary periodic orbits in the classical model of the n-electron atom is demonstrated.

INTRODUCTION

The classical helium atom is considered where the two electrons are initially at rest and the same distance r from the nucleus. With these very special initial conditions the three-body problem becomes a one-body problem. It is shown in this paper that for each negative value of the potential energy it is possible to find at least one initial position for the electrons, with the nucleus at the origin, such that when released they execute periodic motion, provided only the Coulomb interaction is considered. Furthermore, it is shown that the corresponding result is true for any atom when considered classically. Finally angular momentum, magnetic forces, and radiation effects are dealt with.

Proof of the Existence of Periodic Nonplanetary Orbits for the n-Electron Classical Atom.

Consider an n-electron atom $(2\leq n\leq 104)$ where the location of the kth electron in spherical polar coordinates is

$$(r,\theta,2\pi(k-1)/n),\qquad k=1,2,\cdots,n. \tag{1}$$

Call this initial configuration mode one. If n is even and greater than or equal to four, two other modes are possible. They are given by

kth electron at

$$\begin{aligned}&(r,\theta,2\pi(k-1)/n), && k=1,3,\cdots,n-1\\ &(r,\pi-\theta,2\pi(k-1)/n), && k=2,4,\cdots,n\end{aligned} \tag{2}$$

or

kth electron at

$$\begin{aligned}&(r,\theta,2\pi(k-1)/n), && k=1,3,\cdots,n-1\\ &(r,\pi-\theta,2\pi(k-2)/n), && k=2,4,\cdots,n.\end{aligned} \tag{3}$$

These three modes for $n=4$ are pictured in Figs. 1, 2, and 3, respectively. Notice that in mode two and three the nucleus remains at rest while it is only approximately at rest in mode one. There are probably no other one body modes. The total potential energy of the configuration in mode one is

$$V_n=-A(n^2/r)(1-K_n/\sin\theta), \tag{4}$$

with

$$\begin{aligned}K_n&=(2\pi)^{-1}\Big[\sum_{j=1}^{n/2}(\sin j\pi/n)^{-1}(\pi/n)\Big]-(4n)^{-1} && n\text{ even}\\ &=(2\pi)^{-1}\Big[\sum_{j=1}^{(n-1)/2}(\sin j\pi/n)^{-1}(\pi/n)\Big] && n\text{ odd}\end{aligned} \tag{5}$$

and $A=e^2/4\pi\epsilon_0$. K_n is a monotonic increasing unbounded function of n with $K_2=0.125$ and $K_{180}<0.903$. Hence for each $n\leq104$ it is possible to find a region in the $r-\theta$ plane where $V_n<0$. This property of V_n fails for some large n. For the other two modes the form and functional dependence of the total potential energy is the same