9 TGR homeworks — November 28th, 2018

9.1 Given a system $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ of non-empty subsets of a finite set $X = \{1, 2, \dots, n\}$ (tj. $S_i \subseteq X$).

We call a *transversal* any k-touple $(x_1, x_2, \dots, x_k)$ of element of X if

1. For every $i \neq j$ we have $x_i \neq x_j$.
2. For every i we have $x_i \in S_i$.

Prove or disprove: *The system $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ has a transversal if and only if for every $I \subseteq \{1, 2, \dots, k\}$ we have*

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|.$$

Hint: Use the Hall theorem for a suitable graph.

9.2 Given a bipartite graph G with sides X and Y its and a maximum matching P_{max}. For every edge $e \in P_{max}$, we choose one end vertex of e (and put it into a set A) as follows:

Let $e = \{x, y\} \in P_{max}$

we choose the vertex y (and put it in A) if there exists a path $a_1, e_1, a_2, e_2, \dots, e_{2k+1}, y$, where

- $a_1 \in X$ is free in P_{max},
- for $i > 0$ it holds that $e_{2i-1} \notin P_{max}$, $i > 0$, $e_{2i} \in P_{max}$.

If such a path does not exist we choose (and put to A) the vertex x.

Prove or disprove: *The set A constructed above is a vertex cover of G.*

9.3 Let G be a simlpe without loops and v its vertex. We say that v is *critical* if $\beta_0(G \setminus v) < \beta_0(G)$.

Prove or disprove:

A vertex v is critical in G if and only if it belongs to some vertex cover with the smallest possible number of vertices.