Discrete Convolution

Defination

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

For the well defined functions $f(t)$ and $g(t)$

Suppose $x(n)$ and $y(n)$ are two finite sequences then their convolution defined as

$$x(n) * y(n) = \sum_{r=0}^{n} x(r)\, y(n-r)$$

take $z(n) = x(n) * y(n)$

Suppose $\{x(n)\}$ is a sequence of duration N_1 and $\{y(n)\}$ is a sequence of duration N_2 then $\{z(n)\}$ duration is $N_1 + N_2 - 1$.

For $\{x(n)\} = \{[x(0), x(1), \dots x(n-1)]\}$

Same for y.

$z(n) = x(0)\, y(n) + x(1)\, y(n-1) \dots$
$x(n)\, y(0)$.

$z(0) = x(0)\, y(0)$

$z(1) = x(0)\, y(1) + x(1)\, y(0)$

1. $x(n) = \{1, \underset{\uparrow}{2}, 3, 1\}$ $y(n) = \{2, 1, \underset{\uparrow}{1}, 2\}$.

$z(0) = 2$

$z(1) = 5$

$z(6) : x(0)\,y(6) + 1,5 + 2,4 + 3,3 + 4,2 + 5,1 + 6,0$

$= 2$.

$z(5) = 0,5 + 1,4 + 2,3 + 3,2 + 4,1 + 5,0 = 4 + 2 \quad 2 + 2 = 4$

$z(4) = 0,4 + 1,3 + 2,2 + 3,1 + 4,0 = 1 + 2 + 4 = 7$

$z(3) = 0,3 + 2,1 + 1,2 + 0,3.$
$= 8$

$z(2) = 0,2 + 1,1 + 2,1$
$= 7$.

2, 5, 7, 8, 7, 5, 2

2. $x(n) = \{2, 1, 2, 1\}$ and $y(n) = \{1, 2, 3\}$.
$\quad\quad\quad\;\; 0 \;\; 1 \;\; 2 \;\; 3 \quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; 0 \;\; 1 \;\; 2$

$z(0) = 2$

$4 + 3 - 1 = 6$.

$z(1) = (0,1) + (1,0)$
$= 2 \times 2 + 1 \times 1$
$= 5$

$z(2) = (0,2) + (1,1) + (2,0)$
$6 + 2 + 2$
$= 10$

$z(3) = (0,3) + (1,2) + (2,1) + (3,0)$
$6 + 3 + 4 + 1$
$= 8$

$z(4) = \cancel{(0,4)} + (1,3) + (2,2) + (3,1) + \cancel{(4,0)}$
$= 8$

$z(5) =$
2, 5, 10, 8, 8, 3.

Periodic Sequence

A sequence $x(n)$ is said to be periodic sequence with period N if $x(n+N) = x(n)$

Result.

If $x(n)$ is a periodic sequence of period N and $y(n)$ is also a periodic sequence with period N then their convolution $z(n) = x(n) * y(n)$ is also a periodic N

defination $z(n) = \sum_{r=0}^{N-1} x(r)\, y(n-r)$.

$z(0) = x(0)\, y(0) + x(1)\, y(0-1) + \dots + x(N-1)\, y(0-N+1)$

$z(1) = x(0)\, y(1) + y(1-1)\, x(1) + \dots x(N-1)\, y(1-N+1)$.

- Periodic convolution
- Circular convolution
- Cyclic convolution.

3. $\{x(n)\} = \{1,2,2,1\}$ $\{y(n)\} = \{2,1,1,2\}$.

$x(0) = 1$ $x(2) = 2$ $y(0) = 2$ $y(2) = 1$

$x(1) = 2$ $x(3) = 1$ $y(1) = 1$ $y(3) = 2$

$N = 4$

$z(0) = x(0)\, y(0)$ $= 2 + 4 + 2 + 1$

$+ x(1)\, y(-1)$ $= 9$.

$+ x(2)\, y(-2)$

$+ x(3)\, y(-3)$

$z(1) = x(0)\, y(1) + x(1)\, y(0) + x(2)\, y(-1) + x(3)\, y(-2)$

$= 10$

$z(2) = x(0)\, y(2) + x(1)\, y(1) + x(2)\, y(0) + x(3)\, y(-1)$

$= 9$

$z(3) = x(0)\, y(3) + x(1)\, y(2) + x(2)\, y(1) + x(3)\, y(0)$

$= 8$

Matrix Multiplication Method (MMM)

Consider the two periodic sequences $\{x(n)\} = \{x(0), x(1), x(2), x(3)\}$ and $\{y(n)\} = \{y(0), y(1), y(2), y(3)\}$ each of period $N=4$

Their convolution $\{z(n)\} = \{z(0), z(1), z(2), z(3)\}$.

As $z(n) = x(n) * y(n)$

Take

$$z = \begin{pmatrix} z(0) \\ z(1) \\ z(2) \\ z(3) \end{pmatrix} \quad Y = \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{pmatrix}$$

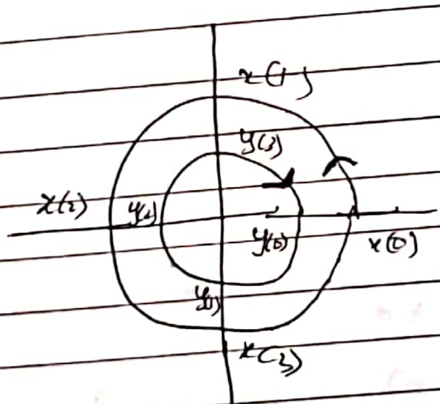
15.75	14.75
17.5	16
17.25	16.75
15.5	16
17.5	17.5

$$X = \begin{pmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(3) \\ x(2) & x(1) & x(0) & x(2) \\ x(3) & x(2) & x(1) & x(0) \end{pmatrix}$$

$$z = X \times Y$$

Circular Representation Method (CRM)

Consider $\{x(n)\}$ $\{y(n)\}$ period = 4



$$z(0) = x(0).y(0) + x(1)\,y(3) + x(2)\,y(2) + x(3)\,y(1)$$

$$z(1) = x(0)\,y(1) + x(1)\,y(0) + x(2)\,y(3)$$
$$x(3)\,y(2)$$

↳ For this Rotate inner circle by one unit anticlockwise.

Keep x Const and keep rotating y in the anti

Discrete Fourier Transform

Suppose $\{x(n)\}$ is assumed to be a finite sequence of duration N then the Discrete Fourier Transform of $x\{(n)\}$ denotes as DFT $\{x(n)\}$ and defined as.

$$\mathrm{DFT}\{x(n)\} = X(K) \quad \text{for } K = 0, 1, \ldots N-1$$

$$\mathrm{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{2\pi i n k}{N}}$$

Phase factor

$$W_N = e^{-\frac{2\pi i}{N}}$$

$$\mathrm{DFT}\{x(n)\} = \sum_{n=0}^{N-1} x(n)\, W_N^{nK}$$

Inverse formula

$$\mathrm{IDFT}\{X(K)\} = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{\frac{2\pi i n k}{N}}$$

$$\mathrm{IDFT}\{X(K)\} = x(n) \quad \text{for } n = 0, 1, \ldots N-1$$

$$x(n) = \mathrm{IDFT}\{X(K)\} = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-nK}$$

PTO

Properties.

Linear Property $DFT\{c_1 x(n) + c_2 y(n)\} = c_1 DFT\{x(n)\} + c_2 DFT\{y(n)\}$

Shifting property

$$DFT\{x(n-m)\} = W_N^{mk} DFT\{x(n)\}$$

period N.

Proff

$$DFT\{x(n-m)\} = \sum_{n=0}^{N-1} x(n-m) W_N^{nk}$$

put $r = n-m$ in the RHS of the sum

$n = r+m$

$$DFT\{x(n-m)\} = \sum_{r+m=0}^{r+m=N-1} x(r) W_N^{(r+m)k}.$$

$$DFT\{x(n-m)\} = W_N^{mk} \sum_{r=-m}^{r=m+N-1} x(r) W_N^{rk} \rightarrow ①$$

Consider

$$W_N = e^{-\frac{2\pi i}{N}}$$

$$W_N^{rk} = \left(e^{-\frac{2\pi i}{N}}\right)^{rk}$$

$$= e^{-\frac{2\pi i rk}{N}}$$

$$W_N^{(r+N)k} = e^{-\frac{2\pi i (r+N)k}{N}}$$

$$W_N^{(r+N)k} = W_N^{rk} \text{ (split and } e^{-2\pi i} \text{ is 1)} \rightarrow ②$$

From ② ① becomes.

$$\sum_{r=-m}^{r=m+N-1} x(r) W_N^{rk} = \sum_{r=0}^{r=N-1} x(r) W_N^{rk}.$$

$$DFT\{x(n-m)\} = W_N^{mk} \sum_{r=-m}^{r=m+N-1} x(r) W_N^{rk}.$$

$$= W_N^{mk} \sum_{r=0}^{r=N-1} x(r) W_N^{rk}.$$

Convolution Property.

$$DFT\{\underbrace{x(n) * y(n)}\} = DFT\{x(n)\} DFT\{y(n)\}.$$

Both are periodic.

Examples

1. $DFT\{1, -1, 1, -1\}$

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk}.$$

$$W_4 = e^{-\frac{2\pi i}{4}}$$

$$= e^{-\frac{\pi i}{2}}$$

$$X(k) = x(0) + x(1) W_4^{k} + x(2) W_4^{2k} + x(3) W_4^{3k}.$$

$$= \cos\frac{\pi}{2} - i \sin\frac{\pi}{2}$$

$$W_4 = -i$$

$1 - i, \; i^2 - t$ $\qquad i^3 = i^2 i = -i$

$X(0) = x(0) + x(1) + x(2) + x(3) \qquad = 0$

$0 \leftarrow X(1) = x(0) + x(1) W_4^1 + x(2) W_4^2 + x(3) W_4^3$

$4 \leftarrow X(2) = x(0) + x(1) W_4^2 + x(2) W_4^4 + (3) W_4^6$

$0 \leftarrow X(3) = x(0) = x(1) W_4^3 + x(2) W_4^6 + x(3) W_4^9$

$X(K) = \{0, 0, 4, 0\}$

<u>Fast Fourier Transform</u>

Fast Fourier Transform is an algorithm to compute the DFT

Suppose $\{x(n)\}$ is a finite sequence of duration N then DFT $\{x(n)\}$ requires N^2 numbers of operation.

In the same case, FFT requires $N \log_2 N$ num of operation.

Suppose $N = 16$

$N^2 = 256$

$N \log N = 16 \log_2 16$
$= 64.$

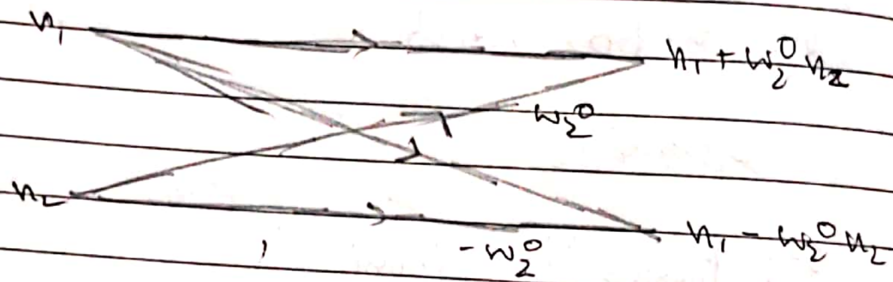
CON

→ Applicable for only radix 2 ($2^{\text{something}}$).

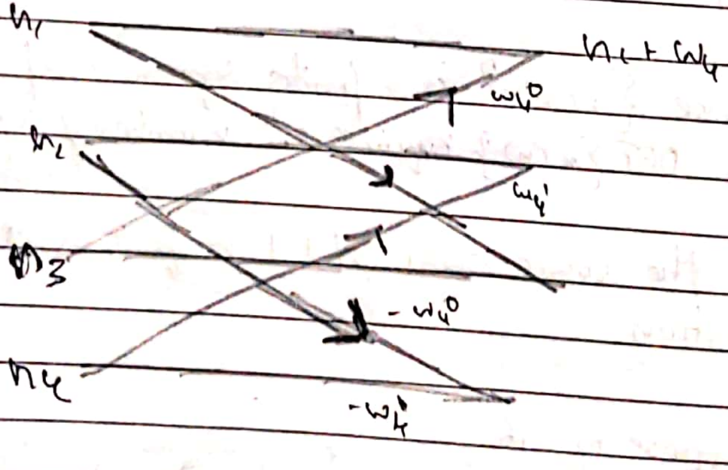
FFT Algo

- Decimation-in-Time (DIT) Algorithm
- Decimation in Frequency (DIF) Algorithm

Two point formula



Four Point Formula



Balance factor only ~~on~~ for ~~xx~~ raising

Decimation Process:

$\{x(n)\} = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

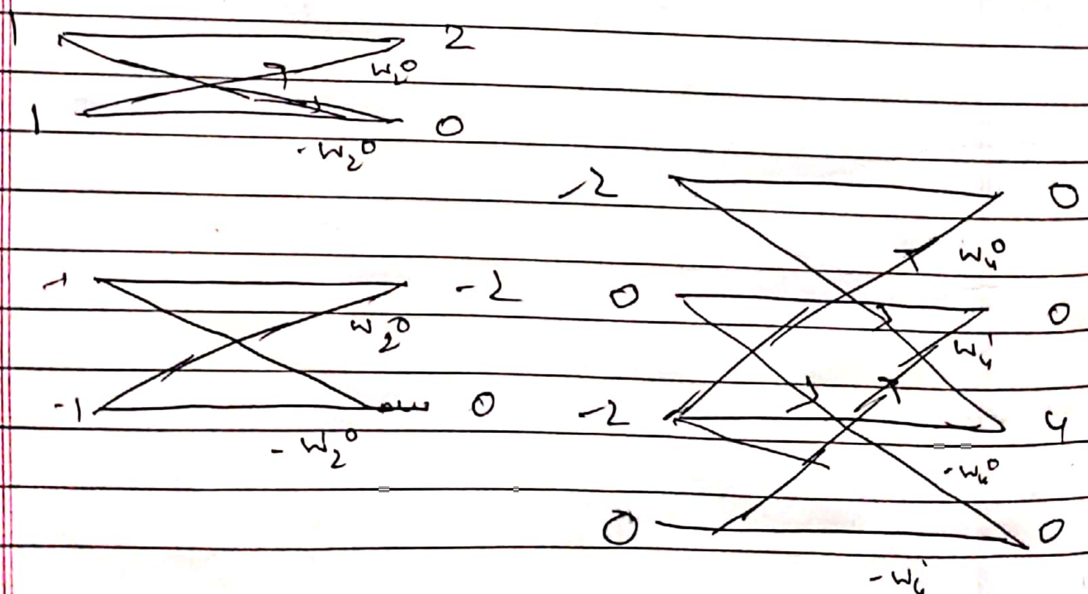
$\{x(0), x(2), x(4), x(6)\}$ $\{x(1), x(3), x(5), x(7)\}$

$\{x(0), x(4)\}$ $\{x(2), x(6)\}$ $\{x(1), x(5)\}$ $\{x(3), x(7)\}$

1. Find the Discrete Fourier Transform $x(n) = \{1, -1, 1, -1\}$ using DIT-FFT.

$\{1, -1, 1, -1\}$.

$\{1, 1\}$ $\{-1, -1\}$.



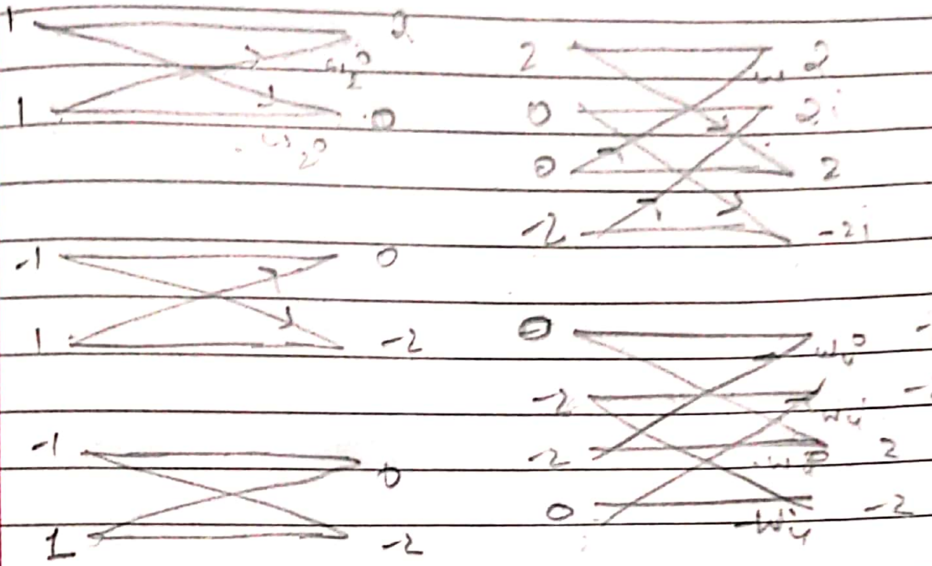
$W_2 = -1$

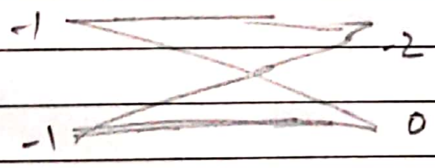
$W_4 = -i$ $W_8 = \frac{1-i}{\sqrt{2}}$

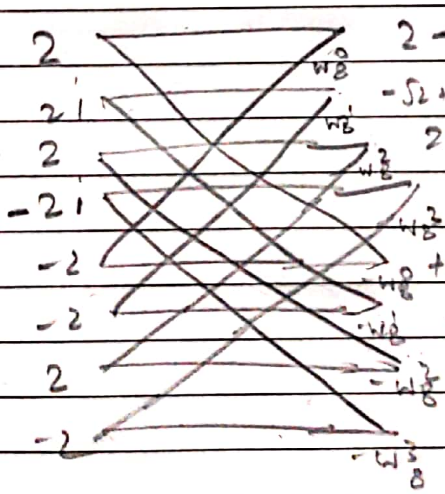
2. $\{1, -1, -1, -1, 1, 1, 1, -1\}$

$\{1 \ -1 \ 1 \ 1\}$ $\{-1 \ -1 \ 1 \ -1\}$

$\{1, 1\}$ $\{-1, 1\}$ $\{-1, 1\}$ $\{-1, -1\}$







$2 - 2i(1) = 2 - 2i = 0$

$-\sqrt{2} + i(2+\sqrt{2})$

$2 - 2i$

$\sqrt{2} + i(-2 + \sqrt{2})$

$+4$

$\cancel{-2\sqrt{2} - \sqrt{2}i}$ $\sqrt{2} + i(2 - \sqrt{2})$

$2 + 2i$

$-\sqrt{2} + i(2 + \sqrt{2})$.

$-2 +$

N