Polynomials and Hankel Matrices

Miroslav Fiedler
Czechoslovak Academy of Sciences
Institute of Mathematics
Žitná 25
115 67 Praha 1, Czechoslovakia

Submitted by V. Pták

ABSTRACT

Compatibility of a Hankel $n \times n$ matrix H and a polynomial f of degree m, $m \leqslant n$, is defined. If $m = n$, compatibility means that $HC_f^T = C_f H$ where C_f is the companion matrix of f. With a suitable generalization of C_f, this theorem is generalized to the case that $m < n$.

INTRODUCTION

By a Hankel matrix [5] we shall mean a square complex matrix which has, if of order n, the form (α_{i+k}), $i, k = 0, \dots, n-1$.

If $H = (\alpha_{i+k})$ is a singular $n \times n$ Hankel matrix, the H-polynomial $\varphi_H(x)$ of H was defined [3] as the greatest common divisor of the determinants of all $(r+1) \times (r+1)$ submatrices of the matrix

$$H_x = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n & x \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n & \cdots & \alpha_{2n-2} & x^{n-1} \end{pmatrix}, \tag{1}$$

where r is the rank of H. In other words, φ_H is that polynomial for which the $n \times (n+1)$ matrix

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & \varphi_H(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

52 Vanderbilt Ave., New York, NY 10017 0024-3795/85/$3.30

is the Smith normal form [6] of H_x. It has also been shown [3] that φ_H is a (nonzero) polynomial of degree at most r.

It is known [4] that to a nonsingular $n \times n$ Hankel matrix $H = (\alpha_{i+k})$ a linear pencil of polynomials of degree at most n can be assigned as follows:

$$f(x) \equiv f_0 + f_1 x + \cdots + f_n x^n \tag{2}$$

belongs to this pencil iff either $n = 1$, or $n > 1$ and

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-2} & \alpha_{n-1} & \cdots & \alpha_{2n-2} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} = 0. \tag{3}$$

Here we shall say, even for the case that $H = (\alpha_{i+k})$ is a singular $n \times n$ Hankel matrix, that H is compatible with the polynomial (2) if (3) is satisfied.

We shall investigate the compatibility of polynomials and (Hankel) matrices; in particular, we shall show that, given a nonzero polynomial f of degree at most n, H is a Hankel matrix compatible with f iff a relation of the form

$$H\Gamma_f^T = \Gamma_f H$$

is satisfied, where Γ_f is an $n \times n$ matrix depending on f only.

1. NOTATION AND PRELIMINARIES

For a monic polynomial of degree $m \geqslant 1$

$$f(x) \equiv x^m - u_{m-1}x^{m-1} - \cdots - u_1 x - u_0, \tag{4}$$

we denote by C_f the companion matrix of f:

$$C_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ u_0 & u_1 & u_2 & \cdots & u_{m-2} & u_{m-1} \end{pmatrix}.$$

Let us recall that for such f, the infinite companion matrix C_f^∞ was defined [8] as the infinite matrix with m columns whose first m rows form the identity matrix whereas each further row is a linear combination of the preceding m rows with coefficients $u_0, u_1, \dots, u_{m-1}$. For $n \geqslant m$, the finite section of C_f^∞ with the first n rows will be called $C_f^{(n)}$.

We shall also need the concept of the Vandermonde matrix associated to a polynomial (with ordered roots). Let f of (4) satisfy

$$f(x) \equiv \prod_{i=1}^{s} (x - t_i)^{m_i}, \qquad t_i \text{ distinct}, \quad m_i \geqslant 1, \quad i = 1, \dots, s. \tag{5}$$

For any $p \geqslant m = \sum_{i=1}^{s} m_i$, we denote by $V_f^{(p)}$ the $p \times m$ matrix

$$V_f^{(p)} = \left(S_{m_1}^{(p)}(t_1), S_{m_2}^{(p)}(t_2), \dots, S_{m_s}^{(p)}(t_s) \right),$$

where

$$S_m^{(p)}(t) = (q_{ij}), \qquad i = 0, \dots, p-1, \quad j = 0, \dots, m-1, \tag{6}$$

$$q_{ij} = \binom{i}{j} t^{i-j}. \tag{7}$$

The square matrix $V_f^{(m)}$ will be denoted by V_f. It is well known that V_f is nonsingular. If we denote by V_f^∞ the infinite Vandermonde matrix (it has m columns, and its kth row is the last row of $V_f^{(k)}$, $k = 1, 2, \dots$), then the following two lemmata are clear:

LEMMA (1, 1). *For $k \geqslant m+1$, the kth row of V_f^∞ is a linear combination of the preceding m rows with coefficients $u_0, u_1, \dots, u_{m-1}$.*

LEMMA (1, 2). *We have*

$$V_f^\infty = C_f^\infty V_f \quad \text{and} \quad V_f^{(k)} = C_f^{(k)} V_f, \quad k = 1, 2, \dots .$$

Let us denote by $J_k(t)$ the Jordan $k \times k$ block with t along the diagonal and ones in the superdiagonal. For $f(x)$ from (4) with ordered roots $t_1, \dots, t_s$ as in (5), let J_f be the block diagonal matrix

$$J_f = \operatorname{diag}\left(J_{m_1}(t_1), \dots, J_{m_s}(t_s) \right).$$

The following is well known:

LEMMA (1,3). *J_f is the Jordan normal form of C_f:*

$$C_f V_f = V_f J_f.$$

In the following lemma, a (rectangular) $p \times q$ Hankel matrix means a matrix

$$(\alpha_{i+j}), \qquad i = 0,\dots,p-1, \quad j = 0,\dots,q-1.$$

LEMMA (1,4). *Let $B = (\beta_{i+j})$, $i = 0,\dots,p-1$, $j = 0,\dots,q-1$ be a $p \times q$ Hankel matrix; let r, s be positive integers, t a complex number. Then,*

$$H = S_p^{(r)}(t) B \left(S_q^{(s)}(t)\right)^T$$

is an $r \times s$ Hankel matrix:

$$H = (\gamma_{i+j}), \qquad i = 0,\dots,r-1, \quad j = 0,\dots,s-1,$$

where

$$\gamma_m = \sum_{k=0}^{r+s} \binom{m}{k} \beta_k t^{m-k}, \qquad m = 0,\dots,r+s-2. \tag{8}$$

Proof. Compute the entry h_{ij} of H:

$$\begin{aligned} h_{ij} &= \sum_{u=0}^{i} \sum_{v=0}^{j} \binom{i}{u} t^{i-u} \beta_{u+v} \binom{j}{v} t^{j-v} \\ &= \sum_{k=0}^{i+j} \beta_k t^{i+j-k} \sum_{\substack{u,v \\ u \geq 0,\, v \geq 0, \\ u+v=k}} \binom{i}{u}\binom{j}{v} \\ &= \sum_{k=0}^{i+j} \binom{i+j}{k} \beta_k t^{i+j-k}, \end{aligned}$$

which implies (8). ∎

In the next lemma, we say that a Hankel matrix $B=(\beta_{i+j})$, $i, j=0,\dots,k-1$ is upper triangular if $\beta_k=\beta_{k+1}=\cdots=\beta_{2k-2}=0$. Similarly, B is a lower triangular Hankel matrix if in the same notation, $\beta_0=\beta_1=\cdots=\beta_{k-2}=0$.

LEMMA (1,5). *A $k\times k$ matrix B is upper triangular Hankel iff for the Jordan block $J_k(0)$ (the shift matrix),*

$$BJ_k^T(0)=J_k(0)B.$$

Also, B is lower triangular Hankel iff

$$BJ_k(0)=J_k^T(0)B.$$

REMARK (1,6). Observe that (3) can also be written in the form

$$F_f^{(n-1)}[\alpha]=0, \tag{9}$$

where for $k=1,2,\dots$

$$F_f^{(k)}=\begin{pmatrix} f_0 & f_1 & \cdots & f_n & 0 & \cdots & 0 & 0\\ 0 & f_0 & \cdots & f_{n-1} & f_n & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & f_{n-1} & f_n \end{pmatrix} \tag{10}$$

has k rows (and $n+k$ columns) and

$$[\alpha]=(\alpha_0,\alpha_1,\dots,\alpha_{2n-2})^T. \tag{11}$$

We shall also need a well-known result [6]:

LEMMA (1,7). *If A is an $m\times m$ matrix, B an $n\times n$ matrix such that A and B have no eigenvalue in common, then the only $m\times n$ matrix X satisfying*

$$AX-XB=0$$

is the zero matrix.

2. RESULTS

The following theorem is partly known [1,2].

THEOREM (2,1). *Let*

$$f(x) \equiv \prod_{i=1}^{s} (x - t_i)^{n_i},$$

t_i distinct, $n_i \geq 1$, be a polynomial of degree n; let H be an $n \times n$ matrix. Then the following are equivalent:

(i) *H is a Hankel matrix compatible with f;*
(ii) *$HC_f^T = C_f H$;*
(iii) *$V_f^{-1}H(V_f^{-1})^T$ is a block diagonal matrix with blocks of dimensions $n_1, \dots, n_s$ which are upper triangular Hankel matrices.*

Proof. Let $f(x) \equiv f_0 + f_1 x + \cdots + f_n x^n$, $f_n = 1$.
(i) $\to$ (ii): If $H = (\alpha_{i+k})$ satisfies (3), then

$$HC_f^T = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & -(\alpha_0 f_0 + \cdots + \alpha_{n-1} f_{n-1}) \\ \alpha_2 & \alpha_3 & \cdots & \alpha_n & -(\alpha_1 f_0 + \cdots + \alpha_n f_{n-1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \alpha_{n+1} & \cdots & \alpha_{2n-2} & -(\alpha_{n-1} f_0 + \cdots + \alpha_{2n-2} f_{n-1}) \end{pmatrix}.$$

By (3), HC_f^T is symmetric, so that

$$HC_f^T = \left(HC_f^T\right)^T = C_f H.$$

(ii) $\to$ (iii): By Lemma (1,3) $C_f V_f = V_f J_f$. Therefore,

$$\begin{aligned} J_f V_f^{-1} H\left(V_f^{-1}\right)^T &= V_f^{-1} C_f H\left(V_f^{-1}\right)^T \\ &= V_f^{-1} H C_f^T \left(V_f^{-1}\right)^T \\ &= V_f^{-1} H\left(V_f^{-1}\right)^T J_f^T. \end{aligned}$$

Since J_f is block diagonal and the eigenvalues of any two blocks are mutually distinct, $V_f^{-1}H(V_f^{-1})^T = D$ is, by Lemma (1,7), also block diagonal with blocks of dimensions $n_1, \ldots, n_s$. Since the ith block D_i of D satisfies $J_{n_i}(t_i)D_i = D_i(J_{n_i}(t_i))^T$, D_i is, by Lemma (1,5), upper triangular Hankel.

(iii) $\to$ (i): Assuming (iii),

$$H = V_f D V_f^T, \tag{12}$$

where $D = \operatorname{diag}(D_i)$, D_i upper triangular Hankel. By Lemma (1,4), H is a sum of Hankel matrices and hence a Hankel matrix. It follows from (12) and Lemma (1,4) that the left-hand side of (3) can be written as

$$\left(S_{n_1}^{(n-1)}(t_1), \ldots, S_{n_s}^{(n-1)}(t_s)\right) D \left(S_{n_1}^{(n+1)}(t_1), \ldots, S_{n_s}^{(n+1)}(t_s)\right)^T [f]$$

$$= V_f^{(n-1)} D \left(V_f^{(n+1)}\right)^T [f],$$

where $[f] = (f_0, f_1, \ldots, f_n)^T$. Since $(V_f^{(n+1)})^T [f] = 0$, (3) is satisfied and H is compatible with f. ∎

In the next theorem, the general case will be considered.

THEOREM (2,2). *Let m, n be integers, $n \geqslant m \geqslant 1$. Let $f(x)$ be a monic polynomial of degree m,*

$$f(x) = \prod_{i=1}^{s} (x - t_i)^{m_i}.$$

For t complex, denote by $\Gamma_f(t)$ the $n \times n$ matrix

$$\Gamma_f(t) = \begin{pmatrix} C_f & 0 \\ ZC_f - J_{n-m}^T(t) Z & J_{n-m}^T(t) \end{pmatrix},$$

where Z is the $(n-m) \times m$ matrix for which

$$\begin{pmatrix} I \\ Z \end{pmatrix} = C_f^{(n)}.$$

If H is an $n \times n$ matrix, then the following are equivalent:

(i) *H is a Hankel matrix compatible with f (as a polynomial of degree at most n);*

(ii) *for any t for which $f(t) \neq 0$, $H\Gamma_f^T(t) = \Gamma_f(t)H$;*
(iii) *there exists a number t for which $f(t) \neq 0$ such that*

$$H\Gamma_f^T(t) = \Gamma_f(t)H;$$

(iv) *if*

$$\tilde{V}_f = \left(V_f^{(n)}, P\right)$$

where

$$P = \begin{pmatrix} 0 \\ I_{n-m} \end{pmatrix},$$

then $\tilde{V}_f^{-1}H(\tilde{V}_f^{-1})^T$ is block diagonal, $D = \operatorname{diag}(D_1, \ldots, D_s, D_0)$, where D_i of order m_i is upper triangular Hankel, $i = 1, \ldots, s$, and D_0 of order $n - m$ is lower triangular Hankel;

(v) *H is a Hankel matrix which is either nonsingular with f belonging to the linear pencil of polynomials* (3), *or singular with f completely divisible by the H-polynomial φ_H of H (this means: the multiplicity of each root of φ_H, including that of infinity if φ_H is considered as a polynomial of degree r, the rank of H, is less than or equal to the multiplicity of this as root of f, including infinity if f is considered as a polynomial of degree n).*

Proof. (i) $\to$ (ii): If $m = n$, the assertion holds by Theorem (2,1). Thus let $m < n$. Observe that $\Gamma_f(t)$ can be written as

$$\Gamma_f(t) = \begin{pmatrix} I_m & 0 \\ Z & I_{n-m} \end{pmatrix} \begin{pmatrix} C_f & 0 \\ 0 & J_{n-m}^T(t) \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -Z & I_{n-m} \end{pmatrix}. \tag{13}$$

Let H_0 be the $m \times m$ upper-left-corner submatrix of H. Since $f_{m+1} = \cdots = f_n = 0$ in (3), it follows that H_0 is a Hankel matrix compatible with f (as a polynomial of degree at most m). By Theorem (2,1), $V_f^{-1}H_0(V_f^{-1})^T = D$ is a block diagonal matrix whose diagonal blocks have orders $m_1, \ldots, m_s$ and are upper triangular Hankel matrices. Therefore, the $n \times n$ matrix H_1 defined as

$$H_1 = C_f^{(n)}H_0\left(C_f^{(n)}\right)^T \tag{14}$$

can be written as $C_f^{(n)}V_f D V_f^T (C_f^{(n)})^T$, which is, by Lemma (1,2), equal to $V_f^{(n)} D (V_f^{(n)})^T$. Thus it is, by Lemma (1,4), a Hankel matrix.

If we form an analogous product using H_1 as on the left-hand side of (3), we see easily that it equals $V_f^{(n-1)} D (V_f^{(n+1)})^T [f]$, which is zero. Thus H_1 is compatible with f.

The parameters α_k of H coincide with those of H_1 for $k = 0, \ldots, 2m-2$. However, both matrices H, H_1 are compatible with f. Therefore, (9) implies that if $\beta_0, \beta_1, \ldots, \beta_{2n-2}$ are the parameters of $H - H_1$, then

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_{m-1} & f_m & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & f_0 & \cdots & f_{m-2} & f_{m-1} & f_m & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_0 & \cdots & f_{m-2} & f_{m-1} & f_m & 0 & \cdots & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{2n-2} \end{pmatrix} = 0,$$

where the matrix on the left-hand side has $n - m$ zeros as the last entries in the last row. Thus, $\beta_0, \ldots, \beta_{2m-2}$ being zero, we have $\beta_{2m-1} = \cdots = \beta_{m+n-2} = 0$ as well. Consequently, $H_2 = H - H_1$ is a Hankel matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \tilde{H}_2 \end{pmatrix}, \tag{15}$$

where $\tilde{H}_2$ is a lower triangular Hankel matrix of order $n - m$.

Now, (13) and (14) imply

$$\begin{aligned} H_1 \Gamma^T(t) &= \begin{pmatrix} I_m \\ Z \end{pmatrix} H_0 (I_m, Z^T) \begin{pmatrix} I_m & -Z^T \\ 0 & I_{n-m} \end{pmatrix} \\ &\quad \times \begin{pmatrix} C_f^T & 0 \\ 0 & J_{n-m}(t) \end{pmatrix} \begin{pmatrix} I_m & Z^T \\ 0 & I_{n-m} \end{pmatrix} \\ &= \begin{pmatrix} I_m \\ Z \end{pmatrix} H_0 C_f^T (I_m, Z^T) \\ &= \begin{pmatrix} I_m \\ Z \end{pmatrix} C_f H_0 (I_m, Z^T), \end{aligned}$$

by (ii) of Theorem (2,1). Thus $H_1\Gamma_f^T(t)$ is symmetric, which implies

$$H_1\Gamma_f^T(t) = \Gamma_f(t)H_1. \tag{16}$$

Also, (15) implies

$$H_2\Gamma_f^T(t) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{H}_2 J_{n-m}(t) \end{pmatrix}.$$

Since $\tilde{H}_2 J_{n-m}(t)$ is symmetric by Lemma (1,5) applied to the lower triangular $\tilde{H}_2$, $H_2\Gamma_f^T(t)$ is also symmetric, so that by (16), (ii) follows.

(ii) $\to$ (iii): Obvious.

(iii) $\to$ (iv): As in the proof of (ii) $\to$ (iii) of Theorem (2,1), we get that the matrices

$$\begin{pmatrix} J_f & 0 \\ 0 & J_{n-m}^T(t) \end{pmatrix} \quad \text{and} \quad \tilde{V}_f^{-1} H \left(\tilde{V}_f^{-1}\right)^T$$

commute. Thus D is block diagonal of the form asserted.

(iv) $\to$ (v): By (iv), H is of the form

$$\sum_{i=1}^{s} S_{m_i}^{(n)}(t_i) D_i \left(S_{m_i}^{(n)}(t_i)\right)^T + \tilde{D}_0,$$

where

$$\tilde{D}_0 = \begin{pmatrix} 0 & 0 \\ 0 & D_0 \end{pmatrix}.$$

By Lemma (1,4), H is Hankel. If all D_i, $i = 0, \ldots, s$, are nonsingular, then H is nonsingular and the matrix $\hat{H}$ on the left-hand side of (3) can again be written as

$$\sum_{i=1}^{s} S_{m_i}^{(n-1)}(t_i) D_i \left(S_{m_i}^{(n+1)}(t_i)\right)^T + \hat{D}_0, \tag{17}$$

where

$$\hat{D}_0 = \begin{pmatrix} 0 & 0 \\ 0 & D_0 \end{pmatrix}$$

is $(n-1)\times(n+1)$. In the case that the sum in (17) is missing and $H = D_0$, $\hat{H}$ has first column zero. Since $f(x) \equiv f_0$, (3) is satisfied. Otherwise,

$$\hat{H} = \left(S_{m_1}^{(n-1)}(t_1),\dots, S_{m_s}^{(n-1)}(t_s), P_1\right)D\tilde{V}_f,$$

where

$$P_1 = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

is $(n-1)\times(n-m)$ and

$$\tilde{V}_f = \begin{pmatrix} \left(S_{m_1}^{(n+1)}(t_1)\right)^T \\ \vdots \\ \left(S_{m_s}^{(n+1)}(t_s)\right)^T \\ P_2^T \end{pmatrix}$$

with $P_2^T = (0, I)$ of dimension $(n-m)\times(n+1)$. Since $\tilde{V}_f[f] = 0$ for $[f] = (f_0, f_1,\dots, f_n)^T$ in the notation (2), Equation (3) is satisfied.

Now let H be singular. To prove the assertion in this case, it suffices to show that the H-polynomial of H has the form

$$h(x) \equiv \prod_{i=1}^{s} (x-t_i)^{p_i}$$

and the rank r of H is $\sum_{i=1}^{s} p_i + p_0$, where p_i is the rank of D_i, $i = 0,\dots, s$. This is clearly equivalent to the following: For any $n\times(n-r-1)$ matrix P and any $(n-r)\times n$ matrix Q [and $X = (1, x,\dots, x^{n-1})^T$],

$$\det\begin{pmatrix} 0 & Q & 0 \\ P & H & X \end{pmatrix} \tag{18}$$

is divisible by $h(x)$ and (if nonzero) its degree does not exceed $n - p_0$. To prove this, let us write, for $i = 1,\dots, s$,

$$D_i = \begin{pmatrix} \hat{D}_i & 0 \\ 0 & 0 \end{pmatrix},$$

where $\hat{D}_i$ is $p_i \times p_i$, and

$$D_0 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D}_0 \end{pmatrix},$$

where $\hat{D}_0$ is $p_0 \times p_0$. Then,

$$H = \sum_{i=1}^{s} S_{p_i}^{(n)}(t_i)\hat{D}_i\left(S_{p_i}^{(n)}(t_i)\right)^T + \tilde{D}_0,$$

$$\tilde{D}_0 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D}_0 \end{pmatrix}.$$

Defining $\hat{V} = (S_{p_1}^{(n)}(t_1), \dots, S_{p_s}^{(n)}(t_s), \hat{P})$ where

$$\hat{P} = \begin{pmatrix} 0 \\ I_{p_0} \end{pmatrix},$$

then

$$H = \hat{V}\hat{D}\hat{V}^T \qquad \text{for} \quad \hat{D} = \operatorname{diag}(\hat{D}_1, \dots, \hat{D}_s, \hat{D}_0).$$

Therefore, (18) can be written as

$$\det\begin{pmatrix} 0 & Q & 0 \\ P & H & X \end{pmatrix} = \det\begin{pmatrix} 0 & Q & 0 \\ P & VDV^T & X \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} 0 & 0 & 0 & I \\ P & \hat{V} & X & 0 \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & D\hat{V}^T & 0 \\ 0 & 0 & I \\ 0 & Q & 0 \end{pmatrix}\right)$$

$$= \pm\det(P, \hat{V}, X)\det\begin{pmatrix} D\hat{V}^T \\ Q \end{pmatrix}.$$

Since the first factor is divisible by $h(x)$ and has degree at most $n - p_0$, the same is true of (18).

(v) $\to$ (i): This being trivial for H nonsingular, let $H = (\alpha_{i+k})$ be singular. Then the H-polynomial $\varphi_H(x)$ of H exists, and its degree does not exceed r.

We can thus write

$$\varphi_H(x) = g_0 + g_1 x + \cdots + g_r x^r.$$

Now, the proof of the following assertion is left to the reader: For

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_r & 0 & \cdots & 0 & 0 \\ 0 & g_0 & \cdots & g_{r-1} & g_r & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & g_{r-1} & g_r \end{pmatrix} \tag{19}$$

of dimensions $(n-r) \times r$,

$$GH = 0,$$

which can also be written as

$$\hat{G}[\alpha] = 0, \tag{20}$$

where $\hat{G}$ has a similar form to (19) but is $(2n-r-1)\times(2n-1)$, and $[\alpha]$ is as in (11).

Let $f(x)$ as in (2) be any multiple of $\varphi_H(x)$ by a nonzero polynomial $k(x)$ of degree not exceeding $n-r$:

$$k(x) \equiv k_0 + k_1 x + \cdots + k_{n-r} x^{n-r},$$

$$f(x) = \varphi_H(x) k(x).$$

Defining the $(n-1)\times(2n-r-1)$ matrix

$$K = \begin{pmatrix} k_0 & k_1 & \cdots & k_{n-r} & 0 & \cdots & 0 & 0 \\ 0 & k_0 & \cdots & k_{n-r-1} & k_{n-r} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & k_{n-r-1} & k_{n-r} \end{pmatrix},$$

we obtain easily for $F_f^{(n-1)}$ from (10) that

$$F_f^{(n-1)} = K\hat{G}.$$

By (20), $F_f^{(n-1)}[\alpha] = 0$ and, using Remark (1,6), (3) is satisfied. ■

REFERENCES

1 S. Barnett and M. J. C. Gover, Some extensions of Hankel and Toeplitz matrices, *Linear and Multilinear Algebra* 14:45–65 (1983).
2 B. N. Datta, Application of Hankel matrices of Markov parameters to the solutions of the Routh-Hurwitz and the Schur-Cohn problems, *J. Math. Anal. Appl.* 68:276–290 (1979)
3 M. Fiedler, Quasidirect decompositions of Hankel and Toeplitz matrices, *Linear Algebra Appl.*, to appear.
4 M. Fiedler, Hankel and Loewner matrices, *Linear Algebra Appl.* 58:75 (1984).
5 F. R. Gantmacher, *Matrix Theory*, Chelsea, New York, 1959.
6 M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, *Allyn & Bacon*, *Boston*, 1964.
7 V. Pták, Spectral radius, norms of iterates and the critical exponent, *Linear Algebra Appl.* 1:245–260 (1968).
8 V. Pták, An infinite companion matrix, *Comment. Math. Univ. Carolin.* 19:447–458 (1978).
9 V. Pták, Biorthogonal systems and the infinite companion matrix, *Linear Algebra Appl.* 49:57–78 (1983).

Received 13 February 1984; revised 30 April 1984