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Monomial curve families supporting Rossi's conjecture

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ABSTRACT

In this article, we give a constructive method to form infinitely many families of monomial curves in affine 4-space with corresponding Gorenstein local rings in embedding dimension 4 supporting Rossi's conjecture. Starting with any monomial curve in affine 2-space, we obtain large families of Gorenstein local rings with embedding dimension 4, having non-decreasing Hilbert functions, although their associated graded rings are not Cohen–Macaulay.

1. Introduction

Rossi's conjecture says that the Hilbert function of a Gorenstein local ring of dimension one is non-decreasing. This conjecture is open even for Gorenstein local rings with embedding dimension 4 and, moreover, even for Gorenstein local rings corresponding to numerical semigroups minimally generated by 4 elements, i.e. to monomial curves in affine 4-space. Due to the classical result of Kunz, the local ring corresponding to a monomial curve is Gorenstein if and only if the numerical semigroup corresponding to the monomial curve is symmetric (Kunz, 1970). Recall that a monomial curve, denoted by $C = C(n_1, \dots, n_k)$ is a curve with generic zero $(t^{n_1}, \dots, t^{n_k})$ in the affine K-space $\mathbb{A}^k$ over an algebraically closed field K, where $n_1, \dots, n_k$ are positive integers with $\gcd(n_1, n_2, \dots, n_k) = 1$, and $\{n_1, n_2, \dots, n_k\}$ is a minimal set of generators for the numerical semigroup $\langle n_1, n_2, \dots, n_k\rangle = \{n \mid n = \sum_{i=1}^{k} a_i n_i,\ a_i\text{'s are non-negative integers}\}$, which is the numerical semigroup corresponding to the monomial curve $C = C(n_1, \dots, n_k)$. The local ring corresponding to the monomial

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curve $C = C(n_1, \dots, n_k)$ is $K[[t^{n_1}, \dots, t^{n_k}]]$ and its Hilbert function is defined as the Hilbert function of its associated graded ring, $gr_m(K[[t^{n_1}, \dots, t^{n_k}]])$, which is isomorphic to the ring $K[x_1, \dots, x_k]/I(C)_*$. Here, m is the maximal ideal of the local ring $K[[t^{n_1}, \dots, t^{n_k}]]$, $I(C)$ is the defining ideal of C and $I(C)_*$ is the ideal generated by the polynomials f_* for f in $I(C)$ and f_* is the homogeneous summand of f of least degree. $I(C)_*$ is known as the defining ideal of the tangent cone of C at $(0, \dots, 0)$.

In Herzog (1970), Herzog proved that the numerical semigroup corresponding to the monomial curve $C(n_1, n_2, n_3)$ is symmetric if and only if $C(n_1, n_2, n_3)$ is a complete intersection. However, this is not the case for embedding dimensions greater than 3. Bresinsky showed that, if the numerical semigroup corresponding to a monomial curve in affine 4-space is symmetric (i.e. the local ring corresponding to the curve is Gorenstein), then the monomial curve is either generated by 3 elements (complete intersection case) or by 5 elements (non-complete intersection case) (Bresinsky, 1975). In this case, to prove Rossi's conjecture, both complete intersection and non-complete intersection monomial curves with corresponding Gorenstein local rings should be considered. It must be added however that Rossi's conjecture is even open for Gorenstein local rings corresponding to complete intersection monomial curves in affine 4-space, and in this article, we focus on this case. Up to now, only some partial results had been proved. In Arslan and Mete (2007), we showed that the Hilbert function is non-decreasing for local Gorenstein rings with embedding dimension four corresponding to monomial curves in affine 4-space, under some arithmetic assumptions on the generators of their defining ideals in the non-complete intersection case. In Arslan et al. (2009), by using the technique of nice gluing, we gave infinitely many families of 1-dimensional local rings associated to complete intersection monomial curves with non-decreasing Hilbert functions. In this article, by using non-nice gluing, we give a method to construct infinitely many families of complete intersection monomial curves in affine 4-space supporting Rossi's conjecture. By starting with any monomial curve in affine 2-space, this constructive method gives us the opportunity to form complete intersection monomial curve families in affine 4-space having non-Cohen–Macaulay tangent cones with non-decreasing Hilbert functions.

2. Constructing families of symmetric numerical semigroups

The technique of gluing introduced by Rosales (1997) is very useful for constructing symmetric numerical semigroups. The technique of nice gluing was introduced in Arslan et al. (2009). We first recall the definitions of gluing and nice gluing. These definitions allow us to split Rossi's conjecture in the complete intersection monomial curve case into separate sub-cases.

Definition 2.1. (See Rosales, 1997, Lemma 2.2.) Let $N_1 = \langle m_1, \dots, m_l\rangle$ and $N_2 = \langle n_1, \dots, n_k\rangle$ be two numerical semigroups. Let $p = b_1m_1 + \dots + b_lm_l \in N_1$ and $q = a_1n_1 + \dots + a_kn_k \in N_2$ be two positive integers satisfying $\gcd(p, q) = 1$ with $p \notin \{m_1, \dots, m_l\}$, $q \notin \{n_1, \dots, n_k\}$ and $\{qm_1, \dots, qm_l\} \cap \{pn_1, \dots, pn_k\} = \emptyset$. The numerical semigroup $N = \langle qm_1, \dots, qm_l, pn_1, \dots, pn_k\rangle$ is called a gluing of the semigroups N_1 and N_2. (In the same way, the monomial curve $C = C(qm_1, \dots, qm_l, pn_1, \dots, pn_k)$ can be interpreted as the gluing of the monomial curves $C_1 = C(m_1, \dots, m_l)$ and $C_2 = C(n_1, \dots, n_k)$.)

Remark 2.2. By using the notation in Definition 2.1, if the defining ideals $I(C_1) \subset K[x_1, \dots, x_l]$ of C_1 and $I(C_2) \subset K[y_1, \dots, y_k]$ of C_2 are generated by the sets $G_1 = \{f_1, \dots, f_s\}$ and $G_2 = \{g_1, \dots, g_t\}$ respectively, then the defining ideal of $I(C) \subset K[x_1, \dots, x_l, y_1, \dots, y_k]$ is generated by the set $G = \{f_1, \dots, f_s, g_1, \dots, g_t, x_1^{b_1} \dots x_l^{b_l} - y_1^{a_1} \dots y_k^{a_k}\}$, see Rosales (1997, Theorem 1.4).

Definition 2.3. (See Arslan et al., 2009, Definition 2.3.) Let the notation be as in Definition 2.1, with m_1 smallest among $m_1, \dots, m_l$ and n_1 smallest among $n_1, \dots, n_k$. The numerical semigroup $N = \langle qm_1, \dots, qm_l, pn_1, \dots, pn_k\rangle$ above is called a nice gluing of $N_1 = \langle m_1, \dots, m_l\rangle$ and $N_2 = \langle n_1, \dots, n_k\rangle$, if $p = b_1m_1 + \dots + b_lm_l$ and $q = a_1n_1$ with $a_1 \leqslant b_1 + \dots + b_l$. A gluing which is not nice is called a non-nice gluing.

By using the concept of gluing, Rossi's conjecture for complete intersection monomial curves in affine 4-space can be split into two parts as follows. Due to a result of Delorme (1976), a numerical semigroup corresponding to a complete intersection monomial curve and minimally generated by at least two elements is a gluing of two numerical semigroups, each of which also corresponds to a complete intersection monomial curve. Thus, a complete intersection monomial curve C in affine 4-space is a gluing of one of those:

a) $C_1 = C(m_1, m_2)$ and $C_2 = C(n_1, n_2)$;
b) $C_1 = C(m_1, m_2, m_3)$ and $C_2 = C(1)$.

Rossi's conjecture for complete intersection monomial curves in affine 4-space has been proved only in the following cases in Arslan et al. (2009, Corollary 3.3):

i) C is a nice gluing of $C_1 = C(m_1, m_2)$ and $C_2 = C(n_1, n_2)$;
ii) C is a nice gluing of $C_1 = C(m_1, m_2, m_3)$ and $C_2 = C(1)$;
iii) C is a nice gluing of $C_1 = C(1)$ and $C_2 = C(n_1, n_2, n_3)$, and C_2 has Cohen–Macaulay tangent cone.

Remark 2.4. Note that ii) and iii) are different, because nice gluing is not a symmetric operation.

Rossi's conjecture is open in all the remaining cases, and in this article, we focus on one of these remaining cases: non-nice gluing of $C_1 = C(m_1, m_2, m_3)$ and $C_2 = C(1)$.

We start with any numerical semigroup $\langle n_1, n_2\rangle$. Let S_1 be the numerical semigroup $S_1 = \langle pn_1, pn_2, a_1n_1 + a_2n_2\rangle$, which is a nice gluing of $\langle n_1, n_2\rangle$ and $\langle 1\rangle$, with $p \leqslant a_1 + a_2$ and $\gcd(p, a_1n_1 + a_2n_2) = 1$. (Note that pn_1 is the smallest integer among these generators.) This is the numerical semigroup corresponding to the monomial curve $C_1 = C(pn_1, pn_2, a_1n_1 + a_2n_2)$ with parametrization

$$x_1 = t^{pn_1}, \qquad x_2 = t^{pn_2}, \qquad x_3 = t^{a_1n_1+a_2n_2}$$

and $I(C_1)$ is generated by $x_1^{n_2} - x_2^{n_1}$ and $x_3^p - x_1^{a_1}x_2^{a_2}$. Let d be the smallest integer satisfying $dn_1 > n_2$, then the numerical semigroup

$$S = \big\langle spn_1, spn_2, s(a_1n_1 + a_2n_2), (s-d)pn_1 + pn_2\big\rangle$$

with $\gcd((s-d)pn_1 + pn_2, s) = 1$ is a non-nice gluing of $C_1 = C(pn_1, pn_2, a_1n_1 + a_2n_2)$ and $C_2 = C(1)$, since $s > (s-d) + 1$. Since $dn_1 > n_2$, we have $(s-d)pn_1 + pn_2 < spn_1$, which makes $(s-d)pn_1 + pn_2$ the smallest among the generators. The semigroup S corresponds to the monomial curve $C = C((s-d)pn_1 + pn_2, spn_1, spn_2, s(a_1n_1 + a_2n_2))$ with parametrization

$$y_1 = t^{(s-d)pn_1+pn_2}, \qquad x_1 = t^{spn_1}, \qquad x_2 = t^{spn_2}, \qquad x_3 = t^{s(a_1n_1+a_2n_2)}$$

and $I(C) = \langle x_1^{n_2} - x_2^{n_1}, x_3^p - x_1^{a_1}x_2^{a_2}, y_1^s - x_1^{s-d}x_2\rangle$. All these families are obtained by doing computations in Greuel et al. (2001). Hence, from each monomial curve in 2-space, infinitely many monomial curves in 3-space are obtained by a nice gluing, and each of these curves leads to a different family of monomial curves in 4-space, which will be shown to support Rossi's conjecture. In other words, by starting with any fixed monomial curve in 2-space, we construct infinitely many families in 4-space, each of which contains infinitely many monomial curves supporting Rossi's conjecture. This shows that Gorenstein local rings with embedding dimension 4, which have non-Cohen–Macaulay associated graded rings, but have non-decreasing Hilbert functions are not rare at all. Moreover, all these families can be extended to higher dimensions by using Arslan et al. (2009, Theorem 3.1) to obtain infinitely many new families supporting Rossi's conjecture in higher dimensions.

We can now state our theorem:

Theorem 2.5. *Let $C = C((s-d)pn_1 + pn_2, spn_1, spn_2, s(a_1n_1 + a_2n_2))$ be the complete intersection monomial curve and S be the corresponding symmetric numerical semigroup $\langle (s-d)pn_1 + pn_2, spn_1, spn_2, s(a_1n_1 + a_2n_2)\rangle$ with $n_2 > n_1 > 1$, $\gcd(n_1, n_2) = 1$, $p \leqslant a_1 + a_2$, $\gcd(p, a_1n_1 + a_2n_2) = 1$, $\gcd((s-d)pn_1 + pn_2, s) = 1$, $s > d$ and $d = \lceil \frac{n_2}{n_1} \rceil$. The set $\{f_1, \ldots, f_{i_C+3}\}$ is a minimal standard basis for $I(C)$, where $i_C = \lceil \frac{n_2-n_1}{d-1} \rceil$ and $f_1 = x_1^{n_2} - x_2^{n_1}$, $f_2 = x_3^p - x_1^{a_1}x_2^{a_2}$, $f_3 = y_1^s - x_1^{s-d}x_2$, $f_{j+3} = y_1^{js}x_2^{n_1-j} - x_1^{n_2+j(s-d)}$ for*

$1 \leqslant j \leqslant i_C$ with respect to the negative degree reverse lexicographical ordering with $x_3 > x_2 > x_1 > y_1$. Moreover, the monomial curve C has non-Cohen–Macaulay tangent cone at the origin.

Proof. We apply the standard basis algorithm to the set $G = \{f_1, f_2, \ldots, f_{i_C+3}\}$. (We have shown above that $I(C)$ is generated by $\{f_1, f_2, f_3\}$ and C is a complete intersection monomial curve.) By using the notation in Greuel and Pfister (2002), we denote the leading monomial of a polynomial f by $\mathrm{LM}(f)$, the s-polynomial of the polynomials f and g by $\mathrm{spoly}(f, g)$ and the Mora's polynomial weak normal form of f with respect to G by $NF(f|G)$. Also, T_h denotes the set $\{g \in G\colon \mathrm{LM}(g)|\mathrm{LM}(h)\}$ and $\mathrm{ecart}(h)$ is $deg(h) - deg(\mathrm{LM}(h))$. We need to show that $NF(\mathrm{spoly}(f_m, f_n)|G) = 0$ for all m, n with $1 \leqslant m < n \leqslant i_C + 3$:

- $NF(\mathrm{spoly}(f_1, f_2)|G) = 0$ as $\mathrm{LM}(f_1)$ and $\mathrm{LM}(f_2)$ are relatively prime.
- $\mathrm{spoly}(f_1, f_{j+3}) = y_1^{js}x_1^{n_2} - x_2^j x_1^{n_2+j(s-d)}$, if $1 \leqslant j < i_C$. We set $h_1 = \mathrm{spoly}(f_1, f_{j+3})$, then $T_{h_1} = \{f_3\}$, since $\mathrm{LM}(\mathrm{spoly}(f_1, f_{j+3})) = x_2^j x_1^{n_2+j(s-d)}$. In this case, $\mathrm{spoly}(h_1, f_3) = x_1^{n_2} y_1^{js} - x_1^{n_2+(j-1)(s-d)} x_2^{j-1} y_1^s$. Also, note that $\mathrm{ecart}(f_3) \not> \mathrm{ecart}(h_1)$. Hence, set $h_2 = \mathrm{spoly}(h_1, f_3)$. Again, $T_{h_2} = \{f_3\}$, $\mathrm{ecart}(f_3) \not> \mathrm{ecart}(h_2)$ and $\mathrm{spoly}(h_2, f_3) = x_1^{n_2} y_1^{js} - x_1^{n_2+(j-2)(s-d)} x_2^{j-2} y_1^{2s}$. Continuing in this manner, after j steps, we obtain $\mathrm{spoly}(h_j, f_3) = 0$, so $NF(\mathrm{spoly}(f_1, f_{j+3})|G) = 0$ for $1 \leqslant j < i_C$.
- $NF(\mathrm{spoly}(f_1, f_{i_C+3})|G) = 0$, as $\mathrm{LM}(f_1)$ and $\mathrm{LM}(f_{i_C+3})$ are relatively prime.
- $NF(\mathrm{spoly}(f_2, f_3)|G) = 0$ as $\mathrm{LM}(f_2)$ and $\mathrm{LM}(f_3)$ are relatively prime.
- $NF(\mathrm{spoly}(f_2, f_{j+3})|G) = 0$ as $\mathrm{LM}(f_{j+3})$ and $\mathrm{LM}(f_2)$ are relatively prime for $1 \leqslant j \leqslant i_C$.
- Since $\mathrm{spoly}(f_3, f_1) = f_4$, we set $h = \mathrm{spoly}(f_3, f_1)$ and $T_h = \{f_4\}$. Obviously, $\mathrm{ecart}(f_4) \not> \mathrm{ecart}(h)$ and $\mathrm{spoly}(f_4, h) = 0$, and $NF(\mathrm{spoly}(f_3, f_1)|G) = 0$.
- $\mathrm{spoly}(f_{j+2}, f_3) = f_{j+3}$ for $2 \leqslant j \leqslant i_C$. Hence, we set $h = \mathrm{spoly}(f_{j+2}, f_3)$, then $T_h = \{f_{j+3}\}$. Since $\mathrm{ecart}(f_{j+3}) \not> \mathrm{ecart}(h)$ and $\mathrm{spoly}(f_{j+3}, h) = 0$, $NF(\mathrm{spoly}(f_{j+2}, f_3)|G) = 0$ for $2 \leqslant j \leqslant i_C$.
- $\mathrm{spoly}(f_{i_C+3}, f_3) = y_1^{i_C s} x_2^{n_1-i_C+1} - x_1^{n_2+(i_C-1)(s-d)} y_1^s$. Hence, we set $h = \mathrm{spoly}(f_{i_C+3}, f_3)$, then $T_h = \{f_{i_C+2}\}$. Since $\mathrm{ecart}(f_{i_C+2}) \not> \mathrm{ecart}(h)$ and $\mathrm{spoly}(f_{i_C+2}, h) = 0$, then $NF(\mathrm{spoly}(f_{i_C+3}, f_3)|G) = 0$.
- $\mathrm{spoly}(f_1, f_4) = x_1^{n_2} y_1^s - x_2 x_1^{n_2+s-d}$ and $\mathrm{LM}(\mathrm{spoly}(f_1, f_4)) = x_2 x_1^{n_2+s-d}$. We set $h = \mathrm{spoly}(f_1, f_4)$. Then, $T_h = \{f_3\}$. Since $\mathrm{ecart}(f_3) = d - 1 \not> d - 1 = \mathrm{ecart}(h)$ and $\mathrm{spoly}(f_3, h) = 0$, $NF(\mathrm{spoly}(f_1, f_4)|G) = 0$.
- $\mathrm{spoly}(f_{k+3}, f_{j+3}) = x_1^{n_2+j(s-d)} y_1^{(k-j)s} - x_1^{n_2+k(s-d)} x_2^{k-j}$, if $1 \leqslant j < k < i_C$, so $\mathrm{LM}(\mathrm{spoly}(f_{k+3}, f_{j+3})) = x_1^{n_2+k(s-d)} x_2^{k-j}$. We set $h_1 = \mathrm{spoly}(f_{k+3}, f_{j+3})$, then $T_{h_1} = \{f_3\}$ and note that $\mathrm{ecart}(f_3) \not> \mathrm{ecart}(h_1)$. Then $\mathrm{spoly}(h_1, f_3) = x_1^{n_2+j(s-d)} y_1^{(k-j)s} - x_1^{n_2+(k-1)(s-d)} x_2^{(k-j)-1} y_1^s$. Hence, set $h_2 = \mathrm{spoly}(h_1, f_3)$ and again $T_{h_2} = \{f_3\}$. In this case, $\mathrm{ecart}(f_3) \not> \mathrm{ecart}(h_2)$ and $\mathrm{spoly}(h_2, f_3) = x_1^{n_2+j(s-d)} y_1^{(k-j)s} - x_1^{n_2+(k-2)(s-d)} x_2^{(k-j)-2} y_1^{2s}$. Continuing in this manner, after $k - j$ steps, $\mathrm{spoly}(h_{k-j}, f_3) = 0$ so $NF(\mathrm{spoly}(f_{k+3}, f_{j+3})|G) = 0$ for $1 \leqslant j < k < i_C$.
- $NF(\mathrm{spoly}(f_{i_C+3}, f_{j+3})|G) = 0$, as $\mathrm{LM}(f_{i_C+3})$ and $\mathrm{LM}(f_{j+3})$ are relatively prime for $1 \leqslant j < i_C$.

Thus, the set $\{f_1, \ldots, f_{i_C+3}\}$ is a standard basis of $I(C)$ with respect to the negative degree reverse lexicographical ordering with $x_3 > x_2 > x_1 > y_1$. Hence, it follows that the set $\{f_{1_*}, \ldots, f_{i_C+3_*}\}$ is a basis for the defining ideal $I(C)_*$. Moreover, since y_1 divides $\mathrm{LM}(f_{4*})$, the tangent cone of the associated monomial curve of C is not Cohen–Macaulay by the checking criterion given in Arslan et al. (2009, Lemma 2.7). □

3. Computation of the Hilbert series

In this section, we show that, although the tangent cone of the monomial curve $C = C((s-d)pn_1 + pn_2, spn_1, spn_2, s(a_1n_1 + a_2n_2))$ given in Theorem 2.5 is non-Cohen–Macaulay, it has a non-decreasing Hilbert function. In this way, starting with any relatively prime n_1 and n_2, we can construct infinitely many families supporting Rossi's conjecture.

Theorem 3.1. *Let the notation be as in Theorem 2.5. The local ring*

$$k\left[\left[t^{(s-d)pn_1+pn_2}, t^{spn_1}, t^{spn_2}, t^{s(a_1n_1+a_2n_2)}\right]\right]$$

has a non-decreasing Hilbert function.

To prove this theorem, we need an explicit Hilbert function computation, since the tangent cone of the monomial curve is not Cohen–Macaulay. To do this computation, we recall the following proposition of Bayer and Stillman (1992):

Proposition 3.2. *(See Bayer and Stillman, 1992, Proposition 2.2.) Let $I \subset K[x_1, x_2, \ldots, x_n]$ be a monomial ideal and $I = \langle J, x^A\rangle$ for a monomial ideal J and a monomial x^A. Let $p(I)$ denote the numerator of the Hilbert series of $K[x_1, x_2, \ldots, x_n]/I$, and let $|A|$ denote the total degree of the monomial x^A. Then $p(I) = p(J) - t^{|A|}p(J : x^A)$.*

Proof of Theorem 3.1. Recalling that $i_C = \lceil \frac{n_2-n_1}{d-1} \rceil$, we have $i_C - 1 < \frac{n_2-n_1}{d-1} \leqslant i_C$. This leads to the inequality

$$n_1 + i_C(s-1) - (d-1) < n_2 + i_C(s-d) \leqslant n_1 + i_C(s-1) \tag{3.1}$$

It follows from Theorem 2.5 that $I(C)_*$ is generated by the least homogeneous summands of the elements of the set, $\{x_1^{n_2} - x_2^{n_1}, x_3^p - x_1^{a_1}x_2^{a_2}, y_1^s - x_1^{s-d}x_2, y_1^s x_2^{n_1-1} - x_1^{n_2+(s-d)}, \ldots, y_1^{js}x_2^{n_1-j} - x_1^{n_2+j(s-d)}, \ldots, y_1^{i_C s}x_2^{n_1-i_C} - x_1^{n_2+i_C(s-d)}\}$. Moreover, again by Theorem 2.5 and the inequality (3.1), $\langle LT(I(C)_*)\rangle$ with respect to the degree reverse lexicographical ordering satisfying $x_3 > x_2 > x_1 > y_1$ can be written as,

$$\left\langle LT\left(I(C)_*\right)\right\rangle = \left\langle x_2^{n_1}, x_3^p, x_1^{s-d}x_2, x_2^{n_1-1}y_1^s, x_2^{n_1-2}y_1^{2s}, \ldots, x_2^{n_1-j}y_1^{js}, \ldots, x_1^{n_2+i_C(s-d)}\right\rangle$$

Hence, since the Hilbert function of $k[x_1, x_2, x_3, x_4]/I(C)_*$ is equal to the Hilbert function of $k[x_1, x_2, x_3, x_4]/\langle LT(I(C)_*)\rangle$, it is sufficient to compute the Hilbert function of the latter.

Let I denote the monomial ideal $\langle LT(I(C)_*)\rangle$. We apply Proposition 3.2 to the ideal I with the monomial x_3^p having total degree p, and obtain $p(I) = p(J) - t^p p(J : x_3^p)$, where

$$J = \left\langle x_2^{n_1}, x_1^{s-d}x_2, x_2^{n_1-1}y_1^s, x_2^{n_1-2}y_1^{2s}, \ldots, x_2^{n_1-j}y_1^{js}, \ldots, x_1^{n_2+i_C(s-d)}\right\rangle$$

Since J has no term containing x_3, we get $(J : x_3^p) = J$. Thus,

$$p(I) = p(J) - t^p p(J) = \left(1 - t^p\right)p(J) \tag{3.2}$$

Case I ($i_C = 1$). We continue by choosing the monomial $x_2^{n_1}$, we have $p(J) = p(J_1) - t^{n_1}p(J_1 : x_2^{n_1})$, where $J_1 = \langle x_1^{s-d}x_2, x_1^{n_2+i_C(s-d)}\rangle$ and

$$p(J) = p(J_1) - t^{n_1}p\left(\left\langle x_1^{s-d}\right\rangle\right) = p(J_1) - t^{n_1}\left(1 - t^{s-d}\right) \tag{3.3}$$

Then we choose the monomial $x_1^{n_2+s-d}$ which has total degree $n_2 + s - d$, then $J_2 = \langle x_1^{s-d}x_2\rangle$ and applying Proposition 3.2 again, $p(J_1) = p(J_2) - t^{n_2+s-d}p(J_2 : x_1^{n_2+s-d})$ and

$$p(J_1) = 1 - t^{s-d+1} - t^{n_2+s-d}(1 - t) \tag{3.4}$$

Hence, $p(J) = 1 - t^{s-d+1} - t^{n_2+s-d}(1 - t) - t^{n_1}(1 - t^{s-d})$ and

$$p(I) = \left(1 - t^p\right)\left(1 - t^{s-d+1} - t^{n_2+s-d}(1 - t) - t^{n_1}\left(1 - t^{s-d}\right)\right) \tag{3.5}$$

A direct computation by expanding $p(I)$ shows that the Hilbert series of the ring $k[x_1, x_2, x_3, x_4]/I$ is $\frac{p(I)}{(1-t)^4} = \frac{h(t)}{(1-t)}$, where $h(t)$ is a polynomial with nonnegative coefficients. This proves that Hilbert function of the tangent cone is non-decreasing for every monomial curve defined above with $i_C = 1$.

Case II ($i_C \geqslant 2$). We continue by choosing the monomial $x_2^{n_1}$, we have $p(J) = p(J_1) - t^{n_1} p(J_1 : x_2^{n_1})$, where

$$J_1 = \langle x_1^{s-d} x_2, x_2^{n_1-1} y_1^s, x_2^{n_1-2} y_1^{2s}, \dots, x_2^{n_1-j} y_1^{js}, \dots, x_1^{n_2+i_C(s-d)} \rangle$$

In this case, $p(J) = p(J_1) - t^{n_1} p(\langle x_1^{s-d}, y_1^s \rangle)$ and

$$p(J) = p(J_1) - t^{n_1}(1 - t^{s-d})(1 - t^s) \tag{3.6}$$

Then we choose the monomial $x_2^{n_1-1} y_1^s$ which has total degree $n_1 + s - 1$ and

$$J_2 = \langle x_1^{s-d} x_2, x_2^{n_1-2} y_1^{2s}, \dots, x_2^{n_1-j} y_1^{js}, \dots, x_1^{n_2+i_C(s-d)} \rangle$$

Applying Proposition 3.2 again, $p(J_1) = p(J_2) - t^{n_1+s-1} p(\langle x_1^{s-d}, y_1^s \rangle)$ and $p(J_1) = p(J_2) - t^{n_1+s-1}(1 - t^{s-d})(1 - t^s)$. Continuing in the same manner, $J_l = \langle J_{l+1}, x_2^{n_1-l} y_1^{ls} \rangle$ for $1 \leqslant l \leqslant i_C - 1$ and we have

$$\begin{aligned} p(J_l) &= p(J_{l+1}) - t^{n_1+(s-1)l} p(\langle x_1^{s-d}, y_1^s \rangle) \\ &= p(J_{l+1}) - t^{n_1+(s-1)l}(1 - t^{s-d})(1 - t^s) \end{aligned} \tag{3.7}$$

for $1 \leqslant l \leqslant i_C - 2$ and

$$p(J_{i_C-1}) = p(J_{i_C}) - t^{n_1+(s-1)(i_C-1)}(1 - t^{s-d}) \tag{3.8}$$

Finally, choosing the monomial $x_1^{n_2+i_C(s-d)}$, we have $J_{i_C+1} = \langle x_1^{s-d} x_2 \rangle$ and

$$\begin{aligned} p(J_{i_C}) &= p(J_{i_C+1}) - t^{n_2+i_C(s-d)} p(J_{i_C+1} : x_1^{n_2+i_C(s-d)}) \\ &= (1 - t^{s-d+1}) - t^{n_2+i_C(s-d)}(1 - t) \end{aligned} \tag{3.9}$$

By using the inequality (3.1) $n_2 + i_C(s - d)$ is substituted with $n_1 + i_C(s - 1) - m$, where m is an integer satisfying $0 \leqslant m \leqslant d - 1$. Then, starting with $p(J_{i_C})$ and substituting backwards in (3.7), (3.6) and (3.2), we obtain $p(J_1)$ and using the above computations:

$$\begin{aligned} p(I) = (1 - t^p)\big[&(1 - t^{s-d+1}) - t^{n_1+i_C(s-1)-m}(1 - t) - t^{n_1+(s-1)(i_C-1)}(1 - t^{s-d}) \\ &- t^{n_1+(s-1)(i_C-2)}(1 - t^{s-d})(1 - t^s) - t^{n_1+(s-1)(i_C-3)}(1 - t^{s-d})(1 - t^s) - \cdots \\ &- t^{n_1+(s-1)}(1 - t^{s-d})(1 - t^s) - t^{n_1}(1 - t^{s-d})(1 - t^s)\big] \end{aligned} \tag{3.10}$$

Hence, the Hilbert series of the ring $k[x_1, x_2, x_3, x_4]/I$ is $\frac{p(I)}{(1-t)^4}$.

First, we divide $p(I)$ by $(1 - t)$ and group the terms containing $t^{n_1}(1 - t^{s-d})(1 - t^s)$ to obtain:

$$\begin{aligned} h_1(t) = (1 + t + \cdots + t^{p-1})\big[&(1 - t^{s-d+1}) - t^{n_1+i_C(s-1)-m}(1 - t) - t^{n_1+(i_C-1)(s-1)}(1 - t^{s-d}) \\ &- t^{n_1}(1 - t^s)(1 - t^{s-d})\big[1 + t^{s-1} + t^{2(s-1)} + \cdots + t^{(i_C-3)(s-1)} + t^{(i_C-2)(s-1)}\big]\big] \end{aligned}$$

Again, we divide $h_1(t)$ by $(1 - t)$ to obtain,

$$\begin{aligned} h_2(t) = (1 + t + \cdots + t^{p-1})\big[&(1 + t + \cdots + t^{s-d}) - t^{n_1+i_C(s-1)-m} - t^{n_1+(i_C-1)(s-1)} \\ &\cdot (1 + t + \cdots + t^{s-d-1}) - t^{n_1}(1 + t + \cdots + t^{s-1})(1 - t^{s-d})\big[1 + t^{s-1} + t^{2(s-1)} + \cdots \\ &+ t^{(i_C-3)(s-1)} + t^{(i_C-2)(s-1)}\big]\big] \end{aligned}$$

By rewriting $1 + t + \cdots + t^{s-d-1} + t^{s-d} - t^{n_1+i_C(s-1)-m} - t^{n_1+(i_C-1)(s-1)}(1 + t + \cdots + t^{s-d-1})$ as $(1 + t + \cdots + t^{s-d-1})(1 - t^{n_1+(i_C-1)(s-1)}) + t^{s-d}(1 - t^{n_1+(i_C-1)(s-1)+d-1-m})$, we have,

$$h_2(t) = \left(1+t+\cdots+t^{p-1}\right)\left[\left(1+t+\cdots+t^{s-d-1}\right)\left(1-t^{n_1+(i_C-1)(s-1)}\right)\right.$$
$$+t^{s-d}\left(1-t^{n_1+(i_C-1)(s-1)+d-1-m}\right)-t^{n_1}\left(1+t+\cdots+t^{s-1}\right)\left(1-t^{s-d}\right)$$
$$\left.\cdot\left[1+t^{s-1}+t^{2(s-1)}+\cdots+t^{(i_C-3)(s-1)}+t^{(i_C-2)(s-1)}\right]\right]$$

Again, we divide $h_2(t)$ by $(1-t)$ to obtain,

$$h_3(t) = \left(1+t+\cdots+t^{p-1}\right)\left[\left(1+t+\cdots+t^{s-d-1}\right)\left(1+t+\cdots+t^{n_1+(i_C-1)(s-1)-1}\right)\right.$$
$$+t^{s-d}\left(1+t+\cdots+t^{n_1+(i_C-1)(s-1)+d-2-m}\right)-t^{n_1}\left(1+t+\cdots+t^{s-1}\right)$$
$$\left.\cdot\left(1+t+\cdots+t^{s-d-1}\right)\left[\left(1+t^{s-1}+\cdots+t^{(i_C-3)(s-1)}+t^{(i_C-2)(s-1)}\right)\right]\right]$$

By doing some rearrangements, we can rewrite $h_3(t)$ as,

$$h_3(t) = \left(1+t+\cdots+t^{p-1}\right)\left[P(t)-N(t)\left(t^{n_1}+t^{n_1+1}+\cdots+t^{n_1+s-d-1}\right)\right]$$

Here, $P(t)=P_0(t)+\cdots+P_l(t)+\cdots+P_{s-d}(t)$, where $P_l(t)=t^l+t^{l+1}+\cdots+t^{n_1+(i_C-1)(s-1)+l-1}$ for $0\leqslant l\leqslant s-d-1$ and $P_{s-d}(t)=t^{s-d}+t^{s-d+1}+\cdots+t^{n_1+i_C(s-1)-1-m}$ and $N(t)=1+t+\cdots+t^{s-2}+2t^{s-1}+t^s+\cdots+t^{2(s-1)-1}+2t^{2(s-1)}+t^{2(s-1)+1}+\cdots+2t^{3(s-1)}+t^{3(s-1)+1}+\cdots+2t^{(i_C-3)(s-1)}+t^{(i_C-3)(s-1)+1}+\cdots+t^{(i_C-2)(s-1)-1}+2t^{(i_C-2)(s-1)}+t^{(i_C-2)(s-1)+1}+\cdots+t^{(i_C-1)(s-1)-1}+t^{(i_C-1)(s-1)}$.

Since the Hilbert series of $k[x_1,x_2,x_3,x_4]/I$ is $\frac{p(I)}{(1-t)^4}=\frac{h_3(t)}{1-t}$, our aim is to show that the polynomial $h_3(t)$ has no terms with negative coefficients, which implies that $k[x_1,x_2,x_3,x_4]/I$ has a non-decreasing Hilbert function. Note that, in our case, it is sufficient to show that all the terms of the polynomial $N(t)(t^{n_1}+t^{n_1+1}+\cdots+t^{n_1+s-d-1})$ are contained in $P(t)$. For $0\leqslant l\leqslant s-d-1$, $P_l(t)$ contains some terms of $t^{n_1+l}N(t)$. After those cancellations, each $P_l(t)-t^{n_1+l}N(t)$ contains only the negative terms

$$-t^{n_1+s-1+l}-t^{n_1+2(s-1)+l}-\cdots-t^{n_1+(i_C-2)(s-1)+l}-t^{n_1+(i_C-1)(s-1)+l}$$

for $0\leqslant l\leqslant s-d-1$, but all these negative terms are already contained in $P_{s-d}(t)$, because $-t^{n_1+s-1}$ is the term with smallest degree and $-t^{n_1+(i_C-1)(s-1)+s-d-1}=-t^{n_1+i_C(s-1)-d}$ is the term with highest degree, and $P_{s-d}(t)$ contains all the monomials having degrees between $s-d$ and $n_1+i_C(s-1)-1-m$. Obviously, $s-d<n_1+s-1$ as $d\geqslant 2$ and $n_1+i_C(s-1)-1-m\geqslant n_1+i_C(s-1)-d$ as $0\leqslant m\leqslant d-1$.

Hence, the polynomial $P(t)-N(t)(t^{n_1}+t^{n_1+1}+\cdots+t^{n_1+s-d-1})$, has no terms with negative coefficients. As a result of this, $h_3(t)$ is a polynomial with no terms having negative coefficients. Hence, the Hilbert function of $k[x_1,x_2,x_3,x_4]/I$ is non-decreasing. □

Theorem 2.5 and Theorem 3.1 can be used to give an easily implementable algorithm to obtain infinitely many families of local rings supporting Rossi's conjecture.

Remark 3.3. Note that for fixed natural numbers n_1 and n_2, there are infinitely many possible choices for a_1, a_2 and p satisfying the requirements in Step 2. Each of these choices corresponds to a different nice gluing and results with a different family, first k elements of which are given in Step 3.

The following example shows how the algorithm works.

Example 3.4. We start with the monomial curve $C(n_1,n_2)=C(8,27)$ in 2-space, and show how the algorithm constructs the first 2 local rings in a family. Thus, the input is $n_1=8$, $n_2=27$ and $k=2$. We can choose any of the infinitely many nice gluings of $C(8,27)$ and $C(1)$ to construct a new family. As an example, choose $a_1=2$, $a_2=1$ and $p=2$. In this case, all the requirements of Step 1 of the algorithm are satisfied. At Step 2, we have $d:=\lceil\frac{n_2}{n_1}\rceil=4$ and $s:=d+1=5$. If $s=5$ and $s=6$, $\gcd((s-d)pn_1+pn_2,s)=\gcd(16s-10,s)\neq 1$. If $s=7$, then $\gcd(16s-10,s)=1$ and we obtain the first local ring $K[\![t^{102},t^{112},t^{378},t^{301}]\!]$. If $s=8$, then $\gcd(16s-10,s)\neq 1$ again. Lastly, for $s=9$, we

Algorithm 1: Constructing families supporting Rossi's conjecture

INPUT: The natural numbers n_1, n_2 with $\gcd(n_1, n_2) = 1$ and $n_2 > n_1 > 1$ (defining the initial monomial curve $C(n_1, n_2)$) and another natural number k.

OUTPUT: First k Gorenstein local rings $K[[t^{m_1}, t^{m_2}, t^{m_3}, t^{m_4}]]$ in a family with non-Cohen–Macaulay associated rings and non-decreasing Hilbert functions

Step 1. Determine integers a_1, a_2, p such that,

- $(a_1, a_2) \neq (0, 1)$
  $(a_1, a_2) \neq (1, 0)$
- $a_1 n_1 + a_2 n_2 \neq p n_1$
  $a_1 n_1 + a_2 n_2 \neq p n_2$
- $p \leqslant a_1 + a_2$
- $\gcd(p, a_1 n_1 + a_2 n_2) = 1$

Step 2. Set

- $d := \lceil \frac{n_2}{n_1} \rceil$
- $s := d + 1$
- $i := 0$

Step 3.

while $i < k$ **do**
  if $\gcd((s-d)pn_1 + pn_2, s) = 1$, **then**
    Set $R := K[[t^{(s-d)pn_1+pn_2}, t^{spn_1}, t^{spn_2}, t^{s(a_1n_1+a_2n_2)}]]$;
    Return(R);
    $i := i + 1$;
  end if
  $s := s + 1$;
end while

have $\gcd(16s - 10, s) = 1$ and the second local ring $K[[t^{134}, t^{144}, t^{486}, t^{387}]]$ is obtained. These are the first two local rings of the family of Gorenstein local rings

$$K\left[\left[t^{16(s-4)+54}, t^{16s}, t^{54s}, t^{43s}\right]\right]$$

with $s \geqslant 5$ and $\gcd(16s - 10, s) = 1$, corresponding to the family of complete intersection monomial curves

$$C_s = C\big((s-4)16 + 54, 16s, 54s, 43s\big)$$

This example can be generalized by choosing the triple (a_1, a_2, p) with $a_1 = p$, $a_2 = 1$ and $\gcd(p, 3) = 1$. In this case, all the requirements of Step 1 in Algorithm 1 are satisfied. This leads to the complete intersection monomial curve family having 2 parameters $C_{s,p} = C((s-4)8p + 27p, 8sp, 27sp, (8p+27)s)$ with corresponding local rings $K[[t^{(s-4)8p+27p}, t^{8sp}, t^{27sp}, t^{(8p+27)s}]]$, where $\gcd(p, 3) = 1$ and $\gcd(8p(s-4) + 27p, s) = 1$. The ideal $I(C_{s,p})$ is generated by the set $\{x_1^{27} - x_2^8, x_3^p - x_1^p x_2, y_1^s - x_1^{s-4} x_2\}$. As $i_C = \lceil \frac{n_2 - n_1}{d-1} \rceil = 7$, it follows from Theorem 2.5 that the set $\{x_1^{27} - x_2^8, x_3^p - x_1^p x_2, y_1^s - x_1^{s-4} x_2, x_2^7 y_1^s - x_1^{23+s}, y_1^{2s} x_2^6 - x_1^{19+2s}, y_1^{3s} x_2^5 - x_1^{15+3s}, \dots, y_1^{7s} x_2 - x_1^{7s-1}\}$ is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $x_3 > x_2 > x_1 > y_1$. By using Theorem 3.1, the Gorenstein local rings

$$K\left[\left[t^{(s-4)8p+27p}, t^{8sp}, t^{27sp}, t^{(8p+27)s}\right]\right]$$

have non-decreasing Hilbert functions for all s and p with $s > 4$, $\gcd(p, 3) = 1$ and $\gcd(s, 5p) = 1$, although $k[x_1, x_2, x_3, x_4]/I(C_{s,p})_*$ is not Cohen–Macaulay, where $I(C_{s,p})_*$ is generated by the set

$$\left\{x_2^8, x_3^p, x_1^{s-4} x_2, x_2^7 y_1^s, y_1^{2s} x_2^6, y_1^{3s} x_2^5, \dots, x_1^{7s-1}\right\}$$

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