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Proving isometry for homogeneous Einstein metrics on flag manifolds by symbolic computation

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ABSTRACT

The question whether two Riemannian metrics on a certain manifold are isometric is a fundamental and also a challenging problem in differential geometry. In this paper we ask whether two non-Kähler homogeneous Einstein metrics on a certain flag manifold are isometric. We tackle this question by reformulating it into a related question on a parametric system of polynomial equations and answering it by carefully combining Gröbner bases and geometrical arguments. Using this technique, we are able to prove the isometry of such metrics.

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1. Introduction

One of the important benefits of homogeneous geometry is that various non-linear problems can be reduced to more tractable algebraic ones. Recall that a homogeneous manifold is a smooth manifold M on which a semisimple Lie group G acts transitively. Then M is diffeomorphic to the quotient

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space $M = G/K$, where G is a Lie group and K is the isotropy subgroup of a fixed point of M. A Riemannian homogeneous manifold is a homogeneous space $M = G/K$ equipped with a G-invariant Riemannian metric g. If $\mathfrak{g}$, $\mathfrak{k}$ are the Lie algebras of the Lie groups G and K respectively, then the geometry of the space $M = G/K$ is essentially determined by the pair $(\mathfrak{g}, \mathfrak{k})$. The idea goes back to F. Klein and his Erlangen program.

A Riemannian manifold (M, g) is called *Einstein* if the Ricci tensor Ric_g of g satisfies the equation $\mathrm{Ric}_g = e \cdot g$, for some $e \in \mathbb{R}$. The equation is a difficult PDE equation and has its origins in general relativity. Finding Einstein metrics on a Riemannian or pseudo Riemannian manifold is a central problem in global differential geometry. If M is compact then Einstein metrics of volume 1 can be characterized variationally as the critical points of the scalar curvature functional $T(g) = \int_M S_g \,\mathrm{dvol}_g$ on the space $\mathcal{M}_1$ of Riemannian metrics of volume 1. Also, for the case of a compact homogeneous space $M = G/K$ a G-invariant Einstein metric is precisely a critical point of T restricted to the set of G-invariant metrics of volume 1.

As a consequence, the Einstein equation reduces to a system of non-linear algebraic equations which, even though still complicated, it is more manageable, and sometimes it can be solved explicitly. Thus, most known examples of Einstein manifolds are homogeneous.

A generalized flag manifold is an adjoint orbit of a compact semisimple Lie group G, or equivalently a compact homogeneous space of the form $M = G/K = G/C(S)$, where $C(S)$ is the centralizer of a torus S in G. We refer to Alekseevsky and Perelomov (1986) and Arvanitoyeorgos and Chrysikos (2010) and references therein for more information on flag manifolds. Einstein metrics on generalized flag manifolds have been studied by several authors (cf. Arvanitoyeorgos and Chrysikos, 2010 for a comprehensive review).

Even though the problem of finding all invariant Einstein metrics on M can be facilitated by use of certain theoretical results (e.g. the work Graev, 2006 on the total number of G-invariant complex Einstein metrics) it remains a difficult one, especially when the number of isotropy summands increases. This difficulty increases when we pass from exceptional flag manifolds to classical flag manifolds, because in the later case the Einstein equation reduces to parametric algebraic systems which are difficult to handle.

There is no standard method to solve such algebraic systems. However, there are previous successful attempts on finding Einstein metrics on the generalized flag manifolds $SO(2n)/(U(p) \times U(n-p))$ (Arvanitoyeorgos and Chrysikos, 2010; Arvanitoyeorgos et al., 2010) and $Sp(n)/(U(p) \times U(n-p))$ (Arvanitoyeorgos et al., 2011). In particular for the latter space, by combining the works Arvanitoyeorgos and Chrysikos (2010) and Arvanitoyeorgos et al. (2011) we have shown that the generalized flag manifold $Sp(n)/(U(p) \times U(n-p))$ ($n \geqslant 3$ and $1 \leqslant p \leqslant n-1$) admits precisely six $Sp(n)$-invariant Einstein metrics. Four of them are Kähler metrics, and two are non-Kähler.

An additional important geometrical question is whether two or more Einstein metrics found are isometric or not. It is interesting to note that even though proving that two invariant Einstein metrics are not isometric is relatively easy (e.g. by computing appropriate scalar invariants – see Arvanitoyeorgos and Chrysikos, 2010), showing that two metrics are isometric is not easy in general. In the present paper we are able to show the following:

Theorem 1 *(Main result). The two non-Kähler–Einstein metrics on the generalized flag manifold $Sp(n)/(U(p) \times U(n-p))$ ($n \geqslant 3$ and $1 \leqslant p \leqslant n-1$) are isometric.*

The proof of the above theorem reduces to an algebraic claim (Theorem 2) as follows. The Einstein condition reduces to a parametric system of four polynomial equations with four unknowns x_1, x_2, x_3, x_4 (cf. Besse, 2008, p. 263). Due to previous works we know the number of positive solutions of this system, and which of these correspond to Kähler or non-Kähler–Einstein metrics. We show that the non-Kähler–Einstein metrics are related by an element of the Weyl group of $Sp(n)$, that is the group of all permutations and sign changes on n variables (cf. Samelson, 1990, p. 76).

The proof of this algebraic claim is based on an analysis of a parametric system of algebraic equations based on computations of Gröbner bases (Cox et al., 2007) where emphasis is given on the choice of lexicographic order for the variables.

2. Algebraic reformulation

In order to be able to formulate the parametric system of equations as well as our algebraic claim, we need some preliminary information. Let $M = G/K$ be a homogeneous space with $o = eK$ the identity coset of G/K. Let B denote the Killing form of $\mathfrak{g}$. Assume that G is compact and simple so $-B$ is a positive definite inner product on $\mathfrak{g}$. With respect to $-B$ we consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This is a reductive decomposition of $\mathfrak{g}$, that is $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$, and as usual we identify the tangent space T_oM with $\mathfrak{m}$.

Let $\chi : K \to \mathrm{Aut}(T_oM)$ be the isotropy representation of K on T_oM. Since χ is equivalent to the adjoint representation of K restricted on $\mathfrak{m}$, the set of all G-invariant symmetric covariant 2-tensors on G/K can be identified with the set of all $\mathrm{Ad}(K)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant metrics on G/K is identified with the set of $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$.

Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ be a $(-B)$-orthogonal $\mathrm{Ad}(K)$-invariant decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $\mathrm{Ad}(K)$-modules $\mathfrak{m}_i$ $(i = 1, \ldots, s)$. Then, a G-invariant Riemannian metric on M (or equivalently, an $\mathrm{Ad}(K)$-invariant inner product $\langle\, ,\rangle$ on $\mathfrak{m} = T_oM$) is given by

$$g = \langle\, ,\rangle = x_1 \cdot (-B)|_{\mathfrak{m}_1} + \cdots + x_s \cdot (-B)|_{\mathfrak{m}_s}, \tag{1}$$

where $(x_1, \ldots, x_s) \in \mathbb{R}^s_+$. Since $\mathfrak{m}_i \neq \mathfrak{m}_j$ as $\mathrm{Ad}(K)$-representations, any G-invariant metric on M has the above form.

Similarly, the Ricci tensor Ric_g of a G-invariant metric g on M, as a symmetric covariant 2-tensor on G/K is given by

$$\mathrm{Ric}_g = r_1 x_1 \cdot (-B)|_{\mathfrak{m}_1} + \cdots + r_s x_s \cdot (-B)|_{\mathfrak{m}_s},$$

where $r_1, \ldots, r_s$ are the components of the Ricci tensor on each $\mathfrak{m}_i$, that is $\mathrm{Ric}_g|_{\mathfrak{m}_i} = r_i x_i \cdot (-B)|_{\mathfrak{m}_i}$. These components have a useful description in terms of the structure constants c_{ij}^k first introduced in Wang and Ziller (1986). Let $\{X_\alpha\}$ be a $(-B)$-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, that is $X_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $X_\alpha \in \mathfrak{m}_i$ and $X_\beta \in \mathfrak{m}_j$). Set $A_{\alpha\beta}^\gamma = B([X_\alpha, X_\beta], X_\gamma)$ so that $[X_\alpha, X_\beta]_{\mathfrak{m}} = \sum_\gamma A_{\alpha\beta}^\gamma X_\gamma$, and $c_{ij}^k = \sum (A_{\alpha\beta}^\gamma)^2$, where the sum is taken over all indices α, β, γ with $X_\alpha \in \mathfrak{m}_i$, $X_\beta \in \mathfrak{m}_j$, $X_\gamma \in \mathfrak{m}_k$ (where $[\, ,]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component). Then c_{ij}^k is non-negative, symmetric in all three entries, and independent of the $(-B)$-orthonormal bases chosen for $\mathfrak{m}_i$, $\mathfrak{m}_j$ and $\mathfrak{m}_k$ (but it depends on the choice of the decomposition of $\mathfrak{m}$).

Proposition 1. *(See Park and Sakane, 1997.) Let $M = G/K$ be a homogeneous space of a compact simple Lie group G and let $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $\mathrm{Ad}(K)$-submodules. Then the components $r_1, \ldots, r_s$ of the Ricci tensor of a G-invariant metric (1) on M are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{i,j} \frac{x_k}{x_i x_j} c_{ij}^k - \frac{1}{2d_k} \sum_{i,j} \frac{x_j}{x_k x_i} c_{ki}^j \quad (k = 1, \ldots, s),$$

where $d_k = \dim \mathfrak{m}_k$.

In view of Proposition 1, a G-invariant metric $g = (x_1, \ldots, x_s) \in \mathbb{R}^s_+$ on M, is an Einstein metric with Einstein constant e, if and only if it is a positive real solution of the system

$$\frac{1}{2x_k} + \frac{1}{4d_k} \sum_{i,j} \frac{x_k}{x_i x_j} c_{ij}^k - \frac{1}{2d_k} \sum_{i,j} \frac{x_j}{x_k x_i} c_{ki}^j = e, \quad 1 \leqslant k \leqslant s.$$

We now turn our attention to the generalized flag manifold $M = Sp(n)/(U(p) \times U(n-p))$ $(n \geqslant 3,$ $1 \leqslant p \leqslant n-1)$ obtained in Arvanitoyeorgos and Chrysikos (2010).

The isotropy representation of M decomposes into a direct sum $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$, which gives rise to a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ of $\mathfrak{m} = T_oM$ into four irreducible inequivalent

$\mathrm{ad}(\mathfrak{k})$-submodules. The dimensions $d_i = \dim \mathfrak{m}_i$ $(i = 1, 2, 3, 4)$ of these submodules can be obtained by use of Weyl's formula (Arvanitoyeorgos and Chrysikos, 2010, pp. 204–205, p. 210) and are given by

$$d_1 = 2p(n-p), \qquad d_2 = (n-p)(n-p+1), \qquad d_3 = 2p(n-p), \qquad d_4 = p(p+1). \tag{2}$$

According to (1) a G-invariant metric on $M = G/K$ is given by

$$\langle\ ,\ \rangle = x_1 \cdot (-B)|_{\mathfrak{m}_1} + x_2 \cdot (-B)|_{\mathfrak{m}_2} + x_3 \cdot (-B)|_{\mathfrak{m}_3} + x_4 \cdot (-B)|_{\mathfrak{m}_4}, \tag{3}$$

for positive real numbers x_1, x_2, x_3, x_4. We will denote such metrics by $g = (x_1, x_2, x_3, x_4)$.

It is known (Nishiyama, 1984) that if $n \neq 2p$ then M admits two non-equivalent G-invariant complex structures J_1, J_2, and thus two non-isometric Kähler–Einstein metrics which are given (up to scale) by (see also Arvanitoyeorgos and Chrysikos, 2010, Theorem 3)

$$\begin{aligned}
g_1 &= (n/2, n+p+1, n/2+p+1, p+1),\\
g_2 &= (n/2, n-p+1, 3n/2-p+1, 2n-p+1),\\
g_3 &= (n/2+p+1, n+p+1, n/2, p+1),\\
g_4 &= (3n/2-p+1, n-p+1, n/2, 2n-p+1).
\end{aligned} \tag{4}$$

If $n = 2p$ then M admits a unique G-invariant complex structure with corresponding Kähler–Einstein metric (up to scale) $g = (p, p+1, 2p+1, 3p+1)$ (cf. also Arvanitoyeorgos and Chrysikos, 2010, Theorem 10 where all isometric Kähler–Einstein metrics are listed).

The Ricci tensor of M is given as follows:

Proposition 2. *(See Arvanitoyeorgos and Chrysikos, 2010.) The components r_i of the Ricci tensor for a G-invariant Riemannian metric on M determined by (3) are given by*

$$\left.\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{c_{12}^3}{2d_1}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + \frac{c_{13}^4}{2d_1}\left(\frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3} - \frac{x_3}{x_1x_4}\right),\\
r_2 &= \frac{1}{2x_2} + \frac{c_{12}^3}{2d_2}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\right),\\
r_3 &= \frac{1}{2x_3} + \frac{c_{12}^3}{2d_3}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\right) + \frac{c_{13}^4}{2d_3}\left(\frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4}\right),\\
r_4 &= \frac{1}{2x_4} + \frac{c_{13}^4}{2d_4}\left(\frac{x_4}{x_1x_3} - \frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4}\right).
\end{aligned}\right\} \tag{5}$$

By taking into account the explicit form of the Kähler–Einstein metrics (4) and substituting these in (5) we find that the values of the triples c_{ij}^k are given by

$$c_{12}^3 = \frac{p(n-p)(n-p+1)}{2(n+1)}, \qquad c_{13}^4 = \frac{p(p+1)(n-p)}{2(n+1)}. \tag{6}$$

A G-invariant metric $g = (x_1, x_2, x_3, x_4)$ on $M = G/K$ is Einstein if and only if there is a positive constant e such that $r_1 = r_2 = r_3 = r_4 = e$ or equivalently

$$r_1 - r_3 = 0, \qquad r_1 - r_2 = 0, \qquad r_3 - r_4 = 0. \tag{7}$$

In Arvanitoyeorgos et al. (2011, pp. S22–S27) it was shown that if $x_1 = x_3$ then system (7) has no real solutions and if $x_1 \neq x_3$ it has exactly six positive solutions. Four of these solutions are given by (4).

In the rest of the paper we assume that $x_1 \neq x_3$.

Since the Ricci tensor is invariant under homotheties we normalize our equations by letting the Einstein constant $e = 1$ and consider in addition the equation $r_1 = 1$. Thus the system of equations $r_1 = r_2 = r_3 = r_4 = 1$ is equivalent to the following system:

$$\begin{cases} x_1x_4(n-p+1)+x_3x_4(n-p+1)-2(n+1)x_2x_4+(p+1)x_1x_2+(p+1)x_2x_3=0, \\ 4(n+1)x_3x_4(x_2-x_1)+(n+p+1)x_4\left(x_1^2-x_2^2\right)-(n-3p+1)x_3^2x_4 \\ \quad +(p+1)x_2\left(x_1^2-x_3^2-x_4^2\right)=0, \\ 4(n+1)x_1x_2(x_4-x_3)+(2n-p+1)x_2\left(x_3^2-x_4^2\right)+(2n-3p-1)x_1^2x_2 \\ \quad +(n-p+1)x_4\left(x_3^2-x_1^2-x_2^2\right)=0, \\ x_1^2x_4(n-p+1)-x_2^2x_4(n-p+1)-x_3^2x_4(n-p+1)-8(n+1)x_1x_2x_3x_4 \\ \quad +4(n+1)x_2x_3x_4+(p+1)x_1^2x_2-(p+1)x_2x_3^2-(p+1)x_2x_4^2=0. \end{cases} \tag{8}$$

The problem we need to solve, and which answers our original geometrical question, can be formulated as follows:

Theorem 2 *(Algebraic reformulation of main result). Let* **s** *be the system of equations $r_1=r_2=r_3=r_4=1$, where r_i $(i=1,2,3,4)$ are defined by (5), (2) and (6). Let $n\geqslant 3$ and $1\leqslant p\leqslant n-1$. Then the positive real solutions of* **s** *other than the solutions (4) have the form $x_1=\gamma$, $x_2=\beta$, $x_3=\delta$, $x_4=\alpha$ and $x_1=\delta$, $x_2=\beta$, $x_3=\gamma$, $x_4=\alpha$, for appropriate positive numbers α, β, γ, δ.*

3. Proof of the algebraic reformulation

An attempt to solve such a system by *Mathematica* or *Maple* will cause difficulties of various sorts. This is a parametric system, computations run too long and the expression of the solutions is large, even the partial results such as $p=2$ take four pages long in the Mathematica file. Note that the parameters vary in open and closed intervals ($n\geqslant 3$ and $1\leqslant p\leqslant n-1$) and this requires the system to be studied by considering cases. Also, since we have to investigate whether the system has positive real solutions or not, it is still hard to show the positiveness. That is why we also use some additional geometrical arguments. Finally, since we need elements of Gröbner bases with lower degree (in our case it turns to be is 2), we have to choose the suitable lex order for the variables x_1, x_2, x_3, x_4.

Our analysis is divided into three cases: $2p-n<0$, $2p-n>0$ and $2p=n$.

Case 1. $2p-n<0$.

We compute a Gröbner basis for Eqs. (8) using lex order with $x_3>x_1>x_2>x_4$ for $x_1x_2x_3x_4\neq 0$ and an element of this Gröbner basis is

$$\begin{aligned} &\big(-2(n+1)x_4+p+1\big)\big(2(n+1)x_4-2n+p-1\big)\big(4n^2(n+1)^3(2n-p+1)x_4^2 \\ &\quad +2n(n+1)\big(n^4-2n^3p+3n^3-9n^2p^2-15n^2p+2n^2+10np^3-14np \\ &\quad -2p^4+6p^3+6p^2-2p\big)x_4+(p+1)(2p-n) \\ &\quad \times\big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)\big)=0. \end{aligned} \tag{9}$$

If $-2(n+1)x_4+p+1=0$ then we compute a Gröbner basis for Eqs. (8) and $-2(n+1)x_4+p+1=0$ using lex order with $x_3>x_1>x_4>x_2$ for $x_1x_2x_3x_4\neq 0$ and $(n-p)(1+n-p)\neq 0$. Then we obtain the system of equations

$$\begin{aligned} &-1-n-p+2(n+1)x_2=0, \qquad 2(n+1)x_4-(p+1)=0, \\ &1+p-2(p+1)x_2+2px_4=0, \qquad (2x_1-x_2-x_4)(2x_1-x_2+x_4)=0, \\ &x_1-x_2+x_3=0. \end{aligned}$$

Therefore we get the solutions

$$x_2=\frac{n+p+1}{2(n+1)}, \qquad x_4=\frac{p+1}{2(n+1)}, \qquad x_1=\frac{n+2p+2}{4(n+1)}, \qquad x_3=\frac{n}{4(n+1)}$$

and

$$x_2=\frac{n+p+1}{2(n+1)},\qquad x_4=\frac{p+1}{2(n+1)},\qquad x_1=\frac{n}{4(n+1)},\qquad x_3=\frac{n+2p+2}{4(n+1)},$$

which are (up to scale) solutions g_3, g_1 in (4).

Similarly, if $2(n+1)x_4-2n+p-1=0$ then we obtain the solutions

$$x_2=\frac{n-p+1}{2(n+1)},\qquad x_4=\frac{2n-p+1}{2(n+1)},\qquad x_1=\frac{3n-2p+2}{4(n+1)},\qquad x_3=\frac{n}{4(n+1)}$$

and

$$x_2=\frac{n-p+1}{2(n+1)},\qquad x_4=\frac{2n-p+1}{2(n+1)},\qquad x_1=\frac{n}{4(n+1)},\qquad x_3=\frac{3n-2p+2}{4(n+1)},$$

which are (up to scale) solutions g_4, g_2 in (4).

We now consider the case that $(-2(n+1)x_4+p+1)(2(n+1)x_4-2n+p-1)\neq 0$.

We set

$$A=n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2. \tag{10}$$

Claim 1. *The quadratic equation of x_4 in (9) has a negative root and a positive root.*

Indeed, we note that the top coefficient of this equation is $4n^2(n+1)^3(2n-p+1)>0$ and that

$$A=4np\big(n^2-2p^2\big)+9n\big(p(n-p)\big)+n^4+3n^3+3n^2+4np+n+4\big(p^2-1\big)p^2>0,$$

and this shows Claim 1.

Now we compute a Gröbner basis for Eqs. (8) including the quadratic equation in (9) using lex order with $x_3>x_1>x_2>x_4$ for $x_1x_2x_3x_4\neq 0$. Then we get the following equations:

$$\begin{aligned}
&4n^2(n+1)^3(2n-p+1)x_4^2+2n(n+1)\big(n^4-2n^3p+3n^3-9n^2p^2-15n^2p+2n^2\\
&\quad+10np^3-14np-2p^4+6p^3+6p^2-2p\big)x_4+(p+1)(2p-n)\\
&\quad\times\big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)=0, &&(11)\\
&(2p-n)\big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)\\
&\quad+2n(n+1)(n-p)(n+p+1)^2x_2-2n(n+1)p(2n-p+1)^2x_4=0, &&(12)\\
&-16n(n+1)^2(n-p)(n+p+1)^2x_1^2+8(n+1)(n+p+1)\big(n^4+4n^3p+3n^3\\
&\quad+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)x_1\\
&\quad-16n(n+1)^3(2n-p+1)(n+p+1)x_1x_4+2n(n+1)\big(4n^4+7n^3p+15n^3\\
&\quad-9n^2p^2+15n^2p+20n^2+4np^3-15np^2+8np+11n-2p^4-8p^2+2\big)x_4\\
&\quad+\big(-n^2-5np-3n+2p^2-4p-2\big)\big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3\\
&\quad-9np^2+4np+n+4p^4-4p^2\big)=0, &&(13)\\
&-2(n+1)n(n-p)(n+p+1)(x_1+x_3)-2(n+1)^2n(2n-p+1)x_4\\
&\quad+n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2=0. &&(14)
\end{aligned}$$

For the positive root $x_4=\alpha$ of Eq. (11) (cf. Claim 1) we see that (since $2p-n<0$) the solution $x_2=\beta$ of Eq. (12) is also positive.

Claim 2. *The two solutions of the quadratic equation (13) are positive.*

We set

$$a_2 = -16n(n+1)^2(n-p)(n+p+1)^2, \tag{15}$$

$$a_1 = 8(n+1)(n+p+1)\big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np + n+4p^4-4p^2\big) - 16n(n+1)^3(2n-p+1)(n+p+1)x_4, \tag{16}$$

$$a_0 = 2n(n+1)\big(4n^4+7n^3p+15n^3-9n^2p^2+15n^2p+20n^2+4np^3-15np^2 + 8np+11n-2p^4-8p^2+2\big)x_4 + \big(-n^2-5np-3n+2p^2-4p-2\big) \times \big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big). \tag{17}$$

In order to show this claim it suffices to prove that for $x_4 = \alpha$, the following three inequalities are satisfied:

(i) $a_0 < 0$, (ii) $a_1 > 0$, (iii) $a_1^2 - 4a_2a_0 > 0$.

For the first one, we solve the quadratic equation of x_4 in (9) and obtain that

$$\alpha = \frac{1}{4n(n+1)^2(2n-p+1)}\big(-n^4+2n^3p-3n^3+9n^2p^2+15n^2p-2n^2-10np^3 + 14np+2p^4-6p^3-6p^2+2p+\big(n^2+n-p(p+1)\big)\sqrt{A}\big).$$

Substituting $x_4 = \alpha$ into (17) we see that

$$a_0 = -\frac{(n-p)(n+p+1)}{2(n+1)(2n-p+1)}\big(-\big(4n^4+7n^3p+15n^3-9n^2p^2+15n^2p+20n^2 + 4np^3-15np^2+8np+11n-2p^4-8p^2+2\big)\sqrt{A}+8n^6+33n^5p+49n^5 + 3n^4p^2+121n^4p+118n^4-68n^3p^3-39n^3p^2+156n^3p+143n^3+24n^2p^4 - 164n^2p^3-112n^2p^2+80n^2p+92n^2+12np^5+82np^4-88np^3-80np^2 + 12np+30n-4p^6+44p^4-12p^2+4\big).$$

We set

$$B = 4n^4+7n^3p+15n^3-9n^2p^2+15n^2p+20n^2+4np^3-15np^2 + 8np+11n-2p^4-8p^2+2$$

and

$$C = 8n^6+33n^5p+49n^5+3n^4p^2+121n^4p+118n^4-68n^3p^3-39n^3p^2+156n^3p + 143n^3+24n^2p^4-164n^2p^3-112n^2p^2+80n^2p+92n^2+12np^5+82np^4 - 88np^3-80np^2+12np+30n-4p^6+44p^4-12p^2+4.$$

Then we have that

$$a_0 = -\frac{(n-p)(n+p+1)}{2(n+1)(2n-p+1)}(-B\sqrt{A}+C).$$

Now we see that

$$B = 4(n-p)^4+(23p+15)(n-p)^3+\big(36p^2+60p+20\big)(n-p)^2 + \big(23p^3+60p^2+48p+11\big)(n-p)+4p^4+15p^3+20p^2+11p+2$$

and

$$\begin{aligned}C &= 8(n-p)^6 + (81p+49)(n-p)^5 + \big(288p^2+366p+118\big)(n-p)^4\\ &\quad + \big(434p^3+935p^2+628p+143\big)(n-p)^3\\ &\quad + \big(288p^4+935p^3+1064p^2+509p+92\big)(n-p)^2\\ &\quad + \big(81p^5+366p^4+628p^3+509p^2+196p+30\big)(n-p)\\ &\quad + 8p^6+49p^5+118p^4+143p^3+92p^2+30p+4,\end{aligned}$$

and thus, since $p<n$, we obtain that $B>0$ and $C>0$ for $p<n$. Now we see that

$$\begin{aligned}C^2 - B^2A &= 8(n+1)(3n+4)(2n-p+1)(n+p+1)\\ &\quad \times \big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)^2.\end{aligned}$$

Thus we get inequality (i). For inequality (ii) we substitute $x_4=\alpha$ into (17) and we see that

$$a_1 = 4(n+1)(n-p)(n+p+1)^2\big(-\sqrt{A}+3n^2+6np+6n-6p^2+2\big).$$

We have

$$\begin{aligned}&\big(-\sqrt{A}+3n^2+6np+6n-6p^2+2\big)\big(\sqrt{A}+3n^2+6np+6n-6p^2+2\big)\\ &\quad = 8\big(n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)\\ &\quad = 8\big((n-p)^4+(8p+3)(n-p)^3+\big(18p^2+18p+3\big)(n-p)^2\\ &\qquad + \big(8p^3+18p^2+10p+1\big)(n-p)+\big(p^4+3p^3+3p^2+p\big)\big)>0\end{aligned}$$

for $p<n$. Noting that $3n^2+6np+6n-6p^2+2>0$, we obtain (ii).

Finally, for inequality (iii) we see from (15), (16) and (17) that

$$\begin{aligned}a_1^2-4a_2a_0 &= 64(n+1)^2(n+p+1)^2\big(16n^8x_4^2-16n^7px_4^2-22n^7px_4+80n^7x_4^2-6n^7x_4\\ &\quad +4n^6p^2x_4^2-16n^6p^2x_4+7n^6p^2-72n^6px_4^2-126n^6px_4+8n^6p\\ &\quad +164n^6x_4^2-30n^6x_4+n^6+90n^5p^3x_4+18n^5p^3+16n^5p^2x_4^2+48n^5p^2x_4\\ &\quad +48n^5p^2-128n^5px_4^2-260n^5px_4+34n^5p+176n^5x_4^2-58n^5x_4+4n^5\\ &\quad -76n^4p^4x_4-36n^4p^4+180n^4p^3x_4+13n^4p^3+24n^4p^2x_4^2+224n^4p^2x_4\\ &\quad +98n^4p^2-112n^4px_4^2-246n^4px_4+55n^4p+104n^4x_4^2-54n^4x_4+6n^4\\ &\quad +20n^3p^5x_4-40n^3p^5-156n^3p^4x_4-205n^3p^4+86n^3p^3x_4-143n^3p^3\\ &\quad +16n^3p^2x_4^2+260n^3p^2x_4+57n^3p^2-48n^3px_4^2-106n^3px_4+39n^3p\\ &\quad +32n^3x_4^2-24n^3x_4+4n^3+108n^2p^6+36n^2p^5x_4+198n^2p^5-96n^2p^4x_4\\ &\quad -55n^2p^4-20n^2p^3x_4-160n^2p^3+4n^2p^2x_4^2+116n^2p^2x_4-6n^2p^2\\ &\quad -8n^2px_4^2-16n^2px_4+10n^2p+4n^2x_4^2-4n^2x_4+n^2-72np^7-56np^6\\ &\quad +16np^5x_4+112np^5-16np^4x_4+64np^4-16np^3x_4-40np^3+16np^2x_4\\ &\quad -8np^2+16p^8-32p^6+16p^4\big).\end{aligned}\tag{18}$$

Substituting $x_4=\alpha$ into (18) we obtain that

$$a_1^2-4a_2a_0 = \frac{64(n+1)(n-p)^2(n+p+1)^3}{2n-p+1}\big(P-Q\sqrt{A}\,\big),$$

where

$$\begin{aligned}P ={}& n^7 + 6n^6p + 12n^6 + 6n^5p^2 + 40n^5p + 47n^5 - 16n^4p^3 - 3n^4p^2 + 90n^4p + 89n^4 \\ & - 12n^3p^4 - 64n^3p^3 - 50n^3p^2 + 90n^3p + 92n^3 + 24n^2p^5 + 7n^2p^4 - 80n^2p^3 - 76n^2p^2 \\ & + 40n^2p + 53n^2 - 8np^6 + 30np^5 + 40np^4 - 28np^3 - 40np^2 + 6np + 16n - 10p^6 \\ & + 14p^4 - 6p^2 + 2\end{aligned}$$

and

$$\begin{aligned}Q ={}& n^5 + 4n^4p + 6n^4 + 12n^3p + 13n^3 - 8n^2p^3 - 9n^2p^2 + 12n^2p + 13n^2 + 4np^4 \\ & - 6np^3 - 12np^2 + 4np + 6n + 3p^4 - 4p^2 + 1.\end{aligned}$$

We see that

$$\begin{aligned}P ={}& (n-p)^7 + (13p+12)(n-p)^6 + \big(63p^2 + 112p + 47\big)(n-p)^5 \\ & + \big(139p^3 + 377p^2 + 325p + 89\big)(n-p)^4 \\ & + \big(139p^4 + 564p^3 + 780p^2 + 446p + 92\big)(n-p)^3 \\ & + \big(63p^5 + 377p^4 + 780p^3 + 728p^2 + 316p + 53\big)(n-p)^2 \\ & + \big(13p^6 + 112p^5 + 325p^4 + 446p^3 + 316p^2 + 112p + 16\big)(n-p) \\ & + p^7 + 12p^6 + 47p^5 + 89p^4 + 92p^3 + 53p^2 + 16p + 2 > 0\end{aligned}$$

and

$$\begin{aligned}Q ={}& (n-p)^5 + (9p+6)(n-p)^4 + \big(26p^2 + 36p + 13\big)(n-p)^3 \\ & + \big(26p^3 + 63p^2 + 51p + 13\big)(n-p)^2 + \big(9p^4 + 36p^3 + 51p^2 + 30p + 6\big)(n-p) \\ & + p^5 + 6p^4 + 13p^3 + 13p^2 + 6p + 1 > 0.\end{aligned}$$

We consider $(P - Q\sqrt{A})(P + Q\sqrt{A})$. Then we obtain that

$$\begin{aligned}(P - Q\sqrt{A})(P + Q\sqrt{A}) ={}& 4(n+1)(p+1)(n-p+1)(2n-p+1)(n+p+1) \\ & \times \big(n^4 + 4n^3p + 3n^3 + 9n^2p + 3n^2 - 8np^3 - 9np^2 \\ & + 4np + n + 4p^4 - 4p^2\big)^2 > 0,\end{aligned}$$

and hence we get our claim $a_1^2 - 4a_2a_0 > 0$. This concludes the proof of Claim 2.

Claim 3. *For the solutions $x_4 = \alpha$ and $x_2 = \beta$, the quadratic equation (13) has two solutions which satisfy the system of Eqs. (11), (12), (13) and (14).*

Indeed, we compute a Gröbner basis for Eqs. (8) including the quadratic equation (11) by using other lex order with $x_1 > x_3 > x_2 > x_4$ for $x_1x_2x_3x_4 \neq 0$. This Gröbner basis contains the following quadratic equation of x_3:

$$\begin{aligned}&-16n(n+1)^2(n-p)(n+p+1)^2x_3^2 + 8(n+1)(n+p+1)\big(n^4 + 4n^3p + 3n^3 + 9n^2p + 3n^2 \\ &\quad - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big)x_3 - 16n(n+1)^3(2n-p+1)(n+p+1)x_3x_4 \\ &\quad + 2n(n+1)\big(4n^4 + 7n^3p + 15n^3 - 9n^2p^2 + 15n^2p + 20n^2 + 4np^3 - 15np^2 + 8np + 11n \\ &\quad - 2p^4 - 8p^2 + 2\big)x_4 + \big(-n^2 - 5np - 3n + 2p^2 - 4p - 2\big)\big(n^4 + 4n^3p + 3n^3 + 9n^2p \\ &\quad + 3n^2 - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big) = 0. \qquad (19)\end{aligned}$$

We see that Eq. (19) is the same as the quadratic equation (13) of x_1. Since the solutions x_1 and x_3 are not equal Claim 3 follows.

If we denote the solutions of the quadratic equation (13) by γ, δ, then either $x_1 = \gamma$, $x_3 = \delta$ or $x_1 = \delta$, $x_3 = \gamma$. Therefore, the two positive solutions of system (8) other than the solutions (4) have the form $x_1 = \gamma$, $x_2 = \beta$, $x_3 = \delta$, $x_4 = \alpha$ or $x_1 = \delta$, $x_2 = \beta$, $x_3 = \gamma$, $x_4 = \alpha$, and Theorem 2 has been proven for this case.

Case 2. $2p - n > 0$.

We compute a Gröbner basis for Eqs. (8) using lex order with $x_3 > x_1 > x_4 > x_2$ for $x_1x_2x_3x_4 \neq 0$ and an element in this Gröbner basis is

$$\begin{aligned}
&\big(2(n+1)x_2 - n + p - 1\big)\big(-2(n+1)x_2 + n + p + 1\big)\big(4n^2(n+1)^3(n+p+1)x_2^2 \\
&\quad - 2n(n+1)\big(2n^4 + 2n^3p + 6n^3 - 9n^2p^2 + 3n^2p + 6n^2 + 2np^3 - 18np^2 - 2np \\
&\quad + 2n + 2p^4 + 6p^3 + 6p^2 - 2p\big)x_2 + (n-p+1)(n-2p) \\
&\quad \times \big(n^4 + 4n^3p + 3n^3 + 9n^2p + 3n^2 - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big)\big) = 0. \qquad (20)
\end{aligned}$$

If $2(n+1)x_2 - n + p - 1 = 0$ we obtain as before the solutions

$$x_2 = \frac{n-p+1}{2(n+1)}, \qquad x_4 = \frac{2n-p+1}{2(n+1)}, \qquad x_1 = \frac{3n-2p+2}{4(n+1)}, \qquad x_3 = \frac{n}{4(n+1)}$$

and

$$x_2 = \frac{n-p+1}{2(n+1)}, \qquad x_4 = \frac{2n-p+1}{2(n+1)}, \qquad x_1 = \frac{n}{4(n+1)}, \qquad x_3 = \frac{3n-2p+2}{4(n+1)},$$

which are solutions g_4, g_2 in (4).

Similarly for $-2(n+1)x_2 + n + p + 1 = 0$, we obtain the solutions

$$x_2 = \frac{n+p+1}{2(n+1)}, \qquad x_4 = \frac{p+1}{2(n+1)}, \qquad x_1 = \frac{n+2p+2}{4(n+1)}, \qquad x_3 = \frac{n}{4(n+1)}$$

and

$$x_2 = \frac{n+p+1}{2(n+1)}, \qquad x_4 = \frac{p+1}{2(n+1)}, \qquad x_1 = \frac{n}{4(n+1)}, \qquad x_3 = \frac{n+2p+2}{4(n+1)}$$

which are solutions g_3, g_1 in (4).

We now consider the case $(2(n+1)x_2 - n + p - 1)(-2(n+1)x_2 + n + p + 1) \neq 0$. By the same way as in Case 1 we see that the quadratic equation in (20) has a negative root and a positive root.

Now we compute a Gröbner basis for Eqs. (8) including the quadratic equation in (20) using lex order with $x_3 > x_1 > x_4 > x_2$ for $x_1x_2x_3x_4 \neq 0$. Then we get the following equations:

$$\begin{aligned}
&4n^2(n+1)^3(n+p+1)x_2^2 - 2n(n+1)\big(2n^4 + 2n^3p + 6n^3 - 9n^2p^2 + 3n^2p + 6n^2 \\
&\quad + 2np^3 - 18np^2 - 2np + 2n + 2p^4 + 6p^3 + 6p^2 - 2p\big)x_2 + (n-p+1)(n-2p) \\
&\quad \times \big(n^4 + 4n^3p + 3n^3 + 9n^2p + 3n^2 - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big) = 0, \qquad (21) \\
&(2p-n)\big(n^4 + 4n^3p + 3n^3 + 9n^2p + 3n^2 - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big) \\
&\quad + 2n(n+1)(n-p)(n+p+1)^2x_2 - 2n(n+1)p(2n-p+1)^2x_4 = 0, \qquad (22) \\
&-16n(n+1)^2p(2n-p+1)^2x_1^2 + 8(n+1)(2n-p+1)\big(n^4 + 4n^3p + 3n^3 + 9n^2p + 3n^2 \\
&\quad - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big)x_1 - 16n(n+1)^3(2n-p+1)(n+p+1)x_1x_2 \\
&\quad + 2n(n+1)\big(4n^4 + 7n^3p + 15n^3 - 9n^2p^2 + 15n^2p + 20n^2 + 4np^3 - 15np^2 + 8np \\
&\quad + 11n - 2p^4 - 8p^2 + 2\big)x_2 + \big(-4n^2 + np - 7n + 2p^2 + 4p - 2\big)\big(n^4 + 4n^3p + 3n^3 \\
&\quad + 9n^2p + 3n^2 - 8np^3 - 9np^2 + 4np + n + 4p^4 - 4p^2\big) = 0, \qquad (23)
\end{aligned}$$

$$-2(n+1)np(2n-p+1)(x_1+x_3)-2(n+1)^2n(n+p+1)x_2$$
$$+n^4+4n^3p+3n^3+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2=0. \tag{24}$$

For the positive root $x_2=\alpha$ of the quadratic equation (21) we see that the solution $x_4=\beta$ of Eq. (22) is also positive.

Claim 4. *For the solution $x_4=\alpha$ and $x_2=\beta$, the quadratic equation (23) gives solutions for x_1 and x_3 of the system of Eqs. (21), (22), (23) and (24).*

Indeed, we compute a Gröbner basis for Eqs. (8) including the quadratic equation (21) by using other lex order with $x_1>x_3>x_4>x_2$ for $x_1x_2x_3x_4\neq 0$. Then we obtain the following quadratic equation of x_3:

$$\begin{aligned}&-16n(n+1)^2p(2n-p+1)^2x_3^2+8(n+1)(2n-p+1)\big(n^4+4n^3p+3n^3+9n^2p+3n^2\\&\quad-8np^3-9np^2+4np+n+4p^4-4p^2\big)x_3-16n(n+1)^3(2n-p+1)(n+p+1)x_3x_2\\&\quad+2n(n+1)\big(4n^4+7n^3p+15n^3-9n^2p^2+15n^2p+20n^2+4np^3-15np^2+8np\\&\quad+11n-2p^4-8p^2+2\big)x_2+\big(-4n^2+np-7n+2p^2+4p-2\big)\big(n^4+4n^3p+3n^3\\&\quad+9n^2p+3n^2-8np^3-9np^2+4np+n+4p^4-4p^2\big)=0.\end{aligned} \tag{25}$$

We see that Eq. (25) is the same as the quadratic equation (23) of x_1. Since the solutions x_1 and x_3 are not equal, Claim 4 follows.

By the same method as in Case 1 we can show that two solutions of the quadratic equation (23) are positive. If we denote the solutions of the quadratic equation (23) by γ, δ, then either $x_1=\gamma$, $x_3=\delta$ or $x_1=\delta$, $x_3=\gamma$. Therefore, the two positive solutions of system (8) other than the solutions (4) have the form $x_1=\gamma$, $x_2=\beta$, $x_3=\delta$, $x_4=\alpha$ or $x_1=\delta$, $x_2=\beta$, $x_3=\gamma$, $x_4=\alpha$, and this proves Theorem 2 for this case.

Case 3. $n=2p$.

We compute a Gröbner basis for Eqs. (8) using lex order with $x_3>x_1>x_4>x_2$ for $x_1x_2x_3x_4\neq 0$ and this basis contains the element

$$\begin{aligned}&(2p+1)\big(2(2p+1)x_2-3p-1\big)\big(2(2p+1)x_2-p-1\big)\\&\quad\times\big(2(2p+1)^2x_2-(p+1)(3p+1)\big)=0.\end{aligned} \tag{26}$$

If $(2(2p+1)x_2-3p-1)=0$ or $(2(2p+1)x_2-p-1)=0$ then we obtain solutions (4) as before.

We now consider the case $(2(2p+1)x_2-3p-1)(2(2p+1)x_2-p-1)\neq 0$. We compute a Gröbner basis for Eqs. (8) and the equation $(2(2p+1)^2x_2-(p+1)(3p+1))=0$ using lex order with $x_3>x_1>x_4>x_2$ for $x_1x_2x_3x_4\neq 0$ and $2p+1\neq 0$. Then we get the following equations:

$$2(2p+1)^2x_2-(p+1)(3p+1)=0, \qquad x_4-x_2=0,$$
$$\begin{aligned}&-32px_1x_2+12px_1+16px_2^2+2px_2-3p+8x_1^2\\&\quad-16x_1x_2+4x_1+8x_2^2-2x_2-1=0,\end{aligned} \tag{27}$$
$$-8px_2+3p+2x_1-4x_2+2x_3+1=0. \tag{28}$$

We substitute the value $x_2=\frac{(p+1)(3p+1)}{2(2p+1)^2}$ into Eq. (27) and obtain the quadratic equation

$$8(2p+1)^3x_1^2-4(3p+1)(2p+1)^2x_1+p(3p+1)(3p+2)=0. \tag{29}$$

We solve Eq. (29) and get

$$x_1 = \frac{(2p+1)(3p+1) - \sqrt{(p+1)(2p+1)(3p+1)}}{4(2p+1)^2} \tag{30}$$

and

$$x_1 = \frac{(2p+1)(3p+1) + \sqrt{(p+1)(2p+1)(3p+1)}}{4(2p+1)^2}. \tag{31}$$

We substitute the values $x_2 = \frac{(p+1)(3p+1)}{2(2p+1)^2}$ and (30) into Eq. (28) and get

$$x_3 = \frac{(2p+1)(3p+1) + \sqrt{(p+1)(2p+1)(3p+1)}}{4(2p+1)^2} \tag{32}$$

and from the values $x_2 = \frac{(p+1)(3p+1)}{2(2p+1)^2}$ and (31) we obtain

$$x_3 = \frac{(2p+1)(3p+1) - \sqrt{(p+1)(2p+1)(3p+1)}}{4(2p+1)^2}. \tag{33}$$

Therefore we see that the solutions are of the form $x_1 = \gamma$, $x_2 = \beta$, $x_3 = \delta$, $x_4 = \beta$ or $x_1 = \delta$, $x_2 = \beta$, $x_3 = \gamma$, $x_4 = \beta$, and this concludes the proof of Theorem 2.

4. Discussion

The aim of the present work was to show that the two non-Kähler homogeneous Einstein metrics on the generalized flag manifold $M = Sp(n)/(U(p) \times U(n-p))$ are isometric. Well-known Lie theory provides an interesting interplay between geometry and algebra, so the problem reduces to the study of a parametric algebraic system of equations and in particular into analysing the positive real solutions of this system.

Some positive solutions of this system (corresponding to homogeneous Einstein metrics which are in addition Kähler metrics) are known from standard results in differential geometry and this facilitates our study.

The study of such parametric algebraic systems is achieved by computing Gröbner bases combined with elementary geometrical arguments. The previously known solutions confirm that our computations are right. The programs used are Risa/Asir (Kobe distribution) (http://www.openxm.org) and Wolfram *Mathematica*©8. The use of the particular lexicographic orders chosen is motivated to some degree by the well-known representation of the flag manifold M in terms of "painted Dynkin diagrams", a point that we did not get into at the present work. It is important to note that if we had chosen different lex orders, then Eq. (11) could be of higher degree (degree 4) and not quadratic as we have computed in Arvanitoyeorgos et al. (2011). Then it would be more difficult to confirm that the two non-Kähler homogeneous Einstein metrics are isometric.

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