ratio test indicates the absolute convergence of the series solution (9) for any arbitrarily large closed interval of the variable ξ. Furthermore, the series formed by multiplying each term of equation (9) by the quantities $\cos(x\xi)$ or $\xi^{-1}\cos(x\xi)$ retain the absolutely convergent property.

The convergence of the series solution given by equation (10) may be examined in like fashion; and this series and the series formed by multiplying each term of equation (10) by the quantities $\sin(x\xi)$ and $\xi^{-1}\sin(x\xi)$ are found to be absolutely convergent for arbitrarily large intervals of the variable ξ.

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CLASS NUMBER OF A DEFINITE QUATERNION WITH PRIME DISCRIMINANT*

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1. *Introduction.*—According to Eichler, normal division algebras over the field of rationals which can have class numbers greater than 1 are definite quaternions, i.e., normal division algebras of fourth order ramified at infinity. He also obtained[1] a formula for the class numbers of definite quaternions, which in the special case of prime discriminants takes the following form:

$$h = {}^1/_3\cdot(1 - (-3/p)) + {}^1/_4\cdot(1 - (-4/p)) + {}^1/_{12}\cdot(p - 1).$$

Here p is the finite prime at which the quaternion is ramified. Now Deuring observed[2] that the class number of a definite quaternion ramified at p is equal to the number of birationally distinct elliptic curves of characteristic p having no points of order p. The absolute invariants of such elliptic curves are called "supersingular." If, therefore, we can count the number of supersingular invariants of characteristic p, we will get a new proof of the Eichler formula. Deuring thought that such a direct computation was *nicht leicht*. In this paper we shall show that a direct computation is actually possible. The crucial point is that the so-called Hasse invariant satisfies a differential equation of the Gauss-Legendre type. Thus the Eichler formula is proved purely algebraically and, in fact, in a conceptual way. This seems to be a remarkable case, at least at present, where the class number formula is proved algebraically without using Dirichlet's method.

2. *Hasse Invariant.*—We first note that the case $p = 2$, i.e., the case of Hamilton quaternion, can be treated directly. Therefore, we can assume that p is odd. Let A be an elliptic curve, i.e., an Abelian variety of dimension 1, of characteristic p. Then A is isomorphic to a plane elliptic curve defined by the equation $Y^2 = X(1 - X)(\lambda - X)$ with the point at infinity as neutral element. Here the parameter λ is different from 0, 1, and ∞. Deuring calculated[3] the Hasse invariant of this model, and he got

$$A(\lambda) = (-1)^r \sum_{i=0}^{r} \binom{r}{i}^2 \lambda^i,$$

with $r = {}^1/_2 \cdot (p - 1)$. The absolute invariant of the model is given by

$$j = 2^8 \frac{(1 - \lambda(1 - \lambda))^3}{\lambda^2(1 - \lambda)^2}.$$

Now, as we can see, the polynomial $A(\lambda)$ is a solution of the following differential equation:

$$\lambda(1 - \lambda)d^2K/d\lambda^2 + (1 - 2\lambda)dK/d\lambda - {}^1/_4 \cdot K = 0.$$

In the classical case, this is the differential equation for the periods of the elliptic differential of the first kind of the model considered as normal functions of the parameter λ. Moreover, the regular integral around $\lambda = 0$ is given by a hypergeometric series. In the case of characteristic p, the difference between regular and nonregular solutions disappears, because we have $A(1 - \lambda) = (-1)^r A(\lambda)$. Hence the elliptic differential of the first kind has only one period, and that is $A(\lambda)$ up to an arbitrary differential constant. This version of Hasse invariant has not yet been explicitly remarked. At any rate, since r roots of $A(\lambda)$ are different from 0, 1, and ∞, from the differential equation, it follows that the r roots are all simple.

Now consider the correspondence between the affine j-line and the λ-space, which is an affine line minus 0, 1. The Galois group of the equation is known, i.e., the 6 conjugates of λ over j are the following: λ, $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$, $(\lambda - 1)/\lambda$. Hence, if we exclude the case $p = 3$ temporarily, to j different from 0 and 12^3 correspond 6 different values of λ. In case $j = 0$, we get only 2 values of λ, and these are the roots of the equation $1 - \lambda + \lambda^2 = 0$. In case $j = 12^3$, we get 3 values of λ, and these are -1, ${}^1/_2$, and 2.

3. *Concluding the Proof.*—We know[4] that the elliptic curve has no points of order p if and only if its Hasse invariant is 0, i.e., if and only if λ is one of the r simple roots of $A(\lambda)$. Let Σ be the set of the corresponding supersingular invariants. In the case $p = 3$ we have $\Sigma = \{0\}$. If we exclude this case, we have the following 4 alternatives: (1) Σ does not contain 0, 12^3; (2) Σ contains 0 but not 12^3; (3) Σ contains 12^3 but not 0; (4) Σ contains 0 and 12^3. Since the number of elements of Σ is the class number h of the definite quaternion ramified at p, we get $6h$, $6(h - 1) + 2$, $6(h - 1) + 3$, $6(h - 2) + 2 + 3$ for r according to the above 4 cases.

Then p is congruent modulo 12 to 1, 5, 7, 11. Therefore, the 4 cases are actually possible, and we get

$$h = \begin{cases} {}^1/_{12}\cdot(p-1) & p \equiv 1 \mod 12 \\ {}^1/_{12}\cdot(p-5)+1 & p \equiv 5 \mod 12 \\ {}^1/_{12}\cdot(p-7)+1 & p \equiv 7 \mod 12 \\ {}^1/_{12}\cdot(p-11)+2 & p \equiv 11 \mod 12. \end{cases}$$

If we remark that the number of supersingular invariants for $p = 3$ is 1, we see that we can express these results by the Eichler formula stated in the introduction.

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NOTE TO MY PAPER "ON MEMBRANES AND PLATES,"

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1. We follow the notation of the paper quoted in the title:[1] D is a given simply connected domain, C its boundary curve; u is the minimizing function of the quotient

$$\frac{\int_D\int (\nabla^2 u)^2 d\sigma}{\int_D\int u^2 d\sigma}, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } C; \tag{1}$$

we *assume* that $0 \leqq u \leqq 1$ in D (the part $u \leqq 1$ of the assumption is, of course, immaterial); C_ρ is the set of curves $u = \rho$, $0 \leqq \rho \leqq 1$; $A(\rho) = \pi R^2$ is the area of the domain $\rho \leqq u \leqq 1$;

$$G = |\text{grad } u|, \qquad \int_{C_\rho} G ds = P(\rho), \qquad \int_{C_\rho} G^{-1}(\nabla^2 u)^2 ds = Q(\rho).$$

2. In a recent communication Dr. P. R. Beesack, of McMaster University, has called my attention to the fact that the "symmetrized" function $\bar{u} = f(\rho)$, introduced by the formulas (6_1) and (6_2) of the paper quoted, is *not* eligible for the circular plate $\bar{D}$ of radius R_0, $A(0) = \pi R_0^2$. This is due to the convergence of $A(\rho)$ to 0 as $\rho \to 1$. The argument must be modified as follows.

3. We conclude as in III (4) of the previous paper that

$$[Q(\rho)]^{1/2}|A'(\rho)|^{1/2} \geqq - \int_{C_\rho} G^{-1}\nabla^2 u \, ds;$$