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Symbolic computation of strongly nonlinear periodic oscillations

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ABSTRACT

Based on Wu's elimination method and homotopy analysis method (HAM), an algorithm is proposed to compute accurate analytic approximation of periodical oscillations with high nonlinearity. A Maple package is developed for periodically oscillating systems of center and limit cycle types, which delivers accurate approximations of frequency, mean of motion and amplitude of oscillation automatically. Since HAM is valid for highly nonlinear problems, the package can be used to find accurate approximate solutions of nonlinear oscillation systems with strong nonlinearity. For systems with physical parameters, it can provide possible constraint conditions on parameters. Several examples are given to illustrate the validity and effectiveness of the algorithm and the Maple package. This package is freely available online, which provides an easy-to-use tool for scientist and engineer to solve accurate approximations of periodic oscillations of dynamic systems with high nonlinearity.

1. Introduction

From macroscopic celestial bodies to microcosmic particles, nonlinear oscillations can be observed and experienced anywhere anytime. Nonlinear oscillation theory and methodology have already been extensively applied in many areas; and to some extent, nonlinear oscillation is a foundation of nonlinear science. Considerable interests have been focused on the study of analytic solutions of nonlinear oscillations, for they may give more insights into internal aspects of nonlinear problems. Foremost among the analytic techniques are perturbation methods in terms of a

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small/large physical parameter (see Dyke, 1975; Nayfeh, 1981; Hinch, 1991; Bush, 1992). The basic premise of perturbation methods is to transfer a nonlinear problem to a set of infinite number of linear sub-problems by means of a small/large *physical* parameter. Unfortunately, many nonlinear equations do not contain such types of small/large physical parameters. Besides, perturbation approximations are generally only accurate enough for weakly nonlinear problems, but often break down when the physical parameters become large. In addition, restricted by such types of small/large physical parameters, the perturbation techniques cannot provide freedom to choose the equation type of the linear sub-problems. This can regularly result in major difficulties when solving nonlinear oscillation problems. To overcome the restrictions of perturbation methods, some non-perturbation methods, such as the artificial small parameter method, the δ-expansion method and Adomian's decomposition method, have been developed (see Lyapunov, 1992; Karmishin et al., 1990; Adomian, 1976). However, it is a pity that these perturbation and non-perturbation methods cannot provide a convenient way to adjust and control the convergence of approximation series.

Liao (1992) proposed an analytic approximation method for strongly nonlinear problems, namely the homotopy analysis method (HAM), which is based on the concept of homotopy in topology and thus has a sound mathematical base. The details of HAM are described in two of Liao's books (see Liao 2003, 2012). Compared with other analytic techniques, HAM has some advantages. Firstly, of all that are based on homotopy in topology, HAM does *not* need any small/large *physical* parameters. Therefore, its use is valid in a greater number of nonlinear problems. Secondly, HAM introduces an auxiliary parameter, called convergence-control parameter, which provides a simple way to control and guarantee the convergence of approximation solutions (see Liao 2003, 2008, 2009, 2012). Thirdly, HAM provides great freedom to choose the proper equation type and base functions to approximate the solution of nonlinear problems efficiently. In addition, compared to numerical schemes, HAM also has some advantages. Firstly, using HAM, accurate approximations of the required solution can be obtained, as well as possible dependent relations of the physical parameters from the system under consideration. These relations are helpful for better understandings the inner aspect of nonlinear problems. Secondly, multiple solutions of nonlinear problems can be gained by means of HAM.

HAM has been successfully applied to solve many nonlinear problems in science and engineering (see Liao, 1992, 1999, 2002, 2003, 2004, 2005, 2008, 2009 and 2012; Hayat and Sajid, 2007; Sajid et al., 2007; Abbasbandy 2006, 2007a, 2007b; Liu and Li, 2008, Liu et al., 2009). This is especially true for several solutions that were not identified using analytic or even numerical techniques, as presented by Liao (2005). All of these show the validity and potential of HAM, especially for strongly nonlinear problems.

The above mentioned advantages of HAM imply that it is possible to develop some general symbolic computation packages for some types of nonlinear problems. In fact, Liao (see Liao, 2012) has already issued a HAM-based Mathematica package **BVPh 1.0** for highly nonlinear boundary-value/eigen-value problems, which is freely available on the website http://numericaltank.sjtu.edu.cn/. This success has encouraged the development of a package for nonlinear initial problems with periodic solutions in this paper. This is the motivation of this work.

The paper is organized as follows. A brief introduction is given in Section 1. The considered problems are described in Section 2. The basic ideas and mathematical formulas are given in Section 3. Some key points are explained in detail to improve the computational efficiency of the algorithm in Section 4. The Maple package **NOPH** is described briefly in Section 5. Several different types of examples are presented in Section 6 to illustrate the effectiveness of the package. Finally, some concluding remarks and discussions are given in Section 7.

2. Problem description

Consider periodic solutions of nonlinear oscillations governed by

$$\ddot{U}_i(t) = f_i\big[\mathbf{U}(t), \dot{\mathbf{U}}(t), \ddot{\mathbf{U}}(t)\big], \quad 1 \leqslant i \leqslant \kappa, \tag{1}$$

where the dependent variable $\mathbf{U}(t)$ has κ components $U_1(t), \ldots, U_\kappa(t)$, the independent variable t denotes time, the dot denotes differentiation with respect to t, and $f_i[\mathbf{U}(t), \dot{\mathbf{U}}(t), \ddot{\mathbf{U}}(t)]$ is either an

algebraic or rational function of $\mathbf{U}(t), \dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$. Note that Eq. (1) is rather general which can be used to describe lots of periodic oscillations of nonlinear problems in science and engineering. In this article, the focus is mainly on single conservative oscillations and self-excited oscillations. The former correspond to periodic oscillations of the center type, and the latter to periodic oscillations of the limit cycle type.

Self-excited oscillators can only appear as relaxation oscillation when the coefficient of the main nonlinear term is large enough. However, for very strong nonlinear oscillators, it needs to calculate higher order approximants to get a proper value of $\hbar$, which can be seen in Section 3. As the intermediate expressions "swells" so fast, the current version of the package NOPH introduced in Section 5 does not work for a relaxation oscillator. It should be emphasized that Eq. (1) does not necessarily contain small/large physical parameters.

Let $\Omega = \{\omega_1, \ldots, \omega_\kappa\}$ denote the set of frequencies of the periodic oscillations. For simplicity, we consider the case when $k = 1$, i.e., Ω contains single ω, in what follows. For multidimensional systems, if ω_i $(1 \leqslant i \leqslant \kappa)$ are different, we could reduce Ω to $\{\omega\}$ by the pretreatments: $\omega_2 = p_2\omega, \ldots, \omega_\kappa = p_\kappa\omega$. In this case, p_i $(i = 2, \ldots, \kappa)$ can be looked upon as positive parameters, and they will be determined by using the second strategy shown on page 78 in this paper. It should be pointed out that according to the property of oscillations, the algorithm can only deal with integer p_i $(2 \leqslant i \leqslant \kappa)$, and it may not work well for rational or irrational ratios p_i $(2 \leqslant i \leqslant \kappa)$.

By the way, we should mention that the method of normal forms is also an effective method to investigate nonlinear systems. The basic idea underlying the method of normal forms is the use of "local" coordinate transformations to "simplify" the equations describing the dynamics of the system under consideration. Theoretically speaking, both of the methods can apply to nonlinear systems. As far as nonlinear oscillators are concerned, it is obvious that for coupled systems, especially with different frequencies, as mentioned above for the case of rational or irrational ratios p_i $(2 \leqslant i \leqslant \kappa)$, the method of normal forms has an advantage over HAM because of its much simpler and straightforward calculation procedure. For the rest HAM has great advantages over the method of normal forms, mainly because HAM provides an easy way to adjust and control the convergence of solution series. Meanwhile, HAM provides great freedom to choose better base functions to approximate nonlinear problems more efficiently.

Define the mean of motions

$$\delta_i = \frac{1}{T}\int_0^T U_i(t)\,dt, \quad i = 1, \ldots, \kappa, \tag{2}$$

where $T = 2\pi/\omega$ is the period of oscillation. Note that if f_i in Eq. (1) contains only odd nonlinearity about $U_i(t)$, the mean of motion δ_i is equal to zero.

Without loss of generality, let us consider nonlinear periodic oscillations with initial conditions

$$\begin{aligned} &U_i(0) = a_i + \delta_i, \qquad \dot{U}_i(0) = \omega b_i, \quad i = 1, \ldots, \kappa - 1, \\ &U_\kappa(0) = a_\kappa + \delta_\kappa, \qquad \dot{U}_\kappa(0) = 0, \end{aligned} \tag{3}$$

where a_i, b_i and δ_i are unknowns to be determined later. Physically, the frequency ω can be regarded as a time-scale. Hence, introducing the transformations

$$\tau = \omega t, \qquad U_i(t) = \delta_i + u_i(\tau), \quad 1 \leqslant i \leqslant \kappa,$$

Eqs. (1) and (3) become

$$\omega^2 \ddot{u}_i(\tau) = f_i\big[\delta + \mathbf{u}(\tau), \omega\dot{\mathbf{u}}(\tau), \omega^2\ddot{\mathbf{u}}(\tau)\big], \tag{4}$$

subject to initial conditions

$$u_\kappa(0) = a_\kappa, \qquad \dot{u}_\kappa(0) = 0, \qquad u_i(0) = a_i, \qquad \dot{u}_i(0) = b_i, \quad 1 \leqslant i \leqslant \kappa - 1, \tag{5}$$

in which $\mathbf{u}$ and δ have κ components $u_1(\tau), \ldots, u_\kappa(\tau)$ and $\delta_1, \ldots, \delta_\kappa$, respectively. In Eqs. (4) and (5), apart from the unknown functions $u_i(\tau)$ and $i = 1, \ldots, \kappa$, there are 3κ unknown physical quantities,

including the frequency ω, the mean of motion δ_i $(1 \leqslant i \leqslant \kappa)$, the amplitudes a_j, b_j $(1 \leqslant j \leqslant \kappa - 1)$ and a_κ. All of these unknowns can be determined by the so-called "rule of solution expression" in the frame of HAM, as described later.

Due to the importance of nonlinear oscillations, several dozen papers are published each year to investigate analytic solutions of nonlinear oscillations with the forms of (1) and (3). However, no program has been reported so far for nonlinear oscillations that automatically delivers analytic solutions. In this paper, based on HAM together with Wu's elimination method, an effective and efficient algorithm is proposed to construct analytic approximate solutions of nonlinear oscillations.

In addition, a Maple package **NOPH** (a package for Nonlinear Oscillation Problems by HAM) has been developed, which can be used to automatically derive accurate approximations of $U_i(t)$ $(1 \leqslant i \leqslant \kappa)$, and all the above mentioned unknown physical quantities. The effectiveness of this package is demonstrated with different types of examples. The package is free to be downloaded from the website: http://numericaltank.sjtu.edu.cn/NOPH.htm.

3. Basic idea

Although HAM has been widely applied while solving nonlinear problems in science and engineering, it is not very easy to apply to a system of coupled nonlinear oscillations. The reason lies in that one has to solve κ coupled nonlinear differential equations together with a set of 3κ highly nonlinear algebraic equations related to the unknown physical quantities ω, δ_i $(1 \leqslant i \leqslant \kappa)$, a_j, b_j $(1 \leqslant j \leqslant \kappa - 1)$ and a_κ. In Sections 3 and 4, HAM is combined with Wu's elimination method to develop an efficient algorithm.

In general, a system of coupled nonlinear oscillators is more complicated. Here, the focus is only on periodic solutions, which can be expressed in terms of the base functions

$$\big\{\cos(m\tau), \sin(m\tau) \,\big|\, m = 1, 2, 3, \ldots\big\}. \tag{6}$$

This is the so-called "rule of solution expression".

To search for periodic solutions, initial estimates were chosen

$$u_{\kappa,0}(\tau) = a_{\kappa,0}\cos(\tau), \qquad u_{i,0}(\tau) = a_{i,0}\cos(\tau) + b_{i,0}\sin(\tau), \quad i = 1, \ldots, \kappa - 1, \tag{7}$$

where $a_{i,0}$ and $b_{i,0}$ are unknowns to be determined later, and an auxiliary linear operator

$$\mathcal{L}\big[\Phi(\tau, q)\big] = \omega_0^2\left[\frac{\partial^2 \Phi(\tau, q)}{\partial \tau^2} + \Phi(\tau, q)\right] \tag{8}$$

with property $\mathcal{L}[C_1\sin(\tau) + C_2\cos(\tau)] = 0$, in which $q \in [0, 1]$ is the embedding-parameter, ω_0 is the initial estimate of the frequency ω, $\Phi(\tau, q)$ is a function of τ and q, while C_1 and C_2 are integral constants.

Based on Eq. (4), the following nonlinear operators are defined

$$\begin{aligned}&\mathcal{N}_i\big[\boldsymbol{\Phi}(\tau, q), \Omega(q), \boldsymbol{\Delta}(q)\big]\\ &\quad = \Omega^2(q)\frac{\partial^2 \boldsymbol{\Phi}(\tau, q)}{\partial \tau^2} - f_i\left[\boldsymbol{\Delta}(q) + \boldsymbol{\Phi}(\tau, q), \Omega(q)\frac{\partial \boldsymbol{\Phi}(\tau, q)}{\partial \tau}, \Omega^2(q)\frac{\partial^2 \boldsymbol{\Phi}(\tau, q)}{\partial \tau^2}\right],\end{aligned} \tag{9}$$

where $\Omega(q)$ and $\boldsymbol{\Delta}(q)$ are functions of q, $\boldsymbol{\Phi}(\tau, q)$ and $\boldsymbol{\Delta}(q)$ have κ components, corresponding to the function $\mathbf{u}(\tau)$ and the mean of motion δ, respectively.

HAM is based on continuous variations $\boldsymbol{\Phi}(\tau, q)$, $\Omega(q)$ and $\boldsymbol{\Delta}(q)$, as the embedding-parameter q varies from 0 to 1, $\boldsymbol{\Phi}(\tau, q)$ varies from the initial estimate $\mathbf{u}_0(\tau)$ to the exact solution $\mathbf{u}(\tau)$. So does $\Omega(q)$ from the initial estimate ω_0 to the exact frequency ω, and $\boldsymbol{\Delta}(q)$ from the initial estimate δ_0 to the exact mean of motion δ. Then, with the aid of homotopy in topology, the following equation is constructed (called the zero-order deformation equation)

$$(1 - q)\mathcal{L}\big[\boldsymbol{\Phi}(\tau, q) - \mathbf{u}_0(\tau)\big] = q\hbar\mathcal{N}_i\big[\boldsymbol{\Phi}(\tau, q), \Omega(q), \boldsymbol{\Delta}(q)\big], \tag{10}$$

subject to initial conditions

$$\Phi_i(0,q)=\bar{a}_i(q),\qquad \left.\frac{\partial\Phi_j(\tau,q)}{\partial\tau}\right|_{\tau=0}=\bar{b}_j(q),$$
$$\left.\frac{\partial\Phi_\kappa(\tau,q)}{\partial\tau}\right|_{\tau=0}=0,\quad 1\leqslant i\leqslant\kappa,\ 1\leqslant j\leqslant\kappa-1, \tag{11}$$

where $\bar{a}_i(q)$ and $\bar{b}_j(q)$ are functions of q, corresponding to a_i and b_j, respectively.

When $q=0$, it follows from (7) and (10) that

$$\boldsymbol{\Phi}(\tau,0)=\mathbf{u}_0(\tau),\qquad \Omega(0)=\omega_0,\qquad \boldsymbol{\Delta}(0)=\delta_0. \tag{12}$$

When $q=1$, since $\hbar\neq 0$, the zero-order deformation equation (10) is equivalent to

$$\mathcal{N}_i\big[\boldsymbol{\Phi}(\tau,1),\Omega(1),\boldsymbol{\Delta}(1)\big]=0,\quad 1\leqslant i\leqslant\kappa, \tag{13}$$

which is exactly the same as the original equation (4), provided

$$\boldsymbol{\Phi}(\tau,1)=\mathbf{u}(\tau),\qquad \Omega(1)=\omega,\qquad \boldsymbol{\Delta}(1)=\delta. \tag{14}$$

Thus, according to (12) and (14), as the embedding-parameter q increases from 0 to 1, $\boldsymbol{\Phi}(\tau,q)$ varies continuously from the initial estimate $\mathbf{u}_0(\tau)$ to the exact solution $\mathbf{u}(\tau)$. So does $\Omega(q)$ from the initial estimate ω_0 to the exact frequency ω, and $\boldsymbol{\Delta}(q)$ from the initial estimate δ_0 to the exact mean of motion δ.

According to Taylor's theorem and using (12), $\Phi_i(\tau,q)$, $\Delta_i(q)$, $\Omega(q)$, $\bar{a}_i(q)$ and $\bar{b}_i(q)$ are expanded in the power series of q as follows

$$\Phi_i(\tau,q)=u_{i,0}(\tau)+\sum_{n=1}^{+\infty}u_{i,n}(\tau)q^n,$$
$$\Omega(q)=\omega_0+\sum_{n=1}^{+\infty}\omega_n q^n,\qquad \Delta_i(q)=\delta_{i,0}+\sum_{n=1}^{+\infty}\delta_{i,n}q^n,$$
$$\bar{a}_i(q)=a_{i,0}+\sum_{n=1}^{+\infty}a_{i,n}q^n,\qquad \bar{b}_j(q)=b_{j,0}+\sum_{n=1}^{+\infty}b_{j,n}q^n, \tag{15}$$

where $1\leqslant i\leqslant\kappa$, $1\leqslant j\leqslant\kappa-1$, and

$$u_{i,n}(\tau)=\frac{1}{n!}\left.\frac{\partial^n\Phi_i(\tau,q)}{\partial q^n}\right|_{q=0},\qquad \delta_{i,n}=\frac{1}{n!}\left.\frac{\partial^n\Delta_i(q)}{\partial q^n}\right|_{q=0},\qquad \omega_n=\frac{1}{n!}\left.\frac{\partial^n\Omega(q)}{\partial q^n}\right|_{q=0},$$
$$a_{i,n}=\frac{1}{n!}\left.\frac{\partial^n\bar{a}_i(q)}{\partial q^n}\right|_{q=0},\qquad b_{j,n}=\frac{1}{n!}\left.\frac{\partial^n\bar{b}_j(q)}{\partial q^n}\right|_{q=0}.$$

Assuming that the parameter $\hbar$ is properly chosen so that the above series are convergent at $q=1$, and due to (12) there is

$$u_i(\tau)=u_{i,0}(\tau)+\sum_{n=1}^{+\infty}u_{i,n}(\tau),\qquad \omega=\omega_0+\sum_{n=1}^{+\infty}\omega_n,$$
$$\delta_i=\delta_{i,0}+\sum_{n=1}^{+\infty}\delta_{i,n},\qquad a_i=a_{i,0}+\sum_{n=1}^{+\infty}a_{i,n},\qquad b_j=b_{j,0}+\sum_{n=1}^{+\infty}b_{j,n}, \tag{16}$$

where $1\leqslant i\leqslant\kappa$ and $1\leqslant j\leqslant\kappa-1$. At the Mth-order approximations, there is

$$\tilde{u}_i(\tau) \approx \sum_{n=0}^{M} u_{i,n}(\tau), \qquad \tilde{\omega} \approx \sum_{n=0}^{M-1} \omega_n,$$
$$\tilde{\delta}_i \approx \sum_{n=0}^{M-1} \delta_{i,n}, \qquad \tilde{a}_i \approx \sum_{n=0}^{M-1} a_{i,n}, \qquad \tilde{b}_j \approx \sum_{n=0}^{M-1} b_{j,n}, \quad 1 \leqslant i \leqslant \kappa,\ 1 \leqslant j \leqslant \kappa - 1. \tag{17}$$

Differentiating the zero-order deformation equation (10) n times with respect to q, then dividing them by $n!$, and finally setting $q = 0$, the nth-order deformation equation can be obtained

$$\mathcal{L}\big[u_{i,n}(\tau) - \chi_n u_{i,n-1}(\tau)\big] = \hbar R_{i,n}, \quad 1 \leqslant i \leqslant \kappa, \tag{18}$$

subject to initial conditions

$$u_{\kappa,n}(0) = a_{\kappa,n}, \qquad \dot{u}_{\kappa,n}(0) = 0, \qquad u_{i,n}(0) = a_{i,n}, \qquad \dot{u}_{i,n}(0) = b_{i,n}, \quad 1 \leqslant i \leqslant \kappa - 1, \tag{19}$$

where

$$R_{i,n} = \frac{1}{(n-1)!} \frac{\partial^{n-1} \mathcal{N}_i[\boldsymbol{\Phi}(\tau, q), \Omega(q), \boldsymbol{\Delta}(q)]}{\partial q^{n-1}}\bigg|_{q=0}, \quad 1 \leqslant i \leqslant \kappa, \tag{20}$$

and

$$\chi_n = \begin{cases} 0, & n \leqslant 1, \\ 1, & n > 1. \end{cases} \tag{21}$$

The above nth-order deformation equation (18) is a set of κ linear differential equations about $u_{i,n}(\tau)$ and $1 \leqslant i \leqslant \kappa$. However, apart from κ unknown functions $u_{i,n}(\tau)$, there are 3κ unknown parameters $\omega_{n-1}, a_{i,n-1}, \delta_{i,n-1}$ $(1 \leqslant i \leqslant \kappa)$ and $b_{j,n-1}$ $(1 \leqslant j \leqslant \kappa - 1)$. According to the properties of trigonometric functions, $R_{i,n}$ can be expressed by

$$R_{i,n} = A_0^{i,n} + \sum_{j=1}^{\phi(i,n)} A_j^{i,n} \cos(j\tau) + \sum_{j=1}^{\psi(i,n)} B_j^{i,n} \sin(j\tau), \quad 1 \leqslant i \leqslant \kappa,$$

where the integers $\phi(i,n)$ and $\psi(i,n)$ depend on Eq. (4) and the order n. To search for a periodic solution and avoid the so-called secular terms, $A_0^{i,n}$, $A_1^{i,n}$ and $B_1^{i,n}$, where $1 \leqslant i \leqslant \kappa$, must vanish, i.e.

$$A_0^{i,n} = 0, \qquad A_1^{i,n} = 0, \qquad B_1^{i,n} = 0, \quad 1 \leqslant i \leqslant \kappa. \tag{22}$$

Eqs. (22) provide us with a set of 3κ algebraic equations to determine the above mentioned 3κ unknowns, i.e. $\omega_{n-1}, a_{i,n-1}, b_{j,n-1}$ and $\delta_{i,n-1}$, where $1 \leqslant i \leqslant \kappa$ and $1 \leqslant j \leqslant \kappa - 1$. Then, it is easy to obtain the solutions for the nth-order deformation equation

$$u_{i,n}(\tau) = C_1^{i,n} \sin(\tau) + C_2^{i,n} \cos(\tau) + \sum_{j=2}^{\phi(i,n)} \frac{A_j^{i,n}}{(1-j^2)} \cos(j\tau) + \sum_{j=2}^{\psi(i,n)} \frac{B_j^{i,n}}{(1-j^2)} \sin(j\tau),$$

where $1 \leqslant i \leqslant \kappa$, the 2κ integral constants $C_1^{i,n}$ and $C_2^{i,n}$ are determined by the 2κ initial conditions given in (19).

It is to be stressed that Eq. (4) is more general, which include single conservative oscillations and self-excited oscillations, etc. For single conservative oscillation the base functions (6) can be simplified as $\{\cos(m\tau) \mid m = 1, 2, 3, \ldots\}$, since in this case, all coefficients of sine functions become zero. Thus Eqs. (22) are reduced to

$$A_0^{1,n} = 0, \qquad A_1^{1,n} = 0,$$

which are used to determine ω_{n-1} and $\delta_{1,n-1}$. Note that different types of oscillations have different characteristics, for example, the amplitude for a single conservative oscillation is fixed, or it is even

known sometimes, but the amplitude for a self-excited oscillation system is unknown and needs to be determined. The problem can be removed when the conservative oscillation amplitude is unknown, as this means it can be considered to be a parameter and treated as such.

Following the above steps, a family of solution series can be obtained in the auxiliary parameter $\hbar$. How can a proper value of $\hbar$ be chosen to guarantee a fast enough convergence of the solution series for any given physical parameters? The optimal value of $\hbar$ is determined by minimizing the squared residual error of Eq. (4), i.e.

$$\Lambda(\hbar) = \sum_{i=1}^{\kappa} \int_0^{2\pi} \epsilon_i(\hbar, \tau)^2 \, d\tau,$$

where $\epsilon_i(\hbar, \tau) = \mathcal{N}_i[\tilde{\mathbf{u}}(\tau), \tilde{\omega}, \tilde{\delta}]$, $\tilde{\mathbf{u}}(\tau) = \{\tilde{u}_1(\tau), \ldots, \tilde{u}_\kappa(\tau)\}$, $\tilde{\delta} = \{\tilde{\delta}_1, \ldots, \tilde{\delta}_\kappa\}$ and $1 \leqslant i \leqslant \kappa$.

4. Main algorithms

It is well known that computational efficiency is critical in software. To improve the computational efficiency of this code, several important techniques were employed in this algorithm, as described below.

4.1. Solving nonlinear algebraic equations with parameters

It should be pointed out that, Eqs. (22) are linear when $n > 1$, but are 3κ coupled nonlinear algebraic equations when $n = 1$. For a single nonlinear oscillation, it is easy to solve Eqs. (22) for $\omega_0, \delta_{1,0}$ and $a_{1,0}$. However, for coupled nonlinear oscillation, it is difficult to solve the corresponding nonlinear algebraic equations (22) for $\omega_0, \delta_{i,0}, a_{i,0}$ and $b_{j,0}$, where $1 \leqslant i \leqslant \kappa$ and $1 \leqslant j \leqslant \kappa - 1$, which may become a bottleneck for the whole algorithm. To overcome this difficulty and to improve the efficiency of this program, various techniques were applied, apart from the divide-and-conquer technique and freeze technique. Here, the three main strategies are listed below:

- Firstly, Wu's elimination method (see Wu et al., 1984 (in Chinese), 1994 (in English)) is a powerful tool for solving nonlinear algebraic equations. Wang et al. (see Wang 1995, 2004) completely implemented Wu's elimination method and developed a Maple package `CharSets`, in which the `Csolve` solver can be directly used to solve the nonlinear algebraic equations. In this paper's Maple code, Eqs. (22) are solved by means of the package `CharSets`, whose greatest advantage is that it can avoid missing solutions. Besides, since the orders of variables can be specified when solving nonlinear algebraic equations by using the package, the orders of variables can be optimized so as to improve the computational efficiency. In addition, the `CharSets` package is very efficient, due to some excellent techniques having been employed in it, such as decomposing expression, extracting subexpression as well as sorting and specifying the orders of equations to be solved, and so on.
- Secondly, in Eqs. (22), usually the degrees of parameters involved in the governing equations are lower than those of unknowns, so treating these parameters as unknowns might greatly weaken the difficulty of the problem. Doing this it is possible to obtain compact special solutions with parameter constraints.
- Thirdly, even with the above two strategies, for some coupled nonlinear oscillating systems, the obtained Eqs. (22) are still very difficult to deal with. In this case, some extra conditions to simplify Eqs. (22) might be imposed. With this strategy, the complicated algebraic equations (22) could be greatly simplified, and then some special solutions may be obtained.

Here, an example is given to illustrate how to apply these techniques to simplify and solve a nonlinear algebraic equations. Consider a three coupled self-excited oscillator

$$\ddot{U} - \varepsilon\left(1 - U^2\right)\dot{U} + U = \varepsilon\mu(W - U),$$

$$\ddot{V} - \varepsilon\left(1 - V^2\right)\dot{V} + V = \varepsilon\mu(W - V),$$
$$\ddot{W} - \varepsilon\left(1 - W^2\right)\dot{W} + p^2 W = \varepsilon\mu(U + V - 2W), \tag{23}$$

with initial conditions

$$U(0) = a_1, \qquad \dot{U}(0) = b_1, \qquad V(0) = a_2, \qquad \dot{V}(0) = b_2, \qquad W(0) = a_3, \qquad \dot{W}(0) = 0, \tag{24}$$

where ε, μ and p are unknown parameters, and a_1, a_2, a_3, b_1 and b_2 are unknowns to be determined.

As this system only contains odd nonlinearity, $\delta_i = 0$ and $i = 1, 2, 3$. Direct calculations, following the procedure of HAM, yield

$$\begin{cases} \varepsilon\omega_0 b_{1,0}^3 - 4\varepsilon\omega_0 b_{1,0} - 4\omega_0^2 a_{1,0} + 4a_{1,0} + \varepsilon\omega_0 a_{1,0}^2 b_{1,0} + 4\varepsilon\mu a_{1,0} - 4\varepsilon\mu a_{3,0} = 0, \\ -\varepsilon\omega_0 a_{1,0}^3 + 4b_{1,0} - \varepsilon\omega_0 b_{1,0}^2 a_{1,0} + 4\varepsilon\omega_0 a_{1,0} + 4\varepsilon\mu b_{1,0} - 4\omega_0^2 b_{1,0} = 0, \\ \varepsilon\omega_0 b_{2,0}^3 - 4\varepsilon\omega_0 b_{2,0} - 4\omega_0^2 a_{2,0} + 4a_{2,0} + \varepsilon\omega_0 a_{2,0}^2 b_{2,0} + 4\varepsilon\mu a_{2,0} - 4\varepsilon\mu a_{3,0} = 0, \\ -\varepsilon\omega_0 a_{2,0}^3 + 4b_{2,0} - \varepsilon\omega_0 b_{2,0}^2 a_{2,0} + 4\varepsilon\omega_0 a_{2,0} + 4\varepsilon\mu b_{2,0} - 4b_{2,0}\omega_0^2 = 0, \\ \varepsilon\mu a_{1,0} + \omega_0^2 a_{3,0} - p^2 a_{3,0} - 2\varepsilon\mu a_{3,0} + \varepsilon\mu a_{2,0} = 0, \\ -4a_{3,0}\omega_0 + 4\mu b_{2,0} + \omega_0 a_{3,0}^3 + 4\mu b_{1,0} = 0. \end{cases} \tag{25}$$

It is difficult to solve Eqs. (25) for the unknowns ω_0, $a_{1,0}$, $b_{1,0}$, $a_{2,0}$, $b_{2,0}$ and $a_{3,0}$, because the highest degrees of $a_{1,0}$, $b_{1,0}$, $a_{2,0}$, $b_{2,0}$ and $a_{3,0}$ are all 3, and the degree of ω_0 is 2. Due to the known fact that the solution expression of a cubic equation is complicated, it is natural that the solution expressions of Eqs. (25) would be huge. It becomes even worse when the intermediate expressions "swell" so fast that the memory will overflow finally. Therefore, treating the parameters ε, μ and p as unknowns may reduce such types of difficulties. If solving the complicated Eqs. (25) with some or all of $\varepsilon, \mu, p, \omega_0, a_{1,0}, b_{1,0}, a_{2,0}, b_{2,0}$ and $a_{3,0}$ as unknowns, unfortunately, a trivial solution may be obtained with the constraint $\varepsilon = 0$, or it may take too long to obtain the result, or even in some cases the memory could overflow. In order to obtain nontrivial solutions, some extra conditions could be imposed, such as $\{b_{1,0} = 0\}$, to reduce Eqs. (25) as

$$\begin{cases} -\omega_0^2 a_{1,0} + a_{1,0} + \varepsilon\mu a_{1,0} - \varepsilon\mu a_{3,0} = 0, \\ \varepsilon\omega_0 b_{2,0}^3 - 4\varepsilon\omega_0 b_{2,0} - 4\omega_0{}^2 a_{2,0} + 4a_{2,0} + \varepsilon\omega_0 a_{2,0}^2 b_{2,0} + 4\varepsilon\mu a_{2,0} - 4\varepsilon\mu a_{3,0} = 0, \\ -\varepsilon\omega_0 a_{2,0}^3 + 4b_{2,0} - \varepsilon\omega_0 b_{2,0}^2 a_{2,0} + 4\varepsilon\omega_0 a_{2,0} + 4\varepsilon\mu b_{2,0} - 4b_{2,0}\omega_0^2 = 0, \\ \varepsilon\mu a_{1,0} + \omega_0^2 a_{3,0} - p^2 a_{3,0} - 2\varepsilon\mu a_{3,0} + \varepsilon\mu a_{2,0} = 0, \\ -4a_{3,0}\omega_0 + 4\mu b_{2,0} + \omega_0 a_{3,0}^3 = 0, \\ a_{1,0}^2 - 4 = 0. \end{cases} \tag{26}$$

Clearly, the equations in (26) are much simpler than those in (25). Treating the parameters ε, μ and p as unknowns and solving Eqs. (26) with the Maple package `CharSets`, the following solutions are obtained

$$\begin{aligned} &\{\omega_0 = \sqrt{1 + \varepsilon\mu},\ a_{1,0} = \mp 2,\ a_{2,0} = \pm 2,\ b_{1,0} = 0,\ b_{2,0} = 0,\ a_{3,0} = 0\}, \\ &\left\{\mu = \frac{1 - p^2}{2\varepsilon},\ \omega_0 = \sqrt{2 - p^2},\ a_{1,0} = a_{2,0} = \mp 2,\ b_{1,0} = b_{2,0} = 0,\ a_{3,0} = \pm 2\right\}, \\ &\{p = \mp 1,\ \omega_0 = 1,\ a_{1,0} = -2,\ a_{2,0} = -2,\ b_{1,0} = 0,\ b_{2,0} = 0,\ a_{3,0} = -2\}, \\ &\{p = \mp 1,\ \omega_0 = 1,\ a_{1,0} = 2,\ a_{2,0} = 2,\ b_{1,0} = 0,\ b_{2,0} = 0,\ a_{3,0} = 2\}. \end{aligned} \tag{27}$$

As p appears in the form of p^2 in Eqs. (23), it is easy to see that if $U(t), V(t)$ and $W(t)$ are solutions of the systems (23) and (24), then $-U(t), -V(t)$ and $-W(t)$ are also solutions for the system. Therefore, there are three essentially different solutions in (27), given by

$$\{\omega_0 = \sqrt{1+\varepsilon\mu},\ a_{1,0} = 2,\ a_{2,0} = 2,\ b_{1,0} = 0,\ b_{2,0} = 0,\ a_{3,0} = 0\},$$
$$\left\{\mu = -\frac{p^2-1}{2\varepsilon},\ \omega_0 = \sqrt{2-p^2},\ a_{1,0} = 2,\ a_{2,0} = 2,\ b_{1,0} = 0,\ b_{2,0} = 0,\ a_{3,0} = 2\right\},$$
$$\{p = 1,\ \omega_0 = 1,\ a_{1,0} = 2,\ a_{2,0} = 2,\ b_{1,0} = 0,\ b_{2,0} = 0,\ a_{3,0} = 2\}. \tag{28}$$

Obviously, there are parameter constraints for the last two solutions. It is easy to see that under the different conditions imposed, the obtained solutions for the systems (23) and (24) would be different. This point will be discussed again, later, in Section 6.5.3.

4.2. The homotopy-Padé technique

Padé approximant expands a function as a ratio of two power series. When a function is given in the form of $R(x) = (\sum_{k=0}^{M} a_k x^k)/(1 + \sum_{k=1}^{N} b_k x^k)$, then $R(x)$ is said to be a Padé approximant to the series $f(x) = \sum_{k=0}^{\infty} c_k x^k$, if

$$R(0) = f(0), \qquad \frac{d^k}{dx^k} R(x)\bigg|_{x=0} = \frac{d^k}{dx^k} f(x)\bigg|_{x=0}, \quad k = 1, 2, \ldots, M+N. \tag{29}$$

The conditions (29) provide $M+N+1$ equations for the unknowns $a_0, \ldots, a_M$ and $b_1, \ldots, b_N$. The Padé technique can be combined with HAM to generate the so-called homotopy-Padé method, as shown by Liao (see Liao, 2003). The general procedure is as follows. First, employ the traditional Padé technique to the series (15) about the embedding-parameter $q \in [0, 1]$ to obtain the $[m, n]$ Padé approximant

$$TP_{m,n} = \left(\sum_{k=0}^{m} W_k(\tau) q^k\right) \Big/ \left(1 + \sum_{k=1}^{n} W_{m+k}(\tau) q^k\right),$$

where $W_k(\tau), k = 0, 1, \ldots, m+n$, are functions determined by the first $m+n+1$ components $u_{i,j}(\tau)$, $j = 0, 1, \ldots, m+n$. Then, setting $q = 1$, the so-called $[m, n]$ homotopy-Padé approximant was obtained

$$HP_{m,n} = \left(\sum_{k=0}^{m} W_k(\tau)\right) \Big/ \left(1 + \sum_{k=1}^{n} W_{m+k}(\tau)\right).$$

More details can be found in Liao (2003).

It has been found that the homotopy-Padé approximant usually converges faster than the corresponding traditional $[m, n]$ Padé approximant. In many cases, the $[m, m]$ homotopy-Padé approximant does not depend upon the auxiliary parameter $\hbar$. In general, the homotopy-Padé technique can greatly enlarge the convergence region of solution series. Hence, it is employed in the code.

5. The package NOPH

Although the algorithm described in Sections 3 and 4 is relatively simple in principle, the calculations would be very complicated if they had to be performed by hand. In this section, based on the package **CharSets**, a Maple package **NOPH** is presented, which can be used to automatically derive homotopy analysis solutions for nonlinear oscillation equations in the forms of (1) and (3).

In NOPH, the main interface is *main*(*eqns*, *ini_con*, [*h_order*, *sol_order*]), in which *eqns* represents an oscillation equations to be solved and *ini_con* represents the corresponding initial conditions. Note that *eqns* should be put in a list, as should *ini_con*. The other two parameters *h_order* (the default value is 8) and *sol_order* (the default value is 10) are optional, which are used to control the process flow of the program. In order to meet different needs, the computations in the program are divided into two parts: the first part is dependent on the auxiliary parameter $\hbar$, and the second part is free of the auxiliary parameter $\hbar$. They are described as follows:

1. When there are unknown parameters in *eqns* or *ini_con*, just perform the first part computation, namely, calculate the first *h_order* components of the solution series to serve as an approximation of the required solution. In this case, the optional parameter *sol_order* can be omitted, and the program will output the $[h\_order/2, h\_order/2]$ homotopy-Padé approximate expression of the frequency ω as well as the mean of motions $\delta_i, i = 1, \ldots, \kappa$.
2. If there are no unknown parameters in *eqns* or *ini_con*, first obtain the first *h_order* components of the solution series by the first part computation, and further calculate the squared residual error of the input equations to determine the optimal value of $\hbar$. Once the value of $\hbar$ is determined, a more accurate approximate solution (i.e., continue to calculate from $h\_order + 1$ to *sol_order* order) may be obtained by the second part computation. In this case, the program will output different order approximations and the corresponding errors of the frequency ω, as well as the mean of motions $\delta_i, i = 1, \ldots, \kappa$, etc. In addition, comparison graphs of the different order approximations will be produced for the required solutions.

As described in Section 3, there are different characteristics for different types of oscillations. Therefore, the output of the program is flexible and diverse, not only depending on the number of unknown parameters, but also depending upon the type of oscillations. Apart from all the output results listed in the next section, if necessary, users could also obtain components of the required solution series. The components of the solution series that are obtained are stored in a set that is named as 'export'. All these sets are listed as follows:

- **Know_u:** the components set of $U_i(t)$ and $i = 1, \ldots, \kappa$.
- **Know_omega:** the components set of the frequency ω; and **Pade_omega:** the homotopy-Padé approximations set of ω.
- **Know_delta:** the components set of the mean of motion δ_i and $i = 1, \ldots, \kappa$, in accordance; and **Pade_delta:** the homotopy-Padé approximations set of δ_i and $i = 1, \ldots, \kappa$.
- **Know_abc:** the components set of the amplitudes $a_i, b_i, i = 1, \ldots, \kappa - 1$ and a_κ. It should be noted that **Know_abc** is set exclusively for self-excited oscillating systems.

It is very simple to list the contents of the above sets. For example, to display the set **Know_u**, one just needs to input the set name **Know_u** after the command prompt ">", that is,

> *Know_u*;

As an example to show how to use the package NOPH, a single conservative oscillating system is considered that contains only odd nonlinearity

$$\ddot{U} = -U - 5U^3, \tag{30}$$

subject to initial conditions $U(0) = 1/2, \dot{U}(0) = 0$.

To load the package NOPH, one proceeds as follows:

> *restart*: initializing Maple.

> *currentdir*("*D* : /*ect*001*a*"); setting current work directory.

Note that the source program and the sample file should be located in the current work directory.

> *read*"*oscillation.mpl*": reading the program file.

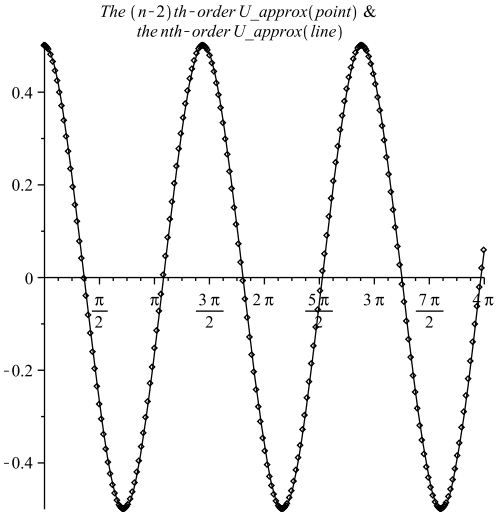
> *with*(*NOPH*); loading the package NOPH.

To solve the system (30), one could run the main procedure as follows:

> *main*([*diff*(*U*(*t*), *t*$2) = −*U*(*t*) − 5 ∗ *U*(*t*)^3], [*U*(0) = 1/2, *D*(*U*)(0) = 0], 4, 7);

The program will output the following results:

1. The optimal value of $\hbar$ is: $\hbar = -1.0$.
2. The comparison graph between 5th and 7th-order approximations of $U(t)$:



3. The HAM approximants of ω and the corresponding relative errors:

i	ω_i	errors
1	1.3919	–
2	1.3852	0.0049
3	1.3852	0.0000
4	1.3851	0.0000
5	1.3851	0.0000

4. The HAM-Padé approximants of ω and the corresponding relative errors:

i	ω_i	errors
1	1.3851	–
2	1.3851	0.0000

The total CPU time is 10.828 seconds. It can be seen from the above figures and tables that the solution series converge very well.

6. Applications of NOPH

In this section, some examples are given to show the effectiveness of the package. More than 30 oscillation equations have been solved using the package NOPH. It should be mentioned that all the results shown in this paper were obtained on a laptop with Intel Core i5-2410M CPU as the processor and a frequency of 2.30 GHz. Due to space limitations, only five different types of nonlinear oscillation systems are considered here.

To demonstrate the convergence of the homotopy analysis solutions obtained in this paper, the solutions obtained are compared with exact solutions or numerical solutions obtained using a Fehlberg fourth-fifth order Runge–Kutta method (rkf45) with degree four interpolant. As this method does not work for a system with unknown parameters, each unknown parameter was assigned the value obtained in this paper.

6.1. Example 1

Consider the free oscillation of a conservative system with odd nonlinearity (Liao, 2003)

$$\ddot{U} + U + \epsilon U^3 = 0,$$

subject to initial conditions $U(0) = a, \dot{U}(0) = 0$.

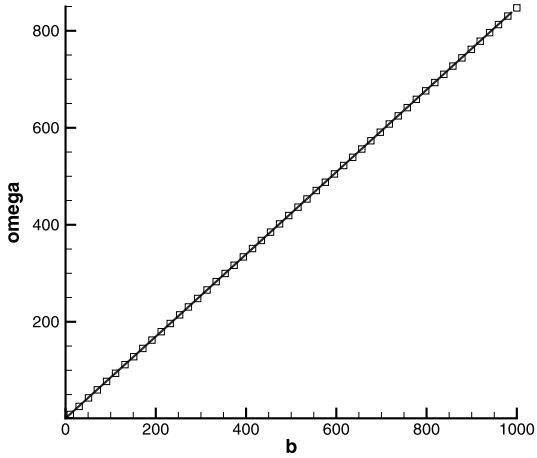


Fig. 1. The homotopy-Padé approximants of ω in Eq. (31) as a function of b. Solid line: [1, 1] homotopy-Padé approximant; Symbols: [2, 2] homotopy-Padé approximant.

For this system, introducing the transformation $V(t) = U(t)\sqrt{\epsilon}$, yields

$$\ddot{V} + V + V^3 = 0, \tag{31}$$

subject to initial conditions

$$V(0) = b, \qquad \dot{V}(0) = 0, \tag{32}$$

where $b = a\sqrt{\epsilon}$ is an unknown parameter.

There is only one parameter b in the system (31) and (32), the package automatically delivers (by $h\_order = 5$) the [1, 1] homotopy-Padé approximation

$$\omega_{P_{1,1}} = \frac{(279b^4 + 768b^2 + 512)\sqrt{3b^2 + 4}}{2(285b^4 + 768b^2 + 512)},$$

and the [2, 2] homotopy-Padé approximation

$$\begin{aligned}\omega_{P_{2,2}} = {} & \big[574\,118\,847b^{14} + 5\,452\,491\,348b^{12} + 22\,121\,717\,760b^{10} + 49\,711\,251\,456b^8 \\ & + 66\,831\,974\,400b^6 + 53\,758\,656\,512b^4 + 23\,957\,864\,448b^2 + 4\,563\,402\,752\big] \\ & /\big[2\sqrt{3b^2 + 4}\big(195\,618\,321b^{12} + 1\,584\,935\,424b^{10} + 5\,335\,944\,192b^8 \\ & + 9\,556\,131\,840b^6 + 9\,602\,662\,400b^4 + 5\,133\,828\,096b^2 + 1\,140\,850\,688\big)\big].\end{aligned}$$

Moreover, the package automatically delivers a figure to compare approximations of ω at different orders. As shown in Fig. 1, the above [1, 1] and [2, 2] homotopy-Padé approximations of ω agree quite well even for $0 \leqslant b < +\infty$.

To further show the convergence of obtained solution series, the case of $b = 1$ is considered. In this case, according to (13) and (14) given in Beléndez (2010), an exact solution for the system (31) and (32) is in the form of $V(t) = \mathrm{cn}(t\sqrt{2}; \frac{1}{4})$, the frequency $\omega = \sqrt{2}\pi/K(1/4)$, where $K(1/4) = \int_0^{2\pi} (1 - \frac{1}{4}\sin^2\theta)^{-\frac{1}{2}} d\theta$ is the complete elliptic integral of the first kind. In this case, all physical parameters are known, when taking $h\_order = 5$ and $sol\_order = 10$, the package automatically delivers $\hbar = -1$. At the same time, the package also outputs approximants of the frequency ω, and a comparison graph for different order approximations of $V(t)$ within 10 seconds. The approximate solutions are compared with the above exact one in Fig. 2.

As shown in Fig. 2, the 8th and 10th-order HAM approximations of $V(t)$ agree very well with the exact solution shown above. Besides, the ith-order approximation of ω and the corresponding $[i, i]$

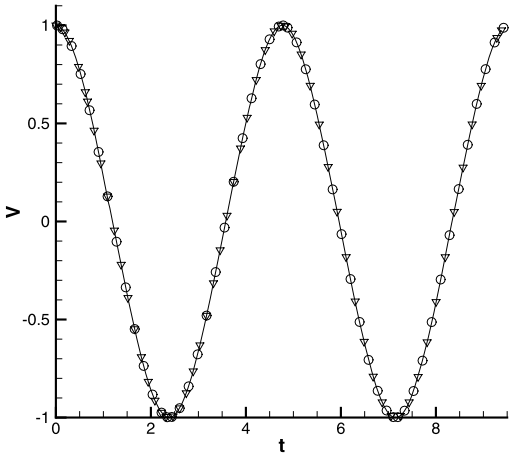


Fig. 2. The comparison of $V(t)$ for Eq. (31) when $b = 1$ and taking $\hbar = -1$. Solid line: 8th-order HAM approximation; Cycles: 10th-order HAM approximation; Gradients: exact solution.

Table 1
The ith-order approximations of ω in Eq. (31) when $b = 1$ and taking $\hbar = -1$.

i	ω_i
1	1.3229
2	1.3178
3	1.3178

Table 2
The $[i, i]$ homotopy-Padé approximations of ω in Eq. (31) when $b = 1$.

i	ω_i
1	1.3178
2	1.3178

homotopy-Padé approximants are given in Tables 1 and 2, respectively. The frequency delivered by the package is in good agreement with the exact one, the homotopy-Padé approximants of ω converge faster than the corresponding homotopy approximants.

6.2. Example 2

For free oscillation of a conservative system with quadratic nonlinearity, the package automatically delivers approximations of both the frequency ω and the mean of motion δ_i $(i = 1, \ldots, \kappa)$. For example, consider a free oscillation with quadratic nonlinearity (Liao, 2003)

$$\ddot{U}(t) + U(t) + \varepsilon U^2(t) = 0, \tag{33}$$

subject to initial conditions $U(0) = 1/2 + \delta$ and $\dot{U}(0) = 0$, where ε is a parameter and δ is the mean of motion.

When ε is unknown, the package automatically delivers (by $h\_order = 7$) within 52 seconds the $[1, 1]$ homotopy-Padé approximants

$$\omega_{P_{[1,1]}} = \frac{((50\epsilon^5 - 212\epsilon^3 + 224\epsilon)\sqrt{4 - 2\epsilon^2} - 37\epsilon^6 + 224\epsilon^4 - 492\epsilon^2 + 384)\sqrt{2}}{(4 - 2\epsilon^2)^{3/4}((54\epsilon^3 - 112\epsilon)\sqrt{4 - 2\epsilon^2} - 35\epsilon^4 + 154\epsilon^2 - 192)},$$

$$\delta_{P_{[1,1]}} = \frac{(3\epsilon^3 - 48\epsilon^2 - 18\epsilon + 96)\sqrt{4 - 2\epsilon^2} - 14\epsilon^3 + 192\epsilon^2 - 48\epsilon^4 + 36\epsilon - 192}{4\epsilon(24\epsilon^2 - 48)\sqrt{4 - 2\epsilon^2} + 7\epsilon^3 - 18\epsilon},$$

the $[2, 2]$ homotopy-Padé approximants

$$\omega_{P_{[2,2]}} = -\big[\big(2\,072\,291\epsilon^{12} - 21\,339\,966\epsilon^{10} + 88\,093\,860\epsilon^8 - 178\,279\,336\epsilon^6$$

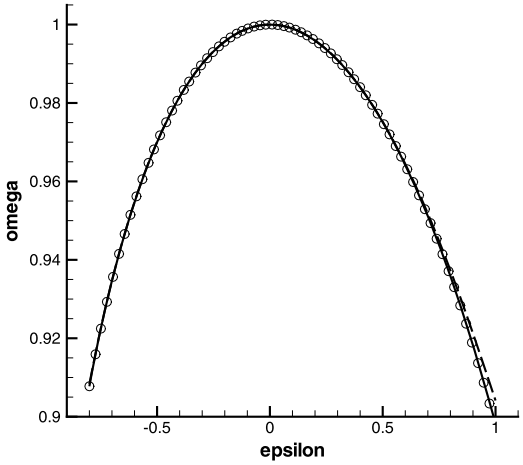


Fig. 3. The HAM-Padé approximants of ω in Eq. (33) as a function of ε. Long dash line: [1, 1] HAM-Padé approximant; Circles: [2, 2] HAM-Padé approximant; Solid line: [3, 3] HAM-Padé approximant.

$$
\begin{aligned}
&+165\,696\,000\epsilon^4-33\,134\,592\epsilon^2-29\,491\,200\big)\sqrt{4-2\epsilon^2}-106\,700\epsilon^{13}\\
&+4\,451\,720\epsilon^{11}-43\,305\,488\epsilon^9+186\,308\,576\epsilon^7-408\,762\,112\epsilon^5+449\,421\,312\epsilon^3\\
&-197\,001\,216\epsilon\big]/\big[\sqrt[4]{16-8\epsilon^2}\big(\big(42\,994\epsilon^{11}+1\,674\,424\epsilon^9-17\,106\,744\epsilon^7\\
&+59\,363\,648\epsilon^5-89\,012\,224\epsilon^3+49\,250\,304\epsilon\big)\sqrt{4-2\epsilon^2}-2\,184\,891\epsilon^{12}\\
&+22\,445\,294\epsilon^{10}-91\,960\,132\epsilon^8+183\,929\,320\epsilon^6-168\,259\,584\epsilon^4+32\,520\,192\epsilon^2\\
&+29\,491\,200\big)\big],\\
\delta_{P_{[2,2]}}=&-\big[\big(1\,935\,360-5\,483\,520\epsilon^5+5\,807\,232\epsilon^3-2\,460\,672\epsilon-27\,113\epsilon^{10}+302\,154\epsilon^8\\
&-1\,405\,356\epsilon^6+3\,347\,640\epsilon^4-4\,025\,088\epsilon^2+58\,248\epsilon^{11}-613\,440\epsilon^9\\
&+2\,591\,424\epsilon^7\big)\sqrt{4-2\epsilon^2}-3\,870\,720+9\,985\,536\epsilon^2+4\,921\,344\epsilon+6\,507\,984\epsilon^6\\
&+11\,261\,952\epsilon^5-10\,895\,184\epsilon^4-11\,775\,744\epsilon^3-37\,999\epsilon^{12}-123\,504\epsilon^{11}\\
&+443\,564\epsilon^{10}+1\,288\,320\epsilon^9-2\,271\,600\epsilon^8-5\,384\,832\epsilon^7\big]/\big[2\big(\big(27\,113\epsilon^{10}\\
&-302\,154\epsilon^8+1\,405\,356\epsilon^6-3\,347\,640\epsilon^4+4\,025\,088\epsilon^2-1\,935\,360\big)\sqrt{4-2\epsilon^2}\\
&+123\,504\epsilon^{11}-1\,288\,320\epsilon^9+5\,384\,832\epsilon^7-11\,261\,952\epsilon^5+11\,775\,744\epsilon^3\\
&-4\,921\,344\epsilon\big)\big],
\end{aligned}
$$

and the [3, 3] homotopy-Padé approximants are omitted here due to space limitations.

It was found that the [2, 2] and [3, 3] homotopy-Padé approximants of the frequency ω and the mean of motion δ agree well, as shown in Figs. 3 and 4.

To show the accuracy of these solutions, the case of $\varepsilon = 1$, for Rand (see Rand, 1990) is considered and presented as an exact solution to Eq. (33) for $\varepsilon = 1$, which has the form $U(t) = A + a\,\mathrm{sn}^2(wt; m)$, as shown in Hu (2006). For convenience of comparison, δ is assigned the same value delivered by the package, namely $\delta = -0.1181$, then $A = 1/2 + \delta = 1/2 - 0.1181 = 0.3819$. Following the steps shown in Hu (2006), $\{a = -0.9147, m = 0.7269, w = 0.5371\}$ is obtained. For the case of $\varepsilon = 1$, there is no unknown parameter and the package automatically derives (by *h_order* = 12 and *sol_order* = 15) $\hbar = -0.4402$, and approximants of ω and δ as well as a comparison graph for different order approximations of $U(t)$ within 820 seconds.

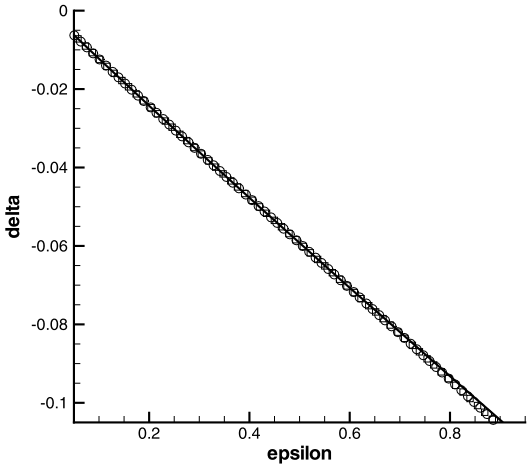


Fig. 4. The HAM-Padé approximants of δ in Eq. (33) as a function of ε. Solid line: [1, 1] HAM-Padé approximant; Circles: [2, 2] HAM-Padé approximant; Squares: [3, 3] HAM-Padé approximant.

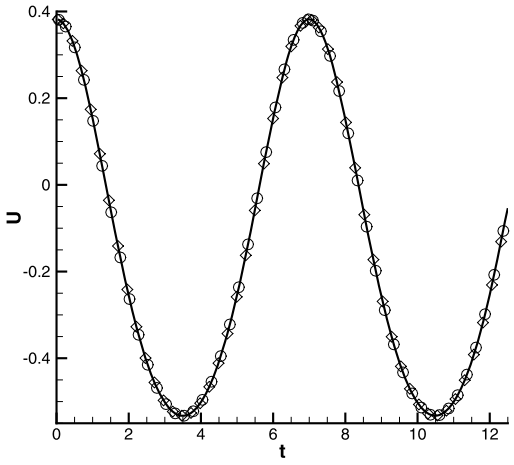


Fig. 5. The comparison of $U(t)$ for Eq. (33) when $\varepsilon = 1$, and taking $\hbar = -0.4402$. Diamonds: 13th-order HAM approximation; Circles: 15th-order HAM approximation; Solid line: exact solution.

As shown in Fig. 5, the 13th and 15th-order HAM approximations of $U(t)$ agree very well with the above exact solution. Table 3 indicates that the frequency ω as well as the mean of motion δ converge quickly, and their homotopy-Padé approximants converge even faster, as shown in Table 4.

6.3. Example 3

The package also works for rational function f with respect to $U, \dot{U}$ and $\ddot{U}$. For example, consider a free periodic oscillator governed by

$$\ddot{U} = -\frac{\lambda U + \varepsilon_1 U \dot{U}^2 + 2\varepsilon_2 U^3 \dot{U}^2 + \varepsilon_3 U^3 + \varepsilon_4 U^5}{1 + \varepsilon_1 U^2 + \varepsilon_2 U^4}, \tag{34}$$

Table 3
The approximations of ω and δ in Eq. (33) when $\varepsilon = 1$ and taking $\hbar = -0.4402$.

i	ω_i	δ_i
1	0.8409	−0.1281
2	0.8781	−0.1205
3	0.8934	−0.1180
4	0.8983	−0.1176
5	0.8991	−0.1178
6	0.8987	−0.1179
7	0.8983	−0.1181
8	0.8981	−0.1181

Table 4
The homotopy-Padé approximations of ω and δ in Eq. (33) when $\varepsilon = 1$.

i	ω_i	δ_i
1	0.9041	−0.1149
2	0.8975	−0.1184
3	0.8981	−0.1180
4	0.8981	−0.1181

subject to initial conditions $U(0) = a$ and $\dot{U}(0) = 0$, where λ is an integer that may take values $-1, 0$ or 1, a is the amplitude and $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ and $\varepsilon_4 > 0$ are physical parameters. Obviously, Eq. (34) only contains odd nonlinearity, so the mean of motion δ equals zero.

In Chen's work (Chen and Chen, 2009), the approximate solution was constructed by a new differential transformation method. To show the effectiveness of their solutions, the authors compared their solutions in four different parameter modes with numerical solutions obtained using the fourth-order Runge–Kutta method; their solutions agreed well with numerical results.

For this example, it is solved using the package in the following three cases.

6.3.1. All physical parameters are unknown

When setting *h_order* = 3, the program automatically delivers the [1, 1] homotopy-Padé approximate expression of ω within 30 seconds. However, the obtained approximate expression is so complicated that it is omitted here due to space limitations.

6.3.2. In the case of $\lambda = 1, \varepsilon_1 = \frac{1}{2}, \varepsilon_2 = \frac{1}{2}, \varepsilon_3 = 1$ *and* $a = 1$

In this case, there is only one unknown parameter ϵ_4. Setting *h_order* = 5, it takes about 910 seconds for the package to deliver the [1, 1] homotopy-Padé approximant

$$\begin{aligned}\omega_{P_{1,1}} = \sqrt{644 + 230\epsilon_4}\big(&282\,621\,400\epsilon_4^4 + 1\,040\,639\,690\epsilon_4^3 - 2\,101\,840\,661\epsilon_4^2 \\ &- 10\,283\,156\,607\epsilon_4 - 7\,846\,822\,521\big)/\big[23\big(289\,246\,200\epsilon_4^4 + 1\,013\,871\,130\epsilon_4^3 \\ &- 2\,120\,380\,083\epsilon_4^2 - 10\,191\,069\,819\epsilon_4 - 7\,768\,422\,279\big)\big],\end{aligned}$$

and the [2, 2] homotopy-Padé approximant

$$\begin{aligned}\omega_{P_{2,2}} = 2\big(&721\,600\,802\,297\,830\,549\,820\,000\,000\epsilon_4^{13} \\ &+ 12\,383\,021\,889\,777\,757\,320\,964\,000\,000\epsilon_4^{12} \\ &+ 77\,058\,516\,168\,389\,190\,627\,707\,250\,000\epsilon_4^{11} \\ &+ 134\,007\,102\,107\,481\,137\,948\,434\,740\,000\epsilon_4^{10} \\ &- 767\,685\,927\,995\,469\,573\,412\,814\,008\,500\epsilon_4^9 \\ &- 4\,746\,263\,062\,725\,126\,846\,856\,252\,962\,150\epsilon_4^8 \\ &- 8\,982\,509\,966\,803\,881\,294\,989\,436\,310\,615\epsilon_4^7 \\ &+ 5\,388\,967\,563\,593\,694\,859\,126\,248\,691\,641\epsilon_4^6 \\ &+ 60\,335\,045\,968\,270\,544\,233\,419\,404\,252\,763\epsilon_4^5 \\ &+ 134\,137\,324\,911\,540\,129\,238\,896\,999\,420\,609\epsilon_4^4 \\ &+ 160\,727\,905\,086\,757\,562\,229\,362\,014\,143\,834\epsilon_4^3\end{aligned}$$

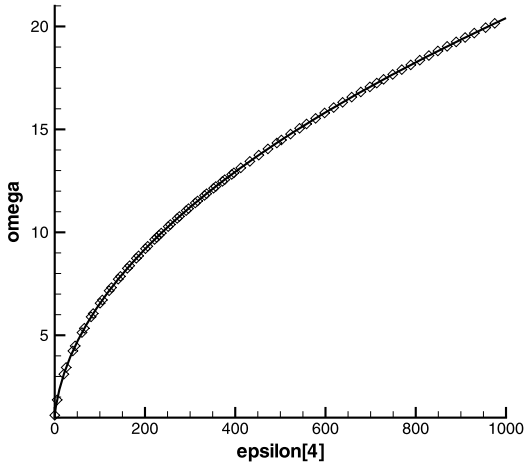


Fig. 6. The homotopy-Padé approximants of ω in Eq. (34) when $a = \lambda = \varepsilon_3 = 1, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Line: [1, 1] homotopy-Padé approximant; Symbols: [2, 2] homotopy-Padé approximant.

$$
\begin{aligned}
&+ 114\,686\,883\,370\,009\,768\,195\,513\,608\,627\,951\epsilon_4^2 \\
&+ 47\,071\,103\,220\,848\,811\,048\,576\,739\,633\,845\epsilon_4 \\
&+ 8\,859\,722\,465\,244\,665\,645\,188\,353\,217\,170\big) \\
&/\big[\sqrt{644 + 230\epsilon_4}\big(147\,948\,325\,632\,742\,461\,644\,000\,000\epsilon_4^{12} \\
&+ 2\,095\,495\,378\,931\,519\,030\,605\,600\,000\epsilon_4^{11} \\
&+ 9\,565\,294\,105\,746\,675\,483\,760\,970\,000\epsilon_4^{10} \\
&- 690\,982\,710\,186\,054\,606\,457\,068\,000\epsilon_4^{9} \\
&- 153\,990\,028\,938\,150\,585\,378\,469\,292\,900\epsilon_4^{8} \\
&- 516\,157\,443\,143\,427\,716\,491\,919\,938\,930\epsilon_4^{7} \\
&- 327\,658\,150\,909\,914\,224\,045\,065\,731\,355\epsilon_4^{6} \\
&+ 2\,033\,311\,501\,959\,887\,749\,602\,831\,227\,901\epsilon_4^{5} \\
&+ 6\,346\,583\,360\,504\,481\,105\,299\,648\,696\,625\epsilon_4^{4} \\
&+ 8\,882\,680\,801\,800\,394\,612\,648\,600\,646\,085\epsilon_4^{3} \\
&+ 7\,006\,874\,832\,650\,635\,013\,575\,763\,874\,036\epsilon_4^{2} \\
&+ 3\,105\,495\,703\,111\,427\,773\,085\,478\,899\,457\epsilon_4 \\
&+ 626\,473\,417\,036\,090\,662\,183\,962\,398\,527\big)\big].
\end{aligned}
$$

The above two approximations agree very well for $\epsilon_4 > 0$, as shown in Fig. 6.

6.3.3. In the case of $a = \lambda = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$

In this case, there are no unknown parameters and the program automatically gives (by setting *h_order* $= 6$, *sol_order* $= 8$) $\hbar = -0.3384$, approximants of ω as well as a comparison graph for different order approximations of $U(t)$ within 2000 seconds. As shown in Fig. 7, the 6th and 8th-order HAM approximations of $U(t)$ agree very well. Meanwhile these curves agree well with the numerical imitating curve of the numerical solution for the given values $a = \lambda = \varepsilon_3 = \varepsilon_4 = 1, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and the absolute error 10^{-7}. Moreover, the frequency ω also converges quickly, as shown in Tables 5

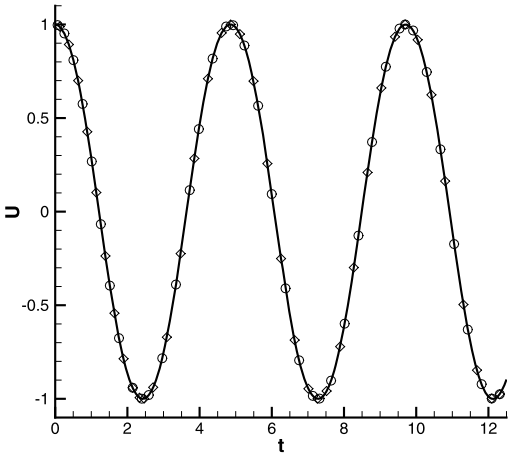


Fig. 7. The HAM approximations of $U(t)$ for Eq. (34) when $a=\lambda=\varepsilon_3=\varepsilon_4=1$, $\varepsilon_1=\varepsilon_2=\frac{1}{2}$ and taking $\hbar=-0.3384$. Diamonds: 6th-order HAM approximation; Cycles: 8th-order HAM approximation; Solid line: numerical approximation.

Table 5
The ith-order HAM approximations of ω in Eq. (34) when $a=\lambda=\varepsilon_3=\varepsilon_4=1$, $\varepsilon_1=\varepsilon_2=\frac{1}{2}$ and taking $\hbar=-0.3384$.

i	ω_i
1	1.2854
2	1.2942
3	1.2944
4	1.2945
5	1.2945

Table 6
The $[i,i]$ homotopy-Padé approximants of ω in Eq. (34) when $a=\lambda=\varepsilon_3=\varepsilon_4=1$ and $\varepsilon_1=\varepsilon_2=\frac{1}{2}$.

i	ω_i
1	1.2944
2	1.2945

and 6. It needs to be emphasized that for the four different modes of the parameters provided in Chen and Chen (2009), the solutions also agree well with the numerical solution obtained using the rkf45, while the comparison graph is omitted due to space limitations.

6.4. Example 4

Consider the Van der Pol equation

$$\ddot{U}(t)+U(t)=\varepsilon\left(1-U(t)^2\right)\dot{U}(t), \tag{35}$$

with initial conditions $U(0)=c_1$ and $\dot{U}(0)=0$, where ε is an unknown parameter.

As there is only one unknown parameter in this system, it takes about 157 seconds for the program to deliver (by $h\_order=7$)

$$\left\{\{\omega_0=1,\ c_{1,0}=2\},\ \{\omega_0=1,\ c_{1,0}=-2\}\right\}.$$

By selecting the first solution, it further outputs:

the $[1,1]$ homotopy-Padé approximation

$$\omega_{P_{1,1}}=\frac{\varepsilon^2+32}{3\varepsilon^2+32}, \tag{36}$$

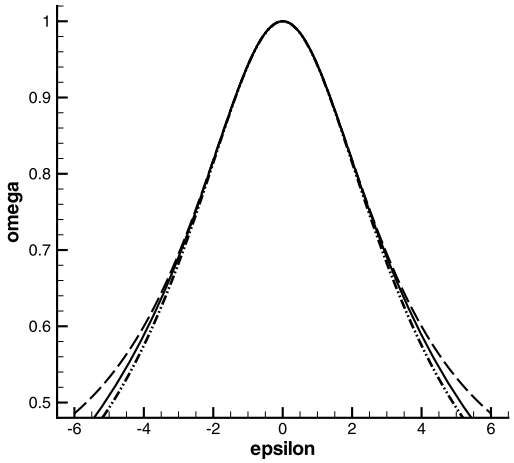


Fig. 8. The homotopy-Padé approximants of ω in Eq. (35) as a function of ε. Long dash line: [1, 1] homotopy-Padé approximant; DashDotDot line: [2, 2] homotopy-Padé approximant; Solid line: [3, 3] homotopy-Padé approximant.

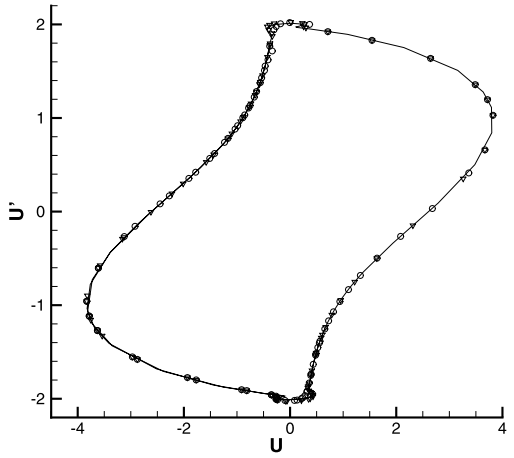


Fig. 9. The approximations of $U(t)$ for Eq. (35) when $\varepsilon = 2$ and taking $\hbar = -0.6655$. Solid line: 12th-order HAM approximation; Deltas: 14th-order HAM approximation; Cycles: numerical approximation.

the [2, 2] homotopy-Padé approximation

$$\omega_{P_{2,2}} = \frac{49\,152 + 9\varepsilon^6 + 16\,640\varepsilon^2 + 960\varepsilon^4}{49\,152 + 45\varepsilon^6 + 19\,712\varepsilon^2 + 1920\varepsilon^4}, \tag{37}$$

and the [3, 3] homotopy-Padé approximation

$$\begin{aligned}\omega_{P_{3,3}} = {} & \big(-3\,020\,544\varepsilon^8 - 1\,970\,749\,440\varepsilon^6 + 729\varepsilon^{12} - 109\,231\,931\,392\varepsilon^4 \\ & - 1\,175\,344\,644\,096 - 2\,382\,205\,288\,448\varepsilon^2 + 178\,848\varepsilon^{10}\big)/\big(-73\,979\,136\varepsilon^8 \\ & - 4\,347\,887\,616\varepsilon^6 + 5103\varepsilon^{12} - 256\,206\,766\,080\varepsilon^4 - 1\,175\,344\,644\,096 \\ & - 2\,455\,664\,328\,704\varepsilon^2 + 598\,752\varepsilon^{10}\big).\end{aligned} \tag{38}$$

Table 7
The homotopy approximations of ω and c_1 in Eq. (35) when $\varepsilon = 2$ by means of $\hbar = -0.6655$.

i	ω_i	$c_{1,i}$
1	1.0000	2.0000
2	0.8336	2.0000
3	0.8195	1.7694
4	0.8156	1.6918
5	0.8149	1.7405
6	0.8192	1.7827
7	0.8215	1.8016
8	0.8234	1.8056
9	0.8245	1.8042
10	0.8246	1.8029
11	0.8245	1.8010
12	0.8242	1.7988
13	0.8238	1.7970
14	0.8232	1.7964

Table 8
The homotopy-Padé approximations of ω and c_1 in Eq. (35) when $\varepsilon = 2$.

i	ω_i	$c_{1,i}$
1	0.8182	2.0000
2	0.8147	1.7665
3	0.8171	1.7665
4	0.8257	1.8012
5	0.8238	1.8017
6	0.8232	1.7957

Moreover, the package automatically outputs a figure to compare approximations of ω at different orders, as shown in Fig. 8. It can be seen that the curves of $[2, 2]$ and $[3, 3]$ homotopy-Padé approximants of ω agree well.

In addition, in the case of $\varepsilon = 2$, there are no unknown parameters in the input system and the package takes 15 186 seconds to automatically output (by *h_order* $= 12$ and *sol_order* $= 14$) $\hbar = -0.6655$, the approximants of ω, c_1, and the comparison graph for different order approximations of $U(t)$. As shown in Fig. 9, the 12th and 14th-order HAM approximations of $U(t)$ agree very well. Meanwhile, these curves agree well with the numerical imitating curve of the numerical solution for the given values $\varepsilon = 2, c_1 = 1.7957$ and the absolute error 10^{-7}. The frequency ω and the amplitude c_1 converge quickly for the given value $\hbar = -0.6655$, as shown in Tables 7 and 8.

6.5. Example 5

Consider two directly coupled van der Pol oscillators

$$\begin{cases} \ddot{U}(t) - \varepsilon\big(1 - U^2(t)\big)\dot{U}(t) + U(t) = \varepsilon\mu\big(V(t) - U(t)\big), \\ \ddot{V}(t) - \varepsilon\big(1 - V^2(t)\big)\dot{V}(t) + p^2 V(t) = \varepsilon\mu\big(U(t) - V(t)\big), \end{cases} \tag{39}$$

with initial conditions

$$U(0) = a_1, \qquad \dot{U}(0) = b_1, \qquad V(0) = c_1, \qquad \dot{V}(0) = 0, \tag{40}$$

where ε, μ and p are the physical parameters.

Similar to the above examples, this system is solved by considering three different cases.

6.5.1. All physical parameters are unknown

In this case there are three unknown parameters ε, μ and p, and the package takes about 20 seconds to deliver 8 group solutions for $\omega_0, a_{1,0}, b_{1,0}$ and $c_{1,0}$

$$\begin{aligned} &\{p = \mp 1,\ \omega_0 = 1,\ a_{1,0} = 2,\ b_{1,0} = 0,\ c_{1,0} = 2\}, \\ &\{p = \mp 1,\ \omega_0 = 1,\ a_{1,0} = -2,\ b_{1,0} = 0,\ c_{1,0} = -2\}, \\ &\{p = \mp 1,\ \omega_0 = \sqrt{1 + 2\varepsilon\mu},\ a_{1,0} = 2,\ b_{1,0} = 0,\ c_{1,0} = -2\}, \\ &\{p = \mp 1,\ \omega_0 = \sqrt{1 + 2\varepsilon\mu},\ a_{1,0} = -2,\ b_{1,0} = 0,\ c_{1,0} = 2\}. \end{aligned} \tag{41}$$

As p appears in the form of p^2 in Eq. (39), it is easy to verify that if $U(t)$ and $V(t)$ are solutions of the systems (39) and (40), then $-U(t)$ and $-V(t)$ are also solutions for the system. Therefore, there are two essentially different solutions in (41) as follows:

$$\{p=1,\ \omega_0=1,\ a_{1,0}=2,\ b_{1,0}=0,\ c_{1,0}=2\},$$
$$\{p=1,\ \omega_0=\sqrt{1+2\varepsilon\mu},\ a_{1,0}=-2,\ b_{1,0}=0,\ c_{1,0}=2\}. \tag{42}$$

Select a solution and input its order, such as 2, then the program further delivers:

the parameter constraint condition: $\{p=1\}$,
the $[1,1]$ homotopy-Padé approximant

$$\omega_{P_{1,1}}=\frac{(\varepsilon^2+64\varepsilon\mu+32)\sqrt{2\varepsilon\mu+1}}{3\varepsilon^2+64\varepsilon\mu+32}, \tag{43}$$

and the $[2,2]$ homotopy-Padé approximant

$$\begin{aligned}\omega_{P_{2,2}}=&\big(16\,640\varepsilon^2+66\,560\varepsilon^3\mu+960\varepsilon^4+294\,912\varepsilon\mu+66\,560\varepsilon^4\mu^2+589\,824\varepsilon^2\mu^2\\&+49\,152+393\,216\varepsilon^3\mu^3+1920\varepsilon^5\mu+9\varepsilon^6\big)\sqrt{2\varepsilon\mu+1}/\big(45\varepsilon^6+393\,216\varepsilon^3\mu^3\\&+589\,824\varepsilon^2\mu^2+78\,848\varepsilon^4\mu^2+78\,848\varepsilon^3\mu+294\,912\varepsilon\mu+3840\varepsilon^5\mu\\&+19\,712\varepsilon^2+49\,152+1920\varepsilon^4\big).\end{aligned} \tag{44}$$

6.5.2. In the case of $\varepsilon=1/2$

From (42), it is known that there is a parameter constraint $p=1$ in each solution. Therefore, letting $\varepsilon=1/2$, there will be only one unknown parameter μ contained in the input system, then the program automatically delivers (by selecting the 2nd solution in (42) and setting *h_order* $=5$) the following results within 590 seconds.

The parameter constraint condition: $\{p=1\}$,
the $[1,1]$ homotopy-Padé approximant

$$\omega_{P_{1,1}}=\frac{(128\mu+129)\sqrt{\mu+1}}{128\mu+131}, \tag{45}$$

and the $[2,2]$ homotopy-Padé approximant

$$\omega_{P_{2,2}}=\frac{(9\,973\,504\mu+9\,703\,424\mu^2+3\,415\,817+3\,145\,728\mu^3)\sqrt{\mu+1}}{3\,468\,845+3\,145\,728\mu^3+9\,752\,576\mu^2+10\,075\,648\mu}. \tag{46}$$

Moreover, the program also provides a figure to compare homotopy-Padé approximants of ω at different orders, as shown in Fig. 10. It indicates that $[1,1]$ and $[2,2]$ homotopy-Padé approximants of ω agree very well.

6.5.3. In the case of $\varepsilon=1/2$, $\mu=8$

In this case, there are no unknown parameters in the input system, for there is the parameter constraint $p=1$ involved in each solution. The program automatically outputs (by setting *h_order* $=$ 7 and *sol_order* $=9$) $\hbar=-0.498$, approximants of ω, a_1, b_1 and c_1, as well as comparison graphs for different order approximations of $U(t)$ and $V(t)$ within 755 seconds. As shown in Fig. 11, the 7th and 9th-order HAM approximations of $U(t)$ and $V(t)$ agree very well. Meanwhile, both agree very well with the corresponding imitating curves of the numerical solutions for the given values $a_1=1.9964, b_1=-0.1234, c_1=-1.9964$ and the absolute error 10^{-7}. Table 9 indicates that the homotopy approximants of the frequency ω and the amplitudes a_1, b_1 and c_1 converge very quickly when setting $\hbar=-0.498$. It is shown from Table 10 that the homotopy-Padé approximants converge even faster than the corresponding homotopy approximants.

To indicate that the solutions obtained from the different solutions in (42) are different, $U(t)$ and $V(t)$ obtained from the 1st and 2nd solutions in (42), respectively, were compared. As shown in Fig. 12, it can be seen that the solutions obtained from different solutions in (42) are indeed different.

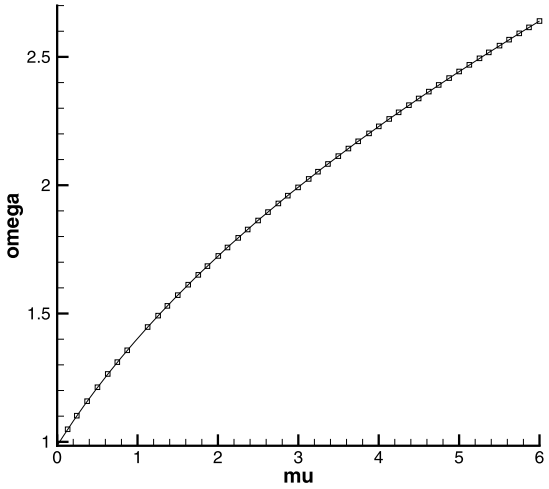


Fig. 10. The homotopy-Padé approximants of ω in Eq. (39) as a function of μ. Solid line: [1, 1] homotopy-Padé approximant; Diamonds: [2, 2] homotopy-Padé approximant.

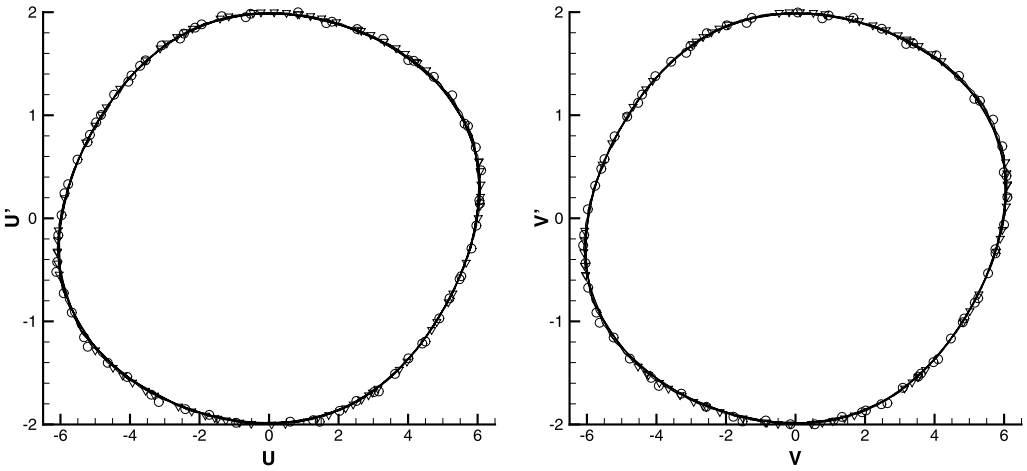


Fig. 11. Approximations of $U(t), V(t)$ for Eq. (39) when $\varepsilon = 1/2, \mu = 8$ by means of $\hbar = -0.498$. Cycles: 7th-order HAM approximation; Gradients: 9th-order HAM approximation; Solid line: numerical solution.

Table 9
The HAM approximants of ω, a_1, b_1, c_1 in Eq. (39) when $\varepsilon = 1/2$, $\mu = 8$ and taking $\hbar = -0.498$.

i	ω_i	$a_{1,i}$	$b_{1,i}$	$c_{1,i}$
1	3.0000	2.0000	−0.1245	−2.0000
2	2.9964	2.0000	−0.1248	−2.0000
3	2.9964	1.9964	−0.1234	−1.9964
4	2.9964	1.9964	−0.1234	−1.9964

Table 10
The HAM-Padé approximants of ω, a_1, b_1, c_1 in Eq. (39) when $\varepsilon = 1/2, \mu = 8$.

i	ω_i	$a_{1,i}$	$b_{1,i}$	$c_{1,i}$
1	2.9964	2.0000	−0.1248	−2.0000
2	2.9964	1.9964	−0.1234	−1.9964

7. Discussions and conclusions

In this paper, based on HAM and Wu's elimination method, an algorithm was proposed to gain accurate analytic approximations of periodical oscillations of dynamic systems with high nonlinearity.

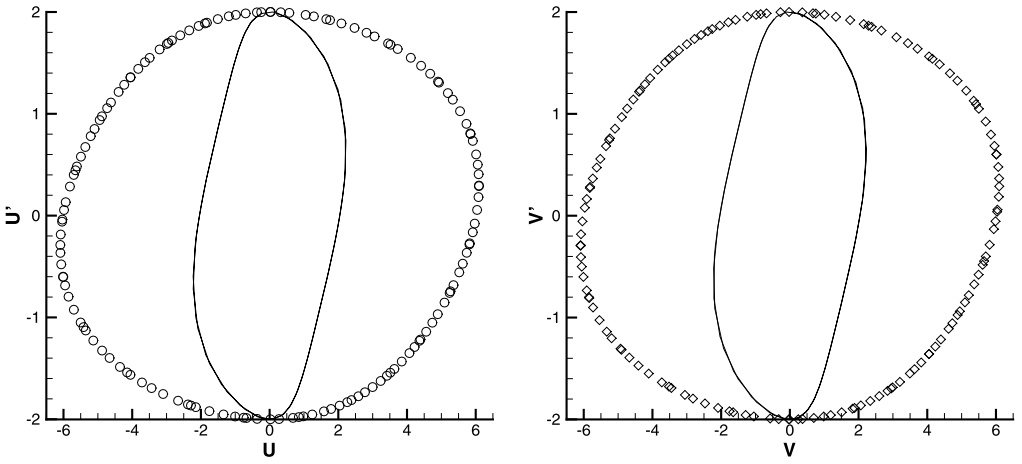


Fig. 12. Different approximations obtained from different solutions in (42) when $\varepsilon = 1/2$, $\mu = 8$.

The package **NOPH** can automatically calculate approximate solutions for various types of nonlinear oscillation equations, as illustrated in this paper. There are two main merits of this algorithm. Firstly, it can gain approximate solutions of highly nonlinear oscillating systems, besides filtering out the corresponding possible constraint conditions on physical parameters. Secondly, it may construct several different solutions simultaneously for the considered system. The package is also user-friendly as one need only input a set of governing equations and initial conditions, and then periodic analytic approximations are returned in a short time. This type of package enables the analysis of periodically nonlinear oscillations in science and engineering through the use of an easy-to-use tool.

Note that f_i in Eq. (1) is a general function with respect to $\mathbf{U}(t), \dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$. However, the current version of the package NOPH only works in cases where f_i is either a polynomial or rational function. Theoretically speaking, f_i could represent more complicated functions, such as a piecewise function, a function involving absolute value function or trigonometric functions, etc. In addition, for n coupled self-excited oscillating systems, n auxiliary parameters $\hbar_i, i = 1, \ldots, n$, can be introduced in the frame of the HAM. In this way, the approximation solution series may converge even faster. In future, these problems will be investigated and further improves to the package NOPH will be made so that the newer versions of NOPH are valid for more complicated periodic oscillations. In addition, forced or parametrically excited oscillators will be investigated by the method of normal forms, to further develop the corresponding automated reasoning program. It should be emphasized that the purpose was to develop an easy-to-use package that is valid for as many periodic oscillations as possible.

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