Maximizing Distance of the Golf Drive: An Optimal Control Study

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Optimal control theory is used to search for the optimal control torques necessary to maximize distance of the golf drive. In the method, a mathematical model of a generalized golf swing is first developed. Film of the author's swing serves to verify the model and to supply parameter values, constraints, and actual torques. The variational formulation of optimal control theory is utilized to establish necessary conditions for optimal control, in which constraint violations are discouraged by inclusion of penalty functions. Finally, the method of steepest ascent is used to compute optimal control torques. Also, comparison of optimal and actual torques is made, and the sensitivity of the results to small changes in model parameter values is investigated.

Introduction

The world of golf is replete with advice. A golfer wishing to improve his game may find himself overwhelmed by books, magazines, and other golfers. Indeed, he may become confused by contradictory advice, thus confounding his search for dependable counsel. It was such a state of uncertainty in the author's mind which led, in part, to [1][1] and to this paper.

In golf, every shot demands the satisfaction of two requirements: (1) distance, and (2) direction. In the case of the drive from the tee with the driver club, it is often desirable to achieve maximum distance. It is this golf shot, that of the drive of maximum distance, which is the subject of this study.

In order to hit a drive of maximum distance, the clubhead must be traveling at maximum speed at impact of club and ball.[2] The proper technique for maximizing clubhead speed is often the source for differences in the various theories of the golf swing. However, most theories do agree that the expenditure of energy should be delayed as long as possible during the downswing.

[1]Numbers in brackets designate References at end of paper.

[2]Actually, many variables affect the distance of a shot. Such variables are of two types: (1) post-impact variables, such as forces on the ball during flight, and (2) pre-impact variables such as clubhead speed, clubhead mass, and mass of ball. The golfer has no control over post-impact variables and, with a given club and ball, can affect only one pre-impact variable clubhead speed.

Contributed by the Automatic Control Division and presented at the Winter Annual Meeting, Houston, Texas, November 30–December 5, 1975, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received at ASME Headquarters, July 31, 1975. Paper No. 75-WA/Aut-12.

Objectives

Theories of the golf swing evolve by trial and error. Golfers experiment with their swings and adopt techniques which improve the shot. Some concern may be given to the dynamics of the swing in justification of a theory, but the primary criterion for justification is the result—the shot itself.

The primary objective of this study is to use the dynamics of the golf swing and optimal control theory to search for the optimal control necessary to maximize clubhead speed at impact, and therefore distance of the drive. The secondary objective is to identify the sensitivity of distance of the drive to small changes in parameters such as mass and length of club.

Description of Problem

The variational formulation of optimal control theory requires a mathematical model of the system and a performance index, which is the criterion for optimization. Once these are provided, necessary conditions for optimal control can be established. Then, with a suitable solution technique, the optimal control can be computed.

Mathematical Model

A model of the golf swing would seemingly be complicated, as swinging requires the coordinated effort of several body parts. However, as discussed in [2], analysis of high-speed films of professional golfers has revealed a certain uniformity and simplicity in their swings. A pause usually occurs at the top of the swing, between backswing and downswing, indicating that the angular velocities of club and body are zero at that time.[3] The backswing can therefore be ignored in modeling the golf swing. During the downswing, the hands move in a circular arc about a "fixed point" between the shoulders. This fact suggests use of a physical pendulum to represent motion of hands, arms, and shoulder mass about the "fixed point." Rotation of arms within shoulder joints and wrist action must also be considered, but the result of such motions is the creation of an effective "hinge point," about which the club rotates. (In [4] discussion is given concerning the two degrees of freedom of rotation in the left wrist). Thus, the club may be represented by a second physical pendulum, rotating about the "hinge point." Further, the planes of rotation of arms and club are nearly coincident, so that the downswing may be modeled as an inclined, double physical pendulum, as shown in Fig. 1. Film of the author swinging at and striking a golf ball with a driver club has assured the validity of the model.

When swinging a club, many different muscles are used, of course, but the results of their action are torques, which are responsible for rotation of body and club. These torques are applied torques, or control torques. In the double pendulum model, control torques are applied about two axes. The first axis is perpendicular to the plane of the swing and passes through the fixed point. The second axis is also perpendicular to the plane of the swing, but passes through the hinge point. The primary objective of this study is to find the optimal application of these control torques.

The equations of motion of the model are now derived, using Lagrange's method. Similar derivations are found in [5] and [6]. The Lagrangian of the system is

$$L = \{I_a\dot{\theta}^2 + m_c[L_a^2\dot{\theta}^2 + l_c^2\dot{\psi}^2 + 2\,L_a l_c\dot{\theta}\dot{\psi}\cos(\psi-\theta)] + I_c\dot{\psi}^2\}/2 + \{m_a g l_a\cos\theta + m_c g[L_a\cos\theta + l_c\cos\psi]\}\sin\phi \quad (1)$$

Using Lagrange's generalized equations of motion, the dynamics of the golf swing can be described by two second-order, nonlinear differential equations:

$$\ddot{\theta}(IA) + \ddot{\psi}(MLS)\cos(\psi-\theta) - \dot{\psi}^2(MLS)\sin(\psi-\theta) + (WLA)\sin\theta\sin\phi = T_a - T_c \quad (2)$$

$$\ddot{\psi}(IC) + \ddot{\theta}(MLS)\cos(\psi-\theta) + \dot{\theta}^2(MLS)\sin(\psi-\theta) + (WLC)\sin\psi\sin\phi = T_c \quad (3)$$

[3] This is only approximately true. Actually, club and body reach zero velocity at different times. However, such differences in timing are slight and do not significantly affect the results. See also [3].

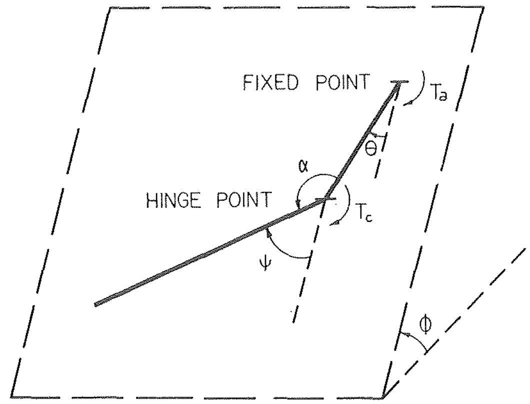

Fig. 1 Double pendulum model of golf swing

Optimal control theory requires the system dynamics to be expressed in state variable form. So, letting

$$x_1 = \theta \quad (4)$$

$$x_2 = \dot{\theta} \quad (5)$$

$$x_3 = \psi \quad (6)$$

$$x_4 = \dot{\psi} \quad (7)$$

the state variable form of equations (2) and (3) becomes

$$\dot{x}_1 = x_2 \quad (8)$$

$$\dot{x}_2 = Q_2\{T_a - T_c + x_4^2(MLS)\sin(x_3 - x_1) - (WLA)\sin x_1\sin\phi - C_2[T_c - x_2^2(MLS)\sin(x_3 - x_1) - (WLC)\sin x_3\sin\phi]\} \quad (9)$$

$$\dot{x}_3 = x_4 \quad (10)$$

$$\dot{x}_4 = Q_4\{T_c - x_2^2(MLS)\sin(x_3 - x_1) - (WLC)\sin x_3\sin\phi - C_4[T_a - T_c + x_4^2(MLS)\sin(x_3 - x_1) - (WLA)\sin x_1\sin\phi]\} \quad (11)$$

Actual values of the parameters are established from three sources: film data of the author's swing, actual measurement, and [7]. Numerical parameter values used in the solution for the optimal control torques are given in Table 1.

Performance Index

The performance index is the quantity to be optimized (minimized or maximized) in an optimal control problem. In general, the performance index is of the form

$$J = h[\mathbf{x}(t_f), t_f] + \int_{t_0}^{t_f} g[\mathbf{x}(t), \mathbf{u}(t), t]dt \quad (12)$$

Nomenclature

C_2 = (1/IC) (MLS) cos $(\psi - \theta)$
C_4 = (IC/IA)C_2
cm = center of mass
g = acceleration of gravity
H = system Hamiltonian function
I_a = moment of inertia of arms pendulum about fixed point
I_c = moment of inertia of club about cm of club
IA = $I_a + m_c L_a^2$
IC = $I_c + m_c l_c^2$
J = performance index
J_a = augmented performance index
L_a = length of arms pendulum
L_c = length of club
l_a = length from fixed point to cm of arms pendulum
l_c = length from hinge point to cm of club
MLS = $m_c L_a l_c$
m_a = mass of arms pendulum
m_c = mass of club
Q_2 = $1/[IA - C_2(MLS)]$
Q_4 = $(IA/IC)Q_2$
T_a = torque on arms pendulum
T_c = torque on club
V = clubhead speed
WLA = $m_a g l_a + m_c g L_a$
WLC = $m_c g l_c$
α = wrist cock angle
θ = angle of arms pendulum
ϕ = inclination of plane of swing
ψ = angle of club
* = superscript, indicates optimal value

Table 1 Parameter Values

Parameter	Parameter Value	
L_a	0.615 m	(2.018 ft)
l_a	0.326 m	(1.069 ft)
m_a	7.312 kg	(0.501 slug)
I_a	1.150 kg.m²	(0.848 slug·ft²)
L_c	1.105 m	(3.625 ft)
l_c	0.753 m	(2.469 ft)
m_c	0.394 kg	(0.027 slug)
I_c	0.077 kg.m²	(0.057 slug·ft²)
ϕ	0.785 rad	

where t_0 is the initial time, t_f is the final time, $\mathbf{x}(t)$ is the system state vector, $\mathbf{u}(t)$ is the control vector, and g and h represent scalar functions.

For maximum distance of the golf drive, clubhead speed at impact is to be maximized. The clubhead speed at impact can be written as

$$V(t_f) = \{L_a^2 x_2^2(t_f) + L_c^2 x_4^2(t_f) + 2L_a L_c x_2(t_f) x_4(t_f) \cos [x_3(t_f) - x_1(t_f)]\}^{1/2} \quad (13)$$

and is of the general form.

Necessary Conditions for Optimal Control

In the optimal control problem, it is desired to find an admissible control $\mathbf{u}^*(t)$ that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] \quad (14)$$

to follow an admissible trajectory $\mathbf{x}^*(t)$ that maximizes (minimizes) the performance index J (note: an asterisk (*) indicates optimal value). Assuming that the state and control regions are not constrained, that the initial conditions $\mathbf{x}(t_0)$ are given, and that both the initial time t_0 and the final time t_f are specified, necessary conditions for optimal control may be derived using the methods of the calculus of variations as in [8]. Defining the Hamiltonian function as

$$H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) \triangleq g(\mathbf{x}, \mathbf{u}, t) + \mathbf{p}^T\mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad (15)$$

where $\mathbf{p}$ is the adjoint, or costate, vector and vectors $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{p}$ are each functions of time, the necessary conditions are

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \mathbf{p}} (\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*, t) \quad (16)$$

$$\dot{\mathbf{p}}^* = \frac{-\partial H}{\partial \mathbf{x}} (\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*, t) \quad (17)$$

$$0 = \frac{\partial H}{\partial \mathbf{u}} (\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*, t) \quad (18)$$

$$0 = \left\{ \frac{\partial h}{\partial \mathbf{x}} [\mathbf{x}^*(t_f), t_f] - \mathbf{p}^*(t_f) \right\}^T \delta \mathbf{x}_f + \left\{ H[\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f] + \frac{\partial h}{\partial t} [\mathbf{x}^*(t_f), t_f] \right\} \delta t_f \quad (19)$$

Before application of these general necessary conditions, consideration must be given to constraints in the golf swing. First, the time to complete the downswing is not fixed, so that t_f is unknown. A scheme is necessary to estimate t_f. Also, the state variables and the control variables are constrained in the true golf swing. To insure proper simulation of impact, angles θ and ψ must be constrained at t_f. Wrist motion is limited, so the difference between angles θ and ψ must also be constrained, from t_0 to t_f. Finally, the control torques are limited by the musculature of the golfer, requiring constraints on torques T_a and T_c at all times.

In order to include these constraints, the performance index is augmented with penalty functions, as in [9]. In the solution process, constraint violations are discouraged by the penalty functions, which subtract from the performance index if violations occur. Thus, with augmentation, the performance index becomes

$$J_a = V(t_f) + \mathbf{s}(t_f)\mathbf{W}\mathbf{s}(t_f)/2 + kx_5^2(t_f)/2 + (1/2) \int_{t_0}^{t_f} \sum_{i=1}^{2} c_i M(c_i) dt \quad (20)$$

Discussion of the augmented performance index is found in the Appendix.

Inclusion of control constraints requires modification of equation (18), which is only valid in the absence of control constraints. Equation (18) becomes

$$0 \geq \frac{\partial H}{\partial \mathbf{u}} (\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*, t) \quad (21)$$

Also, the constrained solution must satisfy Pontryagin's Maximum Principle

$$H(\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*, t) \geq H(\mathbf{x}^*, \mathbf{u}, \mathbf{p}^*, t) \quad (22)$$

Finally, since t_f is free, J_a is extremal with respect to t_f. That is,

$$\frac{\partial J_a}{\partial t_f} = 0 \quad (23)$$

so that the value of t when (23) is satisfied is the estimate of t_f.

Thus J_a includes a cost for violation of each constraint and is to be maximized. Equations (16) and (17) are used to identify the necessary conditions for optimal control, a set of $2n$ first-order differential equations. Initial conditions and equation (19) provide $2n$ constants of integration, or boundary conditions. However, the boundary conditions are split; n conditions $\mathbf{x}(t_0)$ are given at t_0 and n conditions $\mathbf{p}(t_f)$ are given at t_f. To find the optimal control torques T_a^* and T_c^*, a two-point boundary value problem must be solved. An iterative, numerical technique is needed.

Solution Technique

The method of steepest ascent is used to find the optimal control torques, which must satisfy all necessary conditions. The technique is analogous to that used by a climber who always chooses steepest slopes in his route to a mountain top.

First an initial guess of the time history of optimal control torques $\mathbf{u}(t)$ is made. The state trajectory $\mathbf{x}(t)$ is then computed, using the estimated value of t_f. Next, $\mathbf{p}(t_f)$ is computed, so that $\mathbf{p}(t)$ may then be identified by integrating the co-state equations (17) from t_f to t_0. As it is very unlikely that the initial guess of $\mathbf{u}(t)$ will maximize the performance index, the gradient $\frac{\partial H}{\partial \mathbf{u}}(t)$ is computed and specifies the direction of increasing value of J_a. A series of increments in control

$$\Delta \mathbf{u}_i(t) = r_i \frac{\partial H}{\partial \mathbf{u}} (t) \quad (24)$$

is computed, using a set of pre-selected numbers r_i. Numbers r_i are arbitrary step-sizes, and specify amount of change in direction $\frac{\partial H}{\partial \mathbf{u}}(t)$. The control

$$\mathbf{u}_i(t) = \mathbf{u}(t) + \Delta \mathbf{u}_i(t) \quad (25)$$

which maximizes J_a is the updated control. The change in the performance index ΔJ_a is tested and if sufficiently close to zero, J_a has been maximized and $\mathbf{u}(t)$ is optimal. If ΔJ_a is not near

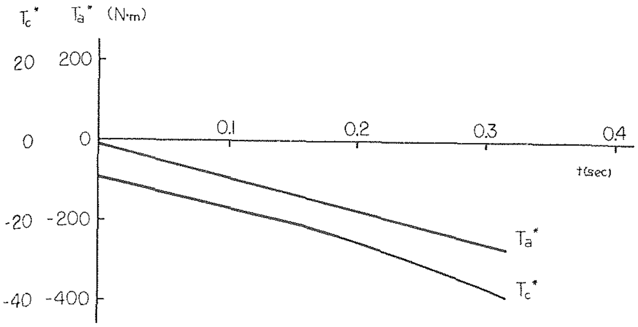

Fig. 2 Optimal control torques

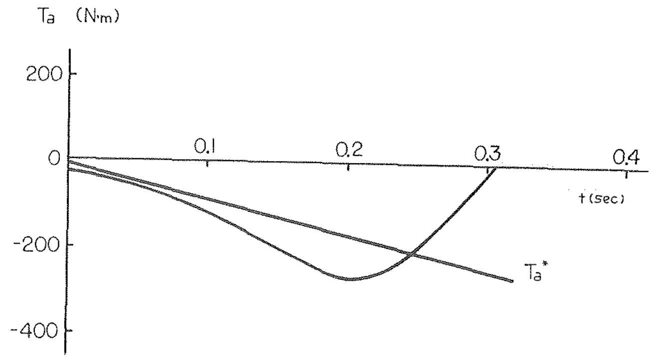

Fig. 3 Actual and optimal torques on arms pendulum

zero, the process is repeated, using the updated $\mathbf{u}(t)$ as the initial guess.

Results

Optimal Control Torques. As in many optimal control problems, computational restrictions and approximations in the mechanization of algorithms prevent absolute satisfaction of all conditions for optimization, forcing the analyst to accept a suboptimal solution. Such is the case in solving for the optimal control torques in the golf swing. Fig. 2 shows the computed optimal control torques. This solution satisfies all necessary conditions with no constraint violations, and the performance index is, for practical purposes, maximized. Thus, Fig. 2 presents a good suboptimal solution.

In Fig. 2, both $T_a{}^*$ and $T_c{}^*$ increase negatively with time. At time zero, when the downswing begins, both $T_a{}^*$ and $T_c{}^*$ are about -9.5 N•m(-7.0 lb•ft). $T_a{}^*$ decreases linearly, reaching the negative limit at 0.315 s, which is the time of impact. $T_c{}^*$ decreases linearly until 0.16 s, and then more rapidly to impact, at which time it also has attained its negative limit. The fact that the absolute values of the optimal control torques increase with time confirms the belief that the expenditure of energy should be delayed as long as possible for maximum distance of the drive.

Comparison of Actual and Optimal Torques. The time histories of the actual torques applied in the author's swing were computed from film data. Figs. 3 and 4 show actual and optimal torques for arms and club, respectively. In general, it may be said that, during the first 0.23 s, too much torque is applied in the actual downswing. In this time period, both T_a and T_c are greater in the actual swing than in the optimal swing. However, in the final 0.085 s before impact not enough torque is applied in the actual swing. Note that the times to complete the two downswings are nearly equal, the optimal time only 0.01 s longer than the actual.

The benefit received by applying optimal control torques can be seen in Fig. 5. Even though actual clubhead speed exceeds optimal clubhead speed for 0.28 s, the optimal speed becomes significantly larger than the actual speed at impact.[4] The difference of 16 m/s (53 ft/s) is equivalent to 101 m (110 yd). A golf ball hit with the optimal swing would carry approximately 293 m (320 yd)! These distance computations are made as in [2], and are only valid for normal weather conditions, which offer no extraordinary aid to the golfer. The distance of the optimal drive indicates that no golfer swings optimally, as tee shots of such great carry have never been achieved. However, the results do suggest that increased strength is not necessarily required for longer drives.

The Optimal Swing. If longer drives come from swinging more optimally, then what can one do to swing more optimally? A partial answer to this question is found in Fig. 6, which shows actual and optimal wrist cock angle α during the downswing. As indicated, the rate of increase of the wrist cock angle is greater in the actual swing than in the optimal swing for 0.26 s, at which time the optimal rate surpasses the actual rate, indicating that the uncocking of the wrists is to be delayed as long as possible for maximum distance.

Sensitivity

The optimal solution is, of course, a function of the parameter values of the mathematical model. To determine the effects of small parameter changes, a sensitivity analysis is necessary.

The method here is the same as that used to compute the optimal torques. The parameter values are simply changed from their original values, and steepest ascent is used to compute: (*a*) optimal control torques, and (*b*) clubhead speed at impact.

The double pendulum model can be completely specified by six independent parameters: (1) MA = mass of arms pendulum (m_a), (2) LARM = length of arms pendulum (L_a), (3) MHEAD = mass of clubhead, (4) MSHAFT = mass per unit length of clubshaft, (5) LCLUB = length of club (L_c), and (6) PHI = inclination of plane of swing (ϕ). Both (*a*) and (*b*) are computed for ten percent increases in each of the six parameters, taken one at a time. For example, (*a*) and (*b*) are first computed with MA increased by ten percent and with all other parameters at their original values. The process is then repeated, each time increasing only one parameter and maintaining the others at their original values.

It is found that both the optimal control torques and clubhead speed at impact are relatively insensitive to ten percent in-

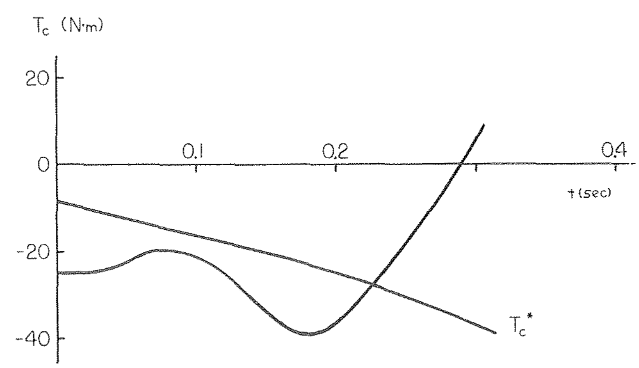

Fig. 4 Actual and optimal torques on club

[4] At impact, optimal speed is 61.9 m/sec (203 ft/sec) and actual speed is 45.7 m/sec (150 ft/sec). According to [10], "An average player moves the clubhead at around 140 feet per second; a good amateur at 150 to 160; a woman at 130; and Jack Nicklaus at 165 to 170. The human limit is thought to be about 180 ft. per sec." See also [11], in which is reported data from a study of backswing length, plane of downswing, and clubhead speed at impact in the swings of twenty professional golfers. Of those pros, George Bayer, at the time one of the longest hitters in golf, generated the greatest clubhead speed with the driver, 104.3 miles per hour (153 ft/sec).

Table 2 Clubhead speed sensitivity

	MA	LARM	MHEAD	MSHAFT	LCLUB	PHI
Original parameter values	7.312 kg	0.615 m	0.204 kg	0.110 kg/m	1.105 m	0.785 rad
Parameter changes (percent)	10.0	10.0	10.0	10.0	10.0	10.0
Relative change in clubhead speed (percent)	−1.28	1.22	−1.31	−1.10	0.45	0.17

creases in parameter values. The optimal torque on the arms pendulum, T_a^*, is not affected at all, and that on the club, T_c^*, only slightly more by the parameter changes. Increases in the two lengths LARM and LCLUB are the only parameter changes which require modification of T_c^*. Less torque T_c^* (1.7 percent less total system angular momentum) is required when LARM is increased, and more torque T_c^* (2.7 percent more total system angular momentum) is necessary when LCLUB is increased. Such modifications of T_c^* are considered necessary so that state variable constraints are t_f are not violated.

Table 2 presents sensitivities of clubhead speed at impact. The effects of ten percent parameter value increases vary in magnitude and direction, but, again, little sensitivity is exhibited. For all six parameters, the relative change in clubhead speed, $\Delta V(t_f)/V(t_f)$ is no more than 1.31 percent, which is equivalent to only some 4.6 m (5.0 yd) of distance.

Conclusion

The sensitivity analysis is valid for the optimal swing only. Also of interest would be sensitivites for the author's actual swing, but no such study was made because of the extensive testing required. However, if it is assumed that the actual sensitivities are of the same order of magnitude as the optimal sensitivities, it appears that only small increases in distance result from parameter value changes. For longer drives from the tee, the golfer is best advised to swing the club more optimally.

APPENDIX

There are four terms in the augmented performance index, presented as equation (20). The first term is clubhead speed at impact, as in equation (13). The remaining three terms are penalty functions and are now defined.

Cost of State Variable Constraint Violations at t_f. For proper impact, angles x_1 and x_3 must simultaneously be zero at t_f. Violation of this constraint is penalized by the quadratic form $\mathbf{s}(t_f)\mathbf{W}\mathbf{s}(t_f)/2$, where

$$\mathbf{s}(t_f) = [x_1(t_f) \quad x_3(t_f)]^T \tag{1a}$$

$$\mathbf{W} = \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} \tag{2a}$$

W is a negative-definite weighting matrix.

Cost of State Variable Constraint Violations from t_0 to t_f. The difference of angles x_1 and x_3 must not exceed those values found in the actual golf swing. Cumulative violations of this constraint are penalized by the quadratic term $kx_5^2(t_f)/2$, where k is a negative scalar and x_5 is an auxiliary state variable computed as

$$\dot{x}_5(t) = d^2S_1(d) + [d_0 - d]^2S_2(d_0 - d) \tag{3a}$$

where

$$d = x_3(t) - x_1(t) \tag{4a}$$

$$d_0 = x_3(0) - x_1(0) \tag{5a}$$

$$S_1(d) = \begin{cases} 0 & d \geq 0 \\ 1 & d < 0 \end{cases} \tag{6a}$$

$$S_2(d_0 - d) = \begin{cases} 0 & d_0 - d \geq 0 \\ 1 & d_0 - d < 0 \end{cases} \tag{7a}$$

$$x_5(0) = 0 \tag{8a}$$

Cost of Control Variable Constraint Violations from t_0 to t_f. Control torques in excess of those found in the actual swing are forbidden. Any violation of this constraint is penalized by the integral $(1/2)\int_{t_0}^{t_f}\sum_{i=1}^{2} c_iM(c_i)dt$, where

$$c_i = (T_{i\max} - T_i)(T_i - T_{i\min}) \tag{9a}$$

$$M(c_i) = \begin{cases} 0 & c_i \geq 0 \\ l_i & c_i < 0 \end{cases} \tag{10a}$$

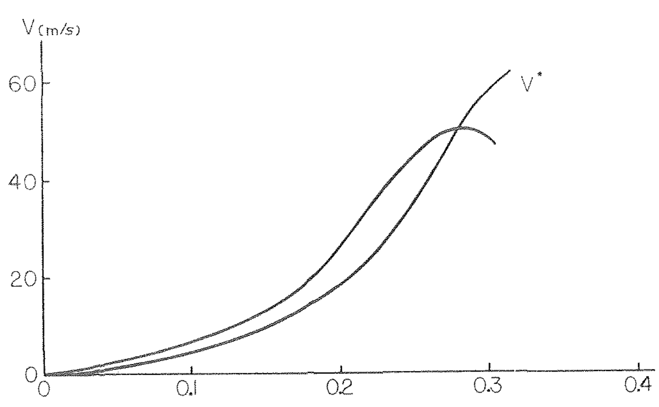

Fig. 5 Actual and optimal clubhead speed

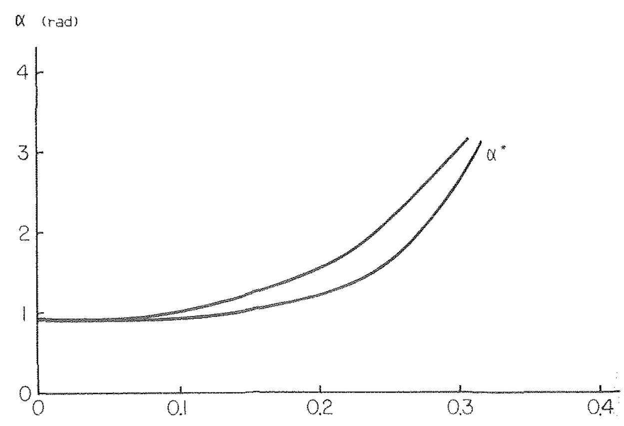

Fig. 6 Actual and optimal wrist cock angle

where $T_1 = T_a$, $T_2 = T_c$, and l_i are positive scalars. Numerical values of the elements of $\mathbf{W}$, scalar k, and scalars l_i are determined by simulation.

Acknowledgment

I wish to thank my thesis advisor Prof. Peter L. Balise and the Department of Mechanical Engineering at the University of Washington for providing me the opportunity to do this research. Also, discussions with Prof. Juris Vagners of the Department of Aeronautics and Astronautics and with Prof. Paul M. Frank were invaluable for me. Prof. Frank was visiting scholar in the Department of Electrical Engineering during the academic year 1974–75, and is from the University of Karlsruhe in West Germany. Finally, I am grateful to Marcia Phillips of Seattle for typewriting this paper.

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