On the double pendulum model of the golf swing

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Abstract

The primary aim of this computational study was to investigate the effect of positioning the ball so that contact between clubhead and ball takes place at the point in the downswing where the clubhead achieves its maximum horizontal component of velocity. The double pendulum model of the downswing was employed and computational results were obtained for a range of 'release angles' (the release angle determines the stage in the downswing at which the wrist joint is allowed to turn freely). The position of the wrist joint and the direction of motion of the clubhead at the instant at which the clubhead makes contact with the ball were also determined. Furthermore, clubhead velocity was found to increase if the release is delayed (the so-called 'late hit'). The energy supplied by the golfer was also investigated and, in particular, its variation with release angle is studied. Using a detailed perturbation analysis of the equations of motion, the results show that for a swing using a 'natural release,' the energy supplied by the golfer was a minimum.

Keywords: golf swing, ball position, contact, energy

Introduction

Most golfers, if asked, will claim that they try to swing the club with the aim of achieving maximum clubhead speed at impact and the degree of success, or otherwise, is dependent on the skill of the player. Several authors have studied the dynamics of the golf swing and, in particular, the downswing has received detailed attention. For example, Daish (1972) modelled the downswing as the motion of a double pendulum with the upper pivot located at a point in the player's spine a few centimetres below the base of the neck and the second pivot at the player's wrist joint, as illustrated in Fig. 1. The turning motion of a player's body was modelled by the application of a torque to the upper lever (upper torso and arms), and any wrist action imparted by the player by a second couple which was applied to the lower lever (the club). The downswing was modelled in two phases. In the first, the wrists are fully cocked ($\alpha = \pi/2$, see Fig. 1) and the upper body rotates as a rigid body under the turning action of the player. In the second, the player allows the wrist joint to turn freely, commonly referred to as the release of the club, or wrist 'uncocking.' The correct anatomical term is wrist adduction; see, for example, McLaughlin & Best (1994). It is, of course, the player's decision at what point in the swing to allow this second phase to begin and this affects the clubhead velocity (and various other parameters) at the instant of contact of the clubhead with the ball. For simplicity, this instant will be referred to as the 'instant of impact' or simply 'impact' throughout this paper. A good review article is provided by Dillman & Lange (1994).

Daish (1972) derives the equations of motion for the double pendulum using a Lagrangian approach

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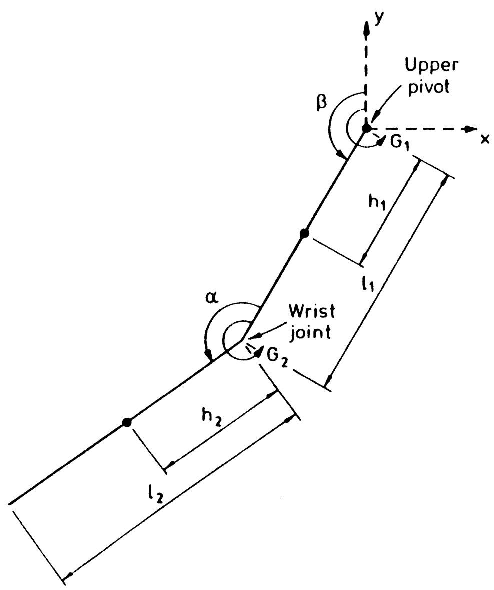


Figure 1 The double pendulum model.

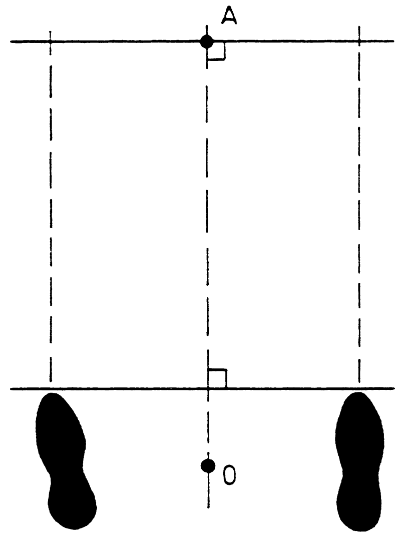


Figure 2 Schematic overhead view of a right-handed golfer. The pivot axis is at O and the point A is the ball position which, in the text, is referred to as 'aligned with the pivot axis.'

and the same equations are also given, for example, by Ramsey (1933). These equations comprise a pair of coupled nonlinear second-order differential equations. Jorgensen (1970) has obtained numerical solutions to these equations and concentrated particularly on investigating the effect of delaying the release of the club (the 'late hit') whereas Lampsa (1975), in an optimal control study, determined the idealised form of the applied torques at both the upper and lower levers to maximise the distance of the golf drive. Both these studies confirmed that a delayed release of the club increases clubhead velocity, a fact which is assumed to be true by most golfers. Jorgensen (1994) has also considered a flexible shaft model and highlighted the difficult problem of modelling the grip of the hands on the club. In particular it was noted that the frequency of oscillation of the shaft was lower using a 'soft' grip compared with a 'vice-like' grip, and reasonable agreement with experimental data was only obtained by a judicious choice of oscillation frequency. For simplicity, a rigid-rod model is used in the present study.

One of the major assumptions of the double pendulum model is that the downswing and follow-through are coplanar. In addition to the early work of Cochran & Stobbs (1968) and Williams (1967), the more recent observational studies of McLaughlin & Best (1994) and Lowe (1994) strongly support this assumption. The observations of Neal & Wilson (1985), however, suggested that the downswing and follow-through are not, in general, coplanar but the validity of the results of these authors has been queried by McLaughlin & Best (1994) due to the small sample size used. A further assumption of the double pendulum model is that the clubhead rotates about a central pivot (the upper pivot of the double pendulum). This was found to be the case for the majority of the low-handicap golfers observed by Lowe (1994). Jorgensen (1970) also assumes that, at impact, the position at which contact is made between clubhead and ball is position A in the overhead view shown in Fig. 2 (see also, Jorgensen 1970; figs 7 and 8). For simplicity, this ball position will be referred to as 'aligned with the pivot axis' and is at the centre of the player's stance if the player stands symmetrically with respect to the pivot axis.

In the present study, rather than assume that the ball is positioned at A (see Fig. 2), the ball is placed at the point in the downswing where the clubhead achieves its maximum horizontal component of velocity. This position is unknown in advance and has to be determined during the course of the computations. It might be regarded as preferable to choose the point A as the point in the downswing where the clubhead achieves its maximum *resultant* velocity but, as noted by Milne (1990), there is also a difficulty in deciding the form of torque input to use in a computational model, and of particular relevance here is the torque G_1 applied to the upper lever (Fig. 1). Jorgensen (1970) assumed that G_1 is constant throughout the swing, Lampsa's (1975) calculations suggested that, optimally, G_1 should be linearly increasing with time and Milne (1990) used a linearly increasing torque up to a maximum value which is maintained until impact, although precise details are not given. All such torque models lead to continuously increasing computed resultant clubhead velocities as the downswing proceeds and consequently the maximum resultant clubhead velocity occurs well past any possible point of contact with the ball and well into the 'follow-through' or upswing. However, it is clear that the *horizontal component* of clubhead velocity must reach a maximum value during the downswing and that this maximum will occur when the clubhead is travelling close to the ground. This impact criterion was therefore employed in our calculations and the consequences of such a choice are investigated in detail and comparisons made with the ball positioned as in Jorgensen (1970). As far as the authors are aware, no experimental studies have been carried out which investigate the points raised in the above discussion. In this study, G_1 is assumed to be constant throughout the downswing as in Jorgensen (1970).

Using a perturbation analysis, it is further shown that a swing with a natural release of the club (neither delayed nor released early) has some interesting mathematical properties which have not been previously investigated. The consequent physical interpretation is that such a swing requires a minimum of energy input from the player compared with a delayed or early release of the club. This fact is confirmed by the detailed numerical calculations.

The double pendulum model

The equations of motion

As described in Daish (1972), Jorgensen (1970) and Lampsa (1975), it is assumed that the swing takes place in a plane inclined at a fixed angle to the horizontal. Furthermore, the effects of air resistance and gravity will be ignored. Jorgensen (1970) considered this latter point in detail and showed that, to a very good approximation, the effect of gravity is equivalent to a small increase in the torques applied at the upper and lower levers. This increase was shown to be effectively constant throughout the swing. The following notation is employed:

m_1 = the mass of the upper lever (arms and upper torso)
l_1 = the length of the upper lever
I_1 = the moment of inertia of the upper lever about its centre of gravity
h_1 = the position of the centre of gravity of the upper lever measured from the upper pivot axis
m_2 = the mass of the lower lever (club)
l_2 = the length of the lower lever (club)
h_2 = the position of the centre of gravity of the club measured from the hinge point (wrist joint)
I_2 = the moment of inertia of the club about its centre of gravity.

The quantities A, B and D, which appear in the equations relevant to the double pendulum model, are defined by the relations

$$A = I_1 + m_1 h_1^2 + m_2 l_1^2 \tag{1}$$

$$B = m_2 h_2 l_1 \tag{2}$$

$$D = I_2 + m_2 h_2^2 \tag{3}$$

and the data employed in all the calculations are that given by Lampsa (1975). The relevant values are given in Table 1 and the club data are appropriate for a driver.

Table 1 The values of the parameters

l_1	0.615 m
l_2	1.105 m
h_2	0.753 m
m_2	0.394 kg
$I_1 + m_1h_1^2$	1.150 kg m^2
I_2	0.077 kg m^2

It should be noted that $I_1 + m_1h_1^2$ (the moment of inertia of the upper lever about the upper pivot) is given as data by Lampsa (1975) (Lampsa's parameter I_a) and thus explicit values for m_1 and h_1 are not required in Eq. (1). Furthermore, an explicit value for I_1 is not given by Lampsa (1975). Thus from Eqs (1), (2) and (3) we find that $A = 1.30$, $B = 0.18$ and $D = 0.30$, all being in units of kg m^{-2}. These values are very similar to those given by Daish (1972).

If β is the angle through which the upper body rotates and α is the angle between the line of the upper arm and the line of the club (as indicated in Fig. 1), then the kinetic energy of the system is T, where

$$T = \tfrac{1}{2}A\dot{\beta}^2 + \tfrac{1}{2}D(\dot{\alpha} + \dot{\beta})^2 - B\dot{\beta}(\dot{\alpha} + \dot{\beta})\cos\alpha. \tag{4}$$

This expression is that given by Daish (1972; eqn (6)) where Daish's angles θ and ϕ are related to α and β by $\alpha = \phi - \theta + \pi$, $\beta = \theta + \pi$. When gravity is neglected, the Lagrangian equations of motion for the upper arm and club are, respectively,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) = \frac{\partial T}{\partial \alpha} + G_2 \text{ and} \tag{5}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) = \frac{\partial T}{\partial \beta} + G_1, \tag{6}$$

where G_1 and G_2 are the couples exerted by the golfer on the upper body and the club, as shown in Fig. 1. It follows from Eqs (4), (5) and (6) that

$$\frac{d}{dt}\left[A\dot{\beta} + D(\dot{\alpha} + \dot{\beta}) - B(\dot{\alpha} + 2\dot{\beta})\cos\alpha\right] = G_1 \tag{7}$$

and

$$\frac{dT}{dt} = G_1\dot{\beta} + G_2\dot{\alpha}. \tag{8}$$

If G_1 and G_2 are given as functions of time then these equations may be integrated numerically to give α and β as functions of time. It is perhaps worth mentioning here that eqs (8) and (9) of Daish (1972; p. 113) [or eqs (5) and (6) of Jorgensen 1970] may be obtained from our Eqs (7) and (8) by carrying out the differentiation with respect to t followed by some algebraic manipulation. However, the form of Eqs (7) and (8) is more convenient for the approach which is described in the following Section.

Initial conditions, the values of G_1 and G_2 and the two phases of the motion

The initial conditions appropriate to Eqs (7) and (8) are

$$\alpha(0) = \frac{\pi}{2},\ \dot{\alpha}(0) = 0, \tag{9}$$

$$\beta(0) = \beta_s,\ \dot{\beta}(0) = 0. \tag{10}$$

The condition $\alpha(0) = \pi/2$ corresponds to the situation where the wrists are 'fully cocked' at the start of the downswing and β_s is used to denote the value of β at $t = 0$. This notation is used to allow for the possibility of different starting positions at the top of the backswing. This point is further considered later in this Section and is also discussed by Jorgensen (1970). Although the equations may be solved for any given time-functions G_1 and G_2, only the case where G_1 is a constant for all time is considered here and G_2 is chosen so that for $0 \le t < t_R$ the angle α is fixed at $\pi/2$ and for subsequent times (i.e. $t \ge t_R$) G_2 is zero. Thus, up to the 'release-time', t_R, the whole system (upper body, arms and club) rotates as a rigid body and after this time the club swings freely at the wrist joint.

For the first phase, the equations become simply

$$\alpha = \pi/2 \tag{11}$$

$$\ddot{\beta} = G_1/(A + D). \tag{12}$$

Thus for $0 \le t < t_R$, Eq. (12) is easily solved to give

$$\beta(t) = \frac{G_1t^2}{2(A + D)} + \beta_s, \tag{13}$$

and, from Eq. (5), it follows that the couple G_2 applied at the wrist joint must be given by

$$G_2(t) = \frac{DG_1}{A+D} - \frac{BG_1^2 t^2}{(A+D)^2}. \tag{14}$$

Starting from a positive value, $G_2(t)$ is a quadratically decreasing function of t which is zero at $t = t_0$ where

$$t_0 = \left(\frac{D(A+D)}{BG_1}\right)^{1/2}. \tag{15}$$

Thus there is a 'natural' stage in the swing at which $G_2(t) = 0$. The corresponding angle $\beta(t_0)$ is given by $\beta(t_0) = D/2B + \beta_s$ which is independent of G_1 and G_2 and depends only on the physical attributes of the player and club and the length of the backswing. A player may choose whether or not to keep the wrists fully cocked ($\alpha = \pi/2$) up to or beyond $t = t_0$ and the initial conditions for the subsequent, nonrigid body, phase of the motion will be determined by such action.

For the second phase, G_1 is still constant and $G_2\dot{\alpha}$ is still zero. This is because G_2 is zero in the second phase and $\dot{\alpha}$ is zero in the first phase. Hence, Eqs (7) and (8) can be integrated and, after some manipulation, they yield

$$\dot{\alpha}^2 = \frac{2G_1(\beta-\beta_s)(A+D-2B\cos\alpha) - G_1^2 t^2}{AD - B^2\cos^2\alpha} \tag{16}$$

$$\dot{\beta} = \frac{(B\cos\alpha - D)\dot{\alpha} + G_1 t}{A+D-2\,B\cos\alpha}. \tag{17}$$

Eq. (7) provides the expression for $\dot{\beta}$ and this has been substituted into the integrated form of the second equation. Eqs (16) and (17) can then be manipulated to give the following second-order differential equation for α:

$$\begin{aligned}&\left[\ddot{\alpha}(AD - B^2\cos^2\alpha) - G_1(B\cos\alpha - D)\right]\\ &\quad\times(A+D-2B\cos\alpha)\\ &= B\sin\alpha\left[G_1^2 t^2 - \dot{\alpha}^2(B\cos\alpha - D)(A - B\cos\alpha)\right].\end{aligned} \tag{18}$$

This equation can be obtained by noting that, from Eq. (16)

$$\beta - \beta_s = \frac{G_1^2 t^2 + (AD - B^2\cos^2\alpha)\dot{\alpha}^2}{2G_1(A+D-2B\cos\alpha)} \tag{19}$$

and substitution of this expression into Eq. (17) gives Eq. (18). Thus it is of interest to note that instead of integrating Eqs (16) and (17) for α and β, the single second-order Eq. (18) for α may be solved and subsequently β may be calculated from Eq. (19) without any further integration (a single second-order differential equation for α may also be derived for the case where G_1 and G_2 are any given functions of t).

For this second phase of the motion, the condition of continuity in α and β provides the necessary starting values. Thus α is $\pi/2$. The value of the 'release-time', t_R, will be variously chosen to represent the different strategies employed by the golfer. Alternatively, this may be thought of as choosing the angle β_R at which the rigid-body phase ends, since, from Eq. (13) it follows that

$$t_R = \left(\frac{2(A+D)(\beta_R - \beta_s)}{G_1}\right)^{1/2}. \tag{20}$$

The corresponding value $\dot{\beta}_R$ is then given by

$$\dot{\beta}_R = \left(\frac{2G_1(\beta_R - \beta_s)}{A+D}\right)^{1/2}. \tag{21}$$

Thus, in the calculations going through both phases one and two, it is assumed that $G_2(t)$ is given by Eq. (14) for $t \leqslant t_R$ and, for $t > t_R$, $G_2(t) = 0$.

It should be noted that Eqs (19), (20) and (21) involve only the difference $\beta_R - \beta_s$. Thus, for $\beta_s > 0$, the same value of $\dot{\beta}_R$ will be achieved as for the case $\beta_s = 0$ but at some later stage (larger value of β_R) in the swing. The subsequent motion for the case $\beta_s > 0$ is thus identical with that for $\beta_s = 0$ but offset by β_s, as also noted by Jorgensen (1970). In all calculations presented here it is assumed that $\beta_s = 0$.

Impact criterion, computed parameters at impact, scaling

As indicated in the Introduction, in addition to solving Eq. (18), the horizontal component of clubhead velocity, v_h, is calculated as the computation proceeds. From Fig. 1 it follows that v_h is given by

$$v_h = -l_1\dot{\beta}\cos\beta + l_2(\dot{\alpha}+\dot{\beta})\cos(\alpha+\beta), \qquad (22)$$

and hence, by differentiating Eq. (22) with respect to t, it follows that v_h will be a maximum when

$$-l_1\ddot{\beta}\cos\beta + l_1\dot{\beta}^2\sin\beta + l_2(\ddot{\alpha}+\ddot{\beta})\cos(\alpha+\beta) - l_2(\dot{\alpha}+\dot{\beta})^2\sin(\alpha+\beta) = 0. \qquad (23)$$

Eq. (18) was solved by the fourth-order Runge–Kutta method subject to the initial conditions described above and the expression on the left-hand side of Eq. (23) was evaluated at each time-step. The computation was halted a few steps beyond the stage where Eq. (23) changed from a positive to a negative value and the time for which Eq. (23) was most closely satisfied was noted. This value of t, which is denoted here by $\bar{t}_i$, is an approximation to the time t_i at which the horizontal component of clubhead velocity reaches a maximum value. In order to improve on this estimate, the data for several steps either side of $t = \bar{t}_i$ were fitted by a cubic spline approximation and the value of $t = t_i$ was determined for which relation (23) is exactly satisfied and the corresponding values of α, β, $\dot{\alpha}$, $\dot{\beta}$, $\ddot{\alpha}$ and $\ddot{\beta}$ were calculated at this time, again using cubic spline interpolation. These values were used to evaluate the left-hand side of Eq. (23) as a checking procedure and, in all cases considered, the left-hand side of Eq. (23) was found to be less than 10^{-5} in magnitude. In addition to evaluating v_h from Eq. (22), the following quantities were also calculated at time $t = t_i$:

(i) the vertical component of clubhead velocity, v_v, where

$$v_v = -l_1\dot{\beta}\sin\beta + l_2(\dot{\alpha}+\dot{\beta})\sin(\alpha+\beta), \qquad (24)$$

(ii) the angle, θ, between the direction of motion of the clubhead and the horizontal

$$\theta = \tan^{-1}\left(\frac{v_v}{v_h}\right), \qquad (25)$$

(iii) the ball position, x_b, relative to the pivot axis (positive forward of the pivot axis)

$$x_b = -l_1\sin\beta + l_2\sin(\alpha+\beta), \qquad (26)$$

(iv) the position of the wrist joint, x_w, relative to the ball position (positive forward of the ball position)

$$x_w = -l_2\sin(\alpha+\beta). \qquad (27)$$

The time step for the fourth-order Runge–Kutta method was chosen, after some computer experiments with various step sizes, as 5×10^{-4} s. For smaller time steps the values determined from Eqs (22) and (24) were found to differ, at most, by approximately ±0.03 m s^{-1} from the values obtained with a time step of 5×10^{-4} s. The values determined from Eqs (26) and (27) can thus be shown (using the standard methods of small error analysis) to be accurate to approximately ±0.0025 m.

It is, of course, possible to rewrite Eqs (7) and (8) in terms of a scaled time, t', where $t' = \sqrt{G_1}$, as described by Jorgensen (1970) (Jorgensen's "zeit" parameter). The resulting equations thus depend only on the ratio G_2/G_1. For the first phase of the swing, Eq. (14) indicates that $G_2(t')/G_1$ is independent of G_1 and, in the second phase, the scaled version of Eq. (18) is also independent of G_1. This implies that, for a given release angle, β_R, any two solutions of Eq. (18), say $\alpha_a(t)$ and $\alpha_b(t)$, corresponding to values $G_1 = G_1^{(a)}$ and $G_1 = G_1^{(b)}$ are such that $\dot{\alpha}_a(t) = \dot{\alpha}_b(t)\sqrt{(G_1^{(a)}/G_1^{(b)})}$ and $\ddot{\alpha}_b(t) = \ddot{\alpha}_b(t)\left(G_1^{(a)}/G_1^{(b)}\right)$. Furthermore, $\alpha_a(t_a) = \alpha_b(t_b)$ at times t_a and t_b which are related by $t_a = t_b\sqrt{(G_1^{(b)}/G_1^{(a)})}$. Thus, for example, if $G_1^{(a)} < G_1^{(b)}$, $t_a > t_b$. Similar remarks apply to the solution for $\beta(t)$.

The above discussion implies that (for a given release angle) the paths traced out by the wrist joint and by the clubhead are independent of the value of G_1 (although the time taken to reach a fixed position on these paths does depend on G_1). Since the impact criterion (23) is also independent of G_1, it follows that the ball position and wrist joint position [Eqs (26) and (27)] for maximum horizontal component of clubhead velocity are the same for all values of G_1. The components of clubhead velocity given by Eqs (22) and (24) and hence the resultant clubhead speed $(v_h^2 + v_v^2)^{1/2}$ vary with G_1 as indicated above.

Numerical results and discussion

In all results presented here $\beta_s = 0$ and $G_1 = 200$ Nm. This value of G_1 was chosen as reasonably typical of an average golfer [the measured torque G_1 given by Lampsa (1975) varies from zero to approximately 400 Nm]. As indicated in the Section on impact criterion, computed parameters and scaling, other values of G_1 simply result in changes to velocities and accelerations, the computed impact positions of the clubhead and wrist joint are independent of G_1. For example, clubhead velocities for $G_1 = 300$ Nm may be obtained by multiplying the velocities given here by $\sqrt{3/2}$.

In Fig. 3(a)–(d), the solid curve is that obtained using the maximum horizontal component of clubhead velocity criterion [relation (23)] and the broken curve is for the ball position aligned with the pivot axis. Figure 3(a) shows the resultant clubhead speed $[=(v_h^2 + v_v^2)^{1/2}]$ at impact plotted against release angle β_R. For the data employed here, the natural release angle is given by $\beta(t_0) \approx 47.75°$ and is indicated by a vertical line in the figure. It is clear from Fig. 3(a) that, for the same value of β_R, the impact clubhead speed is higher using criterion (23)

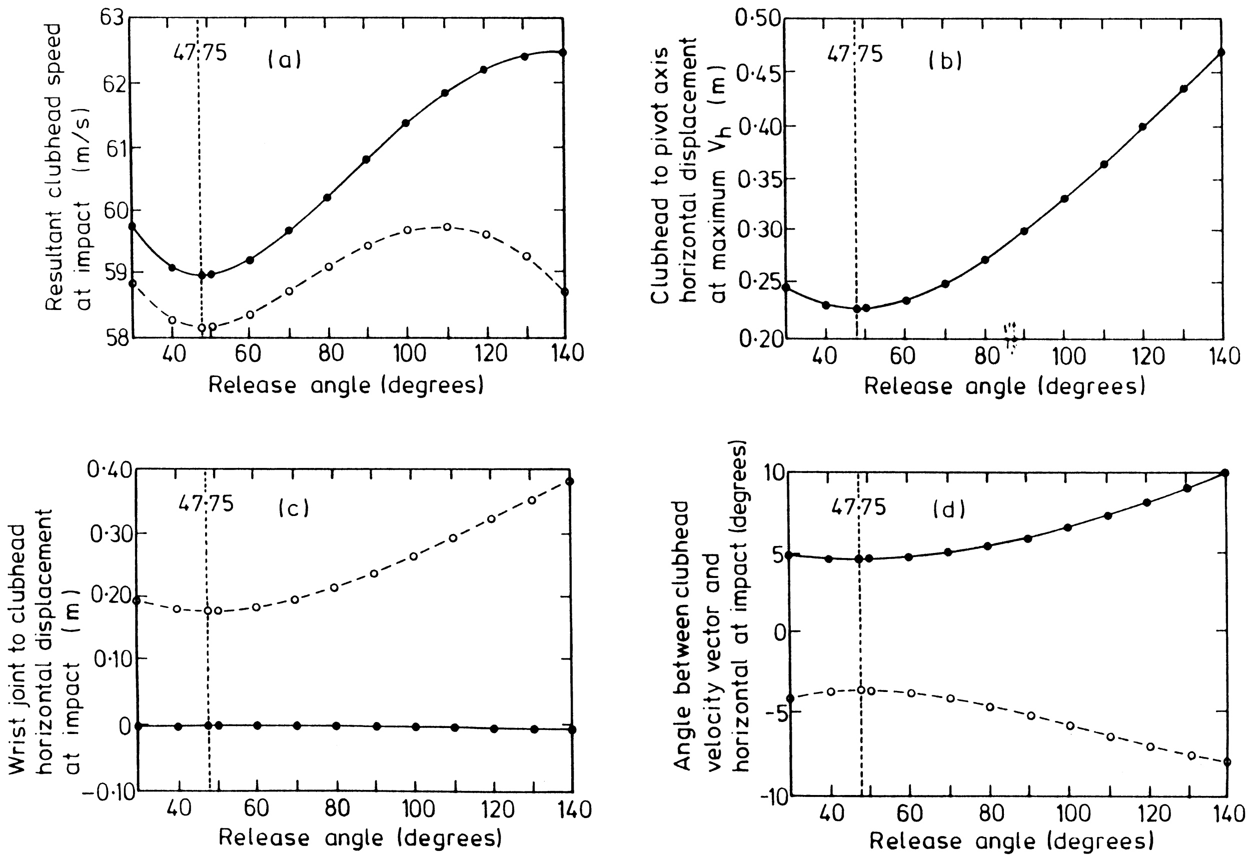


Figure 3 Computed values of various parameters at impact vs. release angle. The solid curves are for the maximum horizontal component of the clubhead velocity impact criterion, the broken curves are for the case where the ball is aligned with the pivot axis. (a) Clubhead speed at impact; (b) clubhead to pivot axis horizontal displacement at maximum v_h (ball position) measured positive forward of the pivot axis; (c) wrist joint to clubhead horizontal displacement, measured positive for the wrist joint ahead of the clubhead; (d) angle between the clubhead velocity vector and horizontal, measured positive upwards.

than for the axis aligned ball position. This increase is of the order of 1.3% for values of β_R near to $\beta(t_0)$ but gradually becomes larger as β_R increases, being approximately 3% for $\beta_R = 100°$. According to Williams (1967), a typical professional golfer releases the club for β_R in the range $60° < \beta_R < 70°$, for which the increase in clubhead speed is of order 1.6% compared with the axis aligned ball position. Jorgensen (1970) considered release angles of $\beta(t_0)$, $2.5\beta(t_0)$ and $3.5\beta(t_0)$ which, for the data used in the present study, correspond to approximately 48°, 119° and 167° and Jorgensen's results (for the axis aligned ball position) show the expected behaviour as the release angle is increased. However, in an actual comparison with real data (Jorgensen's Fig. 6) a release angle of $1.25\beta(t_0)$ ($\approx 60°$) was employed, which is generally consistent with Williams' observations.

Both curves in Fig. 3(a) show that by delaying the release, higher resultant clubhead speeds can be generated at impact. Indeed using criterion (23) with $\beta_R = 70°$, an increase in resultant clubhead speed at impact of approximately 2.5% is obtained, compared with the axis aligned ball position result with a natural release of the club. Whilst this increase is relatively small it turns out that there are other factors which are likely to make the results obtained using (23) preferable in practice. The range of values of β_R considered in the calculations is almost certainly larger than that which can be achieved in practice and, in particular, it is regarded as poor practice to release the club early. The only reason for the inclusion of such cases is to show the existence of the minimum in Fig. 3(a) and it is clear that both curves in Fig. 3(a) show a minimum resultant clubhead speed for the case $\beta_R = \beta(t_0) \approx 47.75°$.

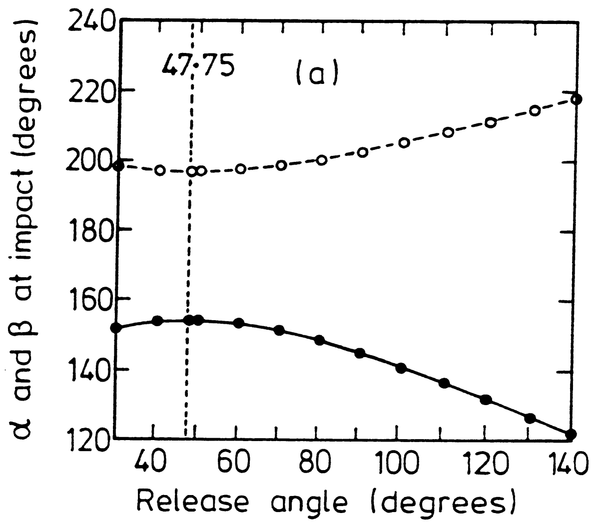
Figure 3(b) shows the ball position, measured in metres forward of the pivot axis, for the computations using the impact criterion given by Eq. (23). The minimum distance ball position (forward of the pivot axis) occurs for a swing with $\beta_R = \beta(t_0)$, the natural release angle, and is about 0.226 m. As β_R increases from $\beta(t_0)$ the ball position moves further away from the pivot axis, having the value of about 0.249 m for $\beta_R = 70°$ and, for $\beta_R = 110°$ the ball position is more than 0.356 m forward of the pivot axis, which is approaching an impracticable (or at least undesirable) situation for most players.

The impact position of the wrist joint, measured in metres ahead of the ball position, is shown in Fig. 3(c) as a function of β_R. For impact criterion (23), for all practical purposes, the wrist joint is level with the ball at impact (the detailed calculations show a very shallow minimum about the natural release angle). This contrasts sharply with the case for the 'aligned with the pivot axis' ball position for which, at impact, the wrist joint is ahead of the ball position by at least 0.178 m. This value is larger than observed in many photographs of impact positions of professional players, such as that of Dai Rees mentioned earlier, where the wrist joint appears to be approximately level with the ball at impact.

The direction of motion of the clubhead relative to the horizontal [the angle θ given by Eq. (25)] is shown in Fig. 3(d) as a function of release angle β_R. For impact criterion (23), all simulated swings result in a small positive value of θ corresponding to a slightly upward motion of the clubhead through impact with a minimum θ, for $\beta_R = \beta(t_0)$, of approximately 4.7° which increases to approximately 5.1° for $\beta_R = 70°$ and about 6.6° for $\beta_R = 100°$. (For such small positive angles θ, there is no possibility of contact with the ground before hitting the ball, assuming typical tee heights.) For a driver, a slightly upward contact is much more desirable than the situation shown by the broken curve in Fig. 3(d) where a downward blow is delivered through impact. Such a contact between clubhead and ball is likely to be more appropriate for irons and this point is further discussed later in this Section.

Figure 4(a) shows plots of $\alpha(t_i)$ (the solid curve) and $\beta(t_i)$ (the broken curve) at the instant of impact, $t = t_i$, between clubhead and ball as functions of the release angle, β_R, for the axis aligned ball position. Figure 4(b) shows similar curves for the maximum horizontal component of clubhead velocity impact criterion given by Eq. (23).

In Fig. 4(a) and (b) it is clear that, for the case of a natural release of the clubhead [for which $\beta_R = \beta(t_0)$] the curves for $\alpha(t_i)$ display a shallow



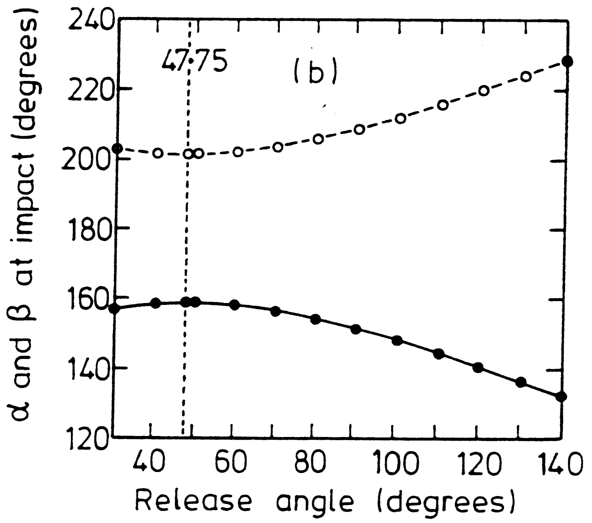


Figure 4 (a) Values in degrees of α (solid curve) and β (broken curve) at impact vs. release angle for the case where the ball is aligned with the pivot axis. (b) Values in degrees of α (solid curve) and β (broken curve) at impact vs. release angle for the maximum horizontal component of clubhead velocity impact criterion.

maximum whereas the curves of $\beta(t_i)$ display a shallow minimum. Consequently the energy input up to impact [$= G_1\beta(t_i)$, see Eq. (A16) of the Appendix], for a swing using a natural release of the club requires the least energy input compared with swings which use other release angles. These energy calculations were checked by calculating the kinetic energy of the system at impact using relation (4) and exact agreement was obtained in every case. In the Appendix, a theoretical analysis is presented which shows that such a minimum will exist if the elapsed time to impact from the start of the downswing is less than 0.28 s. The computed times to impact for all cases presented here varied from approximately 0.248 s to 0.258 s and hence it may be concluded that the theoretical and numerical calculations are in agreement. It should be stressed that the above conclusions are independent of the impact criterion used in the calculations and only depend on impact being achieved within a certain time. Comparison of Fig. 4(a) and (b) also shows that, for a given value of β_R, the values of $\alpha(t_i)$ and $\beta(t_i)$ in Fig. 4(b) are always larger than the corresponding values shown in Fig. 4(a), which is consistent with the fact that contact between clubhead and ball takes place at a later stage in the swing using criterion (23) compared with the pivot axis aligned ball position. This is clearly evident from Fig. 3(b). All impact positions of the upper lever and club (for different release angles, β_R) were found to occur for $197° < \beta < 233°$ and $116° < \alpha < 159°$ and hence the club and the upper lever form a straight line only at some instant after contact is made between clubhead and ball, as illustrated, for example, by the photograph showing the impact position of a professional player (Dai Rees) (Cochran & Stobbs 1968; p. 59).

In practice, most golfers are very familiar with the recommendation that, for a right-handed player using a driver (or other long club), the ball should be placed on the target line within the width of the stance and towards the left foot (often described as 'inside the left heel') rather than in the centre of the stance. Clearly the computed ball positions using impact criterion (23) are generally consistent with this teaching recommendation. Unfortunately, as far as the authors are aware, there is no published scientific data currently available which provides experimentally observed values for any of the specific parameters discussed in Fig. 3(b)–(d). Hence, whilst a detailed comparison of the computed results presented here and data observed in an actual golf swing is very desirable, the lack of suitable experimental evidence makes this impossible to achieve at the present time. Furthermore, detailed comparison with previous computational studies (for example, Jorgensen 1970; Lampsa 1975) is not possible since these authors did not provide numerical results for the parameters presented in Fig. 3(b)–(d).

All the computed results presented here would, of course, be modified if the couple G_2 were not assumed to be zero for $t > t_R$. For example, consider a golfer who delays the release of the clubhead to $\beta_R = 100°$ or possibly later. The results [using Eq. (23)] suggest that, in order to make contact with the ball at a suitable point within the stance, for $t > t_R$ it will be necessary for the golfer to speed up the rotation of the club relative to the upper lever. This may be achieved by applying a (possibly time-dependent) positive torque G_2 for $t > t_R$ which concurs with the remarks of Cochran & Stobbs (1968), who discuss the matter of the 'timing' of this part of the swing. In addition to simply making contact with the ball, the quality of the contact is vitally important as evidenced, for example, by the direction of motion of the clubhead at impact.

As indicated earlier, the data used in the present study is appropriate for a driver but the results also suggest that, for the longer irons, the maximum horizontal component of clubhead velocity might provide a suitable impact criterion, whereas for the shorter irons, for which it is usually desirable that the clubhead is moving on a slightly downward trajectory at the instant of contact with the ball, it is likely that a ball position aligned with the pivot axis is more appropriate. These points are discussed in more detail by Pickering (1998).

Conclusions

It has been shown that the double pendulum model of the golf swing may be used, with an impact criterion which requires that the ball be placed at the point where the horizontal component of clubhead speed is a maximum, to give impact values for resultant clubhead speed, ball and wrist joint positions and direction of travel of the clubhead. The calculated resultant clubhead speeds at impact are higher than those determined with the ball placed in line with the pivot axis. Furthermore, the results indicate that a delayed release of the club results in a higher resultant clubhead speed at impact and this behaviour is in general agreement with the results obtained by earlier workers. However, the computed swings suggest that if release of the club is delayed too far then the point at which the maximum horizontal component of velocity of the clubhead is achieved is further ahead of the pivot axis than is attainable in an actual golf swing. The numerical results also show that a swing which employs a 'natural release' of the club requires the least energy input from the player and the perturbation analysis, presented in the Appendix, confirms that this is the case for both impact criteria used. The lack of published experimental data which is directly relevant to the specific parameters evaluated in this computational study is a considerable obstacle to the evaluation of the model and, in particular, the impact criterion.

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Appendix

The extremal properties of the solution with $t_R = t_0$

It has been commented earlier that the solution in which the release time has its natural value, i.e. $t_R = t_0$ (so that not only $\dot{\alpha}$ but $\ddot{\alpha}$ is also continuous at the moment of release), has curious properties. Specifically, if $\alpha_0(t)$ is this special solution, then $\alpha_0(t) \geq \alpha(t)$ for all 'reasonable' values of t. This Appendix is devoted to proving this observation.

The Eqs (18) and (19) are algebraically awkward to deal with and an approximate pair of equations is devised which, for the parameter values of practical concern, are of more than adequate precision. Let

$$\beta = \frac{G_1 t^2}{2(A + D - 2B\cos\alpha)} + \omega, \tag{A1}$$

then Eq. (16) implies that

$$\dot{\alpha}^2 = \frac{2G_1(A + D - 2B\cos\alpha)\omega}{AD - B^2\cos^2\alpha} \tag{A2}$$

and Eqn (17) implies

$$\dot{\omega} = \dot{\alpha}\frac{[B\cos\alpha - D)(A + D - 2B\cos\alpha) + BG_1 t^2\sin\alpha]}{(A + D - 2B\cos\alpha)^2}. \tag{A3}$$

These equations have the trivial solution in which $\omega = 0$ and α is constant (equal to $\pi/2$ from the initial conditions). Numerical results confirm that ω is always small compared with the first term in Eq. (A1). Because ω is small, $B/(A + D) \approx 0.11$ and $B^2/(AD) \approx 0.08$, Eq. (A2) can be replaced by

$$\dot{\alpha}^2 = \frac{2G_1(A + D)\omega}{AD} \Rightarrow \dot{\alpha}\ddot{\alpha} = \frac{G_1(A + D)\dot{\omega}}{AD} \tag{A4}$$

and Eq. (A3) can be approximated by

$$\dot{\omega} = \dot{\alpha}\frac{[B\cos\alpha - D)(A + D - 2B\cos\alpha) + BG_1 t^2\sin\alpha]}{(A + D)^2}. \tag{A5}$$

Note that approximating the term in square brackets in Eq. (A5) is not advisable since it is zero when $t = t_R = t_0$. Hence the equations may be written

$$\ddot{\alpha} = \frac{G_1}{AD(A + D)}\Big[(B\cos\alpha - D)(A + D - 2B\cos\alpha) + BG_1 t^2\sin\alpha\Big] \tag{A6}$$

$$\beta = \frac{G_1 t^2}{2(A + D - 2B\cos\alpha)} + \frac{AD\dot{\alpha}^2}{2G_1(A + D)}. \tag{A7}$$

Numerical computations show that the solutions of Eqs (A6) and (A7) are within 1.4% of the solutions of Eqs (18) and (19) for $\alpha \leq \pi$ (in all our computations $\alpha \leq \pi$ at impact).

The equation for α can be written as

$$\ddot{\alpha} = -p + q\cos\alpha - r\cos^2\alpha + st^2\sin\alpha, \tag{A8}$$

where

$$p = \frac{G_1}{A}, \quad q = \frac{B(A + 3D)G_1}{AD(A + D)}, \quad r = \frac{2B^2 G_1}{AD(A + D)}, \quad s = \frac{BG_1^2}{AD(A + D)}. \tag{A9}$$

Let $\alpha = \alpha_0 + \alpha_1$, where $\alpha_0(t)$ is the solution when

$$t_R = t_0 = \sqrt{p/s} \tag{A10}$$

and $\alpha_1(t)$ is a small perturbation. The approximate equation for this perturbation is

$$\ddot{\alpha}_1 = -f(t)\alpha_1, \tag{A11}$$

where $f(t) = q \sin \alpha_0 - r \sin 2\alpha_0 - st^2 \cos \alpha_0$. Let $u = \dot{\alpha}_1/\alpha_1$, then it follows that

$$\dot{u} + u^2 + f(t) = 0 \text{ and } \alpha_1 = c \exp\left\{ \int u \mathrm{d}t \right\} \quad \text{(A12)}$$

for some constant, c.

Suppose first that $t_R > t_0$ and consider the value of $u(t_R)$. Set $t_R = t_0 + \Delta$, then

$$\begin{aligned} u(t_R) &= \frac{\dot{\alpha}_1(t_R)}{\alpha_1(t_R)} = \frac{\dot{\alpha}(t_R) - \dot{\alpha}_0(t_R)}{\alpha(t_R) - \alpha_0(t_R)} \\ &= \frac{-\dot{\alpha}_0(t_R)}{\pi/2 - \alpha_0(t_R)} = \frac{\dot{\alpha}_0(t_0 + \Delta)}{\alpha_0(t_0 + \Delta) - \pi/2} \\ &= \frac{\dot{\alpha}_0 + \Delta\ddot{\alpha}_0 + \frac{1}{2}\Delta^2 \dddot{\alpha}_0 + \cdots}{\alpha_0 + \Delta\dot{\alpha}_0 + \frac{1}{2}\Delta^2 \ddot{\alpha}_0 + \frac{1}{6}\Delta^3 \dddot{\alpha}_0 + \cdots - \pi/2}, \end{aligned} \quad \text{(A13)}$$

where, in the last line of Eq. (A13), the argument is t_0. Hence, $u(t_R) = 3/\Delta$, which is indefinitely large when t_R is very close to t_0. Thus, the solution u of Eq. (A12) is required in which $u(t_R)$ is large. The point of interest is the value of t, say T, for which $u(t) \to -\infty$ as $t \to T$. This will imply that $\alpha_1(T) = 0$. Using the stated values for A, B and D and with $G_1 = 200$ Nm (see numerical results and discussion Section) it is found numerically that the value of T is approximately 0.28 s. Thus, for times up to this value, $\alpha_0(t) > \alpha(t)$.

Now consider the case $t_R < t_0$. Set $t_0 = t_R + \Delta$ and consider the value of $u(t_0)$.

$$\begin{aligned} u(t_0) &= \frac{\dot{\alpha}_1(t_0)}{\alpha_1(t_0)} = \frac{\dot{\alpha}(t_0)}{\alpha(t_0) - \pi/2} = \frac{\dot{\alpha}(t_R + \Delta)}{\alpha(t_R + \Delta) - \pi/2} \\ &= \frac{\dot{\alpha} + \Delta\ddot{\alpha} + \cdots}{\alpha + \Delta\dot{\alpha} + \frac{1}{2}\Delta^2\ddot{\alpha} + \cdots - \pi/2} \end{aligned} \quad \text{(A14)}$$

where now, in the last line of Eq. (A14), the argument is t_R. Hence, $u(t_0) = 2/\Delta$ which is again indefinitely large when t_R is very close to t_0. Again, the conclusion is that $\alpha_0(t) > \alpha(t)$ for times up to 0.28 s.

Now consider the behaviour of β. Set $\beta = \beta_0 + \beta_1$, $\alpha = \alpha_0 + \alpha_1$ and substitute these into Eqs (A6) and (A7). This gives the expression

$$\begin{aligned} \beta_1 &= -\frac{G_1 B \alpha_1 t^2 \sin \alpha_0}{(A + D - 2B\cos\alpha_0)^2} + \frac{AD\dot{\alpha}_1\dot{\alpha}_0}{G_1(A + D)} \\ &= -\alpha_1 \left[\frac{BG_1 t^2 \sin \alpha_0}{(A + D - 2B\cos\alpha_0)^2} - \frac{AD\dot{\alpha}_0 u}{G_1(A + D)} \right] \end{aligned} \quad \text{(A15)}$$

for the deviation from β_0, the solution for β when the release time is t_0. Now we know that α_1 is negative up to T and numerical integration shows that the expression in square brackets in Eq. (A15) is positive in this range so that $\beta_1 < 0$ [in Eq. (A15) u is the solution of Eq. (A12) with initial value very large at $t = t_0$]. Hence, $\beta_0(t) < \beta(t)$ for $t \le T$.

The overall work done by the golfer up to impact at $t = t_i$ is given by

$$\int_{\beta_s}^{\beta(t_i)} G_1 \mathrm{d}\beta' = G_1(\beta(t_i) - \beta_s) \quad \text{(A16)}$$

(the couple G_2 does no work since $\alpha = \pi/2$ for $0 \le t \le t_R$ and $G_2 = 0$ for $t > t_R$) and thus it follows that a swing using the 'natural' release (at $t_R = t_0$) requires the least energy input from the player compared with any swing for which $t_R \ne t_0$ (provided that $t_i \le 0.28$ s for the parameters used here). The detailed computations, presented in the numerical results and discussion Section, confirm that this is indeed the case.