12 Series

Syllabus content	Suggested teaching activities
Whole unit	Learners should cover **11 Permutations and combinations** before this topic, to the extent that learners are familiar with factorial notation, combinations and the notation ^nC_r and/or $\begin{pmatrix} n \\ r \end{pmatrix}$. Finding of first terms, common differences and common ratios might involve solving simultaneous equations, therefore we also recommend that learners have studied **6 Simultaneous equations**. As arithmetic and geometric progressions are new to this syllabus, there is little past paper material with which to practice. These topics have been, and are still included in, the Cambridge International AS & A Level Mathematics 9709 syllabus and the Pure Mathematics 1 papers (Paper 1) should be a good source of material for formative assessment. Learners should already be partially familiar with this topic from the syllabus requirements of Cambridge O Level Mathematics or Cambridge IGCSE Mathematics. Learners should be able to manipulate directed numbers, use brackets and extract common factors, expand products of algebraic expressions and manipulate simple algebraic fractions. Learners should also be able to recognise simple arithmetic and geometric sequences from a list of terms and find simple expressions for *n*th terms of such sequences. Learners extend their knowledge of algebraic manipulation to expanding expressions of the form $(a+b)^n$, where n is a positive integer, using the Binomial Theorem. Pascal's Triangle can be used as an initial way of expanding fairly straightforward binomial expansions and these expansions can then be related to the given formula using the work on combinations from **11 Permutations and combinations**. Learners also extend their knowledge of arithmetic and geometric sequences. Work they have covered in Cambridge O Level or Cambridge IGCSE Mathematics for sequences of numbers is translated into algebraic terms. Learners will problem-solve using the structure of each type of sequence. The concept of a series being the sum of the terms of a sequence is introduced and the formulae for these developed. The sum to infinity of appropriate geometric sequence is also considered.

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• use the Binomial Theorem for expansion of $(a+b)^n$ for positive integer n	If learners are not familiar with Pascal's Triangle, introduce this by expanding $(a+b)^2, (a+b)^3$ and $(a+b)^4$ looking at the coefficients of each term when put in an ordered fashion. This will be a good revision of algebraic practice. Ask learners to deduce the expansions of $(a+b)^5$ and $(a+b)^7$. 'The binomial theorem: formulas' is a useful introduction using Pascal's Triangle, which can be found at: www.purplemath.com/modules/binomial.htm . 'The binomial theorem: examples' at: www.purplemath.com/modules/binomial2.htm provides useful examples. Practise some basic expansions by introducing different terms, including negative terms to replace *a* and *b*, making sure that learners understand that if, e.g. $a = 3x$, then $a^4 = (3x)^4$, **not** $3x^4$, which is a **very** common mistake. Show that a better way is needed for expansions where *n* is a large positive integer, by rewriting the numbers in Pascal's triangle in terms of combinations, using the notation ${}^nC_r$ and/or $\binom{n}{r}$. Show learners where the binomial expansion formula comes from so that they can relate it to the work done in **11 Permutations and combinations**. Many examples can be found on the internet. For example, three presentations that follow on from each other, introducing Pascal's Triangle and linking it to combinations resulting in the binomial formula are at: www.khanacademy.org/math/trigonometry/polynomial_and-rational/binomial-theorem. Provide learners with exercises for practice, for example, 'The binomial theorem' worksheet at www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/The%20Binomial%20Theorem.pdf. **(I)**
• use the general term $\binom{n}{r}a^{n-r}b^r$, $0 \leqslant r \leqslant n$ (knowledge of the greatest term and properties of the coefficients is not required)	Ask learners if they can deduce the general term in the expansion of $(a+b)^n$, looking for patterns which can be applied to other expansions. Extend this to expansions of the type $\left(ax+\frac{b}{x}\right)^n$ and introduce the idea of terms independent of *x*. Worksheets involving plenty of examples are available on the internet and may be used for general practice. **(I) (F)** For example, 'The Binomial theorem' at www.kutasoftware.com/FreeWorksheets/Alg2Worksheets/The%20Binomial%20Theorem.pdf

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• recognise arithmetic and geometric progressions	Learners will be familiar with an arithmetic progression (AP) as a number sequence with a common difference between terms, and a geometric progression (GP) as a number sequence with a common multiplier (ratio) between terms. At this level, the problems and formulae are based on the **structure** of the sequences rather than the numbers in the sequence. Therefore, both the language and notation used is more formal. As a whole class starter activity, give learners number sequences and ask them, for example, to identify the first term and difference between the terms (the common difference). Also give them the 1st and 5th terms and ask them to find the term-to-term rule. Matching activities linking a number sequence with its first term and common difference could also be used. Create a Tarsia set of dominoes for this purpose or use a simple two-column arrangement on a whiteboard or handout. Both arithmetic and geometric sequences should be explored at this stage. All of this will be revision of prior knowledge. Use the notation '*a*' and '*d*' for the first term and the difference that is common between the terms of an AP. Similarly use, '*a*' and '*r*' for the first term and the multiplier that is common between the terms of a GP. The first four activities from the 'Sequences and series' resource at www.stem.org.uk/resources/elibrary/resource/32369/sequences-and-series will be helpful to use at various stages. This resources uses subscript notation for terms: a is u_1 and so on. Next, start considering the structure of the sequence. At this stage it is important that learners know that, in an AP, the first term is usually referred to as '*a*' and the common difference is usually given as '*d*' as the formulae they will be using are given in terms of these. (Some texts or websites may use subscript notation such as 1st term is u_1, 2nd term is u_2, ..., *n*th term is u_n and so on.) Learners should understand that they are now going to work with the structure of the sequence as a starting point to solve problems. This helps them understand why the algebra needs to be applied. Once the idea of an AP as a pattern of terms $a,\ a+d,\ a+2d,\ a+3d,\ldots$ has been established, the formula for the *n*th term can be developed. Similarly, it is important for learners to recognise that, in a GP, the first term is usually referred to as '*a*' and the common ratio (multiplier) is usually given as '*r*'. Once the idea of a GP as a pattern of terms $a,\ ar,\ ar^2,\ ar^3,\ldots$ has been established, the formula for the *n*th term can be developed.

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	To help learners deepen their understanding, you could ask them to investigate the development of these patterns of terms in preparation for finding the formulae for the *n*th terms. Free worksheets for learners to practise all the skills they need in these topics – from the basic notation to more complex problem solving – are at: www.kutasoftware.com/freeia2.html . Scroll down to 'Sequences and Series'. (Some questions involve the interpretation of sigma notation which is not required for this syllabus.) **(I)**
• use the formulae for the *n*th term and for the sum of the first *n* terms to solve problems involving arithmetic or geometric progressions	Learners do not need to memorise the formulae as they are given these in the formula sheet at the start of the examination paper. The derivation of each formula is not assessed. However, it is important that learners understand the origins of each formula and know how to use and apply each one. The structure of each *n*th term formulae is relatively simple and most learners should have no difficulty with these. The sum of the first *n* terms formulae are much less intuitive. Develop the formulae for arithmetic progressions first and allow learners the opportunity to practise these before moving on to geometric progressions. 'Arithmetic sequences and sums' at www.mathsisfun.com/algebra/sequences-sums-arithmetic.html provides a simple introduction to AP work (although the sigma notation is not included in this syllabus). The derivation of the formula for the sum to *n* terms is given as a footnote and is useful. There are 10 interactive multiple-choice questions at the bottom of the page for learners to practise their formalised skills. **(I)** The resource 'Sum of n terms of an A.P' at www.geogebra.org/m/nhDAMPgE provides an individual or whole class activity that encourages the correct use of the sum to *n* terms formula: $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$ Ask learners why $(n – 1)d$ is used in the AP formulae, instead of $d(n – 1)$. Some learners may not realise that they are in fact the same expression but that $(n – 1)d$ follows the pattern of the terms in the sequence. Learners will need a lot of practice. Set problem-solving activities for pairs or groups of learners. In these, give learners information about, for example, a specific term and sum and ask them to find the first term and common difference. Learners will need to be able to solve simultaneous equations to be able to do this. Give learners two versions of the formula for the sum to *n* terms of an AP. They need to choose which of these formulae is suitable for the information they have. It is sensible to include plenty of practice with both so that learners work equally well with either version. Learners who have a good understanding of the structure should be able to solve problems more easily. **Extension activity:** 'Risp 8: Arithmetic simultaneous equations' at http://www.s253053503.websitehome.co.uk/risps/risp8.html is an interesting activity linking arithmetic sequences and simultaneous equations. **(I)**

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	When learners have attempted some problems for themselves, summarise the results before moving on to the formulae for a GP. Again, the sum to *n* terms formula will not be intuitive and should be developed even though the derivation of it is not assessed. 'Geometric sequences and sums' at www.mathsisfun.com/algebra/sequences-sums-geometric.html is a simple introduction to GP work (again, the sigma notation is not included in this syllabus). The derivation of the formula for the sum to *n* terms is given as a footnote and is useful. **Extension activity:** The 'Proof sorter – geometric series' activity at https://nrich.maths.org/1398 gives the sum to *n* terms formula as an unordered list that needs to be ordered. This is a challenging and more interactive activity that some learners might enjoy. Again, learners will need a lot of practice. Set problem-solving activities for pairs or groups of learners. In these, give learners information about, for example, a specific term and sum and ask them to find the first term and possible values of the common ratio. Learners will need to be able to solve simultaneous equations to be able to do this. **Extension activity:** Provide learners with examples of terms that are algebraic rather than numerical to add to the challenge. For example: given that the terms $x-2$, $2x-4$, $x+7$ are three consecutive terms of a geometric progression, find the possible values of *x* and of the common ratio. Other challenging questions can be set combining arithmetic and geometric progressions.
• use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression	Even though the derivation of the sum to infinity of a GP is not assessed, learners should have an understanding of where it comes from. Do this by considering some numerical arithmetic progressions, geometric progressions that diverge and geometric progressions that converge and look at their structure. Once learners have seen practical examples of a GP converging only when $\|r\| < 1$, the reasoning can be supported by looking at what happens to the sum to *n* terms formula in these cases. Give learners a variety of sequences and ask them to tell you what happens to the sum as the number of terms increases: Does it increase, does it decrease, does it approach a limit? For some learners this will be intuitive. For other learners, graphing the expressions for the sum to *n* terms for some sequences can help them to visualise this. For example, using Geogebra or other graphing software, consider: AP $y=\frac{x}{2}(2(3)+4(x-1))$ for $x > 0$; as *x* increases then so does *y* AP $y=\frac{x}{2}(2(3)-5(x-1))$ for $x > 0$; as *x* increases then *y* decreases

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	GP $y = \frac{2(1-3^n)}{1-3}$ for $x > 0$; as x increases then so does y GP $y = \frac{2(1-0.5^n)}{1-0.5}$ for $x > 0$; as x increases then y approaches the value 4. Learners might find it helpful to write the formula for the sum to n terms as the difference of two fractions: $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$. The reason for the condition $\|r\| < 1$ is then much clearer. 'Geometric sequences and sums' at www.mathsisfun.com/algebra/sequences-sums-geometric.html is a summary of the effect on the sum to n terms formula when n tends to infinity. There are also 11 interactive multiple-choice questions at the bottom of the page for learners to practise their formalised skills, including the sum to infinity. **(I)** 'Sum of infinite geometric series' at www.geogebra.org/m/j6QR8z3j is an excellent geometric visual presentation that shows what happens to the sum using sectors of a circle. Learners can see the ever-decreasing size of the 'slice' being added to the 'pie'. Once learners are confident with the basis for the sum to infinity, set problems including the formula for this. Practice at problem solving with all the formulae is essential. Learners will have to think carefully about how to solve each one. Learners who have a good understanding and mastery of the skills required should be able to apply them to these problems. Much practice material is available in textbooks or online. For example, an excellent variety of video and worksheet material that can be used to support learning is at: www.mathcentre.ac.uk/; search for 'Arithmetic and geometric progressions'.**(I)** **Extension activity:** • 'Risp 20: When does $S_n = u_n$?' at http://www.s253053503.websitehome.co.uk/risps/risp20.html is a challenging activity linking sums of terms with values of terms. **(I)** • A variety of support material and some excellent material for extension activities is at. https://undergroundmathematics.org/sequences **(I)**

Past and specimen papers
Past/specimen papers and mark schemes are available to download at www.cambridgeinternational.org/support **(I)(F)** 2020 Specimen Paper 1 Q9, Q10 Nov 2017 Paper 12 Q3; Nov 2017 Paper 13 Q7; Nov 2017 Paper 21 Q9 (including 2 Quadratic functions) Jun 2017 Paper 12 Q4; Jun 2017 Paper 21 Q5; Jun 2017 Paper 23 Q6 Mar 2017 Paper 12 Q3 Nov 2016 Paper 11 Q4; Nov 2016 Paper 13 Q4 Jun 2016 Paper 12 Q2; Jun 2016 Paper 21 Q8 Mar 2016 Paper 22 Q5; Mar 2015 Paper 12 Q4