ON THE SECURITY OF SUBSPACE SUBCODES OF REED–SOLOMON CODES FOR PUBLIC KEY ENCRYPTION

by

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Abstract. — This article discusses the security of McEliece-like encryption schemes using subspace subcodes of Reed–Solomon codes, *i.e.* subcodes of Reed–Solomon codes over \mathbb{F}_{q^m} whose entries lie in a fixed collection of \mathbb{F}_{q^-} subspaces of \mathbb{F}_{q^m} . These codes appear to be a natural generalisation of Goppa and alternant codes and provide a broader flexibility in designing code based encryption schemes. For the security analysis, we introduce a new operation on codes called the *twisted product* which yields a polynomial time distinguisher on such subspace subcodes as soon as the chosen \mathbb{F}_{q^-} subspaces have dimension larger than m/2. From this distinguisher, we build an efficient attack which in particular breaks some parameters of a recent proposal due to Khathuria, Rosenthal and Weger.

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1. Introduction

In the late 70's, at the very beginning of public key cryptography, McEliece proposed a public key encryption scheme whose security relies on the hardness of the bounded decoding problem [BMvT78]. However, the system should be instantiated with a public code equipped with an efficient decoding algorithm and which should be computationally indistinguishable from an arbitrary code. In his seminal article [McE78], McEliece proposed to instantiate his system with a binary Goppa code [Gop70, Gop71] (see [Ber73] for a description in English).

One of the major drawbacks of such a proposal is the significant size of the public key. McEliece's historical proposal with binary Goppa codes required a 32.7 kB key for a claimed classical security of 65 bits. In the recent NIST submission *Classic McEliece* [BCL⁺19], a public key size of 261 kB is proposed for a claimed security level of 128 bits. For this reason, there has been many attempts in the last forty years to replace the Goppa codes used in McEliece's scheme by other families

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of codes, in order to reduce the key size. Many of these proposals focus on Generalised Reed-Solomon (GRS) codes, since they benefit from excellent decoding properties. On the other hand, their structure is difficult to hide. Consequently, to our knowledge, all code-based cryptographic schemes using GRS codes or low-codimensional subcodes of GRS codes as trapdoor have been attacked.

The use of GRS codes to replace Goppa codes in McEliece's scheme was initially suggested in Niederreiter's paper [Nie86], but this proposal was attacked by Sidelnikov and Shestakov [SS92] (although an earlier article from Roth and Seroussi [RS85] already explained how to recover the structure of a GRS code). To overcome the attack of Sidelnikov and Shestakov while trying to keep the benefits of GRS codes, several proposals appeared in the literature. Berger and Loidreau proposed to replace the GRS code by a subcode of low codimension [BL05]; Wieschebrink [Wie06] included some random columns in a generator matrix of a GRS code; his approach was enhanced in NIST submission RLCE [Wan16, Wan17], where random columns are included and then "mixed" with the original columns using specific linear transformations; finally Baldi, Bianchi, Chiaraluce, Rosenthal and Schipani (BBCRS) [BBC+16] proposed to mask the structure of a GRS code by right multiplying it by a "partially weight preserving" matrix. All these proposals have been partially or fully broken using attacks derived from a square code distinguisher. The first contribution, due to Wieschebrink [Wie10] broke Berger-Loidreau proposal, Wieschebrink's and BBCRS schemes were attacked in [CGG+14, COTG15] and RLCE in [CLT19].

In summary, forty years of research on the use of algebraic codes for public key encryption boil down to the following observations.

- (1) On one hand, the raw use of GRS codes as well as most of the variants using these codes lead to insecure schemes.
- (2) On the other hand, Goppa codes or more generally alternant codes remain robust decades after they were initially proposed by McEliece.

Alternant codes are nothing but subfield subcodes of GRS codes. Hence, considering the spectrum with (full) GRS codes on one end and their subfield subcodes (i.e. alternant codes) on the other, the intermediary case is that of subspace subcodes of Reed-Solomon codes. This notion of subspace subcodes of Reed-Solomon (SSRS) was originally introduced without any cryptographic motivation by Solomon, McEliece and Hattori [Ste93, MS94, Hat95, HMS98]. An SSRS code is a subset of a parent Reed-Solomon code over \mathbb{F}_{q^m} consisting of the codewords whose components all lie in a fixed λ -dimensional \mathbb{F}_q -vector subspace of \mathbb{F}_{q^m} , for some $\lambda \leq m$. These codes are no longer linear over \mathbb{F}_{q^m} but only over \mathbb{F}_q . The SSRS construction provides long codes with good parameters over alphabets of moderate size, in the spirit of alternant codes [MS86, Chapter 12]. Therefore these codes are interesting from an information-theoretic point of view.

For public key cryptography, two recent works exploring different approaches appeared in the recent years. First, Berger, Gueye, Klamti and Ruatta [BGKR19] proposed a McEliece scheme based on some low–dimensional subcodes of quasi–cyclic subspace subcodes of Reed–Solomon codes. In another line of work, in the article "Encryption Scheme Based on Expanded Reed-Solomon Codes" [KRW19a], Khathuria, Rosenthal and Weger propose an encryption scheme using expanded subspace subcodes of GRS codes instead of Goppa codes. Throughout the document, we will refer to this scheme as the XGRS scheme (where the X stands for expanded).

The use of subspace subcodes of Reed–Solomon codes is of particular interest in code based cryptography since it includes McEliece's original proposal based on Goppa codes on the one hand and encryption based on generalised Reed–Solomon codes on the other hand as the two extremities of a same spectrum. Indeed, starting from Reed–Solomon codes over \mathbb{F}_{q^m} and considering subspace subcodes over subspaces of \mathbb{F}_{q^m} of dimension $1 \le \lambda \le m$, the case $\lambda = m$ is corresponds to GRS codes, while the case $\lambda = 1$ corresponds to alternant codes (which include Goppa codes). The notion of subspace subcodes permits a modulation of the parameter λ . Consequently, they are of particular interest for two reasons.

(1) Subspace subcodes may provide interesting codes for encryption with $\lambda > 1$, providing shorter keys than the original McEliece scheme.

(2) Their security analysis encompasses that of Goppa and alternant codes and may help to better understand the security of McEliece encryption scheme. We emphasise that such a security analysis is of crucial interest since *Classic McEliece* lies among the very few candidates selected by the NIST for the last round of the post-quantum standardisation process.

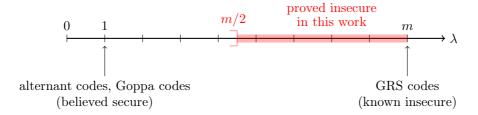


FIGURE 1. The case $\lambda=m$ (i.e. using the whole GRS code as secret key) is known to be insecure. On the other hand, the case $\lambda=1$ corresponding to alternant codes (among which Goppa codes) is a well-studied hard problem. Our attack covers all cases where $m/2 < \lambda \leqslant m$.

Our contribution. — In the present article, we first introduce a general public key cryptosystem relying on subspace subcodes of Reed Solomon codes, which we refer to as the SSRS cryptosystem. We prove that the XGRS cryptosystem of [KRW19a] is in fact a sub-instance of the SSRS scheme. Then, we analyse the security of the SSRS cryptosystem, using alternatively a high-level approach (considering abstract subspaces of \mathbb{F}_{q^m}) or a more constructive one (focusing on explicit descriptions of such codes as \mathbb{F}_q -linear codes using the expansion operator). We present a distinguisher on the SSRS scheme, by introducing a new and original notion, which we called twisted product of codes. Our distinguisher succeeds for any subpace subcode of a GRS code over \mathbb{F}_{q^m} when the subspaces have dimension $\lambda > m/2$ (see Figure 1). Using this distinguisher we derive a polynomial time attack on SSRS when $\lambda > m/2$, which in particular breaks some parameters of the XGRS scheme proposed by its authors, namely the case of subspace subcodes on 2-dimensional subspaces of GRS codes over \mathbb{F}_{q^3} .

Related work. — This work should be related to [BGK19, § VI.B] where it is shown that an encryption scheme based on an expanded GRS code (the full code, not a subspace subcode) is not secure. Next, in the same reference [BGK19, § VI.C], the case of subspace subcodes is discussed and the proposed attack involves a brute force search on the all the bases used for the expansion of each position. This attack has an exponential complexity, while ours runs in polynomial time.

On the other hand, the " $\lambda > m/2$ condition" for the public code to be distinguished from a random one should be compared to the results of [COT14b, COT17], where some classical Goppa codes are attacked using, among others, the square code operation. The considered classical Goppa codes correspond to the parameters m=2 and $\lambda=1$ and hence lie at the very limit of the distinguisher. These Goppa codes got however broken due to peculiar features that give them a larger dimension compared to generic alternant codes of the same parameters (see [SKHN76, COT14a]).

Outline of the article. — We start in Section 2 by fixing the notation and bringing well–known notions of algebraic coding theory that are necessary to follow the article. The notion of subspace subcodes is recalled in Section 3 as well as some of their known properties. We also discuss the way to practically represent subspace subcodes. Some operators play an important role in this article, Section 4 is devoted to them and the way they interact with each other. In Section 5, we present the SSRS cryptosystem and explain why the XGRS scheme is a proper sub-instance. The notion of twisted square code and the corresponding distinguisher are introduced in Section 6. Finally, Section 7 is devoted to the presentation of the attack on SSRS scheme.

2. Notation and prerequisites

In this section, we fix the notation and recall some usual tools of code-based cryptography that are used in the definition of subspace subcodes and in the construction the XGRS cryptosystem.

2.1. Notation. — In this article, q denotes a power of a prime and m a positive integer. The vector space of polynomials of degree less than k over a field \mathbb{F} is denoted by $\mathbb{F}[X]_{\leq k}$. The space of matrices with entries in a field \mathbb{F} with m rows and n columns is denoted by $\mathbb{F}^{m \times n}$. Given an \mathbb{F} -vector space \mathcal{V} and vectors $\mathbf{v}_0, \ldots, \mathbf{v}_{s-1} \in \mathcal{V}$, the subspace spanned over \mathbb{F} by the \mathbf{v}_i 's is denoted by

$$\left\langle \left. oldsymbol{v}_{0}, \ldots, oldsymbol{v}_{s-1} \right.
ight
angle_{\mathbb{F}} \stackrel{\mathrm{def}}{=} \left\{ \sum_{i=0}^{s-1} \lambda_{i} oldsymbol{v}_{i} \, \middle| \, \lambda_{i} \in \mathbb{F} \right\}.$$

Moreover, given a matrix $G \in \mathbb{F}^{k \times n}$, we denote by $\langle G \rangle_{\mathbb{F}}$ the space spanned by the **rows** of G, that is to say the code with generator matrix G, defined as

$$\langle \, oldsymbol{G} \,
angle_{\mathbb{F}} \stackrel{\mathrm{def}}{=} \langle \, oldsymbol{v} \mid oldsymbol{v} \; \mathrm{row} \; \mathrm{of} \; oldsymbol{G} \,
angle_{\mathbb{F}} = \{ oldsymbol{m} \cdot oldsymbol{G} \mid oldsymbol{m} \in \mathbb{F}^k \}.$$

The *support* of a vector is the set of its indices with non-zero entries. An [n, k] code over \mathbb{F} is a linear code over \mathbb{F} of length n and dimension k.

Given two integers a, b with a < b, we denote by [a, b] the interval of integers $\{a, a + 1, ..., b\}$. The cardinality of a finite set U is denoted by |U|. Given a probabilistic event A, its probability is denoted by $\mathbb{P}[A]$ and the mean of a random variable X is denoted by $\mathbb{E}[X]$.

Convention. In this article, any word or finite sequence of length ℓ is indexed from 0 to $\ell - 1$. In particular, codewords of length n are indexed as follows: (x_0, \ldots, x_{n-1}) .

2.2. Reed-Solomon codes. —

Definition 1 (Generalised Reed–Solomon codes). — Let $x \in \mathbb{F}_q^n$ be a vector whose entries are pairwise distinct and $y \in \mathbb{F}_q^n$ be a vector whose entries are all nonzero. The generalised Reed–Solomon (GRS) code with support x and multiplier y of dimension k is defined as

$$\mathbf{GRS}_k(\boldsymbol{x},\boldsymbol{y}) \stackrel{\text{def}}{=} \left\{ (y_0 f(x_0), \dots, y_{n-1} f(x_{n-1})) \mid f \in \mathbb{F}_q[x]_{< k} \right\}.$$

If y = (1, ..., 1) then the code is said to be a *Reed-Solomon* code and denoted as $\mathbf{RS}_k(x)$.

2.3. Component-wise product of codes and square codes distinguisher. —

Notation 2. — The component-wise product of two vectors a and b in \mathbb{F}^n is denoted by

$$\mathbf{a} \star \mathbf{b} \stackrel{\text{def}}{=} (a_0 b_0, \dots, a_{n-1} b_{n-1}).$$

This definition extends to the product of codes, where the *component-wise product* or \star -product of two \mathbb{K} -linear codes \mathscr{A} and $\mathscr{B} \subseteq \mathbb{F}^n$ spanned over a field $\mathbb{K} \subseteq \mathbb{F}$ is defined as

$$\mathscr{A}\star_{\mathbb{K}}\mathscr{B}\stackrel{\mathrm{def}}{=}\left\langle \, oldsymbol{a}\,\star\,oldsymbol{b}\midoldsymbol{a}\in\mathscr{A},\;oldsymbol{b}\in\mathscr{B}
ight
angle _{\mathbb{K}}.$$

When $\mathscr{A}=\mathscr{B}$, we denote by $\mathscr{A}_{\mathbb{K}}^{\star 2}\stackrel{\mathrm{def}}{=}\mathscr{A}\star_{\mathbb{K}}\mathscr{A}$ the square code of \mathscr{A} spanned over \mathbb{K} .

Remark 3. — The field \mathbb{K} in Notation 2 is almost always equal to \mathbb{F} the base field on which the codes are defined. However, it may sometimes be a subfield. For the sake of clarity, we make the value of \mathbb{K} explicit only in the ambiguous cases. The rest of the time we simply write \mathscr{A}^{*2} the square product of a code.

We recall the following result on the generic behaviour of random codes with respect to this operation.

Theorem 4. — ([CCMZ15, Theorem 2.3], informal) For a linear code \mathscr{R} chosen at random over \mathbb{F}_q of dimension k and length n, the dimension of \mathscr{R}^{*2} is typically $\min(n, \binom{k+1}{2})$.

Theorem 4 provides a distinguisher between random codes and algebraically structured codes such as generalised Reed–Solomon codes and their low–codimensional subcodes [Wie10, CGG+14], Reed–Muller codes [CB14], polar codes [BCD+16] some Goppa codes [COT14b, COT17], high–rate alternant codes [FGO+11] or algebraic geometry codes [CMCP14, CMCP17]. For instance, in the case of GRS codes, we have the following result.

Proposition 5. — Let n, k, x, y be as in Definition 1. Then,

$$(\mathbf{GRS}_k(\boldsymbol{x},\boldsymbol{y}))^{\star 2} = \mathbf{GRS}_{2k-1}(\boldsymbol{x},\boldsymbol{y} \star \boldsymbol{y}).$$

In particular, if $k \leq n/2$, then

$$\dim \left(\mathbf{GRS}_k(\boldsymbol{x}, \boldsymbol{y})\right)^{\star 2} = 2k - 1.$$

Thus, compared to random codes whose square have dimension quadratic in the dimension of the code, the square of a GRS code has a dimension which is linear in that of the original code. This criterion allows to distinguish GRS codes of appropriate dimension from random codes. A rich literature of cryptanalysis of code—based encryption primitives involves this operation.

Remark 6. — In an initial version of the XGRS cryptosystem, submitted on ArXiv [KRW19b], such a distinguisher could be applied to the cryptosystem and lead to an attack. The authors changed the cryptosystem in order to avoid such attacks.

2.4. Punctured and shortened codes. — The notions of puncturing and shortening are classical ways to build new codes from existing ones. These constructions will be useful for the attack. We recall here their definition. For a codeword $c \in \mathbb{F}_q^n$, we denote by (c_0, \ldots, c_{n-1}) its entries.

Definition 7 (Punctured code). — Let $\mathscr{C} \subseteq \mathbb{F}_q^n$ and $\mathcal{L} \subseteq [0, n-1]$. The puncturing of \mathscr{C} at \mathcal{L} is defined as the code

$$\mathbf{Pct}_{\mathcal{L}}\left(\mathscr{C}\right) \stackrel{\mathrm{def}}{=} \{(c_i)_{i \in \llbracket 0, n-1 \rrbracket \setminus \mathcal{L}} \text{ s.t. } \mathbf{c} \in \mathscr{C}\}.$$

Similarly, given a matrix M with n columns, one defines $\mathbf{Pct}_{\mathcal{L}}(M)$ as the matrix whose columns with index in \mathcal{L} are removed, so that puncturing a generator matrix of a code yields a generator matrix of the punctured code.

Definition 8 (Shortened code). — Let $\mathscr{C} \subseteq \mathbb{F}_q^n$ and $\mathcal{L} \subseteq [0, n-1]$. The shortening of \mathscr{C} at \mathcal{L} is defined as the code

$$\mathbf{Sh}_{\mathcal{L}}\left(\mathscr{C}\right)\stackrel{\mathrm{def}}{=}\mathbf{Pct}_{\mathcal{L}}\left(\left\{ oldsymbol{c}\in\mathscr{C}\ \mathrm{s.t.}\ \forall i\in\mathcal{L},\ c_{i}=0\right\} \right).$$

Shortening a code is equivalent to puncturing the dual code, as explained by the following proposition.

Proposition 9 ([HP03, Theorem 1.5.7]). — Let \mathscr{C} be a linear code over \mathbb{F}_q^n and $\mathcal{L} \subseteq [0, n-1]$. Then,

$$\mathbf{Sh}_{\mathcal{L}}(\mathbf{Dual}(\mathscr{C})) = \mathbf{Dual}(\mathbf{Pct}_{\mathcal{L}}(\mathscr{C})) \text{ and } \mathbf{Dual}(\mathbf{Sh}_{\mathcal{L}}(\mathscr{C})) = \mathbf{Pct}_{\mathcal{L}}(\mathbf{Dual}(\mathscr{C})),$$

where $\mathbf{Dual}(\mathscr{A})$ denotes the dual of the code \mathscr{A} .

Remark 10. — The notation $\mathbf{Dual}(\mathscr{C})$ to denote the dual code of \mathscr{C} is rather unusual. This code is commonly denoted \mathscr{C}^{\perp} , but it will be more convenient to write it the former way and see duality as an operator on codes.

2.5. Bases and trace map. — In this paper we will consider codes over a finite field \mathbb{F}_{q^m} and codes over its subfield \mathbb{F}_q . A useful and natural tool is the trace map.

Definition 11 (Trace map). — Let q be a prime power and m an integer. The trace map is defined as

$$\mathbf{Tr}: \qquad \left\{ \begin{array}{ccc} \mathbb{F}_{q^m} & \longrightarrow & \mathbb{F}_q \\ x & \longmapsto & \sum_{i=0}^{m-1} x^{q^i}. \end{array} \right.$$

Definition 12 ([LN97, Definition 2.30]). — Let $\mathcal{B} = (b_0, \dots, b_{m-1})$ be an \mathbb{F}_q -basis of \mathbb{F}_{q^m} . There exists a unique basis $\mathcal{B}^* = (b_0^*, \dots, b_{m-1}^*)$, such that :

$$\forall 0 \leqslant i, j \leqslant m-1,$$
 $\mathbf{Tr}(b_i b_j^*) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

This basis will be referred to as the dual basis of \mathcal{B} and denoted \mathcal{B}^* .

Given an \mathbb{F}_q -basis $\mathcal{B} = (b_0, \dots, b_{m-1})$ of \mathbb{F}_{q^m} and x an element of \mathbb{F}_{q^m} . Then the expression of x as an \mathbb{F}_q -linear combination of the elements of \mathcal{B} writes as

(1)
$$x = \mathbf{Tr}(b_0^* x) b_0 + \dots + \mathbf{Tr}(b_{n-1}^* x) b_{n-1}$$

where $\mathcal{B}^* = (b_0^*, \dots, b_{m-1}^*)$ denotes the dual basis of \mathcal{B} .

3. Subspace Subcodes

3.1. Definition and first properties. —

Definition 13 ([HMS98]). — Given a linear code \mathscr{C} defined over a field \mathbb{F}_{q^m} , and a λ -dimensional subspace \mathcal{S} of \mathbb{F}_{q^m} ($0 \leq \lambda \leq m$), the subspace subcode $\mathscr{C}_{|\mathcal{S}}$ is defined to be the set of codewords of \mathscr{C} whose components all lie in \mathcal{S} .

$$\mathscr{C}_{|\mathcal{S}} \stackrel{\text{def}}{=} \{ \boldsymbol{c} \in \mathscr{C} \mid \forall i \in [0, n-1], \ c_i \in \mathcal{S} \} \subseteq \mathbb{F}_{q^m}^n.$$

It is important to note that the code $\mathscr{C}_{|S}$ is an \mathbb{F}_q -linear subspace of \mathbb{F}_q^n which is generally neither \mathbb{F}_{q^m} -linear nor linear over some intermediary extension. Since each entry of a codeword can be represented as λ elements of \mathbb{F}_q , the code could be converted into a code over the alphabet \mathbb{F}_q^{λ} . Such a code would form an additive subgroup over $(\mathbb{F}_q^{\lambda})^n$, therefore this construction is called a *subgroup subcode* by Jensen in [**Jen95**]. In a context of message transmission, this natural way to represent such a subspace subcode is detailed further in § 3.2.

We can generalise this definition with different subspaces for each entry.

Definition 14. — Given a linear code \mathscr{C} of length n over a field \mathbb{F}_{q^m} , and the λ -dimensional subspaces $(\mathcal{S}_0, \ldots, \mathcal{S}_{n-1})$ of \mathbb{F}_{q^m} $(0 \leq \lambda \leq m)$, the subspace subcode $\mathscr{C}_{|(\mathcal{S}_0, \ldots, \mathcal{S}_{n-1})}$ is defined to be the set of codewords of \mathscr{C} such that the i-th components lies in \mathcal{S}_i .

$$\mathscr{C}_{|(\mathcal{S}_0,\ldots,\mathcal{S}_{n-1})} \stackrel{\text{def}}{=} \{ \boldsymbol{c} \in \mathscr{C} \mid \forall i \in [0,n-1], \ c_i \in \mathcal{S}_i \}.$$

Remark 15. — When $S_0 = \cdots = S_{n-1} = \mathbb{F}_q$, then we find the usual definition of *subfield* subcode.

Remark 16. — It is possible to give a more general definition where the S_i 's do not have the same dimension λ . However, such a broader definition would be useless in the present article.

Proposition 17. — Let \mathscr{C} be a linear code of length n and dimension k over \mathbb{F}_{q^m} and $S \subseteq \mathbb{F}_{q^m}$ be a subspace of dimension $\lambda \leqslant m$. Then

(2)
$$\dim_{\mathbb{F}_a} \mathscr{C}_{|\mathcal{S}|} \geqslant km - n(m - \lambda).$$

Proof. — See for instance [Jen95, Theorem 2 (1)].

Example 18. — The following example comes from [HMS98].

Consider \mathscr{C} the Reed–Solomon code over \mathbb{F}_{2^4} of length 15 and dimension 9. This code has minimum distance 7. Any element of \mathbb{F}_{2^4} can be decomposed over the \mathbb{F}_2 -basis $(1, \alpha, \alpha^2, \alpha^3)$, where α is a root of the irreducible polynomial $X^4 + X + 1$. Let \mathcal{S} be the subspace spanned by $(1, \alpha, \alpha^2)$. The code $\mathscr{C}_{|\mathcal{S}}$ is the subset of codewords of \mathscr{C} that have no component in α^3 . Hence, if one uses this code for communication, there is no need to send the α^3 component, since it is always zero.

So this subspace subcode can be seen as an \mathbb{F}_2 -linear code of length 15 over the set of binary 3-tuples. But the code is not a linear code over \mathbb{F}_{2^3} . The minimum distance of $\mathscr{C}_{|S}$ is at least 7, because it cannot be less than the minimum distance of the parent code. The number of codewords in $\mathscr{C}_{|S}$ is 2^{2^2} . As a comparison, one other way to create a code of length 15 over binary 3-tuples is by shortening the generalised BCH code [63, 52, 7] over \mathbb{F}_{2^3} . This gives a $[15, 4, \geqslant 7]$ code over \mathbb{F}_{2^3} which has 2^{12} codewords.

Similarly to the case of subfield subcodes, inequality (2) is typically an equality as explained in the following statement that we prove because of a lack of references.

Proposition 19. Let \mathscr{R} be a uniformly random code among the codes of length n and dimension k over \mathbb{F}_{q^m} . Let $\mathcal{S}_0, \ldots, \mathcal{S}_{n-1}$ be \mathbb{F}_q -subspaces of \mathbb{F}_{q^m} of dimension λ . Suppose that $km > n(m-\lambda)$. Then, for any positive integer ℓ such that $\ell \leq km - n(m-\lambda)$, we have

$$\mathbb{P}\left[\dim_{\mathbb{F}_q} \mathscr{R}_{|(\mathcal{S}_0,\dots,\mathcal{S}_{n-1})} \geqslant km - n(m-\lambda) + \ell\right] \leqslant q^{-\ell} \left(\frac{1}{1 - q^{-mn}} + \frac{1}{q^{km - n(m-\lambda)}}\right).$$

In particular, for fixed values of q, m and λ , this probability is in $O(q^{-\ell})$ when $n \to \infty$.

Proof. — Let G_{rand} be a uniformly random variable among the full rank matrices in $\mathbb{F}_{q^m}^{k \times n}$ and

$$\mathscr{R} \stackrel{\mathrm{def}}{=} \{ oldsymbol{m} oldsymbol{G}_{\mathrm{rand}} \mid oldsymbol{m} \in \mathbb{F}_{q^m}^k \}.$$

The code \mathscr{R} is uniformly random among the set of [n, k] codes over \mathbb{F}_{q^m} ([Cou20, Lemma 3.12]). Let Φ be the \mathbb{F}_q -linear canonical projection

$$\Phi: \mathbb{F}_{q^m}^n \longrightarrow \prod_{i=0}^{n-1} \mathbb{F}_{q^m}/\mathcal{S}_i.$$

Then, $\mathscr{R}_{|(\mathcal{S}_0,\dots,\mathcal{S}_{n-1})}$ is the kernel of the restriction of Φ to \mathscr{R} and hence,

$$\mathbb{E}\left[\left|\mathscr{R}_{|(\mathcal{S}_{0},\ldots,\mathcal{S}_{n-1})}\right|\right] = \mathbb{E}\left[\sum_{\boldsymbol{m}\in\mathbb{F}_{q^{m}}^{k}}\mathbb{1}_{\Phi(\boldsymbol{m}\boldsymbol{G}_{\mathrm{rand}})=0}\right]$$

$$= \sum_{\boldsymbol{m}\in\mathbb{F}_{q^{m}}^{k}}\mathbb{P}\left[\Phi(\boldsymbol{m}\boldsymbol{G}_{\mathrm{rand}})=0\right]$$

$$= 1 + \sum_{\boldsymbol{m}\in\mathbb{F}_{q^{m}}^{k}\setminus\{0\}}\mathbb{P}\left[\Phi(\boldsymbol{m}\boldsymbol{G}_{\mathrm{rand}})=0\right].$$
(3)

Since G_{rand} is uniformly random among the full–rank matrices, then for any $m \in \mathbb{F}_{q^m}^k \setminus \{0\}$, the vector mG_{rand} is uniformly random in $\mathbb{F}_{q^m} \setminus \{0\}$ ([Cou20, Lemma 3.13]) and hence

$$\forall \boldsymbol{m} \in \mathbb{F}_{q^m}^k \setminus \{0\}, \quad \mathbb{P}\left[\Phi(\boldsymbol{m}\boldsymbol{G}_{\mathrm{rand}} = 0)\right] = \frac{|\ker \Phi \setminus \{0\}|}{\left|\mathbb{F}_{q^m}^n \setminus \{0\}\right|}$$
$$= \frac{|\prod_i \mathcal{S}_i| - 1}{q^{mn} - 1}$$
$$= \frac{q^{\lambda n} - 1}{q^{mn} - 1} \leqslant q^{-n(m - \lambda)} \cdot \frac{1}{1 - q^{-mn}}.$$

Thus, applied to (3),

$$\mathbb{E}\left[\left|\mathscr{R}_{|(S_0,\dots,S_{n-1})|}\right|\right] \leqslant 1 + \left|\mathbb{F}_{q^m}^k \setminus \{0\}\right| \cdot q^{-n(m-\lambda)} \cdot \frac{1}{1 - q^{-mn}}$$
$$\leqslant 1 + q^{km - n(m-\lambda)} \cdot \frac{1}{1 - q^{-mn}}.$$

Finally, using Markov inequality, we get

$$\mathbb{P}\left[\dim_{\mathbb{F}_q}\left(\mathscr{R}_{|(\mathcal{S}_0,\dots,\mathcal{S}_{n-1})}\right) \geqslant km - n(m-\lambda) + \ell\right] = \mathbb{P}\left[|\mathscr{R}_{|(\mathcal{S}_0,\dots,\mathcal{S}_{n-1})|}| \geqslant q^{km-n(m-\lambda)+\ell}\right] \\
\leqslant \frac{\mathbb{E}\left[|\mathscr{R}_{|(\mathcal{S}_0,\dots,\mathcal{S}_{n-1})|}\right]}{q^{km-n(m-\lambda)+\ell}} \\
\leqslant q^{-\ell}\left(\frac{1}{1-q^{-mn}} + \frac{1}{q^{km-n(m-\lambda)}}\right).$$

3.2. How to represent subspace subcodes? — For a practical implementation, subspace subcodes may be represented as codes over the subfield \mathbb{F}_q with a higher length. For this sake we introduce the *expansion operator* and give some of its properties.

3.2.1. The Expansion Operator. —

Definition 20 (Expansion of a vector). — For a basis \mathcal{B} of \mathbb{F}_{q^m} , let $\mathbf{ExpVec}_{\mathcal{B}}$ denote the expansion of a vector over the basis \mathcal{B} defined by

$$\begin{cases} \mathbb{F}_{q^m}^{\ell} & \longrightarrow & \mathbb{F}_q^{m\ell} \\ (x_0, \dots, x_{\ell-1}) & \longmapsto & (\mathbf{Tr}(b_0^*x_0), \dots, \mathbf{Tr}(b_{m-1}^*x_0), \dots, \mathbf{Tr}(b_0^*x_{\ell-1}) \dots, \mathbf{Tr}(b_{m-1}^*x_{\ell-1})), \end{cases}$$

where $\mathcal{B}^* = (b_0^*, \dots, b_{m-1}^*)$ denotes the dual basis of \mathcal{B} . Note that we will apply this operator to vectors of different lengths ℓ .

As seen in (1), regarding an element $x \in \mathbb{F}_{q^m}$ as the vector (x) of length 1, let $(x_0, \dots, x_{m-1}) \stackrel{\text{def}}{=} \mathbf{ExpVec}_{\mathcal{B}}((x)) \in \mathbb{F}_q^m$, then $x = \sum_{i=0}^{m-1} x_i b_i$.

Definition 21 (Expansion of a code). — For a linear code \mathscr{C} of length n over \mathbb{F}_{q^m} and a basis \mathcal{B} of \mathbb{F}_{q^m} , denote $\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})$ the linear code over \mathbb{F}_q defined by

$$\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C}) \stackrel{\mathrm{def}}{=} \{ \mathbf{ExpVec}_{\mathcal{B}}(\boldsymbol{c}) \, | \, \boldsymbol{c} \in \mathscr{C} \}.$$

We can also define the expansion operator over matrices.

Definition 22 (Expansion of a matrix). — Given $\mathcal{B} = (b_0, \ldots, b_{m-1})$ an \mathbb{F}_q -basis of \mathbb{F}_{q^m} . Let ExpMat_{\mathcal{B}} denote the following operation.

where

$$\boldsymbol{M}_{i,j} \stackrel{\text{def}}{=} \left(\begin{array}{cccc} \mathbf{Tr}(b_0 b_0^* m_{i,j}) & \mathbf{Tr}(b_0 b_1^* m_{i,j}) & \dots & \mathbf{Tr}(b_0 b_{m-1}^* m_{i,j}) \\ \mathbf{Tr}(b_1 b_0^* m_{i,j}) & \mathbf{Tr}(b_1 b_1^* m_{i,j}) & \dots & \mathbf{Tr}(b_1 b_{m-1}^* m_{i,j}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{Tr}(b_{m-1} b_0^* m_{i,j}) & \mathbf{Tr}(b_{m-1} b_1^* m_{i,j}) & \dots & \mathbf{Tr}(b_{m-1} b_{m-1}^* m_{i,j}) \end{array} \right) \in \mathbb{F}_q^{m \times m},$$

and $\mathcal{B}^* = (b_0^*, \dots, b_{m-1}^*)$ denotes the dual basis of \mathcal{B} (Definition 12).

Remark 23. — Caution, applying $\mathbf{ExpMat}_{\mathcal{B}}$ to an $1 \times n$ matrix returns an $m \times n$ matrix. It is not equivalent to applying ExpVec_B to the vector corresponding to this row.

Remark 24. — ExpMat_{$$B^*$$} $(M) = (\text{ExpMat}_{B}(M^{\mathsf{T}}))^{\mathsf{T}}$.

Proposition 25 ([KRW19a, Proposition 1]). — Let $\mathscr C$ be a linear code of dimension k and length n over \mathbb{F}_{q^m} . Let G denote a generator matrix of \mathscr{C} and H denote a parity-check matrix of \mathscr{C} . Then, for any fixed \mathbb{F}_q -basis \mathcal{B} of \mathbb{F}_{q^m} , the following hold.

- (i) For all $\boldsymbol{x} \in \mathbb{F}_{q^m}^k$, we have $\mathbf{ExpVec}_{\mathcal{B}}(\boldsymbol{x} \cdot \boldsymbol{G}) = \mathbf{ExpVec}_{\mathcal{B}}(\boldsymbol{x}) \cdot \mathbf{ExpMat}_{\mathcal{B}}(\boldsymbol{G})$. (ii) For all $\boldsymbol{y} \in \mathbb{F}_{q^m}^n$, we have $\mathbf{ExpVec}_{\mathcal{B}}((\boldsymbol{H} \cdot \boldsymbol{y}^\intercal)^\intercal)^\intercal = \mathbf{ExpMat}_{\mathcal{B}^*}(\boldsymbol{H}) \cdot \mathbf{ExpVec}_{\mathcal{B}}(\boldsymbol{y})^\intercal$.

Corollary 26. — Let G and H be a generator and a parity-check matrix of $\mathscr C$ respectively. Let $\mathcal B$ denote an \mathbb{F}_q -basis of \mathbb{F}_{q^m} . Then $\mathbf{ExpMat}_{\mathcal{B}}(G)$ and $\mathbf{ExpMat}_{\mathcal{B}^*}(H)$ are respectively a generator matrix and a parity-check matrix of $\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})$.

Definition 27 (Block). — Given a vector $v \in \mathbb{F}_{q^m}^n$, an \mathbb{F}_q -basis \mathcal{B} of \mathbb{F}_{q^m} and a non negative integer i < n, the i-th block of the expanded vector $\mathbf{ExpVec}_{\mathcal{B}}(v) \in \mathbb{F}_q^{mn}$ is the length m vector composed by the entries of index $mi, mi + 1, \dots, mi + m - 1$ of $\mathbf{ExpVec}_{\mathcal{B}}(v)$. It corresponds to the decomposition over \mathcal{B} of the *i*-th entry of \mathbf{v} . We extend this definition to matrices, where the i-th block of an expanded matrix means the $mk \times m$ matrix whose rows correspond to the i-th block of each row of the expanded matrix.

In particular, the expansion in a basis \mathcal{B} of some $x \in \mathbb{F}_{q^m}^n$ is the concatenation of n blocks of length m.

3.2.2. Expansion over various bases. — We have seen in Definition 14 that we could define a subspace subcode with different subspaces for each entry. Similarly, we can define an expansion with regard to a different basis for each entry.

Definition 28. — Given ℓ bases $(\mathcal{B}_0, \dots, \mathcal{B}_{\ell-1})$ of \mathbb{F}_{q^m} , let $\mathbf{ExpVec}_{(\mathcal{B}_i)_i}$ denote the expansion of a vector of length ℓ , such that the i^{th} column is expanded over the basis \mathcal{B}_i :

$$\mathbf{ExpVec}_{(\mathcal{B}_{i})_{i}}(x_{0},\ldots,x_{\ell-1}) = (\mathbf{Tr}(b_{0,0}^{*}x_{0}),\ldots,\mathbf{Tr}(b_{0,m-1}^{*}x_{0}),\ldots,\mathbf{Tr}(b_{\ell-1,0}^{*}x_{\ell-1})\ldots,\mathbf{Tr}(b_{\ell-1,m-1}^{*}x_{\ell-1})),$$
 where $\mathcal{B}_{i} = (b_{i,0},\ldots,b_{i,m-1}).$

Definition 29. — For a linear code \mathscr{C} of length n over \mathbb{F}_{q^m} and n bases $(\mathcal{B}_i)_i$ of \mathbb{F}_{q^m} , denote $\mathbf{ExpCode}_{(\mathcal{B}_i)_i}(\mathscr{C})$ the linear code over \mathbb{F}_q defined by:

$$\mathbf{ExpCode}_{(\mathcal{B}_i)_i}(\mathscr{C}) \stackrel{\mathrm{def}}{=} \{ \mathbf{ExpVec}_{(\mathcal{B}_i)_i}(oldsymbol{c}) \, | \, oldsymbol{c} \in \mathscr{C} \}.$$

Definition 30. — Given n+1 bases $(\mathcal{B}_0,\ldots,\mathcal{B}_{n-1},\bar{\mathcal{B}})$ of \mathbb{F}_{q^m} , let $\mathbf{ExpMat}_{(\mathcal{B}_i)_i}^{\bar{\mathcal{B}}}$ denote the expansion of a matrix

$$\left\{ \left(\begin{array}{cccc} \mathbb{F}_{q^m}^{k \times n} & \longrightarrow & \mathbb{F}_{q}^{mk \times mn} \\ \left(\begin{array}{cccc} m_{0,0} & m_{0,1} & \cdots & m_{0,n-1} \\ m_{1,0} & m_{1,1} & \cdots & m_{1,n-1} \\ \vdots & \vdots & & \vdots \\ m_{k-1,0} & m_{k-1,1} & \cdots & m_{k-1,n-1} \end{array} \right) \begin{array}{c} \longrightarrow & \left(\begin{array}{cccc} \boldsymbol{M}_{0,0} & \boldsymbol{M}_{0,1} & \cdots & \boldsymbol{M}_{0,n-1} \\ \boldsymbol{M}_{1,0} & \boldsymbol{M}_{1,1} & \cdots & \boldsymbol{M}_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{M}_{k-1,0} & \boldsymbol{M}_{k-1,1} & \cdots & \boldsymbol{M}_{k-1,n-1} \end{array} \right)$$

$$\boldsymbol{M}_{i,j} \stackrel{\text{def}}{=} \left(\begin{array}{cccc} \mathbf{Tr}(\bar{b}_{0}b_{j,0}^{*}m_{i,j}) & \mathbf{Tr}(\bar{b}_{0}b_{j,1}^{*}m_{i,j}) & \dots & \mathbf{Tr}(\bar{b}_{0}b_{j,m-1}^{*}m_{i,j}) \\ \mathbf{Tr}(\bar{b}_{1}b_{j,0}^{*}m_{i,j}) & \mathbf{Tr}(\bar{b}_{1}b_{j,1}^{*}m_{i,j}) & \dots & \mathbf{Tr}(\bar{b}_{1}b_{j,m-1}^{*}m_{i,j}) \\ \vdots & \vdots & & \vdots \\ \mathbf{Tr}(\bar{b}_{m-1}b_{j,0}^{*}m_{i,j}) & \mathbf{Tr}(\bar{b}_{m-1}b_{j,1}^{*}m_{i,j}) & \dots & \mathbf{Tr}(\bar{b}_{m-1}b_{j,m-1}^{*}m_{i,j}) \end{array} \right) \in \mathbb{F}_{q}^{m \times m}.$$

With this definition, the properties of Proposition 25 still hold for various bases.

Remark 31. — Note that contrary to the expansion of codes, the expansion of a matrix depends on the choice of a basis $\bar{\mathcal{B}}$ for the vertical expansion. When considering the code spanned by an expansion matrix, different choices of $\bar{\mathcal{B}}$ yield the same code, so we will omit the vertical expansion base in the expansion matrix operator.

3.2.3. Squeezing: the inverse of expansion. — We can define the "inverse" of the expansion operator.

Definition 32 (Squeezing Operator). — Let $\mathcal{B} = (b_0, \dots, b_{m-1})$ be a basis of \mathbb{F}_{q^m} . Let $\mathbf{x} = (x_{0,0}, \dots, x_{0,m-1}, \dots, x_{n-1,0}, \dots, x_{n-1,m-1}) \in \mathbb{F}_q^{mn}$. We define the *squeezed vector* of \mathbf{x} with respect to the basis \mathcal{B} as

SqueezeVec_{$$\mathcal{B}$$} $(x) \stackrel{\text{def}}{=} \left(\sum_{j=0}^{m-1} x_{0,j} b_j, \dots, \sum_{j=0}^{m-1} x_{n-1,j} b_j \right) \in \mathbb{F}_{q^m}^n.$

Let \mathscr{C} be an $[m \times n, k]$ -code over \mathbb{F}_q . We define the *squeezed code* of \mathscr{C} with respect to the basis \mathcal{B} as

$$\mathbf{SqueezeCode}_{\mathcal{B}}(\mathscr{C}) \stackrel{\mathrm{def}}{=} \left\{ \mathbf{SqueezeVec}_{\mathcal{B}}(\boldsymbol{c}) \, | \, \boldsymbol{c} \in \mathscr{C} \right\}.$$

Proposition 33. — Let \mathscr{C} be an [n,k] code over \mathbb{F}_{q^m} . Let $\mathcal{B} = (b_0,\ldots,b_{m-1})$ be a basis of \mathbb{F}_{q^m} . Then the following equality holds.

$$\mathbf{SqueezeCode}_{\mathcal{B}}(\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})) = \mathscr{C}.$$

Finally we can define squeezing over a matrix.

Definition 34 (Squeezing matrices). — Let $\mathcal{B} = (b_0, \dots, b_{m-1})$ be a basis of \mathbb{F}_{q^m} . Let $M \in \mathbb{F}_q^{mk \times mn}$ denote an $mk \times mn$ matrix. Then SqueezeMat_{\mathcal{B}} $(M) \in \mathbb{F}_q^{mk \times n}$ denotes the matrix whose rows are obtained by squeezing each row of the matrix M over \mathcal{B} .

Remark 35. — Note that this matrix does not necessarily have full rank. In particular, if M is obtained by expanding a matrix of rank r over the basis \mathcal{B} , then $\mathbf{SqueezeMat}_{\mathcal{B}}(M)$ will be of rank r. It is also worth noting that for a matrix $M \in \mathbb{F}_{q^m}^{k \times n}$, then $\mathbf{SqueezeMat}_{\mathcal{B}}(\mathbf{ExpMat}_{\mathcal{B}}(M))$ is a $km \times n$ matrix and hence is **not** equal to M but generates the same code.

Remark 36. — Similarly to the expansion operators, we can define the squeezing operators with a different basis for each block.

3.2.4. Representation of subspace subcodes as shortenings of expanded codes. — Let $\mathscr C$ be a code of length n and dimension k over the field $\mathbb F_{q^m}$ and $\mathcal S$ denote an $\mathbb F_q$ -subspace of $\mathbb F_{q^m}$ of dimension $\lambda \leqslant m$. Let $\mathcal B_{\mathcal S} = (b_0, \dots, b_{\lambda-1}) \in \mathbb F_{q^m}^{\lambda}$ be an $\mathbb F_q$ -basis of $\mathcal S$. Then any vector $\mathbf c = (c_0, \dots, c_{n-1}) \in \mathcal S^n$, i.e. whose entries are all in $\mathcal S$ can be expanded as

$$\mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}}}(\boldsymbol{c}) \stackrel{\text{def}}{=} (c_{0,0},\ldots,c_{0,\lambda-1},\ldots,c_{n-1,0},\ldots,c_{n-1,\lambda-1}),$$

where the $c_{i,j}$'s are the coefficients of the decomposition of c_i in the $\mathcal{B}_{\mathcal{S}}$.

Remark 37. — Note that the previous definition makes sense only for vectors in S^n .

Next, the subspace subcode $\mathcal{C}_{|\mathcal{S}}$ can be represented as

$$\mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}_{|\mathcal{S}}) \stackrel{\mathrm{def}}{=} \{ \mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}}}(\boldsymbol{c}) \mid \boldsymbol{c} \in \mathscr{C}_{|\mathcal{S}} \}.$$

Here again, as noticed in Remark 37, the notion is well-defined only for codes with entries in S. Similarly to Definition 27, a *block* refers to a set of the form $[i\lambda, (i+1)\lambda - 1]$. That is to say, a set of $\lambda = \dim S$ consecutive indexes of the expanded code, corresponding to the decomposition of a single entry in S in the basis B_S .

3.2.5. Practical calculation of the expansion of a subspace subcode. — The practical calculation rests first on the following statement.

Lemma 38. — For integers n and $\lambda < m$, denote $\mathcal{J}(\lambda, m)$ the subset of [0, mn-1] consisting of the last $m - \lambda$ entries of each block of length m

(4)
$$\mathcal{J}(\lambda, m) \stackrel{def}{=} \{ im + j, i \in \llbracket 0, n - 1 \rrbracket, j \in \llbracket \lambda, m - 1 \rrbracket \}.$$

Then,

$$\mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}_{|\mathcal{S}}) = \mathbf{Sh}_{\mathcal{J}(\lambda,m)}\left(\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})\right).$$

Let $\mathbf{H} \in \mathbb{F}_{q^m}^{k \times n}$ denote a parity-check matrix of \mathscr{C} . Complete the basis $\mathcal{B}_{\mathcal{S}} = (b_0, \dots, b_{\lambda-1})$ with $m - \lambda$ additional elements $(b_{\lambda}, \dots, b_{m-1}) \in \mathbb{F}_{q^m}^{m-\lambda}$ such that $\mathcal{B} = (b_0, \dots, b_{m-1})$ forms an \mathbb{F}_q -basis of \mathbb{F}_{q^m} . According to Corollary 26, the matrix $\mathbf{ExpMat}_{\mathcal{B}^*}(\mathbf{H})$ is a parity-check matrix of $\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})$ and, from Proposition 9, removing (puncturing) the last $m - \lambda$ columns of each block of this matrix provides a parity-check matrix of $\mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}|_{\mathcal{S}})$.

Remark 39. — Of course, what precedes extends straightforwardly to various subspaces and bases.

3.3. Subspace subcodes of generalised Reed–Solomon codes. — Expanding codes, in particular Reed–Solomon codes, over the base field has been studied since the 1980's. For instance, in [KL85, KL88], Kasami and Lin investigate the weight distribution of expanded binary Reed–Solomon codes. Sakakibara, Tokiwa and Kasahara extend their work to *q*-ary Reed–Solomon codes [STK89].

But the idea behind subspace subcodes, which consists in keeping only the subset of codewords that are defined over a subspace of the field, first appears in a paper by Solomon [Sol93]. In a joint work with McEliece [MS94], they define the notion of trace-shortened codes, which is a special case of subspace subcodes where $\lambda = m-1$ and where the considered subspace $\mathcal S$ is the kernel of the trace map. These articles focus uniquely on Reed–Solomon codes. Still, this point of view turns out to be the most general one since a subspace subcode of a GRS code can always be regarded as a subspace subcode of an RS code by changing the subspaces as explained by the following statements.

Proposition 40. — Let $\mathscr{C} \subseteq \mathbb{F}_{q^m}^n$, $S_0, \ldots, S_{n-1} \subseteq \mathbb{F}_{q^m}$ be \mathbb{F}_q subspaces and let $\mathbf{a} \in (\mathbb{F}_{q^m}^{\times})^n$. Then,

$$(\mathscr{C}\star \pmb{a})_{|(\mathcal{S}_0,\dots,\mathcal{S}_{n-1})}=\mathscr{C}_{|(a_0^{-1}\mathcal{S}_0,\dots,a_{n-1}^{-1}\mathcal{S}_{n-1})}\star \pmb{a}.$$

Corollary 41. — Let $x, y \in \mathbb{F}_{q^m}^n$ be a support and a multiplier and $\mathcal{S}_0, \ldots, \mathcal{S}_{n-1} \subseteq \mathbb{F}_{q^m}$, then

$$\mathbf{GRS}_k(\boldsymbol{x},\boldsymbol{y})_{|(\mathcal{S}_0,...,\mathcal{S}_{n-1})} = \mathbf{RS}_k(\boldsymbol{x})_{|(y_0^{-1}\mathcal{S}_0,...,y_{n-1}^{-1}\mathcal{S}_{n-1})} \star \boldsymbol{y}.$$

The notion is then generalised to any kind of subspace and any code by Jensen in [Jen95] under the name *subgroup subcodes*. In his thesis [Hat95] and in [HMS98], Hattori studies the dimension of subspace subcodes of Reed–Solomon codes. Some conjectures of Hattori are later proved by Spence in [Spe04]. Cui and Pei extend the results to generalised Reed–Solomon codes in [JJ01]. Then, Wu proposes a more constructive approach of these codes using the equivalent of the expansion operator in [Wu11].

The idea of using a various basis arises in [vDT99]. The first use of this notion for cryptography comes from Gabidulin and Loidreau who propose to use subspace subcodes of Gabidulin codes for a rank-metric based cryptosystem in [GL05, GL08]. Then Khathuria, Rosenthal and Weger proposed their code-based cryptosystem in [KRW19a], which is the first proposal to use subspace subcodes for cryptography in Hamming metric. At the same time, Berger, Gueye, Klamti and Ruatta propose another cryptosystem based on subspace subcodes of binary Reed–Solomon codes in [BGK19, BGKR19].

4. Further properties of the expansion operator

We now introduce some properties of the expansion operators. More specifically, in order to analyse the XGRS cryptosystem, we study how this operator behaves with respect to other operations (especially those used in the key generation): puncturing/shortening, computing the dual, changing the expansion basis. We also consider the relation with the square product operation, as this is a natural distinguisher for GRS-based codes.

In this section, for the sake of clarity, all properties will be defined considering the same basis for each entry, but everything works exactly the same way if one considered expansion with a different basis for each entry, as different columns of \mathbb{F}_{q^m} (or blocks of columns of \mathbb{F}_q corresponding to the expansion of same column of \mathbb{F}_{q^m}) do not interact.

4.1. Puncturing and shortening. —

Lemma 42. — Let \mathscr{C} be an [n,k] code over \mathbb{F}_{q^m} . Let \mathscr{L} denote a subset of [0,n-1]. Then the following equalities hold.

$$\begin{split} &\mathbf{ExpCode}_{\mathcal{B}}(\mathbf{Pct}_{\mathcal{L}}\left(\mathscr{C}\right)) = \mathbf{Pct}_{\mathcal{L}'}\left(\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})\right), \\ &\mathbf{ExpCode}_{\mathcal{B}}(\mathbf{Sh}_{\mathcal{L}}\left(\mathscr{C}\right)) = \mathbf{Sh}_{\mathcal{L}'}\left(\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})\right), \end{split}$$

where \mathcal{L}' denotes the set of all columns generated from expanding columns in \mathcal{L}' , that is

$$\mathcal{L}' \stackrel{def}{=} \bigcup_{i \in \mathcal{L}} \{i + j, 0 \le j < m\}.$$

Proof. — The result is straightforward for puncturing. The expansion operation is independent for each column, hence puncturing a column before expanding is equivalent to puncturing the corresponding block of m columns. As for shortening, the shortening operation is the dual of puncturing operation, hence the result is a consequence of the next lemma.

4.2. Dual code. —

Lemma 43 ([Wu11], Lemma 1). — Let \mathcal{B} be a basis and \mathcal{B}^* denote the dual basis. For all $\mathbf{a}, \mathbf{b} \in \mathbb{F}_{q^m}^n$, if \mathbf{a} and \mathbf{b} are orthogonal, i.e. $\mathbf{a} \cdot \mathbf{b}^{\mathsf{T}} = 0$, then $\mathbf{ExpVec}_{\mathcal{B}}(\mathbf{a})$ and $\mathbf{ExpVec}_{\mathcal{B}^*}(\mathbf{b})$ are orthogonal

$$\mathbf{ExpVec}_{\mathcal{B}}(\boldsymbol{a}) \cdot (\mathbf{ExpVec}_{\mathcal{B}^*}(\boldsymbol{b}))^{\mathsf{T}} = 0.$$

Corollary 44. — Let $\mathscr C$ be an [n,k] code over $\mathbb F_{q^m}$. Let $\mathcal B=(b_0,\ldots,b_{m-1})$ be a basis of $\mathbb F_{q^m}$. Then the following equality holds.

$$\mathbf{Dual}(\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C})) = \mathbf{ExpCode}_{\mathcal{B}^*}(\mathbf{Dual}(\mathscr{C})),$$

where \mathcal{B}^* denotes the dual basis of \mathcal{B} .

4.3. Changing the expansion basis. —

Lemma 45. — Let \mathscr{C} be an [n,k] code over \mathbb{F}_{q^m} . Let $\mathcal{B} = (b_0,\ldots,b_{m-1})$ be an \mathbb{F}_q -basis of \mathbb{F}_{q^m} . Let $Q \in \mathbb{F}_q^{m \times m}$ denote an invertible $m \times m$ matrix. The following equality holds.

$$\mathbf{ExpCode}_{\mathcal{B}\cdot (\boldsymbol{Q}^{-1})^\intercal}(\mathscr{C}) = \mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C}) \cdot \begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}.$$

Proof. — Let c be a codeword of \mathscr{C} . We only focus on the first entry of c. Denote $x \in \mathbb{F}_{q^m}$ this entry and $(x_0, \ldots, x_{m-1}) = \mathbf{ExpVec}_{\mathcal{B}}((x)) \in \mathbb{F}_q^m$. By definition, $x = \sum_{i=0}^{m-1} x_i b_i$. Let

 $\mathcal{D} = (d_0, \dots, d_{m-1})$ be the basis $\mathcal{B} \cdot (\mathbf{Q}^{-1})^\mathsf{T}$. For all $i \in [0, m-1]$, we have $b_i = \sum_{j=0}^m d_j q_{i,j}$ where $Q = (q_{i,j})_{0 \le i,j \le m}$. Replacing the b_i 's by this formula, we obtain

$$x = \sum_{i} x_{i} \left(\sum_{j} d_{j} q_{i,j} \right) = \sum_{j} \left(\sum_{i} x_{i} q_{i,j} \right) d_{j}.$$

Therefore,

$$\mathbf{ExpVec}_{\mathcal{B}}(x)\cdot \boldsymbol{Q} = \mathbf{ExpVec}_{\mathcal{D}}(x).$$

This holds for any entry of any codeword $c \in \mathscr{C}$.

Lemma 46. — Let \mathscr{C} be an [n,k] code over \mathbb{F}_{q^m} . Let $(\mathcal{B}_i)_i$ be n bases of \mathbb{F}_{q^m} . Let $(\mathbf{Q}_i) \in (\mathbb{F}_q^{m \times m})^n$ denote n invertible $m \times m$ matrices. The following equality holds.

$$\mathbf{ExpCode}_{(\mathcal{B}_i\cdot (\boldsymbol{Q}_i^{-1})^\mathsf{T})_i}(\mathscr{C}) = \mathbf{ExpCode}_{\mathcal{B}_i}(\mathscr{C})\cdot \begin{pmatrix} \boldsymbol{Q}_0 & & \\ & \ddots & \\ & & \boldsymbol{Q}_n \end{pmatrix}.$$

4.4. Scalar multiplication in \mathbb{F}_{q^m} . —

Lemma 47. — Let $\mathscr C$ be an [n,k] code over $\mathbb F_{q^m}$. Let $(\mathcal B_i)_i$ be n basis of $\mathbb F_{q^m}$. Let a = $(a_0,\ldots,a_{n-1})\in\mathbb{F}_{q^m}^n$ denote a vector of length n over \mathbb{F}_{q^m} . The following equality holds.

$$\begin{split} \mathbf{ExpCode}_{(\mathcal{B}_i)_i}(\mathscr{C}) &= \mathbf{ExpCode}_{(a_i\mathcal{B}_i)_i}(\{(c \star a) \,|\, c \in \mathscr{C}\}) \\ &= \mathbf{ExpCode}_{(a_i\mathcal{B}_i)_i}(\mathscr{C} \star a). \end{split}$$

5. The cryptosystem

Let us first present a generic encryption scheme based on subspace subcodes of GRS codes. This cryptosystem will be referred to as the Subspace Subcode of Reed-Solomon (SSRS) scheme. Then, we present the XGRS scheme [KRW19a] and explain why it is a sub-instance of the SSRS scheme.

5.1. SSRS: a generic scheme based on subspace subcodes of GRS codes. —

5.1.1. Parameters. — The cryptosystem is publicly parametrised by:

- -q a prime power;
- -m an integer;
- $-\lambda$ such that $0 < \lambda < m$;
- -n, k such that $0 \le k < n \le q^m$ and $km > (m \lambda)n$.

5.2. Key generation. —

- Generate a uniformly random vector $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$ with distinct entries.
- Choose n uniformly random λ -dimensional vector subspaces $S_0, \ldots, S_{n-1} \subseteq \mathbb{F}_{q^m}$ with respective bases $\mathcal{B}_{\mathcal{S}_0}, \dots, \mathcal{B}_{\mathcal{S}_{n-1}}$.

 – Let $G_{\text{pub}} \in \mathbb{F}_q^{(km-n(m-\lambda)) \times \lambda n}$ denote a generator matrix of the code

$$\mathscr{C}_{\mathrm{pub}} \stackrel{\mathrm{def}}{=} \mathbf{ExpCode}_{(\mathcal{B}_{\mathcal{S}_0}, \dots, \mathcal{B}_{\mathcal{S}_{n-1}})} \left(\mathbf{RS}_k(\boldsymbol{x})_{| (\mathcal{S}_0, \dots, \mathcal{S}_{n-1})} \right).$$

If G_{pub} is not full-rank, abort and restart the process. See Section 3.2.5 for the practical computation on G_{pub} .

- The public key is G_{pub} and the secret key is $(x, \mathcal{B}_{\mathcal{S}_0}, \dots, \mathcal{B}_{\mathcal{S}_{n-1}})$.

Lemma 48 (Public Key Size). — The public key is a matrix of size $m(n-k) \times \lambda n$ over \mathbb{F}_q . Only the systematic part is transmitted. Hence the public key size in bits is

$$m(n-k)(\lambda n - m(n-k))\log_2(q).$$

5.2.1. Encryption. — Let $m \in \mathbb{F}_q^{mk-(m-\lambda)n}$ be the plaintext. Denote

$$t \stackrel{\text{def}}{=} \lfloor \frac{n-k}{2} \rfloor.$$

Choose $e \subseteq \mathbb{F}_q^{(m-\lambda)n}$ uniformly at random among vectors of $\mathbb{F}_q^{(m-\lambda)n}$ with exactly t non-zero blocks (see Definition 27).

5.2.2. Decryption. — From $\mathbf{y} \in \mathbb{F}_q^{\lambda n}$, construct a vector $\mathbf{y}' \in \mathbb{F}_q^{mn}$ by completing each block of size λ with $m - \lambda$ entries set to zero. That is to say, using the notation $\mathcal{J}(\lambda, m)$ introduced in Equation (4) of Lemma 38, we have $\mathbf{Pct}_{\mathcal{J}(\lambda,m)}(y') = y$ and $y'_i = 0$ for any $i \in \mathcal{J}(\lambda,m)$. Denote

$$y'' = \mathbf{SqueezeVec}_{(\mathcal{B}_i)_i}(y').$$

According to the definition of e, the vector $\mathbf{y}'' \in \mathbb{F}_{q^m}^n$ is at distance t of the code $\mathbf{RS}_k(\mathbf{x})$. Hence, by decoding, one computes the unique $\mathbf{c} \in \mathbf{RS}_k(\mathbf{x})$ at distance $\leq t$ from \mathbf{y}'' and the expansion of c yields $mG_{\rm pub}$.

- **5.3.** The XGRS cryptosystem. We describe here the cryptosystem which we call XGRS, presented in [KRW19a] by Khathuria, Rosenthal and Weger. Next, we explain why XGRS is a sub-instance of the SSRS scheme.
- **5.3.1.** Parameters. The cryptosystem is publicly parametrised by:
 - -q a prime power;
 - -m an integer;
 - $-\lambda$ such that $2 \leq \lambda < m$;
 - -n, k such that $0 \le k < n \le q^m$ and $km > (m \lambda)n$.

q	m	λ	n	k	Public Key Size (kB)
13	3	2	1258	1031	579
7	4	2	1872	1666	844

Table 1. Parameters proposed for the XGRS scheme [KRW19a]

Remark 49. — As suggested by the parameters in Table 1, m is a small integer. The preprint version of the paper [KRW19b] proposed to use m=2 with a slightly modified key generation. The proposed parameters are now m = 3 and m = 4.

5.3.2. Key Generation. —

- Generate uniformly random vectors $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{F}_{q^m}^n \times (\mathbb{F}_{q^m}^{\times})^n$ such that \boldsymbol{x} has distinct entries. Denote $\mathscr{C} = \mathbf{GRS}_k(x, y)$ and let \mathbf{H}_{sec} be a parity-check matrix of \mathscr{C} .
- Choose γ, a primitive element of F_{q^m}/F_q, i.e. a generator of the field extension. We consider the basis B_γ = (1, γ, ..., γ^{m-1}) of F_{q^m}.
 Set H ^{def} = ExpMat_{B_γ*}(H_{sec}) ∈ F_q^{m(n-k)×mn} a parity-check matrix ExpCode_{B_γ}(𝒞).
- For any $i \in [0, n-1]$, choose \mathcal{L}_i a random subset of [(i-1)m, im-1] of size $|\mathcal{L}_i| = m \lambda$. Set $\mathcal{L} = \cup_i \mathcal{L}_i$.
- $\text{ Set } \boldsymbol{H}_{\mathcal{L}} \stackrel{\text{def}}{=} \mathbf{Pct}_{\mathcal{L}} (\boldsymbol{H}) \in \mathbb{F}_q^{m(n-k) \times \lambda n}.$ $\text{ For any } i \in [\![0,n-1]\!], \text{ choose } \boldsymbol{Q}_i \text{ a random } \lambda \times \lambda \text{ invertible matrix. Denote by } \boldsymbol{Q} \text{ the }$ block-diagonal matrix having Q_0, \ldots, Q_{n-1} as diagonal blocks.

 – Denote by S the invertible matrix of \mathbb{F}_q such that $S \cdot H_{\mathcal{L}} \cdot Q$ is in systematic form.

- $\begin{array}{l} \text{ Set } \boldsymbol{H}_{\text{pub}} \stackrel{\text{def}}{=} \boldsymbol{S} \cdot \boldsymbol{H}_{\mathcal{L}} \cdot \boldsymbol{Q}. \\ \text{ The public key is } \boldsymbol{H}_{\text{pub}}, \text{ the private key is } (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Q}, \mathcal{L}, \gamma). \end{array}$

Remark 50. — Compared to the cryptosystem presented in [KRW19a], we omitted the block permutation. Indeed, applying a block permutation after expanding is equivalent to applying the permutation before the expansion and then expanding. As we start with a GRS code chosen uniformly at random, applying a permutation on the columns does not change the probability distribution of the public keys.

5.3.3. Encryption. — Recall that $t \stackrel{\text{def}}{=} \lfloor \frac{n-k}{2} \rfloor$ the error-correcting capacity of a GRS code of length n and dimension k. The message is encoded as a vector $\mathbf{y} \in \mathbb{F}_q^{\lambda n}$ whose support is included in t blocks of length λ , *i.e.* there exist positions $i_0, \ldots, i_{t-1} \in [0, n-1]$, such that

$$\mathbf{Support}(\boldsymbol{y}) \subseteq \bigcup_{0 \le \ell \le t-1} [\![\lambda(i_{\ell}-1), \lambda i_{\ell}-1]\!].$$

The ciphertext is then defined as $c^\intercal = H_{\text{pub}} \cdot y^\intercal$.

- **5.3.4.** Decryption. In order to decrypt the ciphertext, a user knowing the private key should:
 - generate $oldsymbol{H}_{ ext{sec}}$ from $oldsymbol{x}$ and $oldsymbol{y}$
 - compute $\mathbf{c}' = \mathbf{c} \cdot \mathbf{S}^{-1\mathsf{T}}$;
 - compute $c'' = \mathbf{SqueezeVec}_{\mathcal{B}_{\gamma}}(c');$
 - find $y'' \in \mathbb{F}_{q^m}^n$ of weight $|y''| \leqslant t$ such that $c''^{\mathsf{T}} = H_{\text{sec}}y''^{\mathsf{T}}$ (i.e. decode in \mathscr{C});
 - $\text{ compute } oldsymbol{y}' = \mathbf{Pct}_{\mathcal{L}}\left(\mathbf{ExpVec}_{\mathcal{B}_{\gamma}}(oldsymbol{y}'')\right);$
 - finally recover $\boldsymbol{y} = \boldsymbol{y}' \cdot (\boldsymbol{Q}^{-1})^{\mathsf{T}}$.
- **5.3.5.** Relation with the SSRS scheme. To conclude this section, we show that the XGRS scheme is a sub-instance of the SSRS scheme presented in Section 5.1.

Proposition 51. — The XGRS scheme with secret key $(\mathbf{x}, \mathbf{y}, \mathbf{Q}, \mathcal{L}, \gamma)$ is equivalent to the SSRS scheme with secret key $(\mathbf{x}, \mathcal{S}_0, \dots, \mathcal{S}_{n-1})$ where the subspaces \mathcal{S}_i are defined as follows.

- $\ Let \ \mathcal{B}_{i}^{(0)} \stackrel{def}{=} \mathbf{Pct}_{\mathcal{L}_{i}} \left(\mathcal{B}_{\gamma} \right) \in \mathbb{F}_{q^{m}}^{\lambda} \ \ where \ \mathcal{L}_{i} \stackrel{def}{=} \{j mi, \forall j \in \mathcal{L} \cap \llbracket im, (i+1)m 1 \rrbracket \}.$
- $\operatorname{Set} \mathcal{B}_{i}^{(1)} \stackrel{\operatorname{def}}{=} y_{i}^{-1} \mathcal{B}_{i}^{(0)} \cdot (\boldsymbol{Q}_{i}^{-1})^{\mathsf{T}}.$
- S_i is the subspace of \mathbb{F}_{q^m} spanned by the elements of $\mathcal{B}_i^{(1)}$.

Proof. — Let \mathscr{C}_{pub} denote the public code of an instance of the XGRS scheme with private key $(x, y, Q, \mathcal{L}, \gamma)$, *i.e.* \mathscr{C}_{pub} is the code over \mathbb{F}_q that admits the public key $\boldsymbol{H}_{\text{pub}}$ as parity-check matrix. We have

$$egin{aligned} \mathscr{C}_{ ext{pub}} &= \mathbf{Dual} \left(\left\langle \left. oldsymbol{H}_{ ext{pub}} \right
ight
angle_{\mathbb{F}_q}
ight) \ &= \mathbf{Dual} \left(\left\langle \left. oldsymbol{H}_{\mathcal{L}} \cdot oldsymbol{Q}
ight
angle_{\mathbb{F}_q}
ight). \end{aligned}$$

Let us define $Q^{(1)} \stackrel{\text{def}}{=} (Q^{-1})^{\mathsf{T}}$. This is still a block-diagonal matrix composed of n blocks of size $\lambda \times \lambda$. We can rewrite this

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Dual}\left(\left\langle \left. oldsymbol{H}_{\mathcal{L}} \left.
ight
angle_{\mathbb{F}_q}
ight) \cdot oldsymbol{Q}^{(1)}.$$

We can replace $H_{\mathcal{L}}$ by its definition: $\mathbf{Pct}_{\mathcal{L}}\left(\mathbf{ExpMat}_{\mathcal{B}_{\gamma}^*}(H_{\mathrm{sec}})\right)$. Next, we can swap the **Dual** and **Punct** operators according to Proposition 9:

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Sh}_{\mathcal{L}}\left(\mathbf{Dual}\left(\mathbf{ExpCode}_{\mathcal{B}^*_{\gamma}}\left(\left\langle\,oldsymbol{H}_{\mathrm{sec}}\,
ight
angle_{\mathbb{F}_q}
ight)
ight)
ight)\cdot oldsymbol{Q}^{(1)}.$$

We can then swap the **Dual** and **ExpCode** operators according to Corollary 44.

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Sh}_{\mathcal{L}}\left(\mathbf{ExpCode}_{\mathcal{B}_{\gamma}}\left(\mathbf{Dual}\left(\left\langle\,oldsymbol{H}_{\mathrm{sec}}\,
ight
angle_{\mathbb{F}_{a}}
ight)
ight)
ight)\cdotoldsymbol{Q}^{(1)}.$$

Let G_{sec} be a generator matrix of the secret code $\mathbf{Dual}\left(\langle H_{\text{sec}} \rangle_{\mathbb{F}_q}\right)$, *i.e.* a generator matrix of the code $\mathbf{GRS}_k(x, y)$. We have

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Sh}_{\mathcal{L}}\left(\mathbf{ExpCode}_{\mathcal{B}_{\gamma}}\left(\left\langle \left. oldsymbol{G}_{\mathrm{sec}} \right
angle_{\mathbb{F}_{q}}
ight)
ight) \cdot oldsymbol{Q}^{(1)}.$$

Let us denote $Q^{(2)}$ the block-diagonal matrix obtained by replacing each $\lambda \times \lambda$ matrix of $Q^{(1)}$ by the $m \times m$ matrix obtained by inserting "an identity row/column" at the positions corresponding to \mathcal{L} . For instance, if $m = 3, \lambda = 2$ and the first element of \mathcal{L} equals 1, which means that the column 1 is shortened, we add a column and a row in the middle of $Q_0^{(1)}$, *i.e.*

if
$$Q_0^{(1)} = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix}$$
, then $Q_0^{(2)} = \begin{pmatrix} q_{00} & 0 & q_{01} \\ 0 & 1 & 0 \\ q_{10} & 0 & q_{11} \end{pmatrix}$.

Hence, we can write

$$\mathscr{C}_{ ext{pub}} = \mathbf{Sh}_{\mathcal{L}} \left(\left\langle \left. \mathbf{ExpMat}_{\mathcal{B}_{\gamma}}(oldsymbol{G}_{ ext{sec}}) \cdot oldsymbol{Q}^{(2)} \,
ight
angle_{\mathbb{F}_q}
ight).$$

We define $Q_i^{(3)}$ as the matrix obtained from $Q_i^{(2)}$ by permuting the columns so that the inserted columns are the $m - \lambda$ rightmost ones. For instance in the previous example, we would have

$$\boldsymbol{Q}_0^{(3)} = \left(\begin{array}{ccc} q_{00} & q_{01} & 0 \\ 0 & 0 & 1 \\ q_{10} & q_{11} & 0 \end{array} \right).$$

Therefore, $\mathbf{Q}^{(3)} = \mathbf{Q}^{(2)} \mathbf{P}$ where \mathbf{P} is a block–diagonal matrix whose diagonal blocks are $m \times m$ permutations matrices. Then, we replace \mathcal{L} by the set $\mathcal{J}(\lambda, m) = \{mi + j \mid 0 \le i < n, \ \lambda \le j < m\}$. Hence, we get

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Sh}_{\mathcal{J}(\lambda,m)} \left(\left\langle \left. \mathbf{ExpMat}_{\mathcal{B}_{\gamma}}(oldsymbol{G}_{\mathrm{sec}}) \cdot oldsymbol{Q}^{(3)} \right.
ight
angle_{\mathbb{F}_q}
ight).$$

We can apply the basis change explained in Lemma 45.

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Sh}_{\mathcal{J}(\lambda,m)} \left(\left\langle \left. \mathbf{ExpMat}_{\left(\mathcal{B}_{i}^{\prime}\right)_{i}}(G_{\mathrm{sec}}) \right. \right
angle_{\mathbb{F}_{q}}
ight),$$

where $\mathcal{B}_i' \stackrel{\text{def}}{=} \mathcal{B}_{\gamma} \cdot \left((Q_i^{(3)})^{-1} \right)^{\mathsf{T}}$ for all $i \in [0, n-1]$. Finally, we apply Corollary 41 and Lemma 47 to replace the code $\mathbf{GRS}_k(\boldsymbol{x}, \boldsymbol{y})$ by $\mathbf{RS}_k(\boldsymbol{x})$. Hence,

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{Sh}_{\mathcal{J}(\lambda,m)} \left(\left\langle \left. \mathbf{ExpMat}_{\left(\mathcal{B}_{i}\right)_{i}}(G_{\mathrm{sec}}') \right. \right\rangle_{\mathbb{F}_{q}} \right),$$

where G'_{sec} is a generator matrix of $\mathbf{RS}_k(x)$ and $\mathcal{B}_i \stackrel{\text{def}}{=} y_i^{-1} \mathcal{B}'_i$ for all $i \in [0, n-1]$. In other words, $\mathscr{C}_{\text{pub}} = \mathbf{RS}_k(x)|_{(\mathcal{S}_0, \dots, \mathcal{S}_{n-1})}$, where \mathcal{S}_i is the subspace spanned by the λ first elements of \mathcal{B}_i . This is indeed an instance of the SSRS cryptosystem.

6. The twisted-square code and distinguisher

In this section, we first explore the case m=3, $\lambda=2$ to give some insight on the interest of defining the twisted star product of two subspace subcodes. Then, we define this notion for other parameters. In the second part of the section, we focus on the dimension of the twisted square codes and how it can be used as a distinguisher.

For the sake of simplicity, all the results of this section are stated using the same subspace and expansion basis for all blocks but they can be straightforwardly generalised to the case of various subspaces and expansion bases.

6.1. The twisted square product. —

6.1.1. Motivation: the case $\lambda = 2, m = 3$. — In this section, we consider the case $\lambda = 2, m = 3$. Let us introduce a definition that is needed in the sequel.

Definition 52. — Let $S \subseteq \mathbb{F}_{q^m}$ be an \mathbb{F}_q -vector space, we define the square subspace

$$\mathcal{S}^2 \stackrel{\mathrm{def}}{=} \langle ab \mid a, b \in \mathcal{S} \rangle_{\mathbb{F}_q}$$

Lemma 53. — Let S be a subspace of \mathbb{F}_{q^m} of dimension 2. Let $\mathcal{B}_S = (\gamma_0, \gamma_1)$ be a basis of S. Let $a, b \in S$ such that

$$\mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}}}((a)) = (a_0, a_1)$$
 and $\mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}}}((b)) = (b_0, b_1).$

Then,

$$\mathbf{ExpVec}_{\mathcal{B}_{S^2}}((ab)) = (a_0b_0, a_0b_1 + a_1b_0, a_1b_1),$$

where $\mathcal{B}_{\mathcal{S}^2} = (\gamma_0^2, \gamma_0 \gamma_1, \gamma_1^2)$.

Remark 54. — Note that when m=3 and $\dim \mathcal{S}=2$, we have $\mathcal{S}^2=\mathbb{F}_{q^3}$. Indeed, let (γ_0,γ_1) be a basis of \mathcal{S} , if $\gamma_0^2,\gamma_0\gamma_1$ and γ_1^2 were not \mathbb{F}_q -independent, denoting $\zeta \stackrel{\text{def}}{=} \gamma_1/\gamma_0$, then $1,\zeta$ and ζ^2 would not be \mathbb{F}_q -independent either. Hence ζ would have degree $\leqslant 2$ over \mathbb{F}_q . But by definition $\zeta \notin \mathbb{F}_q$.

Consider the SSRS scheme with $m=3, \lambda=2$. The public key is a generator matrix G of $\mathbf{ExpCode}_{(\mathcal{B}_{S_0},\ldots,\mathcal{B}_{S_{n-1}})}\left(\mathscr{C}_{|(S_0,\ldots,S_{n-1})}\right)$. From an attacker's point of view, the spaces S_0,\ldots,S_{n-1} and their bases $\mathcal{B}_{S_0},\ldots,\mathcal{B}_{S_{n-1}}$ are unknown. But, we have access to the entries of G, in particular we have access to the coefficients a_0,a_1 (resp. b_0,b_1) of the decomposition in the basis \mathcal{B}_i of the i-th entry of some codewords of \mathscr{C} . Hence, the coefficients of the product ab in the basis $(\gamma_0^2,\gamma_0\gamma_1,\gamma_1^2)$ of S_i^2 can be computed without knowing neither \mathscr{C} nor the basis \mathcal{B}_i . This motivates the following definition.

Definition 55 (Twisted product). — Let a and b in \mathbb{F}_q^{2n} whose components are denoted

$$\boldsymbol{a} = (a_0^{(0)}, a_0^{(1)}, a_1^{(0)}, a_1^{(1)}, \dots, a_{n-1}^{(0)}, a_{n-1}^{(1)});$$

$$\boldsymbol{b} = (b_0^{(0)}, b_0^{(1)}, b_1^{(0)}, b_1^{(1)}, \dots, b_{n-1}^{(0)}, b_{n-1}^{(1)}).$$

We define the *twisted product* of \boldsymbol{a} and \boldsymbol{b} as

$$\boldsymbol{a}\,\tilde{\star}\,\boldsymbol{b}\stackrel{\mathrm{def}}{=}(a_i^{(0)}b_i^{(0)},a_i^{(0)}b_i^{(1)}+a_i^{(1)}b_i^{(0)},a_i^{(1)}b_i^{(1)})_{0\leqslant i\leqslant n-1}\in\mathbb{F}_q^{3n}.$$

This definition extends to the product of codes, where the *twisted product* of two codes \mathscr{A} and $\mathscr{B} \subseteq \mathbb{F}_q^{2n}$ is defined as

$$\mathscr{A} ilde{\star}\mathscr{B}\stackrel{\mathrm{def}}{=}\langle\,oldsymbol{a}\, ilde{\star}\,oldsymbol{b}\midoldsymbol{a}\in\mathscr{A},\;oldsymbol{b}\in\mathscr{B}\,
angle_{\mathbb{F}_{a}}\,.$$

In particular, $\mathscr{A}^{\tilde{\star}2}$ denotes the twisted square code of a code \mathscr{A} : $\mathscr{A}^{\tilde{\star}2} \stackrel{\text{def}}{=} \mathscr{A}\tilde{\star}\mathscr{A}$.

With this definition, we can rewrite Lemma 53 for vectors in the following way.

Lemma 56. — Let S be a subspace of \mathbb{F}_{q^m} of dimension 2. Let $\mathcal{B}_S = (\gamma_0, \gamma_1)$ be a basis of S. Let $a, b \in \mathbb{F}_{q^m}^n$ such that all their entries lie in S. Then,

(5)
$$\mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}}}(a) \tilde{\star} \mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}}}(b) = \mathbf{ExpVec}_{\mathcal{B}_{\mathcal{S}^2}}(a \star b),$$

where $\mathcal{B}_{\mathcal{S}^2} = (\gamma_0^2, \gamma_0 \gamma_1, \gamma_1^2)$.

Extending this result to codes, we obtain the following theorem.

Theorem 57. — Let $\mathcal{B}_{\mathcal{S}}$ be an \mathbb{F}_q -basis of \mathcal{S} such that $\mathcal{B}_{\mathcal{S}} = (\gamma_0, \gamma_1)$. Let

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}_{|\mathcal{S}}),$$

where \mathscr{C} is an [n,k] code over \mathbb{F}_{q^m} . Then,

(6)
$$\mathscr{C}_{\mathrm{pub}}^{\tilde{\star}2} \subseteq \mathbf{ExpCode}_{\mathcal{B}_{S^2}} \left(\mathscr{C}^{\star 2} \right),$$

where $\mathcal{B}_{S^2} = (\gamma_0^2, \gamma_0 \gamma_1, \gamma_1^2)$. This results generalises straightforwardly to an expansion over various subspaces.

Proof. — This is a consequence of Lemma 56.

Remark 58. — In the sequel, we see that under a reasonable conjecture and some condition, the inclusion in (6) is an equality. See Conjecture 67.

6.1.2. General definition of the twisted square code. —

Remark 59. — This section is a generalisation of the previous definitions and results. A reader only interested in the practical aspects of the attack can skip directly to Section 7.

For arbitrary $\lambda \geq 2$, we have the following definition.

Definition 60 (Twisted square product, general case). — Let a and b in $\mathbb{F}_q^{\lambda n}$ whose components are denoted

$$\mathbf{a} = (a_{0,0}, \dots, a_{0,\lambda-1}, a_{1,0}, \dots, a_{1,\lambda-1}, \dots, a_{n-1,0}, \dots, a_{n-1,\lambda-1}),$$

$$\mathbf{b} = (b_{0,0}, \dots, b_{0,\lambda-1}, b_{1,0}, \dots, b_{1,\lambda-1}, \dots, b_{n-1,0}, \dots, b_{n-1,\lambda-1}).$$

We define the twisted product $\mathbf{a} \,\tilde{\star} \, \mathbf{b} \in \mathbb{F}_q^{\binom{\lambda+1}{2}n}$ of \mathbf{a} and \mathbf{b} such that for any $i \in [0, n-1]$ and for r, s such that $0 \leq r \leq s \leq \lambda - 1$,

$$(\boldsymbol{a}\,\tilde{\star}\,\boldsymbol{b})_{i\binom{\lambda+1}{2} + \binom{s+1}{2} + r} \stackrel{\text{def}}{=} \left\{ \begin{array}{ccc} a_{i,r}b_{i,s} + a_{i,s}b_{i,r} & \text{if} & r < s \\ a_{i,r}b_{i,r} & \text{if} & r = s. \end{array} \right.$$

This definition extends to the product of codes, where the twisted product of two codes \mathscr{A} and $\mathscr{B} \subseteq \mathbb{F}_q^{\lambda n}$ is defined as

$$\mathscr{A} \check{\star} \mathscr{B} \stackrel{\mathrm{def}}{=} \langle \, m{a} \, \check{\star} \, m{b} \mid m{a} \in \mathscr{A}, \, \, m{b} \in \mathscr{B} \,
angle_{\mathbb{F}_a}$$
 .

In particular, $\mathscr{A}^{\tilde{\star}2}$ denotes the twisted square code of a code \mathscr{A} : $\mathscr{A}^{\tilde{\star}2} \stackrel{\text{def}}{=} \mathscr{A}_{\tilde{\star}}\mathscr{A}$.

We are interested in the case where $\mathcal{S}^2 = \mathbb{F}_{q^m}$, because the goal is to reconstruct a fully expanded code. For a random subspace $\mathcal{S} \subseteq \mathbb{F}_{q^m}$ of dimension λ , its dimension is typically min $\left\{\binom{\lambda+1}{2}, m\right\}$. The case $m=3, \lambda=2$ is a special case where $\binom{\lambda+1}{2}=m$. Hence, for other parameters m, λ such that $\binom{\lambda+1}{2}=m$, Theorem 57 generalises straightforwardly. But when $\binom{\lambda+1}{2}>m$, the twisted square code does not correspond to an expanded code. It is as if the code were expanded over a generating family of \mathbb{F}_{q^m} which is not a basis: the vectors are too long. A way to circumvent this and to obtain a result similar to Theorem 57 is to shorten the twisted square code to cancel the useless columns and obtain an expansion over a basis. This yields the following results.

Lemma 61. — Let S be a subspace of \mathbb{F}_{q^m} of dimension λ such that $S^2 = \mathbb{F}_{q^m}$. Let $\mathcal{B}_S = (\gamma_0, \ldots, \gamma_{\lambda-1})$ be a basis of S. Let \mathcal{B}_{S^2} denote the first m elements of $(\gamma_0^2, \gamma_0 \gamma_1, \ldots, \gamma_0 \gamma_i, \gamma_1 \gamma_i, \ldots, \gamma_i^2, \ldots, \gamma_{\lambda-1}^2)$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{F}_{q^m}^n$ whose entries all lie in S. Denote \mathbf{c} the vector of length $\binom{\lambda+1}{2}$ n over \mathbb{F}_q defined as

$$c \stackrel{def}{=} \operatorname{ExpVec}_{\mathcal{B}_{\mathcal{S}}}(a) \, \check{\star} \operatorname{ExpVec}_{\mathcal{B}_{\mathcal{S}}}(b).$$

Let \mathcal{J}_0 denote the set $\mathcal{J}\left(m, \binom{\lambda+1}{2}\right)$, i.e.

(7)
$$\mathcal{J}_0 \stackrel{def}{=} \left\{ \binom{\lambda+1}{2} i + j, i \in \llbracket 0, n-1 \rrbracket, j \in \llbracket m, \binom{\lambda+1}{2} - 1 \rrbracket \right\}.$$

Ιf

- (i) $\mathcal{B}_{\mathcal{S}^2}$ is a basis of \mathbb{F}_{q^m} ;
- (ii) for any $i \in \mathcal{J}_0$, the i-th entry of \mathbf{c} is zero,

then,

(8)
$$\mathbf{Pct}_{\mathcal{J}_0}(\mathbf{c}) = \mathbf{ExpVec}_{\mathcal{B}_{\diamond 2}}(\mathbf{a} \star \mathbf{b}).$$

Proof. — Let c be defined as in the statement. We want to prove that

$$\mathbf{SqueezeVec}_{\mathcal{B}_{S^2}}(\mathbf{Pct}_{\mathcal{J}_0}\left(oldsymbol{c}
ight)) = oldsymbol{a}\staroldsymbol{b}.$$

This is equivalent to Equation (8) because $\mathcal{B}_{\mathcal{S}^2}$ is a basis of \mathbb{F}_{q^m} . Without loss of generality, we only need to focus on the block corresponding to the first entry in \mathbb{F}_{q^m} .

Let $(a_0, \ldots, a_{\lambda-1})$ and $(b_0, \ldots, b_{\lambda-1})$ denote the decomposition of the first entries of \boldsymbol{a} (resp. \boldsymbol{b}) over $\mathcal{B}_{\mathcal{S}}$. The first entry of $\boldsymbol{a} \star \boldsymbol{b}$ is

$$\left(\sum_{i} a_{i} \gamma_{i}\right) \left(\sum_{j} a_{j} \gamma_{j}\right) = \sum_{0 \leqslant i \leqslant j < \lambda} c_{i,j} \gamma_{i} \gamma_{j},$$

where the coefficients $c_{i,j}$ match exactly the definition of the twisted square product, hence correspond to the entries of c.

Let $\mathcal{B}_{\text{full}}$ denote the family $(\gamma_0^2, \gamma_0 \gamma_1, \dots, \gamma_0 \gamma_i, \gamma_1 \gamma_i, \dots, \gamma_i^2, \dots, \gamma_{\lambda-1}^2)$. The last entries of each block of c are equal to zero. This corresponds exactly to the elements of $\mathcal{B}_{\text{full}}$ that are not in $\mathcal{B}_{\mathcal{S}^2}$. We therefore have

$$\mathbf{SqueezeVec}_{\mathcal{B}_{\varsigma^2}}(\mathbf{Pct}_{\mathcal{J}_0}\left(\boldsymbol{c}\right)) = \mathbf{SqueezeVec}_{\mathcal{B}_{\mathrm{full}}}(\boldsymbol{c}) = \boldsymbol{a} \star \boldsymbol{b}.$$

This leads to the following main statement.

Theorem 62. — Let $S \subseteq \mathbb{F}_{q^m}$ be a subspace of dimension λ such that $S^2 = \mathbb{F}_{q^m}$. Let \mathcal{B}_S be an \mathbb{F}_q -basis of S such that $\mathcal{B}_S = (\gamma_0, \ldots, \gamma_{\lambda-1})$. Let

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}_{|\mathcal{S}}),$$

where \mathscr{C} is an [n,k] code over \mathbb{F}_{q^m} . Then,

(9)
$$\mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{C}_{\mathrm{pub}}^{\tilde{\star}2}\right) \subseteq \mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}^2}}\left(\mathscr{C}^{\star 2}\right),$$

where \mathcal{B}_{S^2} and \mathcal{J}_0 are defined as in Lemma 61. This result generalises straightforwardly to an expansion over various subspaces and bases.

Proof. — Compared to Lemma 61 and its proof, one should be careful that $\mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{C}_{\text{pub}}^{\tilde{\star}2}\right)$ is in general not spanned by words of the form $\mathbf{Pct}_{\mathcal{J}_0}\left(a\tilde{\star}b\right)$ with $a,b\in\mathbf{ExpCode}_{\mathcal{B}}(\mathscr{C}_{\text{pub}})$ but by words of the form

$$\mathbf{Pct}_{\mathcal{J}_0}\left(a_0\tilde{\star}b_0+\cdots+a_s\tilde{\star}b_s\right), \quad \text{for} \quad a_0,\ldots,a_s,b_0,\ldots,b_s\in\mathscr{C}_{\mathrm{pub}}.$$

Therefore, one needs to apply the very same reasoning as that of the proof of Lemma 61 replacing $a\tilde{\star}b$ by a sum of such vectors. The proof generalises straightforwardly, since all the involved operators are linear.

Remark 63. — In the sequel, we see that under a reasonable conjecture and some condition, the inclusion in (9) is an equality. See Conjecture 67.

6.2. Dimension of the twisted square of subspace subcodes. —

6.2.1. Typical dimension of the twisted square of a random subspace subcode. —

Lemma 64. — Let $\mathscr{C} \subseteq \mathbb{F}_{q^m}^n$ and $\mathcal{S} \subseteq \mathbb{F}_{q^m}$ be an \mathbb{F}_q -subspace. Then

(10)
$$\dim_{\mathbb{F}_q}(\mathscr{C}_{|\mathcal{S}})_{\mathbb{F}_q}^{\star 2} \leqslant \min\left\{mn, \ m \cdot \dim_{\mathbb{F}_{q^m}} \mathscr{C}^{\star 2}\right\}.$$

Proof. — Let $\mathscr{C} \subseteq \mathbb{F}_{q^m}^n$ and $\mathcal{S} \subseteq \mathbb{F}_{q^m}$ be an \mathbb{F}_q -vector space such that $\mathcal{S}^2 = \mathbb{F}_{q^m}$. Then

$$(\mathcal{C}_{|\mathcal{S}})_{\mathbb{F}_q}^{\star 2} \subseteq \mathscr{C}^{\star 2}.$$

Indeed, it suffices to observe that the result holds on \mathbb{F}_q -generators. Let $a, b \in \mathcal{C}_{|S|}$. Then, $a \star b \in \mathcal{C}^{\star 2}$. In addition, for any $i \in \{0, \dots, n-1\}$, we have $(a \star b)_i \in S^2$. Thus, $a \star b \in (\mathcal{C}^{\star 2})_{|S^2|}$. \square

Similarly to the case of the square code (Theorem 4), we expect that (10) and equivalently (11) is typically an equality. Our experiments using the computer algebra software *Sage* encourages us to establish the following conjecture.

Conjecture 65. — For any positive integer k such that $2k \leqslant n$, any \mathbb{F}_q -subspace $\mathcal{S} \subseteq \mathbb{F}_{q^m}^n$ of dimension $\lambda \geqslant 2$ such that $\mathcal{S}^2 = \mathbb{F}_{q^m}$ and any \mathbb{F}_q -basis $\mathcal{B}_{\mathcal{S}}$ of \mathcal{S} , let \mathscr{R} denote an [n,k] code chosen uniformly at random, then

$$\mathbb{P}\left[\dim_{\mathbb{F}_q}\left(\mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{R}_{|\mathcal{S}})\right)^{\tilde{\star}2} = \min\left\{\binom{\lambda+1}{2}n, \binom{km-n(m-\lambda)+1}{2}\right\}\right] \underset{k \to \infty}{\longrightarrow} 1.$$

Here it is worth noting that in general $\binom{\lambda+1}{2} > m$. Therefore, as already noticed before stating Lemma 61, the twisted square product may represent something which is not an expansion with respect to a basis but a kind of expansion with respect to generators of the S_i 's. A solution to get an expansion again is to proceed as in the statement of Theorem 62 and to shorten this twisted square code at the set \mathcal{J}_0 introduced in (7). That is to say, shortening the code at the $\binom{\lambda+1}{2} - m$ last positions of each block of length $\binom{\lambda+1}{2}$. By this manner and according to Conjecture 65, for a random code \mathscr{R} , we typically have

$$\dim_{\mathbb{F}_q} \mathbf{Sh}_{\mathcal{J}_0} \left(\mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}} (\mathscr{R}_{|\mathcal{S}})^{\tilde{\star}2} \right) =$$

$$\min \left\{ mn, \binom{km - n(m - \lambda) + 1}{2} - n \left(\binom{\lambda + 1}{2} - m \right) \right\}.$$

6.2.2. Typical dimension of the twisted square of a subspace subcode of a RS code. — On the other hand, subspace subcodes of Reed–Solomon codes have a different behaviour. Indeed, Theorem 62 yields the following result.

Corollary 66. — Given a GRS code $\mathscr{C} = \mathbf{GRS}_k(\boldsymbol{x}, \boldsymbol{y})$ with $2k \leqslant n$, and an \mathbb{F}_q -subspace $\mathcal{S} \subseteq \mathbb{F}_{q^m}$ of dimension $\lambda < m$ such that $\mathcal{S}^2 = \mathbb{F}_{q^m}$. Then,

(12)
$$\dim_{\mathbb{F}_q} \left(\mathbf{Sh}_{\mathcal{J}_0} \left(\mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}_{|\mathcal{S}}) \right)^{\tilde{\star}2} \right) \leqslant \min\{mn, m(2k-1)\}.$$

Conjecture 67. — The inequality in Corollary 66 is typically an equality as soon as

$$\binom{\dim_{\mathbb{F}_q} \mathscr{C}_{|\mathcal{S}} + 1}{2} \geqslant \min\{mn, m(2k - 1)\}.$$

6.2.3. The distinguisher. — The twisted product provides us a distinguisher between expanded subspace subcodes of GRS codes and expanded subspace subcodes of random codes. Indeed, assuming Conjecture 65, we obtain the following result.

Theorem 68. — Let k be a positive integer such that 2k < n and $\mathscr{D} = \mathbf{ExpCode}_{\mathcal{B}_{\mathcal{S}}}(\mathscr{C}_{|\mathcal{S}})$, where \mathscr{C} is either a random [n,k] code over \mathbb{F}_{q^m} or an [n,k] GRS code over \mathbb{F}_{q^m} . Suppose also that

$$m(2k-1) < \min \left\{ mn, \binom{km-n(m-\lambda)+1}{2} - n \left(\binom{\lambda+1}{2} - m \right) \right\}.$$

Then, assuming Conjectures 65 and 67 the computation of $\dim_{\mathbb{F}_q} \mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{D}^{\tilde{\star}2}\right)$ provides a polynomial-time algorithm which decides whether \mathscr{C} is an RS code or a random code and succeeds with high probability. This extends straightforwardly to the case of multiple spaces and bases.

Remark 69. — The condition 2k < n is necessary for the distinguisher. Indeed, if $2k \ge n$, the square code of the GRS code spans the whole space $\mathbb{F}_{q^m}^n$. Hence it can not be distinguished from a random code. When this condition is not met, it is sometimes possible to shorten the code so that the shortened code meets this condition. This is addressed in Section 6.2.5.

6.2.4. Experimental results. — Using the computer algebra software Sage, we tested the behaviour of the dimension of the twisted square (shortened at \mathcal{J}_0) of subspace subcodes either of random codes or of RS codes. For each parameter set (see Table 2), we ran more than 100 test and none of them yielded dimensions of the twisted square that was different from the typical value given either by Conjecture 65 or by Corollary 66 together with Conjecture 67.

Code	q	m	λ	n	k	Expected Dimension of $\mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{C}^{\tilde{\star}^2}\right)$	Actual Dimension of $\mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{C}^{\tilde{\star}^2}\right)$
Random	1 7	3	2	120	55	360	360
RS	7	3	2	120	55	327	327
Randon	1 7	5	3	160	75	800	800
RS	7	5	3	160	75	745	745

Table 2. Parameter sets for the tests. The code $\mathscr C$ is the shortening at $m-\lambda$ positions per block of the expansion of a parent code. The parent code is either random or a Reed–Solomon code. Its status is precised in the left–hand column. The column before the last one gives the expected dimension of the twisted square code shortened at $\mathcal J_0$ according to Conjecture 65 for the random case and to Corollary 66 and Conjecture 67 for the RS case. The last column gives the actual dimension computed using Sage of the twisted square code. For each set of parameters, at least 100 tests were run and the actual dimension never differed from the expected one.

Remark 70. — Here again, we discussed the case of a single subspace S with a unique basis B for the sake of simplicity, but the distinguisher straightforwardly extends to the case of multiple spaces of dimension λ whose squares fill in \mathbb{F}_{q^m} together with multiple bases.

6.2.5. Broadening the range of the distinguisher by shortening. — Similarly to the works [CGG+14, COT17], the range of the distinguisher can be broadened by shortening the public code. This can make the distinguisher work in some cases when $2k \ge n$. The idea is to shorten some blocks of length λ (corresponding to a given position of the original code in $\mathbb{F}_{q^m}^n$). For each shortened block the degree k is decreased by 1. Indeed, from Lemma 42 shortening a whole block corresponds to shortening the corresponding position of the parent code over \mathbb{F}_{q^m} .

Let us investigate the condition for this to work. Let s_0 be the least positive integer such that $2(k - s_0) - 1 < n - s_0$, *i.e.*

$$s_0 \stackrel{\text{def}}{=} 2k - n.$$

If one shortens the public code at $s \ge s_0$ blocks, which corresponds to $s(m - \lambda)$ positions, we can apply Theorem 68 on the shortened code. The condition of the theorem becomes (13)

$$m(2(k-s)-1) < \min\left\{m(n-s), \binom{m(k-s)-(n-s)(m-\lambda)+1}{2} - (n-s)\left(\binom{\lambda+1}{2}-m\right)\right\}.$$

Example 71. — Consider the parameters of XGRS in the first row of Table 1. Suppose we shorten s = 820 blocks of the public key (i.e. 1260 positions of the parent GRS code). It corresponds to reduce to n' = n - s = 438 and k' = k - s = 211. The shortened public key will have dimension 195.

Thus, the twisted square of the shortened public key will have typical dimension 1263 while the twisted square of an expanded subspace subcode of a random code would have full length, i.e. 3(n-s) = 1314.

6.2.6. Limits of the distinguisher: the "m/2 barrier". — Suppose that $\lambda \leqslant \frac{m}{2}$ and let $\mathscr C$ be a GRS code of dimension k and $\mathcal S$ a subspace of dimension λ such that the SSRS code reaches the typical dimension (see Propositions 17 and 19), i.e. $\dim_{\mathbb{F}_q} \mathscr C_{|\mathcal S} = km - n(m - \lambda)$.

In order for this dimension to be positive, we must have

$$k > n\left(1 - \frac{\lambda}{m}\right) > \frac{n}{2}$$

This is incompatible with the necessary condition 2k < n (see Remark 69) and can not be overcome by shortening blocks as described in Section 6.2.5. Hence, whenever $\lambda \leq m/2$, the distinguisher is ineffective.

Remark 72. — In [COT14b, COT17] a distinguisher on so-called wild Goppa codes over quadratic extensions is established using the square code operation after a suitable shortening. This corresponds precisely to the case $\lambda = 1$ and m = 2 which, according to the previous discussion, should be out of reach of the distinguisher. The reason why this distinguisher is efficient for these parameters is precisely because the dimension of such codes significantly exceeds the lower bound of Proposition 17 (see [SKHN76, COT14a]).

7. Attacking the SSRS scheme

In this section, we describe how to use these tools to attack the SSRS scheme. For the sake of convenience, we first focus on the parameters with $m=3,\,\lambda=2$ and then discuss the general case.

7.1. The case m=3 and $\lambda=2$. —

7.1.1. Constructing the square code. — Let \mathscr{C}_{pub} be the public code of an instance of the SSRS scheme. This code is described by a generator matrix G_{pub} which is the only data we have access to. We know that there exist unknown spaces S_0, \ldots, S_{n-1} with bases $\mathcal{B}_{S_i} = (b_{i,0}, b_{i,1})$ and an RS code over \mathbb{F}_{q^m} such that

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{ExpCode}_{\left(\mathcal{B}_{\mathcal{S}_{i}}
ight)_{i}}\left(\mathbf{RS}_{k}(oldsymbol{x})_{\left|\left(\mathcal{S}_{i}
ight)_{i}
ight.}
ight).$$

We can compute the generator matrix of the twisted square code $\mathscr{C}^{\tilde{\star}2}_{\text{pub}}$, which according to Theorem 57 and Conjecture 67 is equal with high probability to

$$\mathbf{ExpCode}_{\left(\mathcal{B}_{\mathcal{S}_{i}^{2}}
ight)_{i}}\left(\mathbf{RS}_{2k-1}(oldsymbol{x})
ight),$$

where $\mathcal{B}_{\mathcal{S}_i^2} \stackrel{\text{def}}{=} (b_{i,0}^2, b_{i,0}b_{i,1}, b_{i,1}^2)$. It is important to stress that, at this stage, we do not know the value of \boldsymbol{x} nor the $\mathcal{B}_{\mathcal{S}_i}$ or the $\mathcal{B}_{\mathcal{S}_i^2}$.

7.1.2. Finding the value of x. — We now have access to a fully expanded RS code (and not a subspace subcode) and want to use this to find the value of x. In fact, the authors of [BGK19] propose an algorithm to solve this problem, by using a generalisation of the algorithm of Sidelnikov and Shestakov [SS92] to recover the structure of GRS codes.

Theorem 73. — [BGK19, § IV.B] Let $\mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathbb{F}_{q^m}^n$ be a vector with distinct entries and $\mathcal{B}_0, \dots, \mathcal{B}_{n-1}$ be an n-tuple of \mathbb{F}_q -bases of \mathbb{F}_{q^m} . Let

$$\mathscr{C} = \mathbf{ExpCode}_{(\mathcal{B}_i)_*}(\mathbf{RS}_k(\boldsymbol{x})).$$

There exists a polynomial time algorithm which

takes as inputs \mathscr{C} , three distinct elements $x_0', x_1', x_2' \in \mathbb{F}_{q^m}$ and an \mathbb{F}_q -basis \mathcal{B}_0' of \mathbb{F}_{q^m} ; and returns $x_3', \ldots, x_{n-1}' \in \mathbb{F}_{q^m}^n$ and \mathbb{F}_q -bases $(\mathcal{B}_1', \ldots, \mathcal{B}_{n-1}')$ of \mathbb{F}_{q^m} such that

$$\mathscr{C} = \mathbf{ExpCode}_{(\mathcal{B}'_0, \dots, \mathcal{B}'_{n-1})}(\mathbf{RS}_k((x'_0, \dots, x'_{n-1}))).$$

The principle of the algorithm is very similar to that of Sidelnikov Shestakov. Starting from a systematic generator matrix of an expanded Reed–Solomon code, the hidden structure of the RS code is deduced from relations satisfied by the $m \times m$ blocks of the right hand side of this systematic generator matrix.

Remark 74. — Theorem 73 asserts in particular that the choice of three values of the support together with one basis uniquely determines a pair $(x, (\mathcal{B}_i)_i)$ describing a code $\mathbf{ExpCode}_{(\mathcal{B}_i)_i}(x)$.

Using this Theorem 73, we obtain a vector x' and \mathbb{F}_q -bases \mathcal{B}'_i of \mathbb{F}_{q^3} such that

$$\mathscr{C}^{\tilde{\star}2}_{\mathrm{pub}} = \mathbf{ExpCode}_{(\mathcal{B}'_{i})_{\tilde{s}}}(\mathbf{RS}_{2k-1}(\boldsymbol{x}')).$$

Remark 75. — Note that the value of x' is not necessarily the same as the one contained in the secret key but we are looking for an equivalent secret key, *i.e.* we only need a code description which allows us to decode.

7.1.3. Recovering a secret key. — Once x' is found, there remains to find bases $\mathcal{B}_{\mathcal{S}'_0}, \ldots, \mathcal{B}_{\mathcal{S}'_{n-1}}$ of 2-dimensional subspaces $\mathcal{S}'_0, \ldots, \mathcal{S}'_{n-1} \subseteq \mathbb{F}_{q^3}$ such that

$$\mathscr{C}_{\mathrm{pub}} = \mathbf{ExpCode}_{\left(\mathcal{B}_{\mathcal{S}'_{z}}\right)_{z}}(\mathbf{RS}_{k}(\boldsymbol{x}')).$$

These bases can be obtained by solving a linear system. They are the pairs

$$\mathcal{B}_{\mathcal{S}_0'} = (b_0^{(0)}, b_0^{(1)}), \dots, \mathcal{B}_{\mathcal{S}_{n-1}'} = (b_{n-1}^{(0)}, b_{n-1}^{(1)})$$

such that

$$\mathbf{SqueezeCode}_{\left(\mathcal{B}_{\mathcal{S}_{i}^{\prime}}
ight)_{i}}(\mathscr{C}_{\mathrm{pub}})\subseteq\mathbf{RS}_{k}(oldsymbol{x}^{\prime}),$$

which can be equated as follows. Let H be a parity-check matrix of $\mathbf{RS}_k(x)$ and G_{pub} a generator matrix of \mathscr{C}_{pub} . Let

$$\boldsymbol{B} = \begin{pmatrix} b_0^{(0)} & & & & & & \\ b_0^{(1)} & & & & & \\ & b_1^{(0)} & & & & \\ & b_1^{(1)} & & & & \\ & & \ddots & & & \\ & & & b_{n-1}^{(0)} \\ (0) & & & b_{n-1}^{(1)} \end{pmatrix} \in \mathbb{F}_{q^3}^{2n \times n}.$$

The unknown entries of \boldsymbol{B} are the solutions of the linear system

(14)
$$G_{\text{pub}}BH^{\intercal} = 0.$$

There are

- 2n unknowns in \mathbb{F}_{q^3} which yields 6n unknowns in \mathbb{F}_q ;
- for $(3k-n)(n-k) = O(n^2)$ equations.

Thus, the matrix B is very likely to be the unique solution up to a scalar multiple. From this, we obtain a complete equivalent secret key, which allows to decrypt any ciphertext.

Remark 76. — After presenting a polynomial time recovery of the structure of expanded GRS codes in [BGK19, § IV.B], the extension to expanded SSRS codes is discussed [BGK19, § VI.C]. The suggested approach consists in performing a brute–force search on the expansion bases $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$. But the cost of such an approach is exponential in n and λ . Our use of the twisted square code permits to address the same problem in polynomial time.

7.1.4. Extending the reach of the attack by shortening blocks. — As explained in Section 6.2.5, it may happen that $\mathscr{C}^{\tilde{\chi}2}_{\text{pub}} = \mathbb{F}_q^{3n}$, i.e. the twisted square of the public code equals the whole ambient space. In such a situation, the distinguisher fails and so does the attack. To overcome this issue, it is sometimes possible to shorten a fixed number s of blocks of \mathscr{C}_{pub} and apply the previous attack to this block—shortened code.

More precisely, let $\mathcal{I} \subseteq \llbracket 0, 2n-1 \rrbracket$ be a set of indices corresponding to a union of blocks, i.e. of the form $\mathcal{I} = \{2i_0, 2i_0+1, \ldots, 2i_s, 2i_s+1\}$. We apply the previous algorithm to the code $\mathbf{Sh}_{\mathcal{I}}(\mathscr{C}_{\text{pub}})$ which returns $((x_i')_{i \notin \mathcal{I}}, (\mathcal{B}_i')_{i \notin \mathcal{I}})$ such that

$$\mathbf{Sh}_{\mathcal{I}}\left(\mathscr{C}_{\mathrm{pub}}\right)^{\tilde{\star}2} = \mathbf{ExpCode}_{(\mathcal{B}_{i}^{\prime})_{i\notin\mathcal{I}}}((x_{i}^{\prime})_{i\notin\mathcal{I}}).$$

Recall that the choice of three of the x_i 's and one of the \mathcal{B}_i 's entirely determines the other ones. Then, one can re-apply the same process with another set of blocks \mathcal{I}_1 such that there are at least 3 that are neither in \mathcal{I}_0 nor in \mathcal{I}_1 . This allows to deduce new values for x_i 's for $i \in \mathcal{I} \setminus \mathcal{I}_1$. And we repeat this operation until x' is entirely computed. Then, we proceed as in Section 7.1.3 to recover the rest of the secret key.

- **7.1.5.** Application: attacking some parameters of the XGRS system. The proposed attack permits to break efficiently any parameters of Type I proposed in [KRW19b] (i.e. with $\lambda = 2$ and m=3). Using a Sage implementation, the calculation of $\mathscr{C}_{\mathrm{pub}}^{\tilde{\star}2}$ takes a few minutes. Next, we obtained a full key recovery using the "guess and squeeze" approach described further in Section 7.5 followed by a usual Sidelnikov Shestakov attack. The overall attack runs in less than one hour for keys corresponding to a claimed security level of 256 bits. The previously described approach consisting in applying directly the algorithm of [BGK19, \S VI.B] on $\mathscr{C}^{\star 2}_{\mathrm{pub}}$ has not been implemented but is probably even more efficient.
- **7.2.** The general case. The example of attack presented in Section 7.1 extends to any case where the distinguisher presented in Section 6.2.3 succeeds. The attack generalises straightforwardly up to the following details.
 - According to Theorem 62, the algorithm of [BGK19, § VI.B] should no longer be applied directly on $\mathscr{C}_{\text{pub}}^{\tilde{\star}2}$ but on $\mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{C}_{\text{pub}}^{\tilde{\star}2}\right)$, where \mathcal{J}_0 is defined in Lemma 61 (7).

 – The recovery of the subspaces and bases described in Section 7.1.3 involves a matrix $\mathbf{B} \in$
 - $\mathbb{F}_q^{\lambda n \times n}$ with λn nonzero entries, which will be the unknowns of the system (14). Hence, this system has λn unknowns in \mathbb{F}_{q^m} , i.e. λmn unknowns in \mathbb{F}_q for $(mk-n(m-\lambda))(n-k)=O(n^2)$ equations. As the value of m (and hence λ) remain very small compared to n, there is still in general a unique solution up to a scalar multiple.
- **7.3.** Summary of the attack. The attack can be summarised by the following algorithms, depending on the values of k and n.

Algorithm 1 The attack when 2k - 1 < n

- 1: Compute $\mathbf{Sh}_{\mathcal{J}_0}\left(\mathscr{C}_{\mathrm{pub}}^{\tilde{\star}2}\right)$, where \mathcal{J}_0 is the the union of the last $\binom{\lambda+1}{2}-m$ positions of each block (see Lemma 38(4));
- 2: Apply the algorithm of $[BGK19, \S VI.B]$ to recover a support x of the parent Reed-Solomon code;
- 3: Apply the calculations of Section 7.1.3 to recover the bases \mathcal{B}_i .

Algorithm 2 Attack when $2k-1 \ge n$

- 1: Choose a number s of blocks to shorten satisfying condition (13) so that the distinguisher succeeds.
- 2: Pick a union of s blocks $\mathcal I$ and
 - (a) Compute $\mathbf{Sh}_{\mathcal{J}_0'}\left(\mathbf{Sh}_{\mathcal{I}}\left(\mathscr{C}_{\text{pub}}\right)^{\tilde{\star}^2}\right)$, where \mathcal{J}_0' is the union of the last $\binom{\lambda+1}{2}-m$ positions
 - (b) Apply the algorithm of [**BGK19**, § VI.B] to recover a partial support $(x_i)_{i \notin \mathcal{T}}$;
 - (c) Repeat this process with another \mathcal{I} until you got the whole support x.
- 3: Apply the calculations of Section 7.1.3 to recover the bases \mathcal{B}_i .

Remark 77. — In the case
$$\lambda = 2$$
 and $m = 3$, $\binom{\lambda+1}{2} = 3 = m$ and hence $\mathcal{J}_0 = \emptyset$.

7.4. Complexity. — For the complexity analysis and according to the parameters proposed in **[KRW19b]**, we suppose that m = O(1), $\lambda = O(1)$ and $k = \Theta(n)$.

- **7.4.1.** Step 1, the twisted square computation. First let us evaluate the cost of the computation of the twisted square of the code $\mathscr{C}_{\text{pub}} \subseteq \mathbb{F}_q^{\lambda n}$ of dimension $k_0 \stackrel{\text{def}}{=} (mk n(m \lambda))$.
 - 1. Starting from a $k_0 \times \lambda n$ generator matrix of \mathscr{C}_{pub} , any non ordered pair of rows provides a generator of the twisted square. Hence there are $\binom{k_0+1}{2} = O(n^2)$ generators to compute, each computation costing $n\binom{\lambda+1}{2}$ operations. This is an overall cost of $O(n^3)$ operations in \mathbb{F}_q .
 - 2. Then, deducing a row echelon generator matrix of this twisted square from these $O(n^2)$ generators has the cost of the computation of the row echelon form of a $O(n^2) \times O(n)$ matrix, which requires $O(n^{\omega+1})$ operations in \mathbb{F}_q (see [BCG⁺17, Théorème 8.6]), where $\omega \leq 3$ is the complexity exponent of operations of linear algebra.

Thus, the overall cost of the computation of this twisted square code is $O(n^{\omega+1})$. In addition, in the situation where 2k-1>n, we need to iterate the calculation on a constant number of shortenings of the public code, which has no influence on the complexity exponent.

- **Remark 78.** Similarly to the discussions [COT17, § VI.D] and [CMCP17, § VI.B.4], it is possible to randomly generate O(n) generators of the twisted square code and perform the echelon form on this subset of generators. This provides the whole twisted square code with a high probability, reducing the cost of the calculation to $O(n^{\omega})$ operations in \mathbb{F}_q .
- **7.4.2.** Step 2, recovering x. The second step of the attack, i.e. performing the algorithm of [BGK19, \S VI.B] to recover x is not that expensive. A quick analysis of this algorithm permits to observe that the most time consuming step is the calculation of the systematic form of the generator matrix, which has actually been performed in the previous step. Therefore, this second step can be neglected in the complexity analysis.
- **7.4.3.** Step 3, recovering the bases. Finally, the last step of the attack, consisting in recovering the bases \mathcal{B}_i , consists in a resolution of a linear system of $O(n^2)$ equations and O(n) unknowns, which costs $O(n^{\omega+1})$ operations.

Summary. The overall cost of the attack is of $O(n^{\omega+1})$ operations in \mathbb{F}_q .

7.5. Recovering the bases for arbitrary expanded codes: guess and squeeze. — To conclude this section, we present an alternative approach to detect the hidden structure of expanded codes and recover the expansion bases. This method applies to the expansion of any code. It can in particular apply to the twisted square of SSRS codes. As explained in Section 7.1.5, this is the approach we implemented. The interest of this approach is that it may apply to expansions of codes which are not RS codes and hence may be an interesting tool for other cryptanalyses.

Given a code $\mathscr{C} \subseteq \mathbb{F}_{q^m}^n$ and bases $\mathcal{B}_0, \dots, \mathcal{B}_{n-1}$ of \mathbb{F}_{q^m} , suppose you only know a generator matrix of

$$\mathscr{C}_{\mathrm{exp}} \stackrel{\mathrm{def}}{=} \mathbf{ExpCode}_{(\mathcal{B}_i)}(\mathscr{C}).$$

The objective is to guess the \mathcal{B}_i 's iteratively instead of brute forcing any n-tuple of bases, which would be prohibitive.

- **Step 1.** Shorten \mathscr{C}_{exp} at k-1 blocks (which corresponds to m(k-1) positions). This yields a code whose dimension most of the times equals m. According to Lemma 42, this is the expansion of a code of dimension 1 obtained by shortening \mathscr{C} at k-1 positions.
- **Step 2.** Puncture this shortened code in order to keep only two blocks. We get a [2m, m] code which we call $\mathscr{C}_{\text{exp,tiny}} \subseteq \mathbb{F}_q^{2m}$. This code is the expansion of a [2,1] code called $\mathscr{C}_{\text{tiny}} \subseteq \mathbb{F}_{q^m}^2$ obtained from \mathscr{C} by shortening k-1 positions and puncturing the remaining code in order to keep only 2 positions.
- **Step 3.** Now, for any pair of bases $(\mathcal{B}_0, \mathcal{B}_1)$ of \mathbb{F}_{q^m} , compute

$$\mathbf{SqueezeCode}_{(\mathcal{B}_0,\mathcal{B}_1)}(\mathscr{C}_{\mathrm{exp,tiny}}).$$

The point is that, for a wrong choice of bases, we get a generator matrix with m rows and 2 columns which is very likely to be full rank. Hence a wrong choice provides the trivial

code $\mathbb{F}_{q^m}^2$. On the other hand, a good choice of bases provides the code $\mathscr{C}_{\text{tiny}}$ which has dimension 1. This property permits to guess the bases.

Actually, according to Lemma 47, if one guesses the bases $a_0\mathcal{B}_0, a_1\mathcal{B}_1$ for some $a_0, a_1 \in \mathbb{F}_{q^m}^{\times}$, the squeezing will provide $\mathscr{C}_{\text{tiny}} \star (a_0, a_1)$ which also has dimension 1. Therefore, it is possible to first guess the bases up to a scalar multiple in $\mathbb{F}_{q^m}^{\times}$. Therefore, the cost of computing these two bases is in $O(q^{2m(m-1)})$ operations.

Once the first two bases are known, one can restart the process by with another pair of blocks involving one of the two blocks for which the basis is already known, which requires $O(q^{m(m-1)})$ operations. This yields an overall complexity of $O(q^{2m(m-1)} + nq^{m(m-1)})$ operations in \mathbb{F}_q for this guess and squeeze algorithm.

Remark 79. — Note that in the attack of XGRS scheme, the bases to guess are known to be of the form $(1, \gamma, \gamma^2, \dots, \gamma^{m-1})$ for some generator $\gamma \in \mathbb{F}_{q^m}$. This additional information permits to significantly improve this search and reduce the cost of the calculation of the n bases to $O(q^{2m} + nq^m)$ operations.

Remark 80. — Proceeding this way, only permits to get back the code $\mathscr{C} \star a \subseteq \mathbb{F}_{q^m}^n$ for an unknown vector $\mathbf{a} \in (\mathbb{F}_{q^m}^{\times})^n$. However, this is an important first step. For instance, if \mathscr{C} was a Reed–Solomon, we obtain a generalised Reed–Solomon code whose structure is computable using Sidelnikov and Shestakov attack. It is then possible to decode.

8. Conclusion

We presented a polynomial time distinguisher on subspace subcodes of Reed–Solomon codes relying on a new operation called the *twisted square product*. We are hence able to distinguish SSRS codes from random ones as soon as the dimension λ of the subspaces exceeds $\frac{m}{2}$. From this distinguisher, we derived an attack breaking in particular the parameter set $\lambda = 2$ and m = 3 of the XGRS system [KRW19a].

These results clarify the overview on McEliece encryption scheme based on algebraic codes. On the one hand, we have generalised Reed–Solomon codes, which are known to be insecure since the early 90's. On the other hand, alternant codes seem to resist to any attack except some Goppa codes with an extension degree m=2 [COT17, FPdP14]. The present work provides an analysis of a family of codes including these two cases as the two extremities of a spectrum. Concerning the subspace subcodes lying in between, we show an inherent weakness of SSRS codes when $\lambda > m/2$ (See Figure 1, page 3). The case $\lambda = m/2$ is in general out of reach of our distinguisher, but remains border line as testified by some attacks on the cases $\lambda = 1, m = 2$ in the literature [COT17, FPdP14].

A question which remains open is the actual security of the cases $1 < \lambda < m/2$ which are out of reach of the twisted square code distinguisher. These codes, which include alternant codes, deserve to have a careful security analysis in the near future. Indeed, if they turn out to be resistant to any attack, they could provide an alternative to Classic McEliece [BCL+19] with shorter key sizes. On the other hand, if some of these codes turned out to be insecure, this may impact the security of Classic McEliece which is a crucial question in the near future.

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